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NEUTRON-DEUTERON SCATTERING AT HIGH ENERGIES

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I. Introduction

The value of the n-n cross section at various energies is of fundamental interest to nuclear physics because of the direct bearing on the question of nuclear forces. Unfortunately it is difficult to obtain direct experimental evidence about this cross section. The method of using two beams of neutrons can not yield results because as yet we do not possess beams that are intense enough for this purpose. Thus all our information about the n-n cross section is limited to that obtained by indirect means. The recent development of 100 Mev neutron beams by use of the Berkeley 184-inch cyclotron permits such an indirect way of determining the n-n cross section at high energies.

The fundamental idea of the Berkeley work may be described as follows: At high energies the n-d cross section should, in first approximation, consist only of the sum of the n-p and n-n cross sections. This is based on the assumption that at high energies the wave length of the incident neutron is short compared to the inter-nuclear distance between the nucleons in the deuteron and that the energy of the incoming neutron is very high compared to ϵ , the binding energy of the deuteron in the ground state. In this approximation the difference between the n-d and n-p should then yield the n-n cross section. Indeed such experiments were carried out at Berkeley by Cook, McMillan, Peterson and Sewell[1]

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[1] L. F. Cook, E. M. McMillan, J. M. Peterson and D. C. Sewell, Phys. Rev. 72, 1264 (1947).

with 90 Mev neutrons. Their results may be summarized as follows:

Substance:	Total Cross Section in barns:
D	0.117 ± 0.006
H	0.083 ± 0.004
Difference (n-n)	0.034

This shows a large discrepancy between the inferred n-n and the measured n-p cross section. In consequence it was thought desirable to examine just how accurate it is to consider the n-d cross section as the sum of the n-n and n-p cross section for 90 Mev neutrons. The present thesis in an attempt to estimate these correction terms.

The correction terms will be due to two causes: finite binding of the deteron in the ground state and interference of the waves scattered from the two particles in the deuteron. In order to see whether these correction terms are negligible let us examine some relevant quantities occurring in the problem. The relative wave length of the incoming neutron, or what may by regarded as more significant, this quantity divided by 2π turns out to be $\lambda\!\!\!^{-} = 0.5 \times 10^{-13}$ cm. On the other hand the average "radius" of the deuteron[2] in the ground state is approximately 4×10^{-13} cm. Thus we see that while $\lambda\!\!\!^{-}$ is

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[2] Estimated by following Bethe and Bacher, Rev. of Mod. Phys., 8, 112 (1936) in setting this radius equal to $1/\alpha$ where $\alpha = \hbar/(M\epsilon)^{\frac{1}{2}}$.

small compared to the average separation in the deuteron, it is by no means negligibly small. The corrections due to the binding energy of the deuteron can thus be expected to be of the order of magnitude of the ratio of ϵ to the neutron energy, i.e., the order of a few per cent. It is difficult to form an off-hand estimate of the correction due to interference. However these rough considerations tend to indicate that it is worthwhile to calculate the correction in more detail.

We shall attempt to set up the n-d scattering problem in such a way that the total n-d cross section divides itself naturally into three separate parts:

1. The scattering of the incoming neutron from the proton bound in the field of the other neutron.
2. The scattering of the incoming neutron from the neutron bound in the field of the proton.
3. The interference term.

Since the energies we wish to consider are reasonably high we shall calculate our cross sections by the Born approximation. While it is realized that at 90 Mev this is far from ideal it should serve to give some idea of the correction.

In order to effect the separation into the scattering from the proton and the neutron it will be well to retain the laboratory system of coordinates as far as the description of the three-particle system is concerned. This does not of course preclude the frequent use of relative coordinates between two particles of the three-particle system.

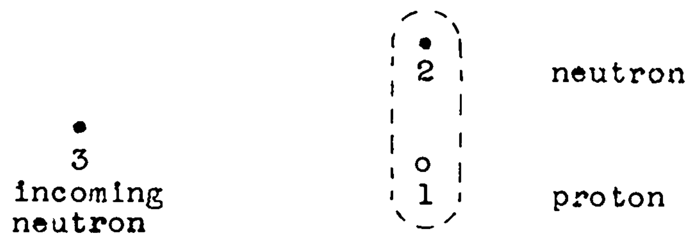
In the next section we shall consider the simplest case, namely the case of Wigner forces, ignoring the effect of the Pauli principle operating between the two neutrons. In the usual calculation with the Born approximation we should expect to represent the final wave functions of the three-particle system as that of three free particles. In our case it will be necessary to consider the final wave function as made up of the product of the final free-particle function of the scattered neutron and the wave function of a very highly-excited deuteron. (Actually it is seen that we really use a modified Hamiltonian in order to work with a free-particle wave function, but this is only a calculational simplification.) The modified picture will insure that we indeed describe the scattering of the incoming neutron from a bound particle, even though the binding after the collision is essentially negligible.

In sections III, IV and V we consider the modifications introduced by more general nuclear forces and also the inclusion of the Pauli principle.

II. Wigner Forces Without Pauli Principle

We shall choose to designate the three particles as indicated in Fig. 1.

Fig. 1



Deuteron

Thus the coordinates of the particles in the laboratory system are designated as r_1, r_2, and r_3 respectively[3]. We shall further introduce relative coordinates between particles 1 and 2, that is let:

$$r = r_1 - r_2 \tag{1}$$

$$R = \tfrac{1}{2}(r_1 + r_2) \tag{2}$$

Further introduce the following momenta in the laboratory system:

Incoming neutron — before collision: p_0
— after collision : p

Deuteron: before collision: zero

after collision — particle 1: p'
— particle 2: p''

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[3] We shall omit to make a distinction between the writing of vectorial and scalar quantities since the particular symbol in question should be clear from the context.

It will be useful to introduce the following combinations of momenta:

$$p_1 = \tfrac{1}{2}(p'-p'') \qquad (3)$$

$$p_2 = p' \quad p'' \qquad (4)$$

These momenta will be recognized as those associated with coordinates r and R respectively.

We define $\psi_i(r,t)$ and $\psi_f(r,t)$ respectively as the initial and final wave function of the three particle system. Further E_f and E_i are total final and initial energies of the system. We shall have no occasion to use E_f in explicit form, but we shall make use of E_f^o; the final energy neglecting the energy of binding between particles 1 and 2. Thus we have

$$E_f^o = \frac{1}{2M}\,(p^2 + p'^2 + p''^2) \qquad (5)$$

$$E_i = \frac{p_o^2}{2M} - \epsilon \qquad (6)$$

Now let us develop our cross section, using the usual time dependent peturbation theory. In this section we shall neglect the Pauli principles; i.e., we shall not antisymmetrize our wave functions. Furthermore we shall neglect treatment of the spin. The nuclear potentials shall be assumed of a straight Wigner type. Now we have the time dependent Schroedinger equation[4] which states that:

$$i\hbar\dot{\psi}(r,t) = (H + V_{nd})\,\psi(r,t) \qquad (7)$$

[4] Of course r is here used to denote a general spacial coordinate and is not the r of equation (1).

Here H is the Hamiltonian corresponding to the kinetic energy of all three particles plus the potential energy between particles 1 and 2. That is

$$H = -\frac{\hbar^2}{2M}\left(\nabla_1^2 + \nabla_2^2 + \nabla_3^2\right) + V_{np}(r_1 - r_2) \qquad (8)$$

The quantity V_{nd} which is made up of the two potential energies $V_{np}(r_1 - r_3)$ and $V_{nn}(r_2 - r_3)$ is regarded as the perturbation in equation (7). Now let

$$\psi(r,t) = e^{-i/\hbar Ht}\phi(r,t) \qquad (9)$$

Then

$$\dot{\phi}(r,t) = -\frac{i}{\hbar} e^{+i/\hbar Ht} V_{nd}\, e^{-i/\hbar Ht}\phi(t) \qquad (10)$$

Thus in first approximation this integral equation (10) has the solution

$$\psi(r,t) = \psi_i(r,t) - \frac{i}{\hbar}\int_0^t e^{-i/\hbar H(t-t')} V_{nd}\, \psi_i(r,t')\,dt' \qquad (11)$$

where

$$\psi_i(r,t) = e^{-i/\hbar E_i t}\phi_i(r) \qquad (12)$$

Now let us ask for the probability of finding the system at the time t in the state "f". This probability is then given by $|b_f|^2$ where

$$b_f = (\psi_f, \psi) \qquad (13)$$

Thus we have from equation (11) and (12) that

$$b_f = e^{i/\hbar (E_f - E_i)t}(\phi_i, \phi_f) - \frac{i}{\hbar}\int_0^t (\phi_f e^{-i/\hbar E_f t},$$
$$e^{-i/\hbar H(t-t')} V_{nd}\, e^{-i/\hbar E_i t'}\phi_i)\,dt' \qquad (14)$$

since ϕ_2 and ϕ_f are orthogonal functions and H is a Hermitian operator we may write:

$$b_f = -\frac{i}{\hbar}(\phi_f, V_{nd}\,\phi_i) \int_o^t e^{+i/\hbar(E_f-E_i)t'}\,dt' \tag{15}$$

$$b_f = -\frac{i}{\hbar}(\phi_f, V_{nd}\,\phi_i) \int_o^t e^{+i/\hbar(E_f-E_i)t'}\,dt' \tag{16}$$

(Eq.16 is identical with Eq.15 and was included inadvertantly)

and finally

$$|b_f|^2 = \frac{2\,|(\phi_f\,|V_{nd}|\,\phi_i)|^2}{(E_f - E_i)^2}\,(1-\cos(E_f-E_i)t/\hbar\,) \tag{17}$$

Form this we then develop our cross section in the usual manner[5] and we find that

$$\sigma_{nd}\left(\frac{\mathcal{V}}{p_o/M}\right)\;\frac{2\pi}{\hbar}\int \left|\,(\phi_f\,|V_{nd}|\,\phi_i)\right.^2 \rho_{E_f}\,\delta(E_f-E_i)dE_f \tag{18}$$

where in equation (19) the symbol $\mathcal{V}$ denotes a large volume to which we normalize and ρ_{E_f} as usual denotes the density of states with energy E_f, i.e., in our case

$$\rho_E\, dE_f = \frac{\mathcal{V}^3 d_{\vec{p}}\, d_{\vec{p}'}\, d_{\vec{p}''}}{h^9} \tag{19}$$

Now let us write the δ function in terms of its integral representation and we find that

$$\sigma_{nd} = \frac{\mathcal{V}\;\;M}{\hbar\;\;p_o}\quad T \tag{20}$$

where

$$T = \frac{\mathcal{V}^3}{h^9}\int \left|\;(\phi_f\,|V_{nd}|\;\phi_i)\right|^2 e^{i\lambda\,(E_f-E_i)}\,d\lambda\; d_p d_{p'} d_{p''} \tag{21}$$

Thus we may write in symbolic form that

$$T = \int \sum_f (\phi_i\,|V_{nd}|\;\phi_f)(\phi_f|V_{nd}|\;\phi_i)e^{i\lambda\,(E_f-E_i)}d\lambda \tag{22}$$

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[5] See for instance Heitler, "The quantum theory of radiation", 2nd edition, page 89.

Thus

$$T = \int \sum_f (\phi_i | V_{nd} \, e^{i\lambda H} | \phi_f)(\phi_f | V_{nd} | \phi_i) e^{-i\lambda E_i} \, d\lambda \qquad (23)$$

Now since the ϕ_f are a complete set of functions we may replace them by any other complete set, say ϕ_f^o. Here the ϕ_f^o are eigen functions of the Hamilton operator H_o, where H_o is

$$H_o = - \frac{\hbar^2}{2M} (\nabla_1^2 + \nabla_2^2 + \nabla_3^2) \qquad (24)$$

Thus H becomes

$$H = H_o + V_{np}(r_1 - r_2) \qquad (25)$$

and equation (23) may be written as

$$T = \int \sum_{f,k} (\phi_i | V_{nd} | \phi_i^o)(\phi_k^o | e^{i\lambda H} | \phi_f^o)(\phi_f^o | V_{nd} | \phi_i) \, e^{-i\lambda E_i} \, d\lambda \qquad (26)$$

Now the following theorem is proved in Appendix A:

$$\left(e^{A+B}\right)_{aa'} = \left(e^{A}\right)_{aa'} + \left(B\right)_{aa'} \left(\frac{e^{a} - e^{a'}}{a - a'}\right) \qquad (27)$$

Where B is small compared to A. Now

$$(\phi_k^o | e^{i\lambda H} | \phi_f^o) = (e^{i\lambda H_o + i\lambda V_{12}})_{kf} \qquad (28)$$

where we have abbreviated $V_{np}(r_1 - r_2)$ by V_{12}

We know that V_{12} is small compared to H_o since we assume a high momentum for the incoming neutron. Then by (27) we have

$$(\phi_k^o | e^{i\lambda H} | \phi_f^o) = (\phi_k^o | e^{i\lambda H_o} | \phi_f^o) + (\phi_k^o | i\lambda V_{12} | \phi_f^o) \frac{e^{i\lambda E_k^o} - e^{i\lambda E_f^o}}{i\lambda (E_k^o - E_f^o)} \qquad (29)$$

However the contribution of the last term to (29) comes only from those portions where $E_k^{\circ} \sim E_f^{\circ}$ and thus

$$(\phi_k^{\circ} \,|\, e^{i\lambda H} \,|\, \phi_f^{\circ}) = (\phi_k^{\circ} \,|\, e^{i\lambda H_0}(1 \;\; i\lambda \; V_{12}) \,|\, \phi_f^{\circ}) \quad (30)$$

Now then

$$T = \int \sum_{i} (\phi_i \,|\, V_{nd} \,|\, \phi_f^{\circ})(\phi_f^{\circ} \,|\, (1 + i\lambda V_{12}) V_{nd} \,|\, \phi_i) e^{i\lambda (E_f^{\circ} - E_i)} \quad (31)$$

