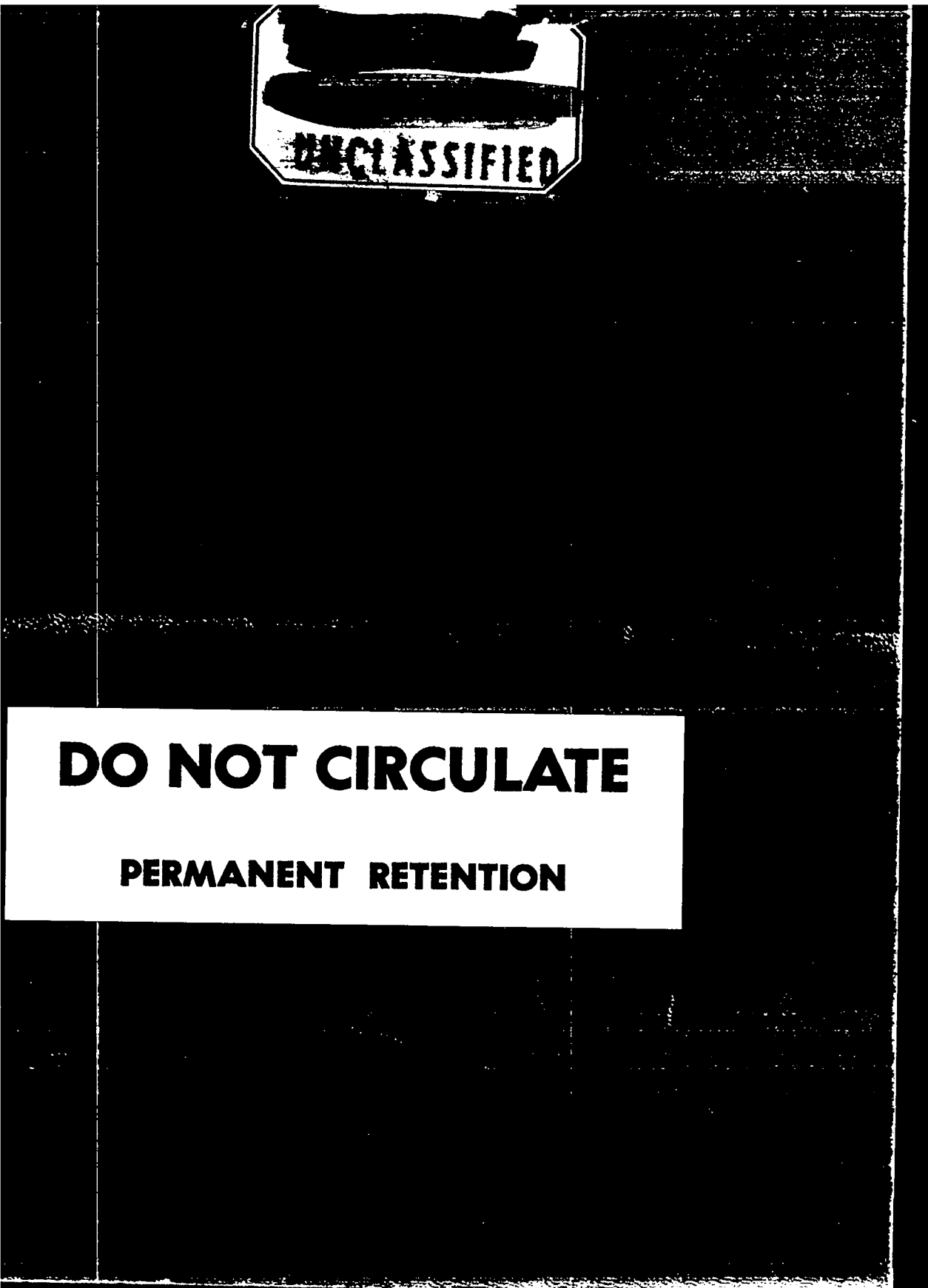

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THE CALCULATION OF CRITICAL MASSES INCLUDING THE EFFECTS OF THE DISTRIBUTION OF NEUTRON ENERGIES

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ABSTRACT

Convenient approximate methods are developed for the calculation of critical sizes and multiplication rates of spherical, active cores surrounded by infinite tampers. Special attention is given to those problems arising from the fact that neutrons of different velocities have different properties. The methods consist essentially of approximating the neutron densities at each velocity by fundamental mode shapes for each velocity. Each one of these shapes forms a source of neutrons of all velocities, and simple considerations of the number and spatial distribution of absorptions from these sources determine the critical size at which equilibrium between production and absorption is achieved. The accuracy obtained is apparently sufficient for all practical purposes. Modifications of the method for systems of shapes other spherical or having finite tampers are discussed.

The Calculation of Critical Masses Including the Effects of the Distribution of Neutron Energies.

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APPENDICES

Introduction

The characteristic neutron problem is that of determining the critical radius of an active, spherical, homogeneous core surrounded by a finite or infinite, homogeneous tamper. The cross sections in general will be energy dependent and the neutron spectrum will not be monochromatic, both because of the spread of the fission spectrum and because of the existence of inelastic scattering in core and tamper. In some cases the neutron energy spectrum can be considered monochromatic, and the problem is then easily soluble by various standard methods. In other cases, two effective neutron energies can be considered to exist and the resulting Boltzmann equation solved by the use of one of the spherical harmonic approximations. In principle, better approximations could be obtained by using three or more effective neutron energies, but the work then becomes prohibitively difficult. Some method of calculation is clearly necessary which will obviate these difficulties and at the same time retain sufficient accuracy to be useful.

The methods described in this report were developed specifically for the easy calculation of critical masses and multiplication rates of hydride assemblies, but their application is of much greater generality. It is believed that they will be found to have great usefulness in the treatment of all neutron problems in which an active, spherical core, with or without hydrogen content, is surrounded by an infinite or finite tamper which may or may not scatter neutrons inelastically.

These methods are not exact, but in all cases of interest thus far investigated their accuracy has been found ample for all practical purposes. Finally, they are extremely simple for computation and, once mastered, quite simple to use in thinking about a wide variety of neutron problems in which the energy spread of the neutrons is an essential feature.

Ch. I. Properties of Systems with Monoenergetic Neutrons.

Section 1 The Fundamental Integral Equation.

We shall find it very useful in our later work to keep in mind certain important features of the simple one-velocity neutron problem. Let us consider an active core surrounded by a tamper. There may be spherical symmetry but this is not necessary We consider the case of isotropic scattering, since this is simple and usually all that is needed. The tamper is characterized by a total cross section and by an absorption cross section. The core is characterized by a total cross section σ, and by a number f, which is the number of extra neutrons emitted per collision. These extra neutrons are emitted isotropically at a single energy.

Suppose a single neutron is emitted isotropically at the point $\vec{x}'$ in the core. We define a quantity $K(\vec{x}' \longrightarrow \vec{x})\, d\vec{x}$ which is the probability that the neutron emitted at $\vec{x}'$ makes its first collision in the core, irrespective of intervening collisions in the tamper, at $\vec{x}$ in the volume element $d\vec{x}$. The function K depends on the properties of the tamper, the radius of the core, and the total cross section of the core.

We imagine a neutron flux density present in the core and tamper, time independent, and defined in the core as the function $\psi(\vec{x})$ We can write the density of collisions per unit time in the core as $\sigma\psi(\vec{x})$. Using the function K , we can also write an expression for this collision density as a sum of collisions arising at the point $\vec{x}$ from neutrons liberated at all other points of the core from collisions and then proceeding to $\vec{x}$ to make first collisions. This obviously gives an integral equation for ψ :

$$\sigma\psi(\vec{x})d\vec{x} = \int(1+f)\sigma\psi(\vec{x}')\,d\vec{x}'\,K(\vec{x}'\to\vec{x})d\vec{x}$$

or

$$\psi(\vec{x}) = (1+f)\int d\vec{x}'\,K(\vec{x}'\to\vec{x})\,\psi(\vec{x}') \qquad (1.1)$$

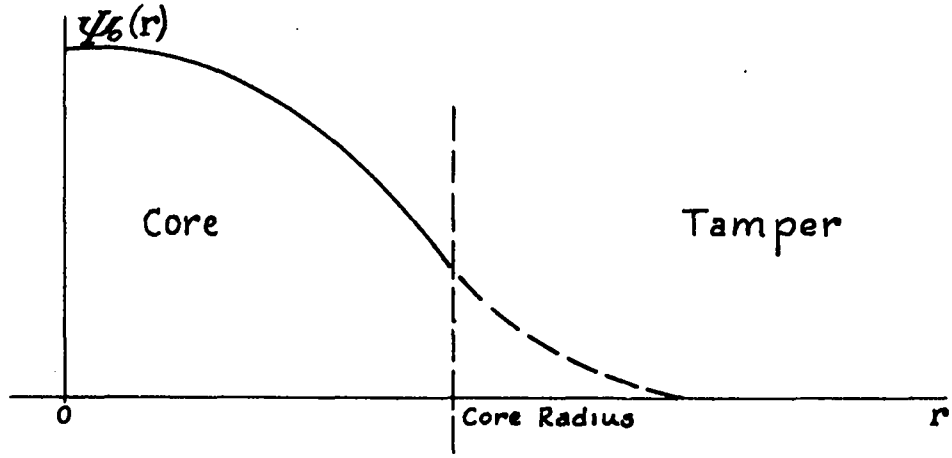
The integration extends only over the core.

Certain general properties of this integral equation follow easily. The kernel K is clearly symmetric. This results from the fact that if one considers the totality of paths from $\vec{x}'$ to $\vec{x}$, it is seen that the reverse of each path is just as likely as the forward path, and the totality of backward paths will then have the same total probability as the totality of forward paths.

Section 2 Character of the eigenfunctions and eigenvalues.

Only for a discrete sequence of values of f can solutions be obtained. There will be a lowest f , called f_o , for which the density ψ_o is everywhere positive. In a system with spherical symmetry it will have the shape (approximately) of

$\frac{\sin kr}{r}$, with kr equal to something between $\frac{\pi}{2}$ and π at the surface of the core. The function ψ_0 will look more or less as follows:



The extension of ψ_0 into the tamper is indicated by the dotted curve. This extension of course has essentially the shape $\frac{e^{-hr}}{r}$, where h is the characteristic decay constant determined by the total and absorption cross sections in the tamper. There will be the usual rapidly decaying transition effects at the core-tamper interface.

The density functions corresponding to higher values of f (f_n being associated with ψ_n) will have one or more nodal surface, the neutron density being positive in some regions and negative in others. Negative neutron densities have physical meaning only as deficiencies below some positive neutron density, but physical reasoning can be used in interpreting the integral equation if negative neutrons are thought of as actual particles whose presence in a region can cancel the presence of an equal number of positive neutrons. From this point of view, it is clear that increasing the number of nodal surfaces in ψ will increase

the value of f required to maintain this ψ. This is so because leakage of neutrons of opposite sign into one another will become more rapid as the density function oscillates more rapidly. A larger net production of neutrons per collision will then be required to compensate these larger losses by leakage.

Section 3 Interpretation of the eigenfunctions and eigenvalues.