By assumption, which we made at the beginning of this section, we may permute V_{12} and V_{nd} since neither of them involve space or spin operators. Further

$$V_{12}\, \phi_i = \left(-\epsilon - \frac{\hbar^2 \nabla_{12}^2}{M} \right) \phi_i \quad (32)$$

So that

$$T = \int \sum_{f} (\phi_i \,|\, V_{nd} \,|\, \phi_f^{\circ})(\phi_f^{\circ} \,|\, V_{nd} \,|\, \left\{ 1 - i\lambda \left[\epsilon + \frac{\hbar^2 \nabla_{12}^2}{M} \right] \right\} \phi_i)$$

$$e^{i\lambda (E_f^{\circ} - E_i)}\, d\lambda \quad (33)$$

However, the operator ∇_{12}^2 does not commute with V_{nd} and we must examine T when V_{nd} is split up into its separate terms. Recall then that

$$V_{nd}^{\;2} \longrightarrow V_{np}V_{np} + V_{nn}V_{nn} + V_{nn}V_{np} + V_{np}V_{nn} \quad (34)$$

and split up T and σ_{nd} correspondingly into

$$T = T_A + T_B + T_{C,A} + T_{C,B} \quad (35)$$

and

$$\sigma_{nd} = \sigma_A + \sigma_B + \sigma_{C,A} + \sigma_{C,B} \quad (36)$$

Now let us concentrate on σ_A. From equation (33) we see that this involves the matrix element

$$L_A = (\phi_f^{\circ} \left| V_{np} \right| (1 - i \lambda \left[\epsilon + \frac{\hbar^2 \nabla_{12}^2}{M} \right]) \ \phi_i) \tag{37}$$

For the evaluation of (37) we now note that L_A can be written in terms of the coordinates r_1, r_3 and $(r_1 - r_2) = r$ as

$$L_A = \frac{1}{V^{3/2}} \int e^{-i/\hbar\,(p_2 \cdot r_1' + p \cdot r_3') + i/\hbar\, p'' \cdot r'}\, V_{np}(r_1' - r_3')\, e^{+i/\hbar\, p_0 \cdot r_3'}$$
$$\left[1 - i\lambda \left(\epsilon + \frac{\hbar^2 \nabla_{12}^2}{M} \right) \right] \chi(r')\, dr_1'\, dr_3'\, dr' \tag{38}$$

Performing partial integration with respect to the coordinate r' we get

$$L_A = \frac{1}{V^{3/2}} \left[1 - i\lambda \left(\epsilon + \frac{p''^2}{M} \right) \right] \int e^{+i/\hbar\,(p \cdot r_3' + p' \cdot r_1' + p'' \cdot r_2')}$$
$$V_{np}(r_1' - r_3')\, e^{-i/\hbar\, p_0 \cdot r_3'}\, \chi(r_1' - r_2')$$
$$dr_1'\, dr_2'\, dr_3' \tag{39}$$

Since only small values of λ contribute to (31) because this expression is oscillatory for large values of λ we may make the following replacement in (39)

$$1 - i\lambda \left(\epsilon + \frac{p''^2}{M} \right) \longrightarrow e^{-i\lambda \left(\epsilon + \frac{p''^2}{M} \right)} \tag{40}$$

Then T_A becomes in virtue of (31), (39), and (40)

$$T_A = \frac{v^3 l}{h^9 V^5} \int \left| \int e^{-i/\hbar\,(p' \cdot r_1 + p \cdot r_3) + i/\hbar\, p'' \cdot r}\, V_{np}(r_1 - r_3) \right.$$
$$\left. e^{+i/\hbar\,(p_0 \cdot r_3 - p'' \cdot r_1)}\, \chi(r)\, dr\, dr_1\, dr_3 \right|^2 e^{\frac{i\lambda}{2M}(p^2 + p'^2 - p''^2 - p_0^2)}$$
$$d\lambda\ dp\ dp'\ dp'' \tag{41}$$

Denote by $\Phi(p'')$ the momentum transform of $\chi(r)$ i.e., let

$$\Phi(p'') = \frac{1}{h^{3/2}} \int e^{+i/\hbar\, p''\cdot r}\, \chi(r)dr \tag{42}$$

then (41) and (20) may be written as follows:

$$\sigma_A = \frac{M}{\hbar\, h^6 p_o V} \int \left| \int e^{-i/\hbar\,(p\cdot r_3 + p'\cdot r_1)} V_{np}(r_1 - r_3) e^{+i/\hbar\,(p_o\cdot r_3 - p''\cdot r_1)} dr_1\, dr_3 \right|^2 \left| \Phi(p'') \right|^2 e^{\frac{i\lambda}{2M}(p^2 + p'^2 - p''^2 - p_o^2)} d\lambda\, dp\, dp'\, dp'' \tag{43}$$

Now let us call $-p'' = p_d$; then we get

$$\sigma_A = \frac{M}{h^6 p_o} \int \left| \int e^{-i/\hbar\,(p\cdot r_3 + p'\cdot r_1)} V_{np}(r_1 - r_3) e^{+i/\hbar\,(p_o\cdot r_3\ p_d\cdot r_1)} dr_1\, dr_3 \right|^2 \left| \Phi(p_d) \right|^2 e^{\frac{i\lambda}{2M}(p^2 + p'^2 - p_d^2 - p_o^2)} d\lambda\, dp\, dp'\, dp'' \tag{44}$$

On the other hand consider now the collision of a neutron of momentum p_o with a proton of momentum p''. The cross section for this process may be written as

$$\sigma_{np}(p_o - p_d) = \frac{M}{\hbar\, h^6 |p_o - p_d| V} \int \left| \int e^{-i/\hbar\,(p\cdot r_3 + p'\cdot r_1)} V_{np}(r_1 - r_3)\, e^{+i/\hbar\,(p_o\cdot r_3 + p_d\cdot r_1)} dr_1\, dr_3 \right|^2 e^{\frac{i\lambda}{2M}(p^2 + p'^2 - p_o^2 - p_d^2)} d\lambda\, dp\, dp'\, dp'' \tag{45}$$

Hence we may express σ_A as

$$\sigma_A = \int \frac{|p_o - p_d|}{p_o}\ \sigma_{np}(p_o - p_d)\ \left| \Phi(p_d) \right|^2 d\vec{p}_d \tag{46}$$

Note that equation (46) is just what we would expect from physical reasoning. It is the average cross section for a mean of neutrons with relative momentum $p_o - p_d$ relative to the proton with an average relative momentum of p_o. The distribution of p_d is just that of the momentum of the proton in the deuteron as we expect.

The techniques in evaluating σ_B are exactly the same as those used in obtaining equation (46) for σ_A. This is so since we are neglecting spin properties and so there is no inherent difference between particle 1 and 2. Hence we shall not give the detials here of obtaining σ_B, but merely state the result which is

$$\sigma_B = \int \frac{|p_o - p_d|}{p_o} \sigma_{nn}(p_o - p_d) \quad \left|\Phi\,(p_d)\right|^2 d\,\vec{p}_d \tag{47}$$

equation (47) is again the result we would expect from physical reasoning.

Before simplifying (46) and (47) for very high p_o we shall first turn our attention to the cross terms. First note that there is a relation between $\sigma_{C,A}$ and $\sigma_{C,B}$. For this purpose look at T in the form of equation (22). Splitting up $V_{nd} = V_{nn} + V_{np}$ in (22) shows that the two cross terms are just complex conjugates of each other. Or that

$$\sigma_{C,A} + \sigma_{C,B} = 2\mathrm{Re}\,\sigma_{C,A} \tag{48}$$

Thus we shall examine only $J_{C,A}$. We find that

$$T_{C,A} = \int \sum_f (\phi_i | V_{nn} | \phi_f^o) L_A e^{i \lambda (E_f^o - E_i)} d\lambda \qquad (49)$$

where L_A is given by (37). By performing the steps analogous to (38)-(40) we find that

$$T_{C,A} = \int \sum_f K M e^{\frac{i\lambda}{2M}(p^2 + p'^2 - p''^2 - p_o^2)} d\lambda \qquad (50)$$

where

$$K = (\phi_i | V_{nn} | \phi_f^o) \qquad (51)$$

$$M = (\phi_f^o | V_{np} | \phi_i) \qquad (52)$$

The important step in solving the cross-term is that of treating K and M separately at this stage. In other words in K the coordinate r_1' is "extraneous" and we must eliminate it. In M the coordinate r_2 is "extraneous" and must be eliminated. In particular then we may write K and M as follows by making use of equation (42)

$$K = \frac{h^{3/2}}{V^{5/2}} \int e^{-i/\hbar\, p_o \cdot r_3'} V_{nn}(r_2' - r_3') e^{+i/\hbar\, (p \cdot r_3' + [p' + p''] \cdot r_2')} \Phi^*(-p')\, dr_2'\, dr_3' \qquad (53)$$

$$M = \frac{h^{3/2}}{V^{5/2}} \int e^{-i/\hbar\, (p \cdot r_3 + [p' + p''] \cdot r_1)} V_{np}(r_1 - r_3) e^{+i/\hbar\, p_o \cdot r_3} \Phi(+p'')\, dr_1 dr_3 \qquad (54)$$

Now let us introduce the following coordinates

$$\left.\begin{aligned} s_1 &= r_1 - r_3 \\ s_2 &= r_2' - r_3' \\ s_3 &= r_3 - r_3' \\ s_4 &= r_1 \end{aligned}\right\} \qquad \begin{aligned} r_1 &= s_4 \\ r_3 &= s_4 - s_1 \\ r_3' &= s_4 - s_3 - s_1 \\ r_2' &= s_4 - s_3 - s_1 + s_2 \end{aligned} \tag{55}$$

Thus we get

$$T_{C,A} = \frac{V'h^3}{h^9V^5}\int e^{-i/\hbar\, p_o\cdot(s_4-s_3-s_1)}\, V_{nn}(s_2)^{+i/\hbar\, p\cdot(s_4-s_3-s_1)}$$

$$e^{+i/\hbar\,(p'+p'')(s_4-s_3-s_1+s_2)}\, e^{-i/\hbar\, p\cdot(s_4-s_1)}\, e^{-i/\hbar\,(p'+p'')\cdot s_4}$$

$$V_{np}(s_1)\, e^{+i/\hbar\, p_o\cdot(s_4-s_1)}\ \Phi^*(-p'')\ \Phi(+p'')\, e^{\frac{i\lambda}{2M}(p^2+p'^2-p''^2-p_o^2)}$$

$$ds_1\, ds_2\, ds_3\, ds_4\, d\lambda\ dp\, dp'\, dp'' \tag{56}$$

Integrating over s_3 and s_4 we get

$$T_{C,A} = \frac{V^3h^6}{h^9V^4}\int \delta\,(p_o-p-p'-p'')\, e^{+i/\hbar\,(p'+p'')(s_2-s_1)}\, V_{nn}(s_2)\, V_{np}(s_1)$$

$$\Phi^*(-p')\ \Phi\,(+p'')\, e^{\frac{i\lambda}{2M}(p^2+p'^2-p''^2-p_o^2)}\, ds_1\, ds_2\, d\lambda\ d_p\, d_n'\, d_p'' \tag{57}$$

Integrating over d_p we get

$$T_{C,A} = \frac{V^3h^6}{h^9V^4}\int e^{\,i/\hbar\,(p'+p'')(s_2-s_1)}\, V_{nn}(s_2)\ V_{np}(s_1)$$

$$\Phi^*(-p')\ \Phi\,(+p'')\, e^{\frac{i\lambda}{2M}\,2(p'^2-p_o\cdot(p'+p''))}\, ds_1\, ds_2\, d\lambda\, d_p\, d_p'\, d_p'' \tag{58}$$

Now the Φ's mean that only very small values of p' and p" will give a contribution, i.e., to first order we have

$$T_{C,A} = \frac{V^3{}_2 \; Mh^6}{h^9 \, \mathcal{V}^4} \int V_{nn}(s_2) V_{np}(s_1) \; \Phi^*(-p') \; \Phi \; (+p'') \qquad \delta \; (p'^2 - |p_o| \; |p' + p''| \; \mu) \tag{59}$$

where μ is cosine of the angle between p_o and $\vec{p'} + \vec{p''}$. Since $\vec{p'} + \vec{p''}$ has no prefered direction we may average over μ . This yields

$$T_{C,A} = \frac{V^3 \; Mh^6}{h^9 \, \mathcal{V}^4} \int V_{nn}(s_2) V_{np}(s_1) \; \Phi^* \; (-p') \; \Phi \; (+p'') \qquad \delta \; (\mu |p_o| \; |p' + p''| \; -p'^2) d\mu \tag{60}$$

Performing the integration over $d\mu$ we get

$$T_{C,A} = \frac{V^3 \; Mh^6}{h^9 \, \mathcal{V}^4} \int \frac{V_{nn}(s_2) V_{np}(s_1) \; \Phi^*(-p') \; \Phi \; (+p'') \; d\vec{s}_1 \; d\vec{s}_2 \; d\vec{p}' \; d\vec{p}''}{|p_o| \; |p' + p''|} \tag{61}$$

Now one can easily show that

$$\int \frac{e^{i/\hbar \; (p' + p'')} d\vec{r}}{r^2} = \frac{\pi \, h}{|p' + p''|} \tag{62}$$

but

$$\chi(r) = \frac{1}{h^{3/2}} \int e^{\; i/\hbar \;\; p \cdot r} \; \Phi(p) d\vec{p} \tag{63}$$

hence

$$T_{C,A} = \frac{V^3 \; Mh^6}{h^9 p_o \mathcal{V}^4} \left(\frac{1}{\pi h}\right) h^3 \left(\int V_{nn}(s_2) V_{np}(s_1) d\vec{s}_1 \; d\vec{s}_2\right) \int \frac{\chi(r) \;\; \chi^* \; (r) d\vec{r}}{r^2} \tag{64}$$

Now let us assume that V_{nn} and V_{np} are related by a simple numerical constant. In particular let

$$V_{nn} = k_1 \;\; V_{np} \tag{65}$$

we then notice that $\int V_{np} V_{np}$ can be expressed as proportional to $\left.\frac{d\sigma_{np}}{d\Omega}\right|_{\theta=0}$. This is so since if we write $\sigma_{np}(0)$ as shorthand for $\sigma(0) = \left.\frac{d\sigma_{np}}{d\Omega}\right|_{\theta=0}$ then

$$\sigma_{np}(0) = \frac{2\pi M^2}{\hbar\, h^3} \int V_{np}(s_1)\, V_{np}(s_2)\, d\vec{s_1}\, d\vec{s_2} \qquad (66)$$

Notice further what the significance of the last part of (64) is; namely that

$$\int \frac{\chi(r)\,\chi^*(r)\,d\vec{r}}{r^2} = \left(\overline{\frac{1}{r^2}}\right) \qquad (67)$$

where the bar as usual denotes an average. Then we may combine all the foregoing to write

$$\sigma^*_{C,A} = \frac{k_1}{2p_o^2} \left(\frac{h^2}{\pi}\right) \left(\overline{\frac{1}{r_d^2}}\right) \sigma_{np}(0) \qquad (68)$$

which is of the form of a deuteron momentum squared over the incident momentum squared multiplied by a cross section. Later we shall notice that this is typical of the form of the correction terms we are looking for.