Another useful interpretation of the eigenvalue $1+f$ exists. The integral equation says that an isotropic source of neutrons distributed spatially in the core as the function ψ_n has the property that all the resulting first collisions are distributed also as ψ_n. Further, it says that if $(1+f_n)$ or ψ_n neutrons are released per unit time per unit volume, or ψ_n first collisions result per unit time per unit volume. This is equivalent to saying that a source density ψ_n gives rise to a density of first collisions $\frac{1}{1+f_n}\psi_n$, or stated more simply, one neutron released in a distribution ψ_n gives $\frac{1}{1+f_n}$ first collisions, having also the spatial dependence of ψ_n.

The eigenfunctions ψ_n, since K is symmetric, have the useful property of being mutually orthogonal when integrated over the core. We shall assume, and it is certainly true for cases of physical interest, that the functions ψ_n form a complete set so that any reasonable function defined over the core can be written as a convergent series of ψ_n.

Section 4 Relation to the Milne Equation.

It is of some interest to derive the relation between the integral equation (1.1) and the more usual way of writing the integral equation for the two medium problem. We let σ^* be the total cross section in the tamper and g be the fraction of tamper collisions which result in absorption. We denote the ordinary Milne kernel between two points $\vec{x}$ and $\vec{x}'$ (core or tamper) by $M(\vec{x}' \to \vec{x})$. This kernel is of course symmetric since it is just the neutron flux density at $\vec{x}$ arising from a unit source at $\vec{x}'$. We write $\psi(\vec{x})$ and $\phi(\vec{x})$ for the neutron densities in the core and tamper respectively and obtain:

$$\psi(\vec{x}) = (1+f)\sigma \int_{core} d\vec{x}' M(\vec{x}' \to \vec{x}) \psi(\vec{x}')$$

$$+(1+g)\sigma^* \int_{tamper} d\vec{x}' M(\vec{x}' \to \vec{x}) \phi(\vec{x}')$$

$$\phi(\vec{x}) = (1+f)\sigma \int_{core} d\vec{x}' M(\vec{x}' \to \vec{x}) \psi(\vec{x}')$$

$$+(1-g)\sigma^* \int d\vec{x}' M(\vec{x}' \to \vec{x}) \phi(\vec{x}') \qquad (4.1)$$

It is clear that if solutions of the equations (4.1) are obtained for two different values of f, these solutions will be

orthogonal with respect to integration over the core only. It is further clear that the set of functions ψ obtained from the equations (4.1) by giving $1+f$ its various allowed values will be identical with the previously defined set of ψ_n . This procedure is definitely contrary to the usual one of increasing $1+f$ and $1-g$ in proportion and thus obtaining solutions orthogonal over both core and tamper. It will be seen that the procedure we have adopted seems to be much better suited to our purposes. Of course, the kernel K can be expressed in terms of the kernel M . If we imagine the core function M to be known, we can use the second of the equations (4.1) to solve in principle for the tamper function ϕ . We can then insert this function ϕ in the first of the two equations to obtain a single integral equation for ψ . We can then identify the kernel of this equation with K .

Section 5 Equivalent eigenvalue problems.

The relations of the one-velocity problem can be written in a somewhat different notation which is convenient for certain purposes. We can imagine that a part σ_a of the total cross section σ results in a process of absorption. We can then ask, how many neutrons ν_n must be liberated for each neutron absorbed in order to maintain the critical condition (in the nth mode). For each nuetron absorbed ν_n are liberated and hence the net number of neutrons liberated per absorption is $\nu_n - 1$, and per collision is $(\nu_n - 1)\frac{\sigma_a}{\sigma}$ since a fraction $\frac{\sigma_a}{\sigma}$ of all collisions

are absorption. But previously we have called f_n the net number of neutrons liberated per collision, so that the relation between our two notations is

$$f_n = (\nu_n - 1)\frac{\sigma_a}{\sigma} \tag{5.1}$$

Another interpretation of ν_n is that for each neutron liberated in the distribution ψ_n, the number $\frac{1}{\nu_n}$ are reabsorbed (with the same distribution) by the cross section σ_a. This idea is made use of in the next section.

This question can also be looked at more formally. If a kernel $P(\vec{x}' \to \vec{x})$ is defined as the density of absorptions at $\vec{x}$ produced by one neutron emitted at $\vec{x}'$, it is easily seen that P can be written as a power series in the kernel K (matrix multiplication) and therefore the kernels K and P both have the ψ_n for their eigenfunctions, and the eigenvalues of P are simply related (by equation 5.1) to the eigenvalues of K.

Section 6. Absorption of neutrons from an arbitrary source.

We are now in a position to answer a question of great interest in our later work. Neutrons are emitted isotropically with a space density $S(\vec{x})$. We wish to determine the resulting density of absorptions $A(\vec{x})$. We write S as a series in the ψ_n:

$$S = \sum_n S_n \psi_n \tag{6.1}$$

It is then clear from our previous work that each ψ_n will reproduce itself in the absorption density, but with a reduced strength $\frac{1}{\nu_n}$. We have:

$$A = \sum_n \frac{1}{\nu_n} S_n \psi_n \qquad (6.2)$$

The series for S is assumed convergent, and the ν_n form an increasing sequence, so that the series for A converges more rapidly than that for S. Physically, the distribution of absorptions is always smoother, that is closer to ψ_0, than the source which gave rise to the absorptions.

The ideas which we have developed will be of great importance for the methods to be presented, but we still need a way of getting numerical values for the f_n or ν_n, and useful shapes for the ψ_n. We shall find that only f_0, ν_0, and ψ_0 are needed, and these can now be obtained with some ease for the case of spherical symmetry with an infinite tamper. Given the tamper constants, the core radius and the total core cross-section, f_0 and hence ν_0 can be obtained by the simple procedures given in LA-173 or LA-234. The eigenfunction ψ_0 will always be approximated by $\frac{\sin kx}{x}$ where k is the infinite medium wave number calculated from the f_0. These approximations will generally be good. Where they are not, care will be taken to point out the reason and possible improvements.

II. The Treatment of Many-Velocity Problems When No Inelastic Scattering is Present in the Tamper.

In attempting to solve critical problems involving a continuum of velocities, we shall begin by finding what problems we can solve exactly. We restrict ourselves in this chapter to problems in which the tamper absorbs and scatters elastically but does not have any cross section for inelastic scattering, fission, absorption, and elastic scattering. All processes are assumed isotropic.

Section 1 Case of constant mean free path.

If, in addition to these requirements, we ask that all cross sections in the tamper and the total cross section in the core be independent of neutron energy, we obtain a problem exactly soluble in terms of the presumably known solutions of the one-velocity problem. To show this we first notice that, because of our restrictions on the cross sections, a one-velocity eigenfunction ψ_0 and eigenvalue $1+f_0$ calculated with the constants at any velocity will be identical with those calculated at any other velocity. This is so because the eigenfunctions and eigenvalues of the one-velocity problem defined by equation (1.1) of Chapter I depend only on the total cross section in the core, the radius of the core, and the properties of the tamper.

We now assert that with these restrictions, if the value of ν is such that the system is critical, neutrons of every velocity will have the same spatial distribution ψ_0. This can be seen by remembering the properties of the eigenfunctions. Neutrons of all velocities will be making collisions which also have the

spatial dependence ψ_0 . From each collision neutrons come out isotropically with an energy distribution determined by the cross sections. We then have an isotropic source of neutrons of all velocities spatially distributed as ψ_0 . The last step in the argument consists of the observation that the first collisions from such a source will also have the shape of ψ_0 . Our assertion is therefore self-consistent.

We can find the value of ν required for criticality in the following way. Suppose that when a neutron of velocity v' makes a collision in the core, a number $S(v' \to v)dv$ are emitted isotropically in the velocity range dv . The function $S(v' \to v)$ will involve the various core cross sections, the spectra of fission neutrons and inelastically scattered neutrons, and the number of neutrons, ν , emitted per fission. Call $\sigma_e(v)$ the cross section for elastic scattering $\sigma_f(v)$ that for capture, $\sigma_c(v)$ that for fission from which ν neutrons are liberated with spectrum $\chi(v)$ and σ_i that for inelastic scattering. If a neutron of velocity v is scattered inelastically, suppose the spectrum resulting is $\phi(v \to v')$. In this case,

$$\sigma = \sigma_e + \sigma_f + \sigma_i + \sigma_c$$

and

$$\sigma(v') \cdot S(v' \to v) = \sigma_e(v')\,\delta(v'-v) + \sigma_f(v')\nu\,\chi(v) + \sigma_i(v')\phi(v' \to v) \quad (1.1)$$

where $\delta(v'-v)$ is Dirac's delta function and insures that neutrons scattered elastically do not alter their velocity.

Let $N(v)$ be the total number of neutrons present in the core per unit range of v, multiplied by v.

The total number of collisions made at velocity v' in the range dv' per second is $N(v')\sigma(v')dv'$ where $\sigma(v')$ is the total core cross section. (actually σ, at present, is considered as independent of v'). From each of these collisions, $S(v' \to v)\,.dv$ neutrons emerge in the velocity range dv and a fraction $\frac{1}{1+f_0}$ of these make first collisions in the core. We can therefore express by an integral equation the fact that all collisions at velocity v arise by the process just described from collisions at all velocities v'. We have:

$$N(v)\sigma(v)dv = \int_0^\infty dv' \frac{S(v' \to v)dv}{1+f_0} N(v')\sigma(v')$$

or

$$N(v)\sigma(v) = \frac{1}{1+f_0} \int_0^\infty dv' S(v' \to v) N(v')\sigma(v') \qquad (1.2)$$

This equation will have a solution only for one value of ν, (which is contained implicitly in the function $S(v' \to v)$ and this value would then just keep the system critical. We could just as well keep ν fixed at its correct value and inquire what the radius of the core must be to make $1+f_0$ such as to permit a solution of the equation.

The calculation can also be made in a somewhat different but equivalent way. This has the advantage of removing the delta function in (11).

Let us consider the elastic cross section as distinct from the rest of the cross section, which we will think of as an absorption cross section $\sigma_a(v)$. The processes included in this absorption cross section are all those which remove neutrons from the velocity v so that $\sigma_a = \sigma_f + \sigma_i + \sigma_c$ We let $\nu(v)$ be the value of ν required to keep a one-velocity problem critical if $\sigma_a(v)$ is considered as absorption, as defined in section 1-5. Suppose that when a neutron of velocity v' suffers an absorption, $S_a(v' \to v)dv$ neutrons come out isotropically in the velocity range dv .

$$\sigma_a(v') S_a(v' \to v) = \sigma_f(v') \nu \chi_f(v) + \sigma_{in}(v') \phi(v' \to v) \tag{1.3}$$

We have for the total number of absorptions per second of neutrons in the velocity range dv , $N(v')\sigma_a(v')dv'$. Each of these emits $S_a(v' \to v)dv$ neutrons, a fraction $\frac{1}{\nu(v)}$ of which are reabsorbed (by equation (5.1) of Chapter I). We can again write an integral equation, expressing the fact that all absorptions at velocity v arise by the above process from absorptions at all velocities. We have:

$$N(v)\sigma_a(v)dv = \int_0^\infty dv' \frac{S_a(v' \to v)dv}{\nu(v)} N(v')\sigma_a(v')$$

or:

$$N(v)\sigma_a(v) = \frac{1}{\nu(v)} \int_0^\infty dv' S_a(v' \to v) N(v')\sigma_a(v') \tag{1.4}$$

The function $\nu(v)$ depends on the radius of the core and the function $S_a(v' \to v)$ contains the real ν for fission. The equation will not in general have a solution, but we can obtain one by properly adjusting either ν or the core radius, in which case we shall have made the system critical.

The equations (1.2) and (1.4) are rigorously equivalent, as can be proved immediately from (1.1) and (1.3) and the relation of f and ν given in (5.1), Chapter I. Equation (1.4) is more convenient to use because of the lack of the delta function in S_a.

The solution of this equation 1.4 is discussed in more detail in section 4, but we can solve it here in a particularly simple special case, namely when there is no inelastic scattering in the core. (We shall also assume $\sigma_c = 0$ but this is really not necessary). In this case, since $\sigma_a = \sigma_f$ Equation 1.4 becomes:

$$N(v)\sigma_f(v) = \frac{\nu\chi(v)}{\nu(v)} \int_0^\infty dv' N(v')\sigma_f(v') \tag{1.5}$$

This equation can be solved by noting that

$$N(v)\sigma_f(v) = c\,\frac{\chi(v)}{\nu(v)} \tag{1.6}$$

where c is a constant, independent of v. Now we substitute this solution in equation (1.5) and obtain:

$$c\,\frac{\chi(v)}{\nu(v)} = c\,\frac{\nu\chi(v)}{\nu(v)} \int_0^\infty dv\,\frac{\chi(v')}{\nu(v')} \tag{1.7}$$

This obviously requires:

$$\frac{1}{\nu} = \int_0^\infty dV \, \frac{\chi(V)}{\nu(V)} \qquad (1.8)$$

This equation has a simple interpretation. In a critical system, the neutrons liberated from a single fission must produce exactly one fission in the first generation. Multiplying equation (1.8) through by ν, we have exactly this statement for the simple system considered. This is so because $\nu\chi(V)$ is the number of neutrons emitted in the velocity range dV in one fission and there is no inelastic scattering. By our previous arguments, a fraction $\frac{1}{\nu(V)}$ of those neutrons emitted at velocity V will be absorbed to produce fission. The total number of fissions coming in one generation from the neutrons released from one fission is then clearly equal to $\int_0^\infty dV \, \frac{\nu\chi(V)}{\nu(V)}$ which must be equal to unity as stated by equation (1.8).

Section 2 Variable cross section. The first lower approximation.

The restriction to constant cross sections in the tamper and constant total cross section in the core, which is necessary for the validity of equations (1) and (2), is of course too stringent to include most cases of interest. We shall, however, use the arguments employed in the derivation of these equations to give a useful approximate treatment of less restricted problems. Suppose that the tamper cross sections and the core total cross sections vary with neutron velocity in addition to the variations permitted in the preceding treatment. We can no longer argue that we know the spatial distribution of neutrons of each velocity.

This is so because the eigenfunctions calculated with the constants at different velocities are no longer the same, and the consistency argument that all neutrons are distributed as the common ψ_0 will therefore fail.

We proceed by defining a set of functions $\psi_n(\vec{x},v)$ which are the eigenfunctions calculated from the constants for neutrons of velocity v. The set of functions for each velocity is assumed complete, and in particular any one of the ψ_n at one velocity can be expanded in a convergent series of the ψ_n for any other velocity. In any reasonable system, a given ψ_n will not vary drastically with neutron velocity. This is especially true of ψ_0, the eigenfunction with no zeros. In a system with spherical symmetry, for example, the function ψ_0 will be approximately of the form $\frac{\sin kr}{r}$ where the phase of the sine function must be at least $\frac{\pi}{2}$ and somewhat less than π, in all interesting cases. Usually conditions will be much less extreme, so that we make the assumption that the change in shape of ψ_0 with velocity is small in a sense to be defined more carefully later.

It can now be seen that a reasonable approximation for the spatial distribution of neutrons of velocity v is to say that it has the shape of $\psi_0(\vec{x},v)$. Suppose that this were strictly true. Neutrons of velocity v' make collisions distributed as $\psi_0(\vec{x},v')$. From these collisions neutrons of all velocities emerge isotropically. Consider those with velocity v. We have an isotropic source of neutrons of velocity v distributed as $\psi(\vec{x},v')$ and must then

calculate the spatial distribution of absorptions from this source. We use the method of Chapter I and expand the source density in the eigenfunctions of velocity v :

$$S(\vec{x}) = \psi_0(\vec{x}, v') = \sum_n S_n \psi_n(\vec{x}, v) \tag{2.1}$$

The density of absorptions, (or first collisions, if any collision is called absorption), will be:

$$A(\vec{x}) = \sum_n \frac{S_n}{\nu_n(v)} \psi_n(\vec{x}, v) \tag{2.2}$$

where the $\nu_n(v)$ are those defined by equation (5.1) of Chapter I. The shape of $A(\vec{x})$ will be very close to that of $\psi_0(\vec{x}, v)$ for two reasons. In the first place, most of the contribution to the expansion of $S(\vec{x})$ comes from the term in $\psi_0(\vec{x}, v)$. This is so because of our assumption that $\psi_0(\vec{x}, v)$ and $\psi_0(\vec{x}, v')$ do not differ much in shape. In the second place, the sequence of the $\nu_n(v)$ increases rapidly as n increases, so that the series for $A(\vec{x})$ converges much more rapidly than that for $S(\vec{x})$. Some numerical illustrations of the validity of these statements can be found in Appendix I.

It is clear then that our assumption of shape $\psi_0(\vec{x}, v)$ for the neutron distribution at velocity v is nearly self-consistent but not completely so. It is exactly so when $\psi_0(\vec{x}, v)$ is independent of v and, as we shall see, is good enough for most purposes when ψ_0 varies in a reasonable way with v .

With this assumption, we can write an equation analogous to equation (1.4) which will however no longer be exact. With the notation used in deriving equation (1.4), and the approximate assumption that neutrons of velocity v emitted from absorptions at velocity v' are emitted in the spatial distribution $\psi_0(\vec{x},v)$, we will have an equation of exactly the same form as equation (1.4):

$$A(v) = \frac{1}{\nu(v)} \int_0^\infty dv' \, S_a(v' \to v) \, A(v') \qquad (2.3)$$

where we have set $A(v) = N(v)\sigma_a(v)$. Thus $A(v)$ has the significance of being the total number of neutrons being absorbed at velocity v.

This is of course no longer an exact equation. It is very nearly correct and can be used for nearly all practical problems. We shall call it the first lower approximation.

Section 3 First upper approximation.

Another equation can be derived by making an approximation no worse than that used in obtaining equation (2.3). We assume that only those absorptions which occur at velocity v' in the distribution $\psi_0(\vec{x},v')$ are to be counted as producing absorptions at velocity v. If $\psi_0(\vec{x},v)$ is assumed to be normalized so that its integral over the core is unity, then we must write:

$$\psi_0(\vec{x},v') = \sum_n c_n \psi_n(\vec{x},v)$$

We are interested in C_o , which is:

$$C_o(v,v') = \frac{\int \psi_0(\vec{x},v')\,\psi_0(\vec{x},v)\,d\vec{x}}{\int \psi_0^2(\vec{x},v)\,d\vec{x}}$$

As before, $N(v')\,\sigma_a(v')\,S_a(v'\to v)\,dv'dv$ neutrons are emitted per second from absorptions at velocities in the range dv to final velocities in the range dv'. With the approximation we are now making, a fraction $C_o(v',v)$ of these form a source for absorptions at velocity v and a fraction $\frac{1}{\nu(v)}$ of these actually are absorbed at velocity v. This reasoning yields the integral equation:

$$N(v)\,\sigma_a(v) = \frac{1}{\nu(v)} \int_0^\infty dv'\,S_a(v'\to v)\,C_o(v,v')\,N(v')\,\sigma_a(v') \qquad (3.1)$$

The equation can be made simpler and more symmetrical if $N(v)\,\sigma_a(v)$ is replaced by $\frac{A(v)}{\sqrt{\int_0^\infty \psi^2(\vec{x},v)\,d\vec{x}}}$ so that, except for the new normalization, $A(v)$ is the number of neutrons absorbed in the shape $\psi_0(x,v)$. In this case it becomes,

$$A(v) = \frac{1}{\nu(v)} \int_0^\infty dv'\,S_a(v'\to v)\,M_o(v,v')\,A(v')\,dv' \qquad (3.2)$$

where

$$M_o(v,v') = \frac{\int \psi_0(\vec{x},v')\,\psi_0(\vec{x},v)\,d\vec{x}}{\sqrt{\int \psi_0^2(\vec{x},v')\,d\vec{x}} \cdot \sqrt{\int \psi_0^2(\vec{x},v)\,d\vec{x}}} \qquad (3.3)$$

and the ψ functions used in calculating M need not be normalized in any particular manner. The equation (3.2) we call the first upper approximation.

We have two equations (2.3) and (3.2) which represent different approximations to the many velocity problem. We shall attempt (the mathematical rigor is not all that may be desired) in Appendix II to show that the two approximations give results which are on opposite sides of the truth. In the very large number of different practical cases which have been tried the equation 2.3 gives a value of the ν needed for criticality, or the critical radius which is too low, and the equation 3.2 gives results too high. Therefore we have a method by which we can bracket the correc answer with confidence. This permits us, for example, to put an upper limit on the error made by using 2.3 (which is the simpler to solve). It is only necessary to investigate the change made on using (3.2).

It is true that the two equations (2.3) and (3.2) in all practical cases tried have given remarkably close results. In appendix III and IV we give a few representative examples. The values of $\nu - 1$ calculated by the two methods rarely differed by more than 4%. Thus the use of equation (2.3) alone can be relied on for great accuracy. If any doubt of the accuracy exists, it may be tested by a comparison with (3.2).