Now let us return to the equations for σ_A and σ_B (equation (46) and (47)) and simplify them for the case of high p_o with which we are concerned. Thus expanding equation (46) around p_o we have

$$\sigma_A = \frac{1}{|p_o|} \int \Big\{ |p_o|\, \sigma_{np}(p_o) - \vec{p_d} \cdot \nabla_{p_o} \left(p_o\, \sigma_{np}(p_o)\right)$$
$$+ \frac{1}{2} (p_d \cdot \nabla_{p_o})^2\, |p_o|\, \sigma(p_o) + \ldots \Big\} \left|\Phi(p_d)\right|^2 d\vec{p_d} \qquad (69)$$

Since $\vec{p_d}$ has no preferred direction, the second term of (69) integrates to zero when the angular integration is performed. The other terms yield

$$\sigma_A = \sigma_{np}(p_o) + \overline{p_d^2}\left(\frac{1}{2p_o}\right) - \frac{1}{3}\nabla^2_{p_o}\left(|p_o|\,\sigma_{np}(p_o)\right)+\ldots \tag{70}$$

where $\overline{p_d^2}$ denotes as usual an average of p_d^2. We may simplify this expression by the use of the standard relation concerned with the Laplacian. Thus

$$\nabla^2_{p_o}\left(|p_o|\,\sigma(p_o)\right) = \frac{1}{|p_o|}\frac{d^2}{dp_o^2}\left(p_o^2\,\sigma(p_o)\right) \tag{71}$$

and hence

$$\sigma_A = \sigma_{np}(p_o) + \frac{1}{6}\left(\frac{\overline{p_d^2}}{p_o^2}\right)\frac{d^2}{dp_o^2}\left(p_o^2\,\sigma_{np}(p_o)\right) \tag{72}$$

Now we may re-express the second terms of (72) which we may call δ_{np}. Differentiating out we find

$$\delta_{np} = \frac{1}{6}\left(\frac{\overline{p_d^2}}{p_o^2}\right)\left\{2\sigma_{np}(p_o) + 4p_o\frac{d\,\sigma_{np}(p_o)}{d\,p_o} + p_o^2\frac{d^2\sigma_{np}(p_o)}{d\,p_o^2}\right. \tag{73}$$

Thus we need to consider whether we can re-express $\frac{d\sigma}{dp}$ as an angular derivative at the energy corresponding to p_o. Indeed this can be done. Consider $\frac{d\sigma(\theta)}{d\Omega}$ which is given by

$$\frac{d\sigma(\theta)}{d\Omega} \sim \left|\int V(r)e^{+i/\hbar(\vec{p_o}-\vec{p})\cdot r}d\vec{r}\right|^2_{p=p_o} \tag{74}$$

Thus if for the sake of brevity we call

$$\frac{d\sigma(\theta)}{d\Omega} = \sigma(\theta) \tag{75}$$

and let x denote the cosine of the angle between $\vec{p}$ and $\vec{p}_o$

$$\sigma(x) = f(p_o^2(1-x)) \tag{76}$$

and

$$\sigma = 2\pi \int_{-1}^{1} \sigma(x)dx \tag{77}$$

hence

$$\frac{d\sigma}{dp_o} = \int_{-1}^{1} \frac{d\sigma(x)}{dp_o} dx \tag{78}$$

but

$$\frac{d\sigma(x)}{dp_o} = 2\, p_o\, (1-x)\, f' \tag{79}$$

whereas

$$\frac{d\sigma(x)}{d\,x} = (-1)p_o^2 f' \tag{80}$$

and thus

$$\frac{d\sigma(x)}{dp_o} = \frac{2(x-1)}{p_o} \frac{d\sigma(x)}{dx} \tag{81}$$

so that

$$\frac{d\sigma}{dp_o} = 4\pi \int_{-1}^{+1} \frac{(x-1)}{p_o} \frac{d\sigma(x)}{dx} dx \tag{82}$$

or

$$\frac{d\sigma}{dp_o} = \frac{-2\sigma}{p_o} + \frac{2\pi}{p_o} 4\sigma(\pi) \tag{83}$$

If we substitute (83) into (73) and do out the indicated operations we find that

$$\delta_{np} = \frac{2}{3} \left(\frac{\overline{p_d^2}}{p_o^2}\right) \left\{ 2\pi\sigma_{np}(\pi) + p_o 2\pi \frac{d\sigma_{np}(\pi)}{d\,p_o} \right\} \tag{84}$$

Now wo still have to re-express $\frac{d\,\sigma_{np}(\pi)}{d\,p_o}$; which is easy to do. From equation (76) we have

$$\sigma(\pi) = f(2p_o^2) \tag{85}$$

thus

$$\frac{d\sigma(\pi)}{dp_o} = 4p_o\, f'\,\pi \tag{86}$$

but from (80)

$$\left.\frac{d\,\sigma(x)}{dx}\right|_{x=-1} = -p_o^2\, f'\pi \tag{87}$$

or

$$\frac{d\,\sigma(\pi)}{dp_o} = -\frac{4}{p_o}\left.\frac{d\,\sigma(x)}{dx}\right|_{x=-1} \tag{88}$$

Thus finally

$$\delta_{np} = \frac{2}{3}\left(\overline{\frac{pd^2}{p_o^2}}\right)\left\{2\pi\,\sigma_{np}(\pi) - 2\pi\,.\,4\left.\frac{d\,\sigma_{np}}{dx}\right|_{x=-1}\right\} \tag{89}$$

We may then collect our two alternate expressions for the case of the nd cross section with the assumption of Wigner potentials and neglect the modifications introduced by the Pauli principle. From equation (68) and (72) we get

$$\begin{aligned}\sigma_{nd} = \sigma_{np} + \sigma_{nn} &+ \frac{1}{6}\left(\overline{\frac{pd^2}{p_o^2}}\right)\frac{d^2}{dp_o^2}\,(p_o^2\,\sigma_{np}) \\ &+ \frac{1}{6}\left(\overline{\frac{pd^2}{p_o^2}}\right)\frac{d^2}{dp_o^2}\,(p_o^2\,\sigma_{nn}) \\ &+ \frac{k_1}{p_o^2}\left(\frac{h^2}{\pi}\right)\left(\overline{\frac{1}{r_d^2}}\right)\sigma_{np}(0)\end{aligned} \tag{90}$$

Alternately from (68) and (89) we get

$$\sigma_{nd} = \sigma_{np} + \sigma_{nn} + \frac{2}{3}\left(\frac{\overline{pd^2}}{p_o^2}\right)\left\{2\pi\ \sigma_{np}(\pi) - 2\pi\ .4\ \frac{d\ \sigma_{np}}{dx}\bigg|_{x=-1}\right\}$$

$$+ \frac{2}{3}\left(\frac{\overline{pd^2}}{p_o^2}\right)\left\{2\pi\ \sigma_{nn}(\pi) - 2\pi\ .4\ \frac{d\ \sigma_{nn}}{dx}\bigg|_{x=-1}\right\}$$

$$+ \frac{k_1}{p_o^2}\left(\frac{h^2}{\pi}\right)\left(\overline{\frac{1}{r_d^2}}\right)\sigma_{np}(0) \tag{91}$$

III. Wigner Forces with Pauli Principle

In this section we shall consider how σ_{nd} is modified if we take the Pauli principle, operating between the two neutrons, into account. For this purpose it will be useful to redevelop the cross section formula in a manner different from the time dependent perturbation theory presented in section II. If we followed the method of section II we would find difficulty in exhibiting which correction terms are of order higher than those we are interested in. This arises from the fact that it is not until a late stage of the previous method that we made use of the fact that V_{12} is small compared to H. In the treatment here we shall make use of this fact as early as possible.

Let us therefore develop our formula from a stationary state perturbation theory. Any wave functions will be understood to include a spin dependent part, but we continue to take the potentials as spin independent.

Let

$$\psi = e^{-i/\hbar\,(E_1 + i\,\tilde{\epsilon}\,)t}\,\phi \qquad (1)$$

i.e., ϕ describes the wave function with the time supressed. Here $\tilde{\epsilon}$ denotes a small imaginary contribution to the energy E_1 and eventually we shall let $\tilde{\epsilon}$ go to zero. In essence then $\tilde{\epsilon}$ will serve as a convergence factor in our integrations. Thus

$$(E_1 + i\,\tilde{\epsilon}\,)\phi = (H + V_{nd})\phi \qquad (2)$$

Thus symbolically

$$\phi = \frac{1}{E_1 + i\tilde{\epsilon} - H}\,\phi \tag{3}$$

If then we add the solution ϕ_1 of the homogeneous equation corresponding to equation (1) we get

$$\phi = \phi_1 + \frac{1}{E_1 - H + i\tilde{\epsilon}}\, V_{nd}\,\phi \tag{4}$$

If we set $\phi = \phi_1$ in the integral equation (4) in accordance with the Born approximation and split up H into H_o and V_{12} we get

$$\phi = \phi_1 + \phi_{sc} \tag{5}$$

with

$$\phi_{sc} = \frac{1}{E_1 - H_o - V_{12} + i\tilde{\epsilon}}\, V_{nd}\,\phi_1 \tag{6}$$

If now we expand for V_{12} small, then

$$\phi_{sc} = \left\{ \frac{1}{E_1 - H_o + i\tilde{\epsilon}} + \frac{1}{E_1 - H_o + i\tilde{\epsilon}}\, V_{12}\, \frac{1}{E_1 - H_o + i\tilde{\epsilon}} \right\} V_{nd}\,\phi_1 \tag{7}$$

Thus we see that ϕ_{sc} gets broken up into a main term and a correction to it.

Now examine the commutability of H_o and V_{12}. We know that

$$(H_o V_{12} - V_{12} H_o)_{EE'} = (E - E')\,(V_{12})_{EE'} \tag{8}$$

where E and E' are eigen values of H_o. Now if we choose any model for V_{12}, (say a Yukawa potential for instance) we see at once that $(V_{12})_{EE'}$ is only significant when $(E - E') < \frac{\overline{p_d}^2}{M}$.

Thus $\left[H_o, V_{12}\right]$ is almost but not quite zero. Consider now however that V_{12} occurs only in the correction term of (7). Thus the non-commutability of V_{12} with H_o is only a correction to the correction term. Hence we shall ignore it in the approximation to which we are working.

Thus

$$\phi_{sc} = \frac{1}{E_i - H_o + i\widetilde{\epsilon}} \left\{ V_{nd} + \frac{1}{E_i - H_o + i\widetilde{\epsilon}} V_{12} V_{nd} \right\} \phi_i \qquad (9)$$

For the purposes of this section we have assumed only Wigner potentials so that V_{12} and V_{nd} commute. Further H_o and V_{nd} commute as far as the second term of (9) is concerned by an analagous argument to that presented for V_{12} and H_o above. Thus ϕ_{sc} becomes

$$\phi_{sc} = \frac{1}{E_i - H_o + i\widetilde{\epsilon}} V_{nd} \left\{ 1 + \frac{1}{E_i - H_o + i\widetilde{\epsilon}} V_{12} \right\} \phi_i \qquad (10)$$

Now we may set

$$V_{12}\phi_i = (-\epsilon - T_{12})\,\phi_i \qquad (11)$$

where T_{12} is the kinetic energy operator corresponding to the potential operator V_{12}. Now re-express ϕ_i as a superposition of plane waves. We have

$$\phi_i = \frac{1}{\sqrt{2}}\, e^{i/\hbar\, p_o \cdot r_3}\, \chi(r_1 - r_2) \qquad (12)$$

thus

$$\phi_i = \frac{1}{V^2} \frac{e^{i/\hbar\, p_o \cdot r_3}}{h^{3/2}} \int e^{i/\hbar\, p_d \cdot (r_1 - r_2)}\, \Phi\,(p_d)\, d\vec{p_d} \qquad (13)$$

hence

$$T_{12}\,\phi_i = T_{12}(p_d)\,\phi_i \qquad (14)$$

where

$$T_{12}(p_d) = \frac{{p_d}^2}{M} \qquad (15)$$

Symbolically we may write

$$\phi_i = \sum_m a_m\, e^{i/\hbar\, p_{dm}(r_1 - r_2)} = \sum_m \phi_{im} \qquad (16)$$

then

$$\phi_{sc} = \sum_m \frac{1}{E_i - H_o + i\tilde{\epsilon}}\, V_{nd} \left\{ 1 - \frac{T_{12}(p_{dm}) + \epsilon}{E_i - H_o + i\tilde{\epsilon}} \right. \qquad (17)$$

or permuting the order of V_{nd} in analogy to the previous argument we find

$$\phi_{sc} = \sum_m \frac{1}{E_i - H_o + i\tilde{\epsilon}} \left\{ 1 - \frac{T_{12}(p_{dm}) + \epsilon}{E_i - H_o + i\tilde{\epsilon}} \right\} V_{nd}\,\phi_{im} \qquad (18)$$