The reason that the two equations give nearly the same results is not far to seek. The equations differ only by the factor $M_0(v, v')$ in equation 3.2. If this factor were exactly 1 the two equations would be identical, and, since they bracket the correct answer, exact. But $M_0(v, v')$ can only be exactly 1 if $\psi_0(x, v) = \psi_0(x, v')$, that is if the shape of the fundamental made were the same at all velocities. This is

the case when the total cross sections are velocity-independant in which case (2.3) is indeed exact as we showed in section 1, equation (1.4). When, however, $\psi_0(x,v)$ differs from $\psi_0(x,v')$, the Schwartz inequality shows that $M_0(v,v')$ must be less than one. However, except when the functions differ to an extreme degree, the amount by which M falls short of unity is very small. Some representative values are given in the accompanying table. The value of M depends only on the shapes of the functions $\psi_0(\vec{x},v)$ and $\psi_0(\vec{x},v')$. These functions depend only on the radius r. If the functions are of shape $1-pr^2$ and $1-qr^2$ respectively, the value of M is

$$M = \frac{1 + xPQ}{\sqrt{1 + xP^2}\sqrt{1 + xQ^2}}$$

where

$$x = \frac{12}{175} = 0.069, \quad P = \frac{p}{1-\frac{3}{5}p} \quad \text{and} \quad Q = \frac{q}{1-\frac{3}{5}q}$$

TABLE

$\psi_0(x,v)$	$\psi(x,v')$	M
1	$1 - 1/3\, r^2$	.994
1	$1 - 1/2\, r^2$	.983
1	$1 - r^2$	.836
$1 - r^2$	$1 - r^2$	1.000
$1 - 1/2\, r^2$	$1 - r^2$	.923
$1 - 1/3\, r^2$	$1 - r^2$	.891
1	$\sin \pi r / \pi r$	.780

If for some velocity the mean free path in the tamper is very much less than that in the core (a very rare case) p is small and ψ_0 is nearly 1 if the tamper is not absorbing. If the paths are more nearly equal, p is near 1/3 and increases as the mean free path in the tamper increases. Absorption in the tamper increases the neutron curvature p . A value of p as high as 1 can occur with a completely absorbing tamper and a core with very small mean free path. In this case ψ is more like a sine wave, $\frac{\sin \pi r}{\pi r}$ rather than $1 - r^2$ but in other cases the parabola is a good enough approximation for the computation of M . In practical cases p will rarely be below 1/3 or as high as 1 so that M will always exceed 0.90. Furthermore, of course not only the extreme values of M appear in (3.2). In the integration over v' the latter variable takes all values and when it is close to v, $M_0(v,v')$ must be very close to 1. Since therefore the M_0 is close to unity over most of the range of integration the results of (3.2) are nearly the same as the results of (2.3). In fact this may be made use of. Equation (3.2) is most easily solved by perturbation methods, considering $1 - M_0(v,v')$ as a perturbation away from the much simpler equation (2.3). Thus a simple formula for the difference of the results of (3.2) and (2.3) is readily found, and appears in appendix II, equation (II-9).

Section 4 Higher approximations

We now give for the sake of completeness a formal scheme of successively better approximations to the correct value of ν . The first of these will be the approximation given by equation (2.3) the second that given by equation (3.2), and further ones will

increase in accuracy and difficulty. The first will be the most convenient because of its ease of application and the second will have some usefulness for estimating accuracy, but higher approximations, while necessary for good accuracy in source multiplication calculations, do not in general increase the accuracy sufficiently to compensate their increased difficulty.

We write an integral equation for $A(\vec{x},v)$, the density of absorptions of neutrons at position $\vec{x}$ in the core v per unit range of v. We denote by $S(v'\to v)$ the function previously called $S_a(v'\to v)$. We define a kernel $P(\vec{x}'\to\vec{x},v)$ as the density of absorptions at $\vec{x}$ when one neutron is released isotropically at $\vec{x}$ with velocity v. This kernel is of course identical with the kernel $P(\vec{x}\to\vec{x}')$ defined in Chapter I, the variable v merely specifying the constants to be used in calculating the kernel. We then see that $A(\vec{x},v)$ satisfied the following integral equation:

$$A(\vec{x},v)=\int d\vec{x}'\int dv' P(\vec{x}'\to\vec{x},v)S(v'\to v)A(\vec{x}',v') \qquad (4.1)$$

The kernel P is a one-velocity diffusion kernel, since it represents the probability that a neutron emitted at a given point at a given velocity suffer only elastic collisions and then be absorbed at another point. We can obtain a form for P by considering a simple one-velocity problem in which the tamper constants are those of velocity v, the total cross section is that of velocity v, and the fission cross section in the core in the core is set equal to the absorption cross section at velocity v. The eigenfunctions at velocity v, $\psi_n(\vec{x},v)$, satisfy a one-velocity integral equation:

$$\psi_n(\vec{x},v) = \nu_n(v) \int d\vec{x}\, P(\vec{x}' \to \vec{x}, v)\, \psi_n(\vec{x}', v) \tag{4.2}$$

This can be seen from the arguments following equation (5.1) of Chapter I. The kernel of such an equation can always be written as a bilinear series of its eigenfunctions. This expansion will be, since P is symmetric:

$$P(\vec{x}' \to \vec{x}, v) = \sum_n \frac{1}{\nu_n(v)} \frac{\psi_n(\vec{x},v)\,\psi_n(\vec{x}',v)}{\int d\vec{x}\, \psi_n^2(\vec{x},v)} \tag{4.3}$$

The approximation to P which we first make is that of setting all the $\nu_n(v)$ equal to $\nu_0(v)$. This gives for P:

$$P(\vec{x} \to \vec{x}, v) = \frac{1}{\nu_0(v)} \sum_n \frac{\psi_n(\vec{x},v)\,\psi_n(\vec{x},v)}{\int d\vec{x}\, \psi_n^2(\vec{x},v)} = \frac{1}{\nu_0(v)}\, \delta(\vec{x}-\vec{x}) \tag{4.4}$$

This approximation will be designated as the first lower approximation (abbreviated L-1) since it will appear that its use will lead to values for the critical radius and critical ν which are smaller than the correct values. If we insert equation (4.4) in equation (4.1) we obtain:

$$A(\vec{x},v) = \frac{1}{\nu_0(v)} \int dv'\, S(v' \to v)\, A(\vec{x},v) \tag{4.5}$$

This is a rather degenerate integral equation, since $\vec{x}$ enters only as a parameter and the kernel is independent of this

parameter. We can then integrate the equation over the core to obtain:

$$A(v) = \frac{1}{\nu_0(v)} \int dv' S(v' \rightarrow v) A(v') \tag{4.6}$$

Here $A(v)$ is merely the space integral of $A(\vec{x}, v)$. Equation (4.6) is seen to be identical with equation (2.3).

The next approximation consists in assuming that all the ν_n are very large except for ν_0. Equation (17) then becomes:

$$P(\vec{x} \rightarrow \vec{x}, v) = \frac{1}{\nu_0(v)} \frac{\psi_0(\vec{x}, v)\, \psi_0(\vec{x}, v)}{\int d\vec{x}\, \psi_0(\vec{x}, v)} \tag{4.7}$$

We will call this the first upper approximation (abbreviated U-1), since its use will always give too high a value for the critical ν or the critical radius. It is plainly an approximation of opposite character to the first lower approximation, since it replaces all ν_n by quantities which are equal to or greater than the correct values, whereas in the first lower approximation, all the ν_n (an increasing sequence) are placed equal to ν_0, the smallest of the sequence.

Combining equations (4.7) and (4.1) we obtain in this approximation:

$$A(\vec{x}, v) = \frac{1}{\nu_0(v)} \frac{\psi_0(\vec{x}, v)}{\int d\vec{x}\, \psi_0^2(\vec{x}, v)} \int d\vec{x}' \int dv' S(v' \rightarrow v) \psi_0(\vec{x}, v) A(\vec{x}, v') \tag{4.8}$$

It immediately follows from equation (4.8) that in this approximation $A(\vec{x},v)$ has the form:

$$A(\vec{x},v)=A_0(v)\,\psi_0(\vec{x},v) \tag{4.9}$$

If we insert the expression (4.9) in equation (4.8), we obtain an integral equation for $A_0(v)$:

$$A_0(v)=\frac{1}{\nu_0(v)}\int dv' S(v'\to v)\frac{\int d\vec{x}\,\psi_0(\vec{x},v)\,\psi_0(\vec{x},v')A_0(v')}{\int d\vec{x}\,\psi_0^2(\vec{x},v)\int d\vec{x}\,\chi_0^2(\vec{x},v')} \tag{4.10}$$

This is of course identical with equation (3.2)

The general nature of our two approximation schemes is now clear. The Nth lower approximation assumes that the first N terms in the bilinear expansion (17) are taken exactly while all the ν_N from ν_N onward are taken equal to ν_{N-1} . The Nth upper approximation takes the first N terms exactly while all further terms are placed equal to zero. This procedure is sufficiently illustrated by working out the second lower approximation. We place:

$$P(\vec{x}'\to\vec{x},v)=\frac{1}{\nu_0(v)}\psi_0(\vec{x},v)\,\psi_0(\vec{x}',v)+\frac{1}{\nu_1(v)}\sum_{n=1}^{\infty}\frac{\psi_n(\vec{x},v)\,\psi_n(\vec{x}',v)}{\int d\vec{x}\,\psi_n^2(\vec{x},v)}$$

$$=\left[\frac{1}{\nu_0(v)}-\frac{1}{\nu_1(v)}\right]\frac{\psi_0(\vec{x},v)\,\psi_0(\vec{x}',v)}{\int d\vec{x}\,\psi_0^2(\vec{x},v)}+\frac{1}{\nu_1(v)}\sum_{n=0}^{\infty}\frac{\psi_n(\vec{x},v)\,\psi_n(\vec{x}',v)}{\int d\vec{x}\,\psi_n^2(\vec{x},v)}$$

$$=\left[\frac{1}{\nu_0(v)}-\frac{1}{\nu_1(v)}\right]\frac{\psi_0(\vec{x},v)\,\psi_0(\vec{x}',v)}{\int d\vec{x}\,\psi_0^2(\vec{x},v)}+\frac{1}{\nu_1(v)}\int(\vec{x}'-\vec{x}) \tag{4.11}$$

With this expression for the kernel P. equation (4.1) becomes:

$$A(\vec{x},v)=\left[\frac{1}{\nu_0(v)}-\frac{1}{\nu_1(v)}\right]\frac{\psi_0(\vec{x},v)}{\int d\vec{x}\,\psi_0^{a}(\vec{x},v)}\int dv' S(v'\to v)\int d\vec{x}'\psi_0(\vec{x}',v)A(\vec{x}',v')$$

$$+\frac{1}{\nu_1(v)}\int dv' S(v'\to v)\,A(\vec{x},v') \qquad (4.12)$$

This equation is not easily transformable into one or two integral equations for one or two functions of velocity alone. If we do not try to treat a continuum of velocities but restrict ourselves to a small number of discrete velocity groups, equation (4.12) is easily soluble.

The convergence of this sequence of approximations is illustrated in Appendix IV by applying them to several two-group diffusion theory problems.

Ch. III. The Treatment of Inelastic Scattering in the Tamper.

In the preceding chapter we have seen that useful approximate methods exist by which neutron properties of some simple systems can be easily calculated, even though a continuum of velocities is necessary in the description. We should now like to extend our treatment to the the case where the tamper possesses an inelastic scattering cross section. We shall see that our position here is not nearly so strong as it was in the simpler systems treated previously. We shall find ourselves unable to treat a continuum of velocities and we must therefore make the approximation of replacing the continuum by discrete energy groups.

Section 1 Two velocity groups. General equations and definitions.

Let us first consider a system in which neutrons of only two velocities are created and destroyed. Suppose that the system consists of a spherical, homogeneous core surrounded by an infinite, homogeneous tamper. Neutrons in the core can be elastically scattered or can be transferred from one velocity to the other by fission or by inelastic scattering. Neutrons in the tamper can be elastically scattered, captured, or transferred from velocity one to velocity two by inelastic scattering. We adopt a point of view similar to that used in the preceding chapter. Neutrons in the core act by means of collisions as a source of neutrons of both velocities, and by studying the absorption from this source, we shall obtain the condition of balance satisfied for criticality. We shall think of the inelastic scattering cross section in the tamper as an absorption cross section for neutrons of velocity one. We shall then have a source of neutrons of velocity two distributed in the tamper as are the neutrons of velocity one. It is then necessary to find what fraction of the neutrons from this source are absorbed in the core, and include these extra absorptions in the balance condition.

To make these ideas more precise let us first consider the situation without the inelastically scattered neutrons. Suppose, for example, that these inelastically scattered were removed and not permitted to produce further fission. Later we can study the actual contribution made by these inelastically scattered neutrons. Thus we have a problem without inelastic scattering in the tamper,

the actual inelastic cross-section being replaced by absorption. In that case, we can use the ideas of Chapter II. In the simplest lower approximation, we are led to write equation (3.1) Chapter II in a form appropriate to just two velocities:

$$A_1 = \frac{1}{\nu_{10}} \left[S(1 \to 1) A_1 + S(2 \to 1) A_2 \right]$$
$$A_2 = \frac{1}{\nu_{20}} \left[S(1 \to 2) A_1 + S(2 \to 2) A_2 \right] \qquad (1.1)$$

Here A_1 and A_2 are the total number of neutrons absorbed in the core at velocity one and two respectively, and ν_{10} (and ν_{20}) are the ν values which make a one velocity system critical which had the cross sections appropriate to velocity one (and two) only. The quantities S are the direct analogue of $S_a(v \to v')$. Thus $S(1 \to 2)$ gives the number of neutrons liberated at velocity two if one is absorbed at velocity one, while $S(1 \to 1)$ gives the number liberated at velocity one under the same circumstances. If a neutron is absorbed at velocity two it has probabilities $S(2 \to 1)$ and $S(2 \to 2)$ to reappear at velocity one and two respectively. The equations are easy to understand. For example, the expression in the square bracket of the first equation is the total number of neutrons liberated at velocity one and of these, as was discussed in Chapter I, the fraction $\frac{1}{\nu_{10}}$ can be expected to return to the core for reabsorption.

Now we must include the effect of the inelastically scattered neutrons. A certain number of neutrons are liberated in the tamper at velocity two and these find their way back to the core and contribute to the absorptions A_2 there.

The neutrons are liberated at various places in the tamper, (the distribution depending on the shape of the tamper solution at velocity 1) and thus have different probabilities of returning to the core. For a simple argument let us define a quantity:

P_{12} as the average probability that a neutron liberated in the tamper by inelastic scattering (from velocity) finds its way into the core and is absorbed there.

Then the term to add to the second of the equations (1.1) is simply P_{12} times the number of neutrons inelastically scattered in the tamper. This number is not hard to find. It is just $\frac{\sigma_{i12}^*}{\sigma_{1a}^*}$ times the total number of neutrons absorbed in the tamper, since each absorption has this probability of resulting in inelastic scattering (σ_{i12}^* is the inelastic cross-section in the tamper, and σ_{1a}^* is the total absorption cross-section of neutrons of velocity one in the tamper and is σ_{i12}^* plus the true capture cross section). The number absorbed in the tamper is just those which are not absorbed in the core. Since of the neutrons liberated in the core a fraction $\frac{1}{\nu_{10}}$ are returned to the core, the remainder, or $1-\frac{1}{\nu_{10}}$ are absorbed in the tamper. In this way we can understand the modification of equations (1) to include the effects of inelastic scattering:

$$A_1 = \frac{1}{\nu_{10}}\left[S(1\to 1)A_1 + S(2\to 1)A_2\right]$$

$$A_2 = \frac{1}{\nu_{20}}\left[S(1\to 2)A_1 + S(2\to 2)A_2\right] + \frac{\sigma_{i12}^*}{\sigma_{1a}^*}\left(1-\frac{1}{\nu_{10}}\right) P_{12}\left[S(1\to 1)A_1 + (2\to 1)A_2\right] \quad (1.2)$$

These are simple equations which prove to be adequate to give a close approximation to the solution of two velocity problems. To use them however, some fairly simple way must be devised for the determination of p_{12} . We shall derive several alternate approximate formulas for this quantity and discuss the validity of the approximations later in this chapter. We shall also be concerned to some extent with the validity of the entire set of equations (1.2), but strictly this presents no really new problem (above those discussed in Chapter II) other than the adequacy of a proposed formula for p_{12} . At the end of this chapter we discuss the extension to a system with three, or more velocity groups.

The determination of p_{12} requires an analysis of the diffusion back to the core of neutrons which have been liberated in the tamper. The dependence of the quantity on the index 1 is solely through the fact that the neutrons of velocity 1 determine the spacial distribution of the source. Let us consider neutrons of velocity two released from any source in the tamper, $S(r)$. We then consider a kernel $Q_2(r \rightarrow r')$ which is the flux density of neutrons (of velocity two) at position r' in the core when a unit source is located at position r in the tamper. The density of absorptions $A(r')$ in the core resulting from the source $S(r)$ is then given by:

$$A(r') = 4\pi \sigma_{2a} \int_0^\infty S(r)\, Q_2(r \rightarrow r')\, r^2\, dr \qquad (1.3)$$

The quantity p_{12} is the number of absorptions which occur in the core in the fundamental mode:

$$p_{12} = \frac{4\pi \int A(r)\psi_{20}(r)r^2dr \cdot 4\pi \int \psi_{20}(r)r^2dr}{4\pi \int \psi_{20}^2(r)r^2dr} \tag{1.4}$$

In the expression (1.3) for A the source S must be taken to be that arising from velocity one (equation below). We are therefore led to study the kernel Q_2 more carefully.

The kernel Q_2 is a diffusion kernel at a single velocity and is therefore symmetric. We use this fact to obtain an expression for it. Let us expand $Q_2(r \rightarrow r')$ as a series in the normal modes at velocity two, $\psi_{2k}(r')$ with coefficients $\phi_{2k}(r)$. We have:

$$Q_2(r \rightarrow r') = Q_2(r' \rightarrow r) = \sum_k \phi_{2k}(r)\psi_{2k}(r') \tag{1.5}$$

Consider a source of neutrons of velocity two in the core, having the form $\psi_{2j}(r')$. We use equation (1.5) and the orthogonality of the ψ_{2k} to obtain as the resulting density of absorptions in the tamper:

$$T_{2k}(r) = 4\pi\sigma_{2a}^* \int_0^a Q_2(r' \rightarrow r)\psi_{2k}(r')r'dr' - 4\pi\sigma_{2a}^* \phi_{2k}(r) \int_0^a \psi_{2k}^2(r)r^2dr \tag{1.6}$$

Now we know that if neutrons are released from a source $\psi_{2k}(r)$ the distribution of absorptions will be $\frac{1}{\nu_{2k}}\psi_{2k}(r)$. It is therefore true that if one net neutron is released in the shape $\psi_{2k}\frac{1}{\nu_{2k}}$ net neutrons are absorbed in the core, and $1 - \frac{1}{\nu_{2k}}$ net neutrons must be absorbed in the tamper.

We now integrate equation (1.6) over the tamper. We obtain the total rate of absorption in the tamper which, by the argument just made, must be equal to $1 - \frac{1}{\nu_{2k}}$ times the total source in the core. This gives:

$$4\pi \int_a^\infty T_{2k}(r)\, r^2 dr = \left(1 - \frac{1}{\nu_{2k}}\right) 4\pi \int_0^a \psi_{2k}(r)\, r^2\, dr \tag{1.7}$$

Combining 1.6 and 1.7 we find for $\phi_{2k}(r)$:

$$\phi_{2k}(r) = \frac{T_{2k}(r)}{4\pi \int_a^\infty T_{2k}(r) r^2 dr} \; \frac{1 - \frac{1}{\nu_{2k}}}{\sigma_{2a}^*} \; \frac{4\pi \int_0^a \psi_{2k}(r) r^2 dr}{4\pi \int_0^a \psi_{2k}^2(r) r^2 dr} \tag{1.8}$$

We can now insert this in equation (1.5) to obtain an expression for the kernel Q_2 in terms of the eigenfunctions and eigenvalues of a one-velocity critical problem:

$$Q_2(r \to r') = \sum_k \frac{T_{2k}(r)}{4\pi \int_a^\infty T_{2k}(r) r^2 dr} \cdot \frac{1 - \frac{1}{\nu_{2k}}}{\sigma_{2a}^*} \cdot \frac{\int_0^a \psi_{2k}(r) r^2 dr}{\int_0^a \psi_{2k}^2(r) r^2 dr} \psi_{2k}(r') \tag{1.9}$$

Now let us obtain the quantity p_{12} defined previously. For the source $S(r)$, we take a function with the shape of the tamper solution appropriate to the fundamental mode for neutrons of velocity one, but normalized so that one neutron is emitted per second. This gives:

$$S(r) = \frac{T_{10}(r)}{4\pi \int_a^\infty T_{10}(r) r^2 dr} \tag{1.10}$$

If we substitute this, and the expression (1.9) for Q into (1.3) to find A, and then use the formula (1.4) we find p_{12} can be expressed as:

$$p_{12}=\frac{\left[4\pi\int_0^a \psi_{20}(r)r^2dr\right]^2}{4\pi\int_0^a \psi_{20}^2(r)r^2dr}\frac{\sigma_{2a}}{\sigma_{2a}^*}\left(1-\frac{1}{\nu_{20}}\right)\frac{4\pi\int_a^\infty T_{10}(r)T_{20}(r)r^2dr}{4\pi\int_a^\infty T_{10}(r)r^2dr\,4\pi\int_a^\infty T_{20}(r)r^2dr} \qquad (1.11)$$

The expression (1.11) is exact.

In the next sections we shall discuss various approximations that can be made to it. Before we do that, however, it may be well to point out that the equations (1.2) in which p_{12} appear are themselves not exact. Their derivation as the first of a sequence of convergent approximations to the exact equations will be undertaken later.

Section 2 Neutron return from tamper to core: First method of approximation.