Now let us remember that T_{12} is small and furthermore call[6]

$$E_i = E_i^{\,o} - \epsilon \qquad (19)$$

then we see that

$$\phi_{sc} = \sum_m \frac{1}{E_i^{\,o} + T_{12}(p_{dm}) - H_o + i\tilde{\epsilon}}\, V_{nd}\,\phi_{im} \qquad (20)$$

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[6] The notation has been chosen in consistency with equation (II-6).

and so

$$\psi_{sc} = e^{-i/\hbar (E_i + i \tilde{\epsilon})t} \phi_{sc} \tag{21}$$

Before proceeding to develop the Pauli cross section it seems desirable at this point to develop the non-Pauli cross section to exhibit agreement with section II. Consider then what the steps are from here on. The probability of finding the system in a certain final state "f" where the deuteron is disrupted and all three particles have certain definite momenta is given by

$$\left| b_f \right|^2 = \left| (\psi_f^{\circ}, \psi_{sc}) \right|^2 \tag{22}$$

or

$$\left| b_f \right|^2 = e^{\frac{2 \tilde{\epsilon} t}{\hbar}} \left| (\phi_f^{\circ}, \phi_{sc}) \right|^2 \tag{23}$$

Thus the total transition probability is given by

$$w = \frac{\partial}{\partial t} \int \left| (\phi_f^{\circ}, \phi_{sc}) \right|^2 \rho_{E_f} \, e^{\frac{2 \tilde{\epsilon} t}{\hbar}} \, dE_f \tag{24}$$

and thus

$$\sigma_{nd} = \frac{1}{6} \sum_i \left(\frac{V}{p_o/M} \right) \frac{\partial}{\partial t} \int \left| (\phi_f^{\circ}, \phi_{sc}) \right|^2 \rho_{E_f} \, e^{\frac{2 \tilde{\epsilon} t}{\hbar}} \, dE_f \tag{25}$$

The insertion of the extra factor $\frac{1}{6} \sum_i$ merely expresses the fact that we must average over the six equally likely initial spin states, which are discussed more fully in Appendix B. Now at this stage we break up V_{nd} as in equation (34). Thus we get

$$\sigma_A = \frac{1}{6} \sum_i \left(\frac{V}{p_o/M}\right) \frac{2\tilde{\epsilon}}{\hbar} \int \left| \left(\phi_f^o, \sum_m \frac{1}{E_i^o + T_{12}(p_{dm}) - H_o + i\tilde{\epsilon}} V_{np}(r_1 - r_3)\, \phi_{im} \right) \right|^2 \rho_{E_f}\, dE_f \tag{26}$$

Examining the matrix element we note that the momentum of coordinate r_2 does not change, i.e.

$$p_{dm} = p'' \tag{27}$$

then

$$\left| \left(\phi_f^o, \sum_m \frac{1}{E_i^o + T_{12}(p_{dm}) - H_o + i\tilde{\epsilon}} V_{np}\,(r_1 - r_3)\, \phi_{im} \right) \right|^2 \tag{28}$$

$$= \left| \left(\phi_f^o, V_{np}\, \phi_i \right) \right|^2 \left| \frac{1}{E_i^o + T_{12}(p'') - E_f^o + i\tilde{\epsilon}} \right|^2$$

now we know that

$$\lim_{\epsilon \to 0} \frac{\epsilon}{x^2 + \epsilon^2} = \pi\, \delta\,(x) \tag{29}$$

then

$$\sigma_A = \frac{1}{6} \sum_i \left(\frac{V}{p_o/M}\right) \frac{2\pi}{\hbar} \int \left| \left(\phi_f^o, V_{np}\, \phi_i \right) \right|^2 \delta\,(E_f^o - E_i - T_{12}(p''))\, \rho_{E_f} dE_f \tag{30}$$

or lastly

$$\sigma_A = \frac{1}{6} \sum_i \left(\frac{V}{p_o/M}\right) \frac{2\pi}{\hbar} \int \left| \left(\phi_f^o, V_{np}\, \phi_i \right) \right|^2 \delta \left(\frac{p^2 + p'^2 - p''^2 - p_o^2}{2M} \right) \rho_{E_f} dE_f \tag{31}$$

in complete agreement with the result of section II, except that the wave functions here include a spin part. We must now examine it.

In section II we proceeded from the analogue of (31)[7] by reducing out the extraneous coordinate r_2. Thus we must here reduce out the extraneous spin coordinate s_2 as well. This operation proceeds as follows. Let

$$\phi = \phi' \eta \tag{32}$$

where η denotes the spin function of a three-particle system which is described more fully in Appendix B. Then call Γ the wave function of a two-particle system and say that

$$\Gamma = \Gamma' \xi \tag{33}$$

where ξ is the spin wave function of a two-particle system. Then by the usual steps for the space part we get[8]

$$\sigma_A = \frac{1}{6} \sum_i \sum_f \left(\frac{V}{p_o/M} \right) \frac{2\pi}{\hbar} \int \left| \left(\Gamma_f' \eta_f , v_{np} \; \Gamma_i' \eta_i \right) \right|^2 \left| \Phi(p'') \right|^2 \delta \left(\frac{p^2 + p'^2 - p''^2 - p_o^2}{2M} \right) \rho_{E_f} \, dE_f \tag{34}$$

now however we have proved in Appendix B that

[7] namely equations (II-41) and (II-43).

[8] In equation (34) the symbol $\sum_f$ denotes summation over the final spin states.

$$\frac{1}{6} \sum_i \sum_f \left| (\eta_f \, | V_{np} | \, \eta_i) \right|^2 = \frac{1}{4} \sum_i \sum_f \left| (\xi_f \, | V_{np} | \, \xi_i) \right|^2 \quad (35)$$

so that indeed

$$\sigma_A = \frac{1}{4} \sum_i \left(\frac{V}{p_o/M} \right) \frac{2\pi}{\hbar} \int \left| (\Gamma(p,p') | V_{np} | \Gamma(p_o,-p'')) \right|^2 \left| \Phi(p'') \right|^2 \varepsilon \left(\frac{p^2 + p'^2 - p''^2 - p_o^2}{2M} \right) \rho_{E_f} E d_f \quad (36)$$

where the $\frac{1}{4} \sum_i$ expresses the fact that we must average over the four equally likely initial spin states of a neutron-proton system and again by the steps outlined in section II we find

$$\sigma_A = \int \frac{|p_o - p_d|}{p_o} \sigma_{np}(p_o - p_d) \left| \Phi(p_d) \right|^2 d\vec{p_d} \quad (37)$$

similarly we could without difficulty carry through the same procedure to evaluate σ_B. Now coordinate r_1 is extraneous and in analogy to equation (27) we get

$$p_{dm} = p' \quad (38)$$

and the entire procedure carries through like that for σ_A. When we come to the interference term we can write it

$$\sigma_{C,A} + \sigma_{C,B} = 2\mathrm{Re} \, \frac{1}{6} \sum_i \left(\frac{V}{p_o/M} \right) \frac{2\tilde{\epsilon}}{\hbar} \int \left(\phi_f^o, \sum_m \frac{1}{E_i^o + T_{12}(p_{dm}) - H_o + i\tilde{\epsilon}} V_{np}(r_1 - r_3) \phi_{im} \right) \left(\phi_f^o, \sum_n \frac{1}{E_i^o + T_{12}(p_{dn}) - H_o + i\tilde{\epsilon}} V_{nn}(r_2 - r_3) \phi_{in} \right) \rho_{E_f} dE_f \quad (39)$$

thus the first matrix element yields

$$p_{dm} = p'' \tag{40}$$

whereas the second one yields

$$p_{dn} = p' \tag{41}$$

which at first sight seems to present complications. Recall however that the interference term is of the order of a correction term; thus we may make approximations in it without changing it to the approximation in which we are interested. In particular set

$$p_{dm} = p_{dn} \tag{42}$$

then

$$\sigma_{C,A} + \sigma_{C,B} = 2 \operatorname{Re} \frac{1}{6} \sum_{i} \left(\frac{V}{p_o/M}\right) \frac{2\pi}{\hbar} \int (\phi_f^{\,o}, V_{np}\, \phi_i)(\phi_f^{\,o}, V_{nn}\, \phi_i)^* \,\delta\left(\frac{p^2 + p'^2 - p''^2 - p_o^2}{2M}\right) \rho_{E_f}\, dE_f \tag{43}$$

which is in complete analogy to the formula obtained in section II. The only difference from here on in the treatment of $\sigma_{C,A} + \sigma_{C,B}$ compared to that in section II is that of the spin which needs to be considered here. Since the potential is spin-independent the spin sum in operation is

$$\frac{1}{6} \sum_i \sum_f (\eta_f \,|\, 1 \,|\, \eta_i)(\eta_i \,|\, 1 \,|\, \eta_f) = \frac{1}{6} \sum_i (\eta_i \,|\, 1 \,|\, \eta_i) \tag{44}$$

But the sum (44) is just unity. Equally well the spin sum which appears in connection with $\sigma_{np}(0)$, namely

$$\frac{1}{4} \sum_i \sum_f (\xi_f \,|\, 1 \,|\, \xi_f)(\xi_i \,|\, 1 \,|\, \xi_f) = 1 \tag{45}$$

thus we find again that

$$\sigma_{C,A} + \sigma_{C,B} = \frac{k_1}{p_0{}^2} \left(\frac{h^2}{\pi}\right) \left(\frac{\overline{1}}{r_d^2}\right) \sigma_{np}(0) \tag{46}$$

Now we are ready to exhibit the effect of antisymmetrizing the wave function ϕ, which we shall denote by $\tilde{\phi}$. When we perform this antisymmetrization in particles 2 and 3 we get

$$\tilde{\phi} = \frac{1}{\sqrt{2}} \left[\phi_i(3,\overline{12}) - \phi_i(2,\overline{13}) + \phi_{sc}(3,\overline{12}) - \phi_{sc}(2,\overline{13}) \right] \tag{47}$$

where the bar denotes the particles in the deuteron. The fact that the normalization is indeed $1/\sqrt{2}$ to an approximation consistent with the solution of our problem is proved in Appendix D.

That equation (47) fulfills the condition we require of it is well illustrated when we consider the asymptotic condition. Consider the case when neutron number 3 is at infinity. Then

$$\tilde{\phi} \sim \frac{1}{\sqrt{2}} \left[\phi_{sc}(3,\overline{12}) - \phi_{sc}(2,\overline{13}) \right] \tag{48}$$

or

$$\tilde{\phi} \sim \frac{1}{\sqrt{2}} \left[\phi_{sc}(3,\overline{12}) - I_{23}\, \phi_{sc}(3,\overline{12}) \right]$$

But indeed (48) is also fulfilled for the limiting case of (47) with neutron 2 at infinity.

$$\tilde{\phi}_{sc} = \frac{1}{\sqrt{2}} \sum_m \frac{1}{E_i^{\,o} + T_{12}(p_{dm}) - H_o + i\tilde{\epsilon}} V_{nd} \; \phi_{im}$$

$$- \frac{1}{\sqrt{2}} \; I_{23} \sum_k \frac{1}{E_i^{\,o} + T_{12}(p_{dk}) - H_o + i\tilde{\epsilon}} V_{nd} \; \phi_{ik} \qquad (49)$$

Then we must examine this for the potential split up as before. Take first the V_{nn} part, i.e., the one belonging to σ_p. We have found before that in this case

$$p_{dm} = p' \qquad (50)$$

now examine p_{dk}; here we have the second term of (49) gives

$$- \sum_k \frac{1}{E_i^{\,o} + T_{13}(p_{dk}) - H_o + i\tilde{\epsilon}} \; V_{nn}(r_2 - r_3)(I_{23} \; \phi_{ik}) \qquad (51)$$

now

$$I_{23} \; \phi_{ik} \sim e^{i/\hbar \, p_o \cdot r_2} \; e^{i/\hbar \, p_{dk}(r_1 - r_3)} \qquad (52)$$

Then on examining the matrix element formed with $\phi_f^{\,o}$ we see that the momentum of coordinate r_1 remains unchanged, just as in the non-Pauli principle case. Hence

$$p_{dk} = p' = p_{dm} \qquad (53)$$

Thus for the n -n portion we effectively have

$$\tilde{\phi}_{sc,B} = \sum_m \frac{1}{E_i^{\,o} + T(p') - H_o + i\,\tilde{\epsilon}} (1 - I_{23}) V_{nn}\, \phi_{im} \tag{54}$$

or in the same manner as before we find that

$$\sigma_B = \frac{1}{6} \sum_i \left(\frac{V}{p_o/M}\right) \frac{2\pi}{\hbar} \int \frac{|(\phi_f^{\,o}, (1-I_{23}) V_{nn} \phi_i)|^2}{2}$$

$$\delta\left(\frac{p^2 + p''^2 - p'^2 - p_o^{\,2}}{2\,M}\right) \rho_{E_f}\, dE_f \tag{55}$$

Now let us reduce $(\phi_f^{\,o} \,|\, (1-I_{23}) V_{nn} \,|\, \phi_i)$ further. For this purpose take the first term **with** the "1" in it. This clearly yields

$$h^{3/2} \int e^{-i/\hbar\,(p \cdot r_3 + p'' \cdot r_2)} \eta_f(s) V_{nn}(r_2 - r_3) e^{+i/\hbar\,(p_o \cdot r_3 - p' \cdot r_2)}$$

$$\eta_i(s)\; \Phi\,(p')\; dr_2 dr_3 \tag{56}$$

where Φ denotes a momentum wave function as usual.

The term with the I_{23} written out yields

$$- \frac{1}{V^{5/2}} \int e^{-i/\hbar\,(p \cdot r_2 + p' \cdot r_1 + p'' \cdot r_2)} \eta_f(I_{23}s) V_{nn}(r_2 - r_3)$$

$$e^{+i/\hbar\, p_o \cdot r_3}\; \chi(r_1 - r_2) \eta_i(s)\; dr_1 dr_2 dr_3 \tag{57}$$

$$= - \frac{h^{3/2}}{V^{5/2}} \int e^{-i/\hbar\,(p \cdot r_2 + p'' \cdot r_3)} \eta_f(I_{23}s) V_{nn}(r_2 - r_3)$$

$$e^{+i/\hbar\,(p_o \cdot r_3 - p' \cdot r_2)} \eta_i(s)\; \Phi\,(p')\; dr_2 dr_3 \tag{58}$$

$$= + \frac{h^{3/2}}{V^{5/2}} \int e^{-i/\hbar\,(p \cdot r_3 + p'' \cdot r_2)} \eta_f(s) - I_{23} V_{nn}(r_2 - r_3)$$

$$e^{+i/\hbar\,(p_o \cdot r_3 - p' \cdot r_2)} \eta_i(s)\; \Phi\,(p')\; dr_2 dr_3 \tag{59}$$

thus

$$(\phi_f^{\circ}|(1-I_{23})V_{nn}|\phi_i) = \frac{h^{3/2}}{V^{5/2}} \int e^{-i/\hbar\,(p\cdot r_3 + p''\cdot r_2)}$$
$$\eta_f(s)(1-I_{23})V_{nn}\ e^{+i/\hbar\,(p_o\cdot r_3 - p'\cdot r_2)}\eta_i(s)\ \Phi(p')\ dr_2 dr_3 \tag{60}$$

If we again use Γ for the wave function of the two-particle system and denote the spatial part of Γ by Γ' then we may write

$$\Gamma'(p_o,-p') = \frac{1}{V}\ e^{+i/\hbar\,(p_o\cdot r_3 - p'\cdot r_2)} \tag{61}$$

then

$$\sigma_B = \frac{1}{6}\sum_i \left(\frac{2\pi V^3}{\hbar h^6 p_o}\right)\int \sum_f \frac{1}{2}\left|(\Gamma'(p,p'')\eta_f|\ (1-I_{23})V_{nn}|\ \Gamma'(p_o,-p')\right|^2$$
$$\left|\Phi(p')\right|^2\ e^{\frac{i\lambda}{2M}(p^2+p''^2-p'^2-p_o^2)}\,d\lambda\ d_p d_p{}' d_p{}'' \tag{62}$$

Now in Appendix C we have proved that

$$\frac{1}{6}\sum_i\sum_f\left|(\eta_f\ |\ (1-I_{23})|\ \eta_i)\right|^2 = \frac{1}{4}\sum_i\sum_f\left|(\xi_f\ |\ (1-I_{23})|\ \xi_i)\right|^2 \tag{63}$$

where ξ is the spin wave function of the two-particle system.

Hence

$$\sigma_B = \frac{1}{4}\sum_i\sum_f\left(\frac{2\pi V^2}{\hbar h^6 p_o}\right)\int \frac{\left|(\Gamma\ (p,p'')\ |\ (1-I_{23})V_{nn}|\ \Gamma\ (p_o,-p'))\right|^2}{2}$$
$$\left|\Phi(p')\right|^2\ e^{\frac{i\lambda}{2M}(p^2+p''^2-p'^2-p_o^2)}\,d\lambda\ d_p d_p{}' d_p{}'' \tag{64}$$

Note that our derivation of equation (55) did not really make any very special assumptions concerning the three- particle system. Hence we may in analogy to (55) write down the easily proved formula for the scattering of a free neutron (2) from another free neutron (3). This then yields the usual scattering formula with V_{nn} replaced by $\frac{1}{\sqrt{2}}(1-I_{23})V_{nn}$; namely

$$\sigma_{nn}^{\text{Pauli}}(p_o-p_d) = \frac{1}{4}\sum_i\sum_f\left(\frac{2\pi\mathcal{V}^3}{\hbar\, h^6|p_o-p_d|}\right)$$

$$\int\frac{\left|(\Gamma(p,p'')\,|\,(1-I_{23})V_{nn}|\,\Gamma(p_o-p_d))\right|^2}{2}$$

$$e^{\frac{i\lambda}{2M}(p^2+p''^2-p_d^2-p_o^2)}\,d\lambda\; d_p d_{p'} d_{p''} \qquad (65)$$

Thus comparing (64) and (65) we find, as expected that

$$\sigma_B = \int\frac{|p_o-p_d|}{p_o}\;\sigma_{nn}^{\text{Pauli}}(p_o-p_d)\;|\Phi(p_d)|^2\;d\vec{p}_d \qquad (66)$$

Now turn to (49) and examine for the n-p part, i.e., the one belonging to σ_A. In this case we find

$$p_{dm} = p'' = p_{dk} \qquad (67)$$

and hence in analogy to equation (55) we find

$$\sigma_A = \frac{1}{6}\sum_i\left(\frac{\mathcal{V}}{p_o/M}\right)\frac{2\pi}{\hbar}\int\frac{\left|(\phi_f^{\,o},(1-I_{23})V_{np}\phi_i)\right|^2}{2}$$

$$\delta\left(\frac{p^2+p'^2-p''^2-p_o^2}{2M}\right)\rho_{E_f}\,dE_f \qquad (68)$$

Now note that $(1-I_{23})$ may be applied to $\phi_f^{\,o}$ in virtue of the commutability of $(1-I_{23})$ with H_o. Further we have that the operator

$$(1-I_{23})^2 = 2(1-I_{23}) \qquad (69)$$

so that in (68) we may replace

$$\frac{\left|(\phi_f^{\,o},(1-I_{23})V_{np}\,\phi_1)\right|^2}{2} \longrightarrow (\phi_f^{\,o},V_{np}\phi_1)(\phi_1,V_{np}(1-I_{23})\,\phi_f^{\,o}) \tag{70}$$

In analogy with (70) break up

$$\sigma_A = \sigma_{A1} + \sigma_{A2} \tag{71}$$

then σ_{A1} is clearly related to the usual n-p cross section and is as was to be expected.

$$\sigma_{A1} = \int \frac{|p_o-p_d|}{p_o}\,\sigma_{np}(p_o-p_d)\;\left|\Phi(p_d)\right|^2\,d\vec{p}_d \tag{72}$$

The term $\sigma_{A,2}$ which is

$$\sigma_{A,2} = \frac{-1}{6}\sum_i\sum_f\left(\frac{M\gamma^4}{\hbar h^9 p_o}\right)\int(\phi_1|V_{np}I_{23}|\phi_f^{\,o})(\phi_f^{\,o}|V_{np}|\phi_1)$$
$$e^{\frac{i\lambda}{2M}(p^2+p'^2-p''^2-p_o^2)}\,d\lambda\,d_p d_p{}' d_p{}'' \tag{73}$$

is a correction term arising in the Pauli principle treatment only in virtue of the binding between particles 1 and 2. We can easily verify that $\sigma_{A,2}$ vanishes for the case of no binding between particles 1 and 2. This is a result we must require physically, since the mere presence of the extra neutron number 2 should not influence the n-p scattering in the case where we have three free particles. For the time being we shall leave $\sigma_{A,2}$ in the form of equation (73) and turn to the evaluation of the interference term.

We may treat the interference term in the manner described earlier in this section. For the matrix element containing V_{np} we find

$$p_{dm} = p'' = p_{dk} \tag{67}$$

whereas for the matrix element containing V_{nn} we find

$$p_{dm} = p' = p_{dk} \tag{53}$$

we are not bothered by the inconsistency between these two matrix elements since the interference term itself is just of the order of a correction term. Thus we may write

$$\sigma_{C,A} + \sigma_{C,B} = 2\mathrm{Re}\ \frac{1}{6} \sum_{i} \left(\frac{V}{p_o/M}\right) \frac{2\pi}{\hbar}$$

$$\int \frac{(\phi_f^{\,o}, (1-I_{23})V_{np}\phi_i)(\phi_f^{\,o}, (1-I_{23})V_{nn}\phi_i)^*}{2}$$

$$\delta\left(\frac{p^2 + p'^2 - p''^2 - p_o^2}{2M}\right) \rho_{E_f}\, dE_f \tag{74}$$

Again if we say that $V_{nn} = k_1 V_{np}$ then we may replace

$$\frac{(\phi_f^{\,o}, (1-I_{23})V_{np}\phi_i)(\phi_f^{\,o}, (1-I_{23})V_{nn}\phi_i)^*}{2} \longrightarrow \tag{75}$$

$$k_1(\phi_f^{\,o}, (1-I_{23})V_{np}(r_1-r_3)\phi_i)(\phi_f^{\,o}, V_{np}(r_2-r_3)\phi_i)^*$$

thus if in accordance with the subdivision of (75) we call

$$\sigma_{C,A} + \sigma_{C,B} = \sigma_{C1} + \sigma_{C2} \tag{76}$$

then σ_{C1} is the usual interference term namely

$$\sigma_{C1} = \frac{k_1}{p_o^2} \left(\frac{h^2}{\pi}\right)\left(\frac{\overline{1}}{r_d^2}\right) \sigma_{np}(0) \tag{77}$$

The term σ_{C2} is an additional term to the usual interference term arising from the introduction of the Pauli principle. Explicitly it is

$$\sigma_{C2} = \frac{-2k_1}{6} \sum_i \sum_f \left(\frac{M\,V^4}{\hbar\, h^9 p_o}\right) \int (\phi_i | V_{np}(r_1-r_3) I_{23} | \phi_f^o)$$
$$(\phi_f^o | V_{np}(r_2-r_3) | \phi_i) e^{\frac{i\lambda}{2M}(p^2+p'^2-p''^2-p_o^2)} d\lambda \, d_p d_p' d_p'' \tag{78}$$

We could develop σ_{C2} into a form similar to (77); namely into a form with $(I_{23}V_{nn})^2$; which would yield a part of $\sigma_{nn}(0)$. Then if we wished to complete $\sigma_{nn}(0)$ we could get part of the term from σ_{C1} by changing it into the n-n form. However we still would have a correction term left over which would be no simpler than (78). Hence there is little advantage in carrying (78) further. It can of course be evaluated for specific models for the potential.

Now let us summarize the results of this section, which are that:

$$\sigma_{nd} = \sigma_{np} + \sigma_{nn} + \frac{1}{6}\left(\frac{\overline{p_d^2}}{p_o^2}\right)\left\{\frac{d^2}{dp_o^2}(p_o^2\,\sigma_{np}) + \frac{d^2}{dp_o^2}(p_o^2\,\sigma_{nn})\right\}$$
$$+ \frac{k_1}{p_o^2}\left(\frac{h^2}{\pi}\right)\left(\frac{\overline{1}}{r_d^2}\right)\sigma_{np}(0) \tag{79}$$
$$- \frac{1}{6}\sum_i\sum_f \left(\frac{MV^4}{\hbar h^9 p_o}\right)\int (\phi_i | V_{np} I_{23} | \phi_f^o) \left\{ (\phi_f^o | V_{np} | \phi_i) \right.$$
$$\left. + 2(\phi_f^o | V_{nn} | \phi_i) \right\} e^{\frac{i\lambda}{2M}(p^2+p'^2-p''^2-p_o^2)} d\lambda$$

The above result is seen to divide itself into the first five familiar terms plus an extra term due to the Pauli principle. This last term has to be evaluated for the specific model under discussion. It is difficult to make any predictions regarding its value, except to say that it is undoubtedly no larger than the correction terms to σ_{nd}.

IV. Spin and Space Exchange Forces Without Pauli Principle

In this section we shall attempt to deal with a general potential of the form[9]

$$V_{np}(r_1-r_3) = (a_1 + b_1\, \sigma_1 \cdot \sigma_3)(c_1 + d_1\, P_{13}) A_{np}(r_1-r_3) \qquad (1)$$

$$V_{nn}(r_2-r_3) = (a_2 + b_2\, \sigma_2 \cdot \sigma_3)(c_2 + d_2\, P_{23}) B_{nn}(r_2-r_3) \qquad (2)$$

where P stands for space exchange.

Now let us develop our expressions by the method of section III. We again get equation (III-9) for ϕ_{sc}. Now however V_{12} and V_{nd} do not commute. Thus

$$\phi_{sc} = \phi_{sc,o} + \phi_{sc,\lambda} \qquad (3)$$

with

$$\phi_{sc,o} = \frac{1}{E_i - H_o + i\tilde{\epsilon}} V_{nd} \left\{ 1 + \frac{1}{E_i - H_o + i\tilde{\epsilon}} V_{12} \right\} \phi_i \qquad (4)$$

and

$$\phi_{sc,\lambda} = \left(\frac{1}{E_i - H_o + i\tilde{\epsilon}} \right)^2 \left[V_{12}, V_{nd} \right] \phi_i \qquad (5)$$

Now we need to evaluate $\left| (\phi_f^{\,o}, \phi_{sc}) \right|^2$. Now we note that, by use of equation (3) we get

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[9] While potentials (1) and (2) are not of the most general form they are of a useful form to exhibit our arguments that follow. As a matter of fact our arguments do carry through with a very general, non-tensor force potential.

$$\left|(\phi_f^{\circ},\phi_{sc})\right|^2 = \left|(\phi_f^{\circ},\phi_{sc,o})\right|^2 + \left|(\phi_f^{\circ},\phi_{sc,\lambda})\right|^2 + 2\mathrm{Re}(\phi_f^{\circ},\phi_{sc,o})(\phi_{sc,\lambda},\phi_f^{\circ}) \tag{6}$$

Now let us introduce an equivalent notation for the cross section, namely

$$\sigma = \sigma_0 + \sigma_\lambda + \sigma(o\lambda) \tag{7}$$

Let us now concentrate on the term arising from $\left|(\phi_f^{\circ},\phi_{sc,o})\right|^2$ i.e., σ_o .