We discuss in this and the next section various approximations which can be made to the quantity p_{12}. There is one simplification which can be made immediately. If we multiply and divide by the core volume we obtain as a factor the combination of integrals:

$$\frac{\left[\int_0^a \psi_{20}(r)r^2dr\right]^2}{\int_0^a \psi_{20}^2(r)r^2dr\cdot\int_0^a r^2dr} \qquad (2.1)$$

We shall hereafter replace this combination by unity. This is a

reasonable approximation. Except near the edge of the core the function $\psi_{20}(r)$ behaves as $\frac{\sin kr}{kr}$. For $k_2 a = 0$ (tamper having large cross section and no capture) the combination is equal to unity, for $k_2 a = \frac{\pi}{2}$ it is 0.98, and for $k_2 a \frac{3\pi}{4}$, it is 0.88. It should be remembered that a large value of $k_2 a$ corresponds to a poor tamper, from which the inelastic return is less important. This approximation therefore seems to be good (and is certainly convenient). If this is done, we have for p_{12}:

$$p_{12} = \frac{\sigma_{2a}}{\sigma_{2a}^*}\left(1-\frac{1}{\nu_{20}}\right)\frac{\int_a^\infty T_{10}(r)\,T_{20}(r)\,r^2dr\cdot\frac{a^3}{3}}{\int_a^\infty T_{10}(r)\,r^2dr\cdot\int_a^\infty T_{20}(r)\,r^2dr} \qquad (2.2)$$

The various approximate methods of estimating p_{12} are simply alternate ways of evaluating the integrals (omitting the subscript 0 for normal mode):

$$\frac{\int_a^\infty T_1(r)\,T_2(r)\,r^2dr}{\int_a^\infty T_1(r)\,r^2dr\int_a^\infty T_2(r)\,r^2dr} \qquad (2.3)$$

This only requires knowledge of the shapes and not the size of the functions T_1 and T_2. When the diffusion theory is valid these shapes are accurately known. They are of the form $e^{-h\frac{r}{r}}$ and $e^{-h_2\frac{r}{r}}$ respectively. The constants h_1 and h_2 are respectively $\sqrt{3\sigma_1^*\sigma_{1a}^*}$ and $\sqrt{3\sigma_2^*\sigma_{2a}}$ respectively. Putting these forms into (2.2) and performing the integrals one finds:

$$p_{12} = \frac{a^3}{3} \frac{h_1^2 h_2^2}{(1+h_1 a)(1+h_2 a)(h_1+h_2)} \frac{\sigma_{2a}}{\sigma_{2a}^*} \left(1 - \frac{1}{\nu_2}\right)$$

(2.4)

This is the first of our forms for p_{12} . It is exact (except for our replacement of the integrals (2.1) by unity) in the diffusion theory. It may also be obtained directly from the differential equations of this theory by studying the diffusion into the core of neutrons distributed in the tamper as $e^{-h_1 \frac{r}{r}}$

The expression (2.4) although only (nearly) exact for the diffusion theory is nevertheless often a very useful approximation when the accurate integral theory must be used. In this theory the functions T_{10} for example behaves asymptotically as $e^{-h_1 \frac{r}{r}}$ and differ from this only near the core surface where transition effects occur. These transition effects, especially when the tamper does not have a large absorption cross section are often quite small, and their effect in the integrals (2.3) is negligible. One change must be made however. The quantities h_1 and h_2 must now be determined from the usual secular equation for the Milne equation in the infinite tamper medium:

$$\frac{\frac{h_1}{\sigma_1^*}}{\tanh^{-1} \frac{h_1}{\sigma_1^*}} = 1 - g_1$$

$$\frac{\frac{h_2}{\sigma_2^*}}{\tanh^{-1} \frac{h_2}{\sigma_2^*}} = 1 - g_2 \qquad (2.5)$$

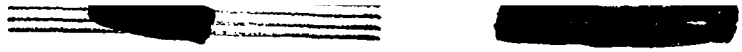
where g_1 and g_2 are the usual ratios of absorption to that

cross sections at the two velocities ($g_1 = \frac{\sigma_{1a}^*}{\sigma_1^*}$ and $g_2 = \frac{\sigma_{2a}^*}{\sigma_2^*}$). The formula may be improved by putting better formulas for T_1 and T_2 into (2.2) in which the transition effects (which has been studied in many one-velocity problems, see LA-53 and LA-258) are more clearly represented. This probably leads to relatively complicated expressions. In the next section we derive another form for p_{12} and show that it takes somewhat better account of these effects than does (2.4).

Section 3 Neutron return from tamper to core: other approximations.

The method we shall adopt to determine p_{12} in this section is to set up an imaginary two velocity problem for which the complete answer is known. Then using our fundamental equations (1.2) for such a problem we shall find the value of p_{12} to solve other problems.

To do this we consider an imaginary problem in which there are two groups. The upper we call group x and the lower simply 2. The constants of the lower velocity are exactly those of the velocity two group for the real problem for which we are trying to solve. The constants of group x however, instead of being those of velocity one, are taken to be simply those of velocity two, with one exception, however. That is we imagine that we add a cross section σ_{ix2}^* of inelastic cross section which transfers neutrons from group x to group 2 (the tamper absorption cross section is therefore $\sigma_{ix2}^* + \sigma_{2a}^* = \sigma_{xa}^*$). Now this transfer from x to 2 really makes no difference, the neutrons still have the same properties. Thus we know that for the whole system we can solve for the critical k.



It is simply ν_{20} .

The point, however, is that p_{12} depends on velocity one only through the shape of T_1 and therefore p_{x2} depends only on the shape of the function T_x representing the distribution of x neutrons in the tamper. But the shape T_x can be made very like that of T_1 by adjusting the value of σ_{ix2} (and hence σ_{xa}^*). We discuss the choice of σ_{ix} later, but first let us see how we can determine p_{x2} which we shall later use as an approximation for p_{12} .

Let the ν required to keep criticality in the upper group be ν_x (which is not ν_2 because of the extra tamper absorption ofxneutrons). We now treat this problem by the pair of equations (2) assuming the exact critical ν for the system is ν_2 . Then since x and 2 neutrons both may be pictured as producing fission from each absorption with this $\nu = \nu_2$,and there is no core inelastic scattering, we have

$$S(x \to x) = \nu_2,\ S(2 \to x) = \nu_2,\ S(2 \to 2) = 0,\ S(x \to 2) = 0 ,$$

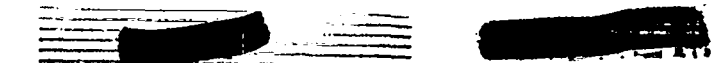
so that equations (1.2) become

$$A_x = \frac{1}{\nu_x}\left[\nu_2 A_x + \nu_2 A_2\right]$$

$$A_2 = 0 + \frac{\sigma_{ix2}^*}{\sigma_{2a}^* + \sigma_{ix2}^*}\left(1 - \frac{1}{\nu_2}\right) p_{x2}\left[\nu_2 A_x + \nu_2 A_2\right] \qquad (3.1)$$

These equations (add them and divide by $A_x + A_2$) require that

$$p_{x2} = \frac{1 - \frac{\nu_2}{\nu_x}}{\nu_2\left(1 - \frac{1}{\nu_2}\right)\frac{\sigma_{ix2}^*}{\sigma_{2a}^* + \sigma_{ix2}^*}} \qquad (3.2)$$



To obtain an expression for p_{12} we must now choose the value of σ^*_{ix2} so that T_x approximates T_1. We shall do this by making the asymptotic forms of these functions the same and hope that the transition effects are nearly similar. Thus we choose $h_x = h_1$. We shall, to indicate this special choice now use the notation 12 for x. We let g_2 be the usual absorption number in the tamper at group two, so that:

$$g_2 = \frac{\sigma^*_{2a}}{\sigma^*_2} \tag{3.3}$$

We can look at the quantity σ^*_{ix2} as the capture cross section that would have to be added to the tamper at velocity two to make the asymptotic rate of decay of neutrons of velocity two match that of neutrons at velocity one in the real two velocity problem. We define g_{12} as the total absorption number necessary at velocity two in the tamper to match the shapes in this way. We have:

$$g_{12} = \frac{\sigma^*_{2a} + \sigma^*_{ix2}}{\sigma^*_2}$$

$$\text{and} \quad g_{12} - g_2 = \frac{\sigma^*_{ix2}}{\sigma^*_2} \tag{3.4}$$

We re-write equation (44) as follows:

$$p_{12} = \frac{\frac{1}{\nu_2} - \frac{1}{\nu_{12}}}{g_{12} - g_2} \cdot \frac{g_{12}}{1 - \frac{1}{\nu_{12}}} \tag{3.5}$$

This is the second of our useful approximations to p_{12} . In it the quantity g_{12} is the absorption number for group-two neutrons in the tamper which would give the same asymptotic exponential solution for region 2 as actually does exist in region one. That is if g_1 is the true absorption number in region one as in (2.5) we find g_{12} by determining h_1 from (2.5) and g_{12} from a similar equation.

$$1-g_1=\frac{\frac{h_1}{\sigma_1^*}}{\frac{\tanh^{-1}h_1}{\sigma_1^*}}$$

$$\text{and}\quad 1-g_{12}=\frac{\frac{h_1}{\sigma_2^*}}{\frac{\tanh^{-1}h_1}{\sigma_2^*}} \qquad (3.6)$$

The quantity ν_{12} is the ν value which would make a system with neutrons of velocity two critical if the actual absorption number in the tamper were g_{12} (instead of g_2).

We shall now derive equation (3.5) by another more formal method which shows more clearly the approximations involved in its derivation. We must first prove a simple orthogonality relation holding between the neutron distributions in certain related one velocity problems. (This theorem and its applications are described in much more detail in LA- 608 .) Let $M(\vec{x}\to\vec{x})\sigma(\vec{x})$ be the probability per unit volume that a neutron released isotropically at $\vec{x}'$ makes its first collisions at $\vec{x}$. The position $\vec{x}$ can be a core point or a tamper point so that σ is a function of position. The kernel M is, of course, symmetrical. Suppose we have two systems, both critical, with the same total cross section as a function of position and which, therefore, have the same $M(\vec{x}'\to\vec{x})$. Let us call the function which is the total cross section multiplied by the total number of neutrons emitted per collision $q(\vec{x})$. We can then write integral equations for ψ and

ψ' , the neutron densities in the two systems:

$$\psi(\vec{x}) = \int d\vec{x}' M(\vec{x}' \to \vec{x})\, q(\vec{x}')\, \psi(\vec{x}')$$

$$\psi'(\vec{x}) = \int d\vec{x}' M(\vec{x}' \to \vec{x})\, q'(\vec{x}')\, \psi'(\vec{x}')$$

We multiply the first of these equations by $q'(\vec{x})\psi'(\vec{x})$ and the second by $q(\vec{x})\psi(\vec{x})$, integrate each over all space and subtract. We obtain:

$$\int d\vec{x}\,[q'(\vec{x}) - q(\vec{x})]\,\psi(\vec{x})\,\psi'(\vec{x}) = \int d\vec{x} \int d\vec{x}' \Big[q'(\vec{x})\,\psi'(\vec{x}) M(\vec{x}' \to \vec{x})$$

$$q(\vec{x}')\,\psi(\vec{x}') - q(\vec{x})\,\psi(\vec{x}) M(\vec{x}' \to \vec{x})\, q'(\vec{x}')\,\psi'(\vec{x}')\Big] \qquad (3.7)$$

In the second term of the right hand side of Eq. 3.7 we interchange the dummy variables of integration and note that, since $M(\vec{x}' \to \vec{x})$ is symmetric, this term now cancels the first. We obtain:

$$\int d\vec{x}\,[q'(\vec{x}) - \bar{q}(\vec{x})]\,\psi'(\vec{x})\,\psi(\vec{x}) = 0 \qquad (3.8)$$

We now specialize this general theorem to the case of interest to us. We consider two systems with the total cross-section in core and tamper equal to σ_2 and σ_2^* , respectively. In the first of these systems we place the absorption number in the tamper equal to g_2 and place the reproduction number in the core equal to f_2 , the f required for criticality. In the other problem we place the absorption number in the tamper equal to the number g_{12} previously defined, and the reproduction number in the core equal

to f_{12}, the f required for criticality with this changed tamper absorption. The quantity $q(x)$ will be equal to $(1+f)\sigma_2$ in the core and $(1-g)\sigma_2^*$ in the tamper. We now write equation (3.8), doing the integrals over the core and tamper separately and obtain, writing ψ_2 and ψ_{12} for the densities in the core T_2 and T_{12} for the densities in the tamper:

$$4\pi(f_{12}-f_2)\sigma_2\int_0^a r^2 dr\,\psi_2(r)\psi_{12}(r) = 4\pi(g_{12}-g_2)\sigma_2^*\int_a^\infty r^2 dr\,T_2(r)T_{12}(r') \tag{3.9}$$