We must first examine $V_{12}\,\phi_1$. It is still true that

$$T_{12}\,\phi_1 = T_{12}(pd)\,\phi_1 \tag{8}$$

Since the interchange operator P_{12} in V_{12} does not change matters, for T_{12} concerns itself only with relative quantities between 1 and 2. Thus we get in analogy to equation (III-31) that

$$\sigma_{o,A} = \frac{1}{6}\sum_i \left(\frac{V}{p_o/M}\right)\left(\frac{2\pi}{\hbar}\right)\int\left|(\phi_f^{\circ},V_{np}\,\phi_1)\right|^2 \delta\left(\frac{p^2+p'^2+p''^2-p_o^2}{2M}\right)\rho_{E_f}dE_f \tag{9}$$

Now we must evaluate $\left|(\phi_f^{\circ},V_{np}\,\phi_1)\right|^2$ further. There is no difficulty concerning the spin, since in Appendix B we have proved that

$$\frac{1}{6}\sum_i\sum_f\left|(\eta_f\,|a_1+b_1\,\sigma_1\cdot\sigma_3|\,\eta_1)\right|^2 = \frac{1}{4}\sum_i\sum_f\left|(\xi_f\,|a_1+b_1\,\sigma_1\cdot\sigma_3|\,\xi_1)\right|^2 \tag{10}$$

Let us now examine the necessary spatial matrix element, namely

$$J = (\phi_f^{o\prime} \mid (c_1 + d_1 P_{13}) A_{np}(r_1 - r_3) \mid \phi_i{}') \tag{11}$$

Let us break up J in accordance with (11) into

$$J = J_c + J_d \tag{12}$$

Then by the familiar arguments

$$J_c = \frac{c_1}{V^{5/2}} \int e^{-i/\hbar (p.r_3 + p'.r_1)} A_{np}(r_1 - r_3) e^{+i/\hbar (p_o.r_3 - p''.r_1)} \Phi(p'') dr_1 dr_3 \tag{13}$$

The new element to consider is J_d.

$$J_d = \frac{d_1}{V^{5/2}} \int e^{-i/\hbar (p'.r_1 + p''.r_2 + p.r_3)} A_{np}(r_1 - r_3) e^{+i/\hbar\, p_o.r_1} \chi(r_2 - r_3) \; dr_1 dr_2 dr_3 \tag{14}$$

or

$$J_d = \frac{d_1}{V^{5/2}} \int e^{-i/\hbar (p.r_3 + p'.r_1)} A_{np}(r_1 - r_3) e^{+i/\hbar (p_o.r_1 - p''.r_3)} \Phi(p'') \; dr_1 dr_3 \tag{15}$$

of finally

$$J = \frac{1}{V^{5/2}} \int e^{-i/\hbar (p.r_3 + p'.r_1)} (c_1 + d_1 P_{13}) A_{np}(r_1 - r_3) e^{+i/\hbar (p_o.r_3 - p''.r_1)} \Phi(p'') dr_1 dr_3 \tag{16}$$

and all steps carry through as usual till we again obtain

$$\sigma_{0,A} = \int \frac{|p_o - p_d|}{p_o} \; \sigma_{np}(p_o - p_d) \; |\Phi(p_d)|^2 \; d\vec{p_d} \tag{17}$$

In an analogous manner we derive the formula for $\sigma_{0,B}$ as

$$\sigma_{0,B} = \int \frac{|p_0 - p_d|}{p_0}\, \sigma_{nn}(p_0 - p_d)\, \left|\Phi(p_d)\right|^2 d\vec{p}_d \tag{18}$$

Now we must discuss the interference term of σ_0 . This term is

$$\sigma_{0,C} = 2\mathrm{Re}\, \frac{1}{6} \sum_i \left(\frac{\mathcal{V}}{p_0/M}\right) \frac{2\pi}{\hbar}$$

$$\int (\phi_f^0, V_{np}\phi_i)(\phi_f^0, V_{nn}\phi_i)^* \,\delta\left(\frac{p^2 + p'^2 - p''^2 - p_0^2}{2M}\right) \rho_{E_f}\, dE_f \tag{19}$$

While this term is straightforward, we can no longer as in section II express it simply in terms of $\sigma_{np}(0)$. However we see by looking at it that given a_1, b_1, a_2 and b_2 we could easily perform the spin sums indicated. We would then proceed by separating the non-space exchange and space exchange terms and evaluating these. Thus given the constants $a_1 \rightarrow d_1$ and $a_2 \rightarrow d_2$ there is no inherent difficulty for $\sigma_{0,C}$. In the absence of definite values it seems of little value to carry the valuation (19) beyond this stage and we shall leave it in this form.

We turn now to the σ_λ term arising from $|(\phi_f^0, \phi_{sc,\lambda})|^2$. Upon examining $\phi_{sc,\lambda}$ as given by (5) we note that on the squaring this term it is of order V^4. Now the chief terms of σ_{nd} are of order V^2; the correction terms in which we are interested are of one order higher, namely V^3; thus we may drop terms of order V^4. Hence we shall set σ_λ equal to zero.

We must now turn to the evaluation of the $\sigma_{(0\lambda)}$ term. This term does have a contribution of order V^3 and so we must retain it. In particular the term with the V^3 contribution reads:

$$\sigma_{(0\lambda)} = \frac{2Re}{6} \sum_i \left(\frac{V}{p_o/M}\right) \frac{2\tilde{\epsilon}}{\hbar}$$

$$\int \left(\phi_f^o, \frac{1}{E_i - H_o + i\tilde{\epsilon}} V_{nd} \phi_i \right) \left(\left(\frac{1}{E_i - H_o + i\tilde{\epsilon}} \right)^2 \right.$$

$$\left. \left[V_{12}, V_{nd} \right] \phi_i, \phi_f^o \right) \rho_{E_f} dE_f \tag{20}$$

By arguments similar to those previously presented it can be shown that equation (20) reduces to

$$\sigma_{(0\lambda)} = -\frac{Re}{6} \sum_i \left(\frac{V}{p_o/M}\right) \frac{2\pi}{\hbar}$$

$$\int (\phi_f^o, V_{nd} \phi_i)(\phi_f^o, \left[V_{12} V_{nd}\right] \phi_i)^* \rho_{E_f} \delta'(E_f^o - E_i) dE_f^o \tag{23}*$$

Performing an integration by parts we find that

$$\sigma_{(0\lambda)} = -\frac{Re}{6} \sum_i \left(\frac{V}{p_o/M}\right) \frac{2\pi}{\hbar}$$

$$\int \frac{\partial}{\partial E_f^o} \left\{ (\phi_f^o, V_{nd} \phi_i)(\phi_f^o, \left[V_{12}, V_{nd}\right] \phi_i)^* \right.$$

$$\left. \rho_{E_f} \right\} \delta(E_f^o - E_i) dE_f^o \tag{24}$$

The evaluation of this term depends on the particular model chosen, since it has few general properties. We shall therefore leave it in the form (24).

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*There are no equations numbered (21) and (22) in this section.

Now we may summarize the results of this section:

$$\sigma_{nd} = \sigma_{np} + \sigma_{nn}$$

$$+ \frac{1}{6} \left(\frac{\overline{p_d{}^2}}{p_o{}^2} \right) \left\{ \frac{d^2}{dp_o{}^2} (p_o{}^2 \ \sigma_{np}) + \frac{d^2}{dp_o{}^2} (p_o{}^2 \ \sigma_{nn}) \right\}$$

$$+ \frac{2\mathrm{Re}}{6} \sum_i \left(\frac{\mathcal{V}}{p_o/M} \right) \frac{2\pi}{\hbar} \int (\phi_f{}^o, V_{np}\, \phi_i)(\phi_f{}^o, V_{nn}\, \phi_i)^* \, \delta\left(\frac{p^2 + p'^2 - p''^2 - p_o}{2M} \right) \rho_{E_f} dE_f \tag{25}$$

$$- \frac{\mathrm{Re}}{6} \sum_i \left(\frac{\mathcal{V}}{p_o/M} \right) \frac{2\pi}{\hbar} \int \frac{\partial}{\partial E_f{}^o} \left\{ (\phi_f{}^o, V_{nd}\, \phi_i) (\phi_f{}^o, \left[V_{12}, V_{nd} \right] \phi_i)^* \rho_{E_f^o} \right\} \delta(E_f{}^o - E_i) dE_f{}^o$$

V. Spin and Space Exchange Forces with Pauli Principle

In this section we shall attempt to deal with the general potential of section IV; - but this time we shall include the modifications due to the Pauli principle.

Then if in analogy to equation (IV-3) we set

$$\tilde{\phi}_{sc} = \tilde{\phi}_{sc,o} + \tilde{\phi}_{sc,\lambda} \tag{1}$$

and we find by antisymmetrizing equations (IV-4) and (IV-5) that

$$\begin{aligned}\tilde{\phi}_{sc,o} = \frac{1}{\sqrt{2}} \sum_m \frac{1}{E_i^{\,o} + T_{12}(p_{dm}) - H_o + i\tilde{\epsilon}} V_{nd}\, \phi_{im} \\ - \frac{1}{\sqrt{2}} I_{23} \sum_k \frac{1}{E_i^{\,o} + T_{12}(p_{dk}) - H_o + i\tilde{\epsilon}} V_{nd}\, \phi_{ik}\end{aligned} \tag{2}$$

and

$$\tilde{\phi}_{sc,\lambda} = \frac{(1-I_{23})}{\sqrt{2}} \left(\frac{1}{E_i^{\,o} - H_o + i\tilde{\epsilon}}\right)^2 \left[V_{12}, V_{nd}\right] \phi_i \tag{3}$$

Now as in section IV (equation IV-6 and IV-7) we break up

$$\sigma = \sigma_o + \sigma_\lambda + \sigma_{(o\lambda)} \tag{4}$$

Now concentrate on the term arising from $\left|(\phi_f^{\,o}, \tilde{\phi}_{sc,o})\right|^2$ i.e., σ_o. This is now carried through in strict analogy to previous work. First we consider the n-n portion. Here equation (III-53) applies to equation (2) and eventually as usual

$$\sigma_{o,B} = \int \frac{|p_o - p_d|}{p_o} \sigma_{nn}^{Pauli}(p_o - p_d) \left|\Phi(p_d)\right|^2 d\vec{p_d} \tag{5}$$

For the n-p portion we find

$$\sigma_{0,A-1} = \int \frac{|p_0 - p_d|}{p_0} \sigma_{np}(p_0 - p_d) \qquad \left|\Phi(p_d)\right|^2 d\vec{p_d} \qquad (6)$$

and

$$\sigma_{0,A-2} = \frac{-1}{6} \sum_i \sum_f \left(\frac{MV^4}{\hbar h^9 p_0}\right) \int (\phi_i | V_{np} I_{23} | \phi_f^0)(\phi_f^0 | V_{np} | \phi_i)$$
$$e^{\frac{i\lambda}{2M}(p^2 + p'^2 - p''^2 - p_0^2)} \, d\lambda \, dp dp' dp'' \qquad (7)$$

The interference term for σ_0 is

$$\sigma_{0,C} = 2\mathrm{Re} \; \frac{1}{6} \sum_i \left(\frac{V}{p_0/M}\right) \frac{}{\hbar}$$
$$\int \frac{(\phi_f^0, (1-I_{23})V_{np}\phi_i)(\phi_f^0, (1-I_{23})V_{nn}\phi_i)^*}{2}$$
$$\delta\left(\frac{p^2 + p'^2 - p''^2 - p_0^2}{2M}\right) \rho_{E_f} dE_f \qquad (8)$$

For reasons already explained in the analogous case of section IV (equation IV-19) it is not worthwhile to express this term in more explicit form.

We turn now to the σ_λ term. For reasons analogous to those given for this term in section IV the σ_λ vanishes to the order we are interested in.

Now as to the $\sigma_{(o\lambda)}$ term. We may write the portions which contribute as

$$\sigma_{(o\lambda)} = \frac{\mathrm{Re}}{6} \left(\frac{V}{p_0/M}\right) \frac{2\tilde{\epsilon}}{\hbar} \int (\phi_f^0, (1-I_{23}) \frac{1}{E_i - H_0 + i\tilde{\epsilon}} V_{nd} \phi_i)$$
$$((1-I_{23}) \left(\frac{1}{E_i - H_0 + i\tilde{\epsilon}}\right)^2 \left[V_{12}, V_{nd}\right] \phi_i, \phi_f^0)$$
$$\rho_{E_f} dE_f \qquad (9)$$

Thus by the arguments presented in section IV

$$\sigma_{(2\lambda)} = - \frac{Re}{12} \sum_i \left(\frac{V}{p_o/M} \right) \frac{2\pi}{\hbar}$$

$$\int \frac{\partial}{\partial E_f^o} \left\{ (\phi_f^o, (1-I_{23}) V_{nd} \, \phi_i)(\phi_f^o, (1-I_{23}) \left[V_{12}, V_{nd} \right] \phi_i)^* \, \rho_{E_f^o} \right\} \delta (E_f^o - E_i) dE_f^o \qquad (10)$$

Again the value of $\sigma_{(2\lambda)}$ must be obtained for the specific model under consideration and so we leave it in the form (10).

Now we may summarize the results of this general section:

$$\sigma_{nd} = \sigma_{nn} + \sigma_{np}$$

$$+ \frac{1}{6} \left(\frac{\overline{p_d^2}}{p_o^2} \right) \left\{ \frac{d^2}{dp_o^2} (p_o^2 \sigma_{np}) + \frac{d^2}{dp_o^2} (p_o^2 \sigma_{nn}) \right\}$$

$$+ \frac{Re}{6} \sum_i \left(\frac{V}{p_o/M} \right) \frac{2\pi}{\hbar} \int (\phi_f^o, (1-I_{23}) V_{np} \, \phi_i)$$

$$(\phi_f^o, (1-I_{23}) V_{nn} \, \phi_i)^* \; \delta \left(\frac{p^2 + p'^2 - p''^2 - p_o^2}{2M} \right) \rho_{E_f} dE_f$$

$$- \frac{Re}{12} \sum_i \left(\frac{V}{p_o/M} \right) \frac{2\pi}{\hbar} \int \frac{\partial}{\partial E_f^o} \left\{ (\phi_f^o, (1-I_{23}) V_{nd} \, \phi_i) \right.$$

$$\left. (\phi_f^o, (1-I_{23}) \left[V_{12}, V_{nd} \right] \phi_i)^* \, \rho_{E_f^o} \right\} \delta (E_f^o - E_i) dE_f^o$$

(11)

VI. Conclusion

Let us first say a word about the general nature of our results. It is worth noting that in all cases considered, including the most complicated one (section V) a certain structure is preserved in our results. In all cases we get the terms

$$\sigma_{np} + \sigma_{nn} + \frac{1}{6}\left(\frac{\overline{p_d^2}}{p_o^2}\right)\frac{d^2}{dp_o^2}\left(p_o^2\,\sigma_{np}\right) + \frac{1}{6}\left(\frac{\overline{p_d^2}}{p_o^2}\right)\frac{d^2}{dp_o^2}\left(p_o^2\,\sigma_{nn}\right) \qquad (1)$$

and interference terms due to straight interference or the Pauli principle.

To get an idea of the order of magnitude of our results we shall first estimate what the correction term to $(\sigma_{nn} + \sigma_{np})$ is from the results of section II; i.e., we choose a model with Wigner forces and neglect the modifications due to the Pauli principle.

We shall compute our correction term from formula (II-91), i.e., from the angular form. For this purpose let us express it as σ_{np} throughout by recalling that from equation (I-65)

$$V_{nn} = k_1\, V_{np} \qquad (2)$$

then

$$\sigma_{nn} = k_1^2\, \sigma_{np} \qquad (3)$$

hence

$$\sigma_{nd} = (1+k_1^2)\,\sigma_{np} + (1+k_1^2)\,\frac{4\pi}{3}\left(\frac{\overline{p_d^2}}{p_o^2}\right)\left\{\sigma_{np}(\pi) - 4\left.\frac{d\,\sigma_{np}}{dx}\right|_{x=-1}\right\} \qquad (4)$$

$$+ \frac{k_1}{p_o^2}\left(\frac{h^2}{\pi}\right)\left(\overline{\frac{1}{r_d^2}}\right)\sigma_{np}(0)$$

Now $\overline{p_d^2}$ is estimated in appendix E to be

$$\left(\frac{\overline{p_d^2}}{M}\right) = 7.46 \text{ Mev} \tag{5}$$

Thus we may write

$$\left(\frac{\overline{p_d^2}}{p_o^2}\right) = \frac{7.46}{2\ E_n} \tag{6}$$

where E_n is the energy of the incident neutron in Mev in the laboratory system. In appendix E we also find an estimate for

$$\left(\overline{\frac{1}{r_d^2}}\right) = 0.282 \times 10^{26}\ \text{cm}^{-2} \tag{7}$$

Thus we may re-express σ_{nd} as

$$\sigma_{nd} = (1+k_1^2)\ \sigma_{np} + (1+k_1^2)\ \frac{15.63}{E_n}\ L + \frac{0.72 \times 10^2}{E_n}\ k_1\ \sigma_{np}(0) \tag{8}$$

where

$$L = \sigma_{np}(\pi) - 4\ \left.\frac{d\ \sigma_{np}}{dx}\right|_{x=-1} \tag{9}$$

Now we shall break up σ_{nd} into

$$\sigma_{nd} = M + C \tag{10}$$

where M is the main term, namely $(1+k_1^2)\ \sigma_{np}$ and C is our correction term. In our numerical work we have exhibited the two parts of C, namely C_1 and C_2 corresponding to equation (8).

Next we must decide what values to choose for k_1. If we look at the experimental data given on page 2 we find that

$$\frac{\sigma_{nn}}{\sigma_{np}} \sim \frac{1}{4} \tag{11}$$

While we know that the n-p cross section cannot be fitted by Wigner forces alone we could still choose the value $k_1 = 0.5$ as a rough indication what to compute for the correction term if we use only Wigner forces. We have tabulated the results for $k_1 = 0.5$ and a value of $E_n = 90.25$ Mev[10] in table 1. In table 1 we have chosen 10^{-26} cm^2 as the unit of area.

Table 1

σ_{np}	=	18.02
$\sigma_{np(o)}$	=	12.58
M	=	22.53
C_1	=	0.18
C_2	=	4.55
C/M	=	21%

From table 1 we note that the indications are that since C is positive the trye n-n cross section is smaller than computed. Hence it might be instructive to examine the case $k_1 = 0.25$, which is summarized in table 2.

Table 2

M	=	19.15
C_1	=	0.11
C_2	=	2.53
C/M	=	14%

---

[10] The cross sections used in this calculation were obtained from declassified report LA-654 by L. Goldstein, entitled "Studies on the Scattering of Neutrons by Protons". We are using the rigorous cross sections obtained from $r_o = 2.8 \times 10^{-13}$ cm.

We shall now have to estimate what the correction term would be for other potentials and the inclusion of the Pauli principle. It is our belief that the correction term in such cases does not exceed the 10% just found. As a matter of fact the indications are that it should be smaller. This argument is substantiated by the fact that σ (o) is so pronounced only in the Wigner case, and we saw that it was due to C_2 term that C was so large.

Now we must ask ourselves whether it is profitable at this stage of the development to compute the additional correction terms by special models. First we must look at the size of the correction term compared to the experimental errors. The experimental errors are at present of the order of 10%; i.e., of the same order of magnitude as our correction term. This means that in order to get a significant answer we would certainly have to know the "correct" potential for our model. We might try to infer this "correct" potential from the n-p scattering experiments at 90 Mev carried out by Segre et al[11]. Unfortunately, as is well known, it has not yet been possible to fit this data unambiguously.

For the time being, therefore, we must leave it at the conclusion that the correction terms are of the order of 10% or less, but may well change the true value of the n-n cross section at 90 Mev.

11 E. Segre, Washington Physical Society meeting, April 29, 1948.

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Note added in proof.

Since the completion of this work, Wu and Ashkin have published some numerical calculations on the subject of n-d Scattering (Physical Review, 73, 986 (1948)). These calculations seem to show agreement for the simplest case, but tend to show that the corrections are probably considerably larger than the 10% estimate made above.

Appendix A:

Theorem:

If $B < A$ then in first approximation

$$\left(e^{A+B}\right)_{aa'} = \left(e^{A}\right)_{aa'} + \left(B\right)_{aa'} \left(\frac{e^{a} - e^{a'}}{a - a'}\right) \quad (1)$$

Proof:

we know that

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (2)$$

now when expanding $(A+B)$ keep only first powers of B, then

$$\left(e^{A+B}\right)_{aa'} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\left(A^n\right)_{aa'} + \sum_{m=0}^{n-1} \sum_{a''a'''} \left(A^{n-1-m}\right)_{aa''} \left(B\right)_{a''a'''} \left(A^m\right)_{a'''a'} \right] \quad (3)$$

but

$$\sum_{a''a'''} \left(A^{n-1-m}\right)_{aa'} \left(B\right)_{a''a'''} \left(A^m\right)_{a'''a'} = a^{n-1-m} (B)_{aa'} a'^m \quad (4)$$

thus

$$\left(e^{A+B}\right)_{aa'} = \left(e^{A}\right)_{aa'} + (B)_{aa'} \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} \frac{a^{n-1-m} a'^m}{n!} \quad (5)$$

now consider $L = \frac{e^{a} - e^{a'}}{a - a'}$; suppose first $a' < a$; then we have

$$L = \sum_{n=0}^{\infty} \frac{a^n - a'^n}{(a-a')n!} \quad (6)$$

or

$$L = \sum_{n=0}^{\infty} \frac{a^n \left[1 - \left(\frac{a'}{a}\right)^n\right]}{n!\ a \left(1 - \frac{a'}{a}\right)} \quad (7)$$

$$L = \sum_{n=o}^{\infty} \sum_{m=o}^{m=n-1} \frac{a^{n-1}}{n!} \left(\frac{a'}{a}\right)^{m} \tag{8}$$

thus

$$\left(e^{A+B} \right)_{aa'} = \left(e^{A}\right)_{aa'} + (B)_{aa'} \left(\frac{e^{a} - e^{a'}}{a - a'}\right) \tag{9}$$

The conditions clearly holds also for $a' > a$ by reversing the grouping; i.e., considering $\left(\frac{a}{a'}\right)$ as a unit. When $a' = a$ the condition is self-evident, since then $L = 1$.

Q.E.D.

Appendix B:

In this appendix we shall concern ourselves with the spin functions appearing in the text and some of their properties.

First of all let us for ready reference write down the spin wave functions of a two-particle system each of spin $\frac{1}{2}$. Let α denote spin $+\frac{1}{2}$ and β denote spin $-\frac{1}{2}$. Thus α_1 means the wave function of particle 1 which has spin $+\frac{1}{2}$. Let ξ denote the spin wave function of the two-particle system with a certain total spin and a given spin projection. Then we can easily verify that the following four functions exist:

Function #	Total Spin	Spin Projection	Wave Function
1	1	-1	$\alpha_1 \quad \alpha_2$
2	1	0	$\frac{1}{\sqrt{2}} \cdot (\beta_1 \alpha_2 + \alpha_1 \beta_2)$
3	1	-1	$\beta_1 \quad \beta_2$
4	0	0	$\frac{1}{\sqrt{2}} (\beta_1 \alpha_2 - \alpha_1 \beta_2)$

If now we combine these wave functions with those of a particle number 3 which has spin $\frac{1}{2}$ we can form spin wave function η describing the three-particle system. In particular we can form one quartet and two doublets; depending on whether the two-particle system is in the triplet or singlet state. These wave functions are summarized in the following table:

Function #	Total Spin of 3-particle system	Spin of 2 particle system	Total Spin projection	Wave Function
1	3/2	1	+ 3/2	$\alpha_1 \alpha_2 \alpha_3$
2	3/2	1	+ $\frac{1}{2}$	$\frac{1}{\sqrt{3}}(\beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3)$
3	3/2	1	- $\frac{1}{2}$	$\frac{1}{\sqrt{3}}(\beta_1\beta_2\alpha_3 + \alpha_1\beta_2\beta_3 + \beta_1\alpha_2\beta_3)$
4	3/2	1	- 3/2	$\beta_1 \beta_2 \beta_3$
5	$\frac{1}{2}$	1	+ $\frac{1}{2}$	$\frac{1}{\sqrt{6}}(\beta_1\alpha_2\alpha_3 - 2\alpha_1\alpha_2\beta_3 + \alpha_1\beta_2\alpha_3)$
6	$\frac{1}{2}$	1	- $\frac{1}{2}$	$\frac{-1}{\sqrt{6}}(\beta_1\alpha_2\beta_3 - 2\beta_1\beta_2\alpha_3 + \alpha_1\beta_2\beta_3)$
7	$\frac{1}{2}$	0	+ $\frac{1}{2}$	$\frac{1}{\sqrt{2}}(\beta_1\alpha_2\alpha_3 - \alpha_1\beta_2\alpha_3)$
8	$\frac{1}{2}$	0	- $\frac{1}{2}$	$\frac{1}{\sqrt{2}}(\beta_1\alpha_2\beta_3 - \alpha_1\beta_2\beta_3)$

Let us now inquire what relation the wave functions $\eta_1 \rightarrow \eta_8$ have to our problem at hand. If we consider our incoming neutron (particle 3) meeting a deuteron (particles 1 and 2) bound in the ground state then only function 1 → 6 can be spin-wave functions describing the initial state of the system. This is so, since for the initial state we require the deuteron to be bound in the ground state, i.e., it must be in the triplet state. Thus if we denote the initial spin-wave function of the three-particle system by η_i, "i" may range from 1 to 6. Further if we denote the final spin-wave function of the three-particle system by η_f, "f" may range from 1 to 8. This is so, since in the final state there is not a priori requirement that particles 1 and 2 be in either the triplet or singlet state.

Now our spin operators which appear in the problem are all of the form $(\gamma_1 + \gamma_2 \sigma_m \cdot \sigma_n)$ where γ_1 and γ_2 are constants and m and n denote two of the particles of our particles 1, 2 and 3.

We shall now proceed to prove some theorems which hold between the η's and ξ's.