We can also use the conservation law to get some other relations. In each problem the total number of extra neutrons produced in the core must be destroyed in the tamper. This condition gives:

$$4\pi f_{12}\sigma_2\int_0^a r^2 dr\,\psi_{12}(r) = 4\pi g_{12}\sigma_2^*\int_a^\infty r^2 dr\,T_{12}(r)$$

$$4\pi f_2\sigma_2\int_0^a r^2 dr\,\psi_2(r) = 4\pi g_2\sigma_2^*\int_a^\infty r^2 dr\,T_{12}(r) \tag{3.10}$$

If we now divide equation (3.9) by each of the equations (3.10), we obtain:

$$\left(\frac{1}{f_2}-\frac{1}{f_{12}}\right)\frac{1}{\sigma_2}\frac{\int_0^a r^2 dr\,\psi_2(r)\psi_{12}(r)}{\int_0^a r^2 dr\,\psi_2(r)\int_0^a r^2 dr\,\psi_{12}(r)} = \left(\frac{1}{g_2}-\frac{1}{g_{12}}\right)\frac{1}{\sigma_2^*}\frac{\int_a^\infty r^2 dr\,T_2(r)T_{12}(r)}{\int_a^\infty r^2 dr\,T_2(r)\int_a^\infty r^2 dr\,T_{12}(r)} \tag{3.11}$$

We now multiply by $\int_0^a r\,dr$ and set the resulting combination of integrals on the left equal to unity, as it will be very nearly in all cases of interest.

We have thus obtained an expression in terms of the eigenvalues of two one-velocity problems an expression for a combination of tamper integrals approximating to the combination required in the equation (2.2) for the quantity p_{12} . The function T_2 , appearing in (2.2), is duplicated exactly in equation (3.11). The function T_1 , appearing in (2.2), is approximated by the function T_{12} . The function T_{12} and T_1 match asymptotically and both have transition effects in the same direction at the interface. By the use of Eq. (3.11) we can, therefore, expect to obtain an expression for p_{12} of greater accuracy than if we had used the asymptotic exponential tamper solutions for T_1 and T_2 .

We combine (3.11) and (2.2) to obtain:

$$p_{12} = \frac{\sigma_{2a}}{\sigma_{2a}^*}\left(1-\frac{1}{\nu_2}\right)\frac{\sigma_2^*}{\sigma_2}\,\frac{\frac{1}{f_2}-\frac{1}{f_{12}}}{\frac{1}{g_2}-\frac{1}{g_{12}}} \qquad (3.12)$$

We can express the quantities f_2 and f_{12} in terms of the quantities ν_2 and ν_{12} previously used by the usual relation (from Chapter I, equation (5.1))

$$\nu_2 = 1+\frac{\sigma_2}{\sigma_{2a}}f_2 \qquad f_2 = \frac{\sigma_{2a}}{\sigma_2}(\nu_2-1)$$

$$\nu_{12} = 1+\frac{\sigma_2}{\sigma_{2a}}f_{12} \qquad f_{12} = \frac{\sigma_{2a}}{\sigma_2}(\nu_{12}-1) \qquad (3.13)$$

The insertion of (3.13) in (3.12) yields, after noting that $\sigma_{2a}^* = g_2\,\sigma_2^*$ and rearranging,

$$p_{12} = \frac{g_{12}}{g_{12}-g_2}\,\frac{\frac{1}{\nu_2}-\frac{1}{\nu_{12}}}{1-\frac{1}{\nu_{12}}}$$

which is seen to be identical with the previously derived equation (3.5).

The foregoing treatment would be expected to be excellent if $\frac{h_1}{\sigma_1^*}$ and $\frac{h_2}{\sigma_2^*}$ are each small compared with unity. The functions T_1 and T_2 will then have essentially the character which they have in diffusion theory. They will consist mostly of the asymptotic exponential solutions with relatively small transition effects at the interface. If we use equation (58) for p_{12}, the size of $\frac{h_2}{\sigma_2^*}$ does not actually concern us since the function T_2 is not approximated in that expression. Suppose, however, we consider what happens as $\frac{h_1}{\sigma_1^*}$ increases. The function T_1 will decay more and more rapidly with a transition effect which becomes relatively larger. This transition effect is not matched by that in the function T_{12} (except qualitatively) so that equation (3.5) may be expected to become less accurate. In many cases of interest the inelastic scattering cross section forms so large a part of the total cross section in the tamper at the higher velocity that $\frac{h_1}{\sigma_1^*}$ may be rather close to unity and this defect in equation (3.5) may therefore be serious.

A further related trouble can also occur. If $\frac{h_1}{\sigma_2^*}$ is greater than one, even the introduction of complete absorption into the tamper at velocity two will not give an asymptotic decay which will match T_{12} to T_1 (there being no meaning to the second equation 78). Even if $\frac{h_1}{\sigma_2^*}$ is less than, but fairly close to one, so much absorption must be introduced at velocity two that the asymptotic part of the function T_{12} becomes unimportant and the match therefore will again be poor.

It is usually true for physical reasons that σ_2^* is greater than σ_1^*, if one refers to the higher energy, and since $\frac{h_1}{\sigma_1^*}$ is necessarily less than one the second trouble will not arise in a reasonable system. Exceptions may occur but this trouble is much more likely to arise if the tamper contains fissionable material so that neutrons entering the tamper at velocity two can have their energy increased to velocity one. We would then be interested in calculating the number of absorptions, taking place in the core at velocity one, of neutrons which have been thus raised in energy in the tamper. This clearly involves the calculation of a quantity p_{21}, which will be just the quantity p_{12} with the one and two interchanged in the definition.

In overcoming the first difficulty it is clearly advantageous to use, if possible, the correct function T_1. We can do this by introducing sufficient absorption, g_{21}, in the tamper at velocity one so that the resulting tamper solution T_{21} duplicates, as closely as possible, the function T_2. By using T_{21} as an approximation for T_2, and the expression (53) for the combination of tamper integrals which we need, we can get an alternative expression for p_{12}. This procedure will, of course, overcome the second difficulty since, h_1 is less than σ_1^*, h_2 is less than σ_2^*, and if h_1 is greater than σ_2^*, it follows that h_2 is less than σ_1^* and the reverse procedure just outlined will give us a reasonable value for the quantity p_{21}.

We proceed to calculate p_{12} in this alternative fashion.

We define g_{21} as the absorption number necessary in the tamper at velocity one to give an asymptotic decay equal to h_2. The eigenvalue f_1 will be the reproduction number necessary in the core at velocity one to maintain criticality for a tamper absorption g_1 and f_{21} will be the corresponding quantity when the tamper absorption is equal to g_{21}. The use of equation (3.11) then gives:

$$\frac{\int_0^a r^2 dr \int_a^\infty r^2 dr\, T_1(r)\, T_{21}(r)}{\int_a^\infty r^2 dr\, T_1(r) \int_a^\infty r^2 dr\, T_{21}(r)} = \frac{\sigma_1^*}{\sigma_1} \frac{\frac{1}{f_1} - \frac{1}{f_{21}}}{\frac{1}{g_1} - \frac{1}{g_{21}}} \qquad (3.14)$$

We then combine equations (3.14) and (2.2) to obtain the desired approximation for p_{12}:

$$p_{12} = \frac{\sigma_{2a}}{\sigma_{2a}^*}\left(1 - \frac{1}{\nu_2}\right) \frac{\sigma_1^*}{\sigma_1} \frac{\frac{1}{f_1} - \frac{1}{f_{21}}}{\frac{1}{g_1} - \frac{1}{g_{21}}} \qquad (3.15)$$

This is our third approximation to p_{12}.

By some obvious manipulation we can obtain the following relation between the p_{12} which is calculated as above and the p_{21} calculated by equation (58) with the interchange of one and two:

$$p_{12} = \frac{\sigma_{2a}}{\sigma_{2a}^*} \frac{\sigma_{1a}^*}{\sigma_{1a}} \frac{\left(1 - \frac{1}{\nu_2}\right)}{\left(1 - \frac{1}{\nu_1}\right)} p_{21} \qquad (3.16)$$

That this relation is valid in general may be seen directly from equation (2.2).

Appendix V, and very simple to calculate. The use of the exact tamper solutions is prohibitively difficult and can be used, therefore, only as a check upon the less accurate expressions. The other two methods are fairly easy and intermediate in accuracy. These two methods depend, however, for their usefulness on the ease and accuracy with which we can obtain the eigenvalues for arbitrary one-velocity problems. With equal mean free paths in core and tamper we can get good accuracy by the use of the extrapolated end point procedure. Reasonably accurate calculations can be made for unequal mean free paths by the methods mentioned in Chapter I.

Section 4 Validity of the approximations inherent in the general equations.

In the last sections of this chapter we shall discuss the technical problem of extending this formalism to systems with three neutron velocities and to systems with finite tampers. Since no essentially new ideas are required for this, it seems advisable at this point to discuss in more detail the accuracy of the entire scheme of equations (2), even granting that p_{12} may be known exactly. We shall consider the equations (1.2) as a first (and most practical) of a series of approximations to the exact equations. To study this in more detail we shall first have to write down the exact equations in the two-velocity case. To this end we shall call the density of absorptions at velocities one and two $A_1(r)$ and $A_2(r)$ respectively and introduce coefficients $S(i \rightarrow k)$ as at the beginning of this chapter.

It is identically satisfied by the formula (2.4) for p_{12} so it gives no new result in this case, but for formula (3.5) it yields the new relation (3.15). An analysis of the transition effects as they affect the accuracy of formulas (3.5) and (3.15) indicates that (when the sizes of h_1, σ_1^*, $h_2\sigma_2^*$ are such that either formula may be used) these formulas err on opposite sides of the truth in many cases. It is possible that this may be true in general, and if this theorem could be established it would permit us to put limits on the error made in calculating p_{12}. In that case we could put the theory with tamper inelastic scattering on the same sure footing as the theory when this scattering is absent. At present writing this has not been done.