<u>Theorem 1</u>:

$$\frac{1}{6} \sum_{i=1}^{6} \sum_{f=1}^{8} \left| (\eta_f \mid \gamma_1 + \gamma_2 \sigma_1 \cdot \sigma_3 \mid \eta_1) \right|^2 = \frac{1}{4} \sum_{i=1}^{4} \sum_{f=1}^{4} \left| (\xi_f \mid \gamma_1 + \gamma_2 \sigma_1 \cdot \sigma_3 \mid \xi_1) \right|^2 \tag{1}$$

here the ξ's are spin functions compounded of particles 1 and 3. Let us first reduce the left side; call it L. Then since the functions η_f are a complete set we may write:

$$L = \frac{1}{6} \sum_{i=1}^{6} (\eta_1 \mid (\gamma_1 + \gamma_2 \sigma_1 \cdot \sigma_3)^2 \mid \eta_1) \tag{2}$$

call

$$z = (\gamma_1 + \gamma_2 \sigma_1 \cdot \sigma_2)^2 \tag{3}$$

Thus the spin operator does not involve particle 2; let us therefore perform the spin integrations over particle 2. Let

$$L = \frac{1}{6} \sum_i L_1 \tag{4}$$

The contribution from the separate terms are as follows:

From i = 1:

$$L_1 = (\xi_1 \mid z \mid \xi_1) \tag{5}$$

From i = 2:

we have

$$(\eta_2 | z | \eta_2) = \frac{1}{3}(\beta_1 \alpha_3 + \alpha_1 \beta_3 | z | \beta_1 \alpha_3 + \alpha_1 \beta_3)$$

$$+ \frac{1}{3}(\alpha_1 \alpha_3 | z | \alpha_1 \alpha_3) \qquad (6)$$

or

$$L_2 = \frac{2}{3}(\xi_2 | z | \xi_2) + \frac{1}{3}(\xi_1 | z | \xi_1) \qquad (7)$$

From i = 3:

we have

$$(\eta_3 | z | \eta_3) = \frac{2}{3}(\xi_2 | z | \xi_2) = \frac{1}{3}(\xi_3 | z | \xi_3) \qquad (8)$$

or

$$L_3 = \frac{2}{3}(\xi_2 | z | \xi_2) + \frac{1}{3}(\xi_3 | z | \xi_3) \qquad (9)$$

From i = 4:

$$L_4 = (\xi_3 | z | \xi_3) \qquad (10)$$

From i = 5:

$$(\eta_5 | z | \eta_5) = \frac{1}{6}(\beta_1 \alpha_3 - 2\alpha_1 \beta_3 | z | \beta_1 \alpha_3 - 2\alpha_1 \beta_3)$$

$$+ \frac{1}{6}(\alpha_1 \alpha_3 | z | \alpha_1 \alpha_3) \qquad (11)$$

now

$$\beta_1 \alpha_3 - 2\alpha_1 \beta_3 = -\frac{1}{2}(\beta_1 \alpha_3 + \alpha_1 \beta_3) + \frac{3}{2}(\beta_1 \alpha_3 - \alpha_1 \beta_3) \qquad (12)$$

thus

$$L_5 = \frac{1}{12}(\xi_2 | z | \xi_2) + \frac{3}{4}(\xi_4 | z | \xi_4) + \frac{1}{6}(\xi_1 | z | \xi_1) \qquad (13)$$

From i = 6:

$$(\eta_6 \,|\, z \,|\, \eta_6) = \tfrac{1}{6}(\alpha_1\beta_3 - 2\beta_1\alpha_3 \,|\, z \,|\, \alpha_1\beta_3 - 2\beta_1\alpha_3)$$

$$+ \tfrac{1}{6}(\beta_1\beta_3 \,|\, z \,|\, \beta_1\beta_3) \qquad (14)$$

hence

$$L_6 = \tfrac{1}{12}(\xi_2 \,|\, z \,|\, \xi_2) + \tfrac{3}{4}(\xi_4 \,|\, z \,|\, \xi_4)$$

$$+ \tfrac{1}{6}(\xi_3 \,|\, z \,|\, \xi_3) \qquad (15)$$

Thus adding all six contributions

$$L = \tfrac{1}{6} \times \tfrac{6}{4} \Big[(\xi_1 \,|\, z \,|\, \xi_1) + (\xi_2 \,|\, z \,|\, \xi_2) + (\xi_3 \,|\, z \,|\, \xi_3)$$

$$+ (\xi_4 \,|\, z \,|\, \xi_4) \Big] \qquad (16)$$

$$L = \tfrac{1}{4} \sum_{i}^{4} (\xi_1 \,|\, z \,|\, \xi_1) \qquad (17)$$

or

$$L = \tfrac{1}{4} \sum_i \sum_f \left| (\xi_f \,|\, \gamma_1 + \gamma_2 \sigma_1 \cdot \sigma_3 \,|\, \xi_1) \right|^2 \qquad (18)$$

Q.E.D.

In the same manner we can prove

Theorem 2:

$$\tfrac{1}{6} \sum_i \sum_f \left| (\eta_f \,|\, \gamma_1 + \gamma_2 \sigma_2 \cdot \sigma_3 \,|\, \eta_i) \right|^2 \qquad (19)$$

$$= \tfrac{1}{4} \sum_i \sum_f \left| (\xi_f \,|\, \gamma_1 + \gamma_2 \sigma_2 \cdot \sigma_3 \,|\, \xi_1) \right|^2$$

where ξ is understood to be the two-particle spin function corresponding to particles 2 and 3. There is no need to give a detailed proof of theorem 2 since the equivalence of particles 1 and 2 as far as spin is concerned is evident from their treatment.

Appendix C:

Theorem

$$\frac{1}{6} \sum_i \sum_f \left| (\eta_f \left| (1-I_{23}) \right| \eta_i) \right|^2 = \frac{1}{4} \sum_i \sum_f \left| (\xi_f \left| (1-I_{23}) \right| \xi_i) \right|^2 \quad (1)$$

Proof:

The operator $(1-I_{23})$ may be written as

$$(1-I_{23}) = (1-S_{23}\ P_{23}) \quad (2)$$

where we have separated I_{23} into spin and space part. Now we wish to show the relation between S_{23} and $\sigma_2 \cdot \sigma_3$. Consider $2\ (1-S_{23})$ operating on a function symmetric in the spin of particles 2 and 3.

$$2(1-S_{23})\eta_s = 0 \quad (3)$$

and when it operates on an antisymmetric function

$$2(1-S_{23})\eta_A = 4\eta_A \quad (4)$$

Now consider what the action of $(\sigma_2 \cdot \sigma_3 + 3)$ is

$$(\sigma_2 \cdot \sigma_3 + 3)\eta_S = 0 \quad (5)$$

$$(\sigma_2 \cdot \sigma_3 + 3)\eta_A = 4\eta_A \quad (6)$$

thus

$$2\ (1-S_{23}) = (\sigma_2 \cdot \sigma_3 + 3) \quad (7)$$

or

$$S_{23} = -\frac{1}{2}\left(1 + \sigma_2 \cdot \sigma_3\right) \tag{8}$$

thus $(1-I_{23})$ may be written as

$$(1-I_{23}) = \left(1 + \frac{1}{2}\left[1 + \sigma_2 \cdot \sigma_3\right] P_{13}\right) \tag{9}$$

but as far as the spin is concerned this is of the form

$$(1-I_{23}) = \gamma_1 + \gamma_2\, \sigma_2 \cdot \sigma_3 \tag{10}$$

and hence theorem 2 of Appendix B may be applied.

Q.E.D.

Appendix D:

In this appendix we shall prove the normalization constant of $\tilde{\phi}$. We know that

$$\tilde{\phi} = \tilde{\phi}_i + \tilde{\phi}_{sc} \tag{1}$$

Now by assumption most of the wave function is still ϕ_i ; so that it will be sufficient to determine the normalization of .
Now

$$\tilde{\phi}_i = C(1-I_{23})\,\phi_i \tag{2}$$

where C is the normalization constant we wish to determine.

Then

$$C^2((1-I_{23})\,\phi_i,\ (1-I_{23})\,\phi_i) = 1 \tag{3}$$

or

$$2C^2(\phi_i,\ \phi_i) - 2C^2(\phi_i, I_{23}\,\phi_i) = 1 \tag{4}$$

We shall now prove that $(\phi_i, I_{23}\,\phi_i)$ is zero to the approximation we are interested in. In particular this means that we must prove that terms arising from $(\phi_i, I_{23}\,\phi_i)$ are not of order $1/p_o^2$ or lower.

Consider now that

$$\begin{aligned}(\phi_i, I_{23}\,\phi_i) = \frac{1}{V^2} \sum_S \int & e^{-i/\hbar\, p_o \cdot r_3}\,\chi(r_1-r_2) \\ & e^{+i/\hbar\, p_o \cdot r_2}\,\chi(r_1-r_3)\,\eta_i(S)\,\eta_i(I_{23}S)\end{aligned} \tag{5}$$

Let

$$\chi(r_1 - r_2) = \frac{1}{h^{3/2}} \int e^{i/\hbar\, p_d(r_1 - r_2)} \; \Phi(p_d)\, d\vec{p_d} \tag{6}$$

then

$$(\phi_i, I_{23}\phi_i) = \frac{1}{h^3\, V^2} \sum_S \int e^{-i/\hbar\,(p_d \cdot r_1 - p_d \cdot r_1)}\, e^{-i/\hbar\, p_o \cdot (r_3 - r_2)}$$
$$e^{+ i/\hbar\, p_d(r_2 - r_3)} \; \Phi^*(p_d) \; \Phi(p_d) \eta_i(S) \eta_i(I_{23}S) \tag{7}$$

or carrying out the integrations

$$(\phi_i, I_{23}\, \phi_i) = \Phi(p_o) \quad \Phi^*(p_o) \sum_S \; \eta_i(S)\eta_i(I_{23}S) \tag{8}$$

Now examine the properties of $|\Phi|^2$. We are interested with what inverse power of p_o it vanishes. Now we know that $\int |\Phi(p_o)|^2 d\vec{p_o}$ must be finite, since in a deuteron there must be a finite total chance of finding the given momentum state. Thus $|\Phi(p_o)|^2$ must go at least as $1/p_o{}^4$ to have the integral converge. Hence to our approximation

$$(\phi_i, I_{23}\, \phi_i) = 0 \tag{9}$$

and hence

$$C = 1/\sqrt{2} \tag{10}$$

Appendix E:

For the sake of completeness we shall describe here how $\overline{p_d^2}$ and $\left(\overline{\frac{1}{r_d^2}}\right)$ were estimated.

1. Calculation of $\overline{p_d^2}$:

Assume a square well potential for the deuteron. The symbols are the conventional ones, and the relations from the Bethe - Bacher article* have been freely used.

Thus

$$-\epsilon = \left(\frac{\overline{p_d^2}}{M}\right) + \overline{V} \tag{1}$$

where

$$\overline{V} = \frac{\int_0^\infty V(r)\psi^2 r^2\, dr}{\int_0^\infty \psi^2 r^2\, dr} \tag{2}$$

then

$$\overline{V} = \frac{-V_o \int_0^{r_o} u^2 dr}{\int_0^{r_o} u^2 dr + \int_{r_o}^\infty u^2 dr} \tag{3}$$

now let

$$u = \sin kr \quad \text{for } r < r_o \tag{4}$$

$$u = \sin kr_o\, e^{-\alpha(r-r_o)} \quad \text{for } r \geq r_o \tag{5}$$

Then we find that

$$\overline{V} = \frac{-V_o}{1 + (1/R)} \tag{6}$$

*Bethe and Bacher, Rev. of Mod. Phys., 8, 112, (1936) where

$$R = \left(r_o - \frac{\sin 2\, kr_o}{2k}\right) \Big/ \left(\frac{\sin^2 kr_o}{\alpha}\right) \tag{7}$$

or

$$R = \frac{\alpha\, r_o}{\sin^2 kr_o} + \frac{\alpha^2}{k^2} = \alpha\, r_o\left(1 + \frac{\epsilon}{V_o - \epsilon}\right) + \frac{\epsilon}{V_o - \epsilon} \tag{8}$$

hence

$$\overline{V} = -V_o \left(\frac{\alpha\, r_o}{1 + \alpha r_o}\right) \left(\frac{V_o - \epsilon}{V_o}\right) - \epsilon \tag{9}$$

hence

$$\left(\frac{\overline{p_d^2}}{M}\right) = -V_o \left(\frac{\alpha r_o}{1 \quad \alpha\, r_o}\right) \left(\frac{V_o - \epsilon}{\epsilon}\right) \tag{10}$$

If we substitute the following values into (10):

$$\begin{aligned} \epsilon &= 2.18 \text{ Mev} \\ V_o &= 21.3 \text{ Mev} \\ r_o &= 2.80 \times 10^{-13} \text{cm} \end{aligned} \tag{11}$$

then $\alpha\, r_o = 0.64$ and

$$\left(\frac{\overline{p_d^2}}{M}\right) = 7.46 \text{ Mev} \tag{12}$$

2. Calculation of $\left(\frac{\overline{1}}{r_d^2}\right)$:

Using the same assumptions as above we may write

$$\left(\frac{\overline{1}}{r_d^2}\right) = B_1 + B_2 \tag{13}$$

where

$$B_1 = b^2 \int_0^{r_o} \frac{\sin^2 kr}{r^2}\, dr \qquad (14)$$

and

$$B_2 = C^2 \int_{r_o}^{\infty} \frac{e^{-2a(r-r_o)}}{r^2}\, dr \qquad (15)$$

and

$$C = \frac{2(V_o-\epsilon)a}{V_o(1+a r_o)} \qquad (16)$$

$$b = \left(\frac{V_o}{V_o-\epsilon}\right)^{1/2} C \qquad (17)$$

we may re-express B_1 and B_2 as

$$B_1 = b^2 k \left[S_i(2kr_o) - \frac{\sin^2 kr_o}{kr_o} \right] \qquad (18)$$

$$B_2 = C^2 \left[\frac{1}{r_o} - 2a\, e^{2ar_o}\, E_1(2a r_o) \right] \qquad (19)$$

Numerical computation of B_1 and B_2 was carried through for two values of r_o:

	$r_o = 1.7 \times 10^{-13}$cm	$r_o = 2.8 \times 10^{-13}$cm
V_o	48.4 Mev	21.3 Mev
B_1	0.450×10^{26} cm^{-2}	0.250×10^{26}cm^{-2}
B_2	0.084×10^{26} cm^{-2}	0.032×10^{26}cm^{-2}
$\left(\overline{\frac{1}{r_d^2}}\right)$	0.534×10^{26} cm^{-2}	0.282×10^{26}cm^{-2}

For the purposes of section VI we have used the value derived from the conventional value of the range, namely from $r_o = 2.8 \times 1$[illegible] cm.

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