We now have four methods for calculating p_{12}. Using equation (2.2) we can first assume that T_1 and T_2 have the shape of the asymptotic exponentials thereby obtaining equation (2.4). We can second assume the correct T_2 and approximate T_1 by T_{12} obtaining equation (3.5). We can third assume T_1 and approximate T_2 by T_{21} obtaining equation (3.5). Finally, with sufficient effort, we could assume the correct functions for both T_1 and T_2. All of these methods coincide essentially in the diffusion limit and the various methods which we have devised merely represent various ways of taking into account, to various extents, the transition effects which occur at the interface except in the diffusion limit. The quantity p_{12} is calculated for some typical cases by each of these methods in Appendix V. The use of the asymptotic exponentials is of the greatest general use since it is reasonably accurate, as shown in

Finally we also need the kernels $P_1(r' \to r)$ and $P_2(r' \to r)$ which are the densities of absorptions in the core at r when one neutron is released at r' at velocities one and two respectively. We obtain by elementary reasoning the integral equations:

$$A_1(r) = 4\pi \int_0^a r'^2 dr' \, P_1(r' \to r)\left[S(1 \to 1)A_1(r') + S(2 \to 1)A_2(r')\right]$$

$$A_2(r) = 4\pi \int_0^a r'^2 dr' P_2(r' \to r)\left[S(1 \to 2)A_1(r') + S(2 \to 2)A_2(r')\right]$$

$$+ 4\pi \int_0^a r'^2 dr' F_{12}(r' \to r)\left[S(1 \to 1)A_1(r') + S(2 \to 1)A_2(r')\right] \qquad (4.1)$$

The first equation states that all absorptions of neutrons of velocity one arise from neutrons liberated at velocity one from previous absorptions. The second equation states that all absorptions of neutrons at velocity two arise either from neutrons liberated at velocity two by previous absorptions or from neutrons liberated at velocity one which were inelastically scattered to velocity two in the tamper. This latter effect is expressed by the last term in which a new quantity $F_{12}(r' \to r)$ appears. We define the kernel $F_{12}(r' \to r)$ as the density of absorptions of neutrons of velocity two taking place at position r when one neutron is emitted per second at velocity one at position r', escapes absorption in the core, is inelastically scattered in the tamper, and finally returns to the core to be absorbed.

The new kernel can easily be expressed exactly in terms of the previously defined kernel $Q_1(r' \to r'')\,\sigma_{1a}^*$. This latter quantity is the density of absorptions in the tamper for one neutron released per second in the core at velocity one at

position r' . A fraction $\frac{\sigma_{i12}^*}{\sigma_{1a}^*}$ of these absorptions yield neutrons of velocity two, so that for one neutron emitted per second at velocity one at position r' , we have a source density of neutrons of velocity two in the tamper given by:

$$S(r'') = Q_1(r' \rightarrow r'')\, \sigma_{i12}^* \tag{4.2}$$

Now we find the density of absorptions at position r in the core resulting from this source. This is, from equation (1.3)

$$F_{12}(r' \rightarrow r) = 4\pi \sigma_{2a}\, \sigma_{i12}^* \int_a^\infty Q_1(r' \rightarrow x'')\, Q_2(x'' \rightarrow r) r^2 dr \tag{4.3}$$

We could now use the expression (1.9) for the kernel Q to obtain an explicit expression for F_{12} .

What approximations can be made to reduce (4.1) to a useful form? The approximations which can be made of the kernels P_1 and P_2 were discussed at considerable length in Chapter II. It was there shown that a very useful approximation, the first lower approximation, was equivalent to replacing $P_1(\vec{x}' \rightarrow \vec{x})$ by $\left(\frac{1}{\nu_{10}}\right) S(\vec{x}' - \vec{x})$ and P_2 by a similar expression. Our problem here is to discuss approximations which can be made to $F_{12}(r' \rightarrow r)$, (with the hope of course of eventually transforming (4.1) into (1.2). We shall discuss this by discussing approximations to $Q_1(r' \rightarrow r)$ and Q_2 since F can be expressed by (4.3) in terms of these quantities.

An exact expression for Q is given by equation (1.9) (we drop the subscript indicating which velocity group we are discussing. The index k represents the mode in question).

$$Q(r-r') = \sum_k \frac{T_k(r')}{4\pi \int_a^\infty T_k(r) r^2 dr} \frac{(1-\frac{1}{\nu_k})}{\sigma_a^*} \frac{\int_0^a \psi_k(r'') r''^2 dr''}{\int_0^a \psi_k^2(r) r^2 dr} \psi_k(r)$$

To make progress we must develop an approximation to it. One way is to observe that the functions $T_k(r)$ are nearly all of the same shape, so that the first factor is nearly independent of the mode k. This is exact in the limit of diffusion theory since $T_k(r)$ are merely simple exponentials characterized only by the tamper properties. In the factor containing these quantities, we therefore omit the subscripts and take this common factor outside the summation sign. It is clear that this procedure will become less accurate as we depart farther from the diffusion limit. This is so because the tamper solutions depend, in the transition effect at the boundary, on the core distribution.

We shall discuss this in much more detail below. Making this approximation we obtain:

$$Q(r-r') = \frac{T(r')}{4\pi \sigma_a^* \int_a^\infty T(r) r^2 dr} \sum_k \left(1-\frac{1}{\nu_k}\right) \frac{\psi_k(r) \int_0^a \psi_k(r'') r''^2 dr''}{\int_0^a \psi_k^2(r) r^2 dr} \tag{4.5}$$

The sum on k may now be performed exactly. We can use the completeness relation for the ψ_k and the bilinear expansion for the one-velocity diffusion kernels defined in Chapter II:

(It has been tacitly assumed that we are interested in neutron distributions which are functions of r only).

$$\sum_k \frac{\psi_k(\vec{x})\psi_k(\vec{x}')}{\int d\vec{x}\,\psi_k^2(\vec{x})} = \delta(\vec{x}-\vec{x}') \tag{4.6}$$

$$\sum_k \frac{1}{\nu_k} \frac{\psi_k(\vec{x})\psi_k(\vec{x}')}{\int d\vec{x}\,\psi_k^2(\vec{x})} = P(\vec{x}'\to\vec{x}) \tag{4.7}$$

We now integrate the expressions (4.6) and (4.7) with respect to one variable over the core to do the sums indicated in (4.5) and obtain:

$$Q(r\to r') = \frac{T(r')}{4\pi\sigma_a^* \int_a^\infty T(r) r^2 dr}\left[1 - 4\pi\int_0^a P(r\to r'') r''^2 dr''\right] \tag{4.8}$$

It is instructive to show that this expression can be understood in a more physical way. Suppose that one neutron per second is released at velocity one from a spherically symmetric source at radius r'. By definition $P_1(x'\to x'')$ are absorbed per second per unit volume at r''. The total number absorbed per second in the core is the volume integral of $P_1(r'\to r'')$ over all points in the core r''. The total number absorbed per second in the tamper must then be equal to:

$$1 - 4\pi\int_0^a P_1(x'\to r'') r''^2 dr'' \tag{4.9}$$

[redacted]

The distribution of these absorptions in the tamper must have, at least in the diffusion limit, just the shape of the previously defined function $T_1(x)$. If we normalize this shape to unit integral and multiply by the total number of absorptions in the tamper (4.9) we obtain the density of absorptions at r' in the tamper resulting from a unit source at r in the core. This is, by definition, the kernel $Q(r' \to r)\sigma_a^*$: in agreement with equation (4.8).

The expression (4.8) for Q can now be inserted into (4.3) to obtain an approximate expression for F_{12} :

$$F_{12}(r' \to r) = \frac{\sigma_{2a}}{\sigma_{2a}^*} \frac{\sigma_{i12}^*}{\sigma_{1a}^*} \frac{4\pi \int_a^\infty T_1(r) T_2(r) r^2 dr}{4\pi \int_a^\infty T_1(r) r^2 dr \cdot 4\pi \int_a^\infty T_2(r) r^2 dr} \tag{4.10}$$

$$\times \left\{1 - 4\pi \int_0^a P_1(r' \to r'') r''^2 dr''\right\}\left\{1 - 4\pi \int_0^a P_2(r \to r'') r''^2 dr''\right.$$

Recalling the expression (2.2) for p_{12} , this can be simplified to

$$F_{12}(r' \to r) = \frac{\sigma_{i12}^*}{\sigma_{1a}^*}\left\{1 - 4\pi \int_0^a P_1(r' \to r'') r''^2 dr''\right\}$$

$$\left\{1 - 4\pi \int_0^a P_2(r \to r'') r''^2 dr''\right\} \cdot \frac{3}{4\pi a^3} \cdot \frac{P_{12}}{(1 - \nu_{20})} \tag{4.11}$$

If in the exact equations (4.1) the (inexact) relation (4.11) is used these equations are seen to depend only on the kernels P . These kernels can be approximated in a series of approximate forms just as in Chapter II. We shall only exhibit the case corresponding to the first lower approximation here. The extension to the higher approximations are obvious by analogy, and an example is given in

Appendix V. It must be pointed out, however, that these approximations cannot converge to the true answer (except in the diffusion theory limit) when there is tamper inelastic scattering. This is because they converge to an exact solution of (4.1) only after the expression (4.11) is used, and this latter is itself inexact. We discuss to some extent the accuracy of 4.11 at the end of this section. Since 4.11 is inexact it is not too sensible to try to solve (4.1) very exactly when (4.11) is used and the method of first lower approximation of Chapter II should be sufficient. In this method we set every eigenvalue in the bilinear expansions for P_1 and P_2 equal to the lowest eigenvalue. Equations 4.3 and 4.4 of Ch. II then yield:

$$P_1(\vec{x}' \to \vec{x}) = \frac{1}{\nu_{10}} \delta(\vec{x}' - \vec{x})$$

$$P_2(\vec{x}' \to \vec{x}) = \frac{1}{\nu_{20}} \delta(\vec{x}' - \vec{x}) \qquad (4.12)$$

A corresponding expression for the kernel F_{12} can be obtained by inserting the expressions 4.12 into equation 4.10. This yields, P_{12}:

$$F_{12}(r' \to r) = \frac{\sigma_{t12}^*}{\sigma_{1a}^*}\left(1 - \frac{1}{\nu_{10}}\right) \frac{P_{12}}{4\pi \int_0^a r^2 dr} \qquad (4.13)$$

With these approximations, equations 4.11 become:

$$A_1(r) = \frac{1}{\nu_{10}}\left[S(1\to 1)A_1(r) + S(2\to 1)A_2(r)\right]$$

$$A_2(r) = \frac{1}{\nu_{20}}\left[S(1\to 2)A_1(r) + S(2\to 2)A_2(r)\right]$$

$$+ \frac{\sigma_{t12}^*}{\sigma_{1a}^*}\left(1 - \frac{1}{\nu_{10}}\right) P_{12} \frac{1}{4\pi \int x^2 dr}\left[S(1\to 1) 4\pi \int_0^a r^2 dr A_1(r)\right.$$

$$\left. + S(2\to 1)\, 4\pi \int_0^a r^2 dr A_2(r)\right] \qquad (4.14)$$

We now integrate each equation over the core and denote by A_1 and A_2 (without arguments) the total number of absorptions taking place in the core at velocities one and two respectively. We obtain two algebraic equations for A_1 and A_2 :

$$A_1 = \frac{1}{\nu_{10}}\left[S(1\to 1)A_1 + S(2\to 1)A_2\right]$$

$$A_2 = \frac{1}{\nu_{20}}\left[S(1\to 2)A_1 + S(2\to 2)A_2\right] + \frac{\sigma_{12}^*}{\sigma_{1a}^*}\left(1-\frac{1}{\nu_{10}}\right)P_{12}\left[S(1\to 1)A_1 + S(2\to 1)A_2\right] \tag{4.15}$$

These equations are identical to the equations (1.2) at the beginning of this chapter. We see that they correspond to several approximations. First, and possibly best, there is the approximation of using first lower approximation. This was discussed in Chapter II. There is further the difficulty that p_{12} can only be calculated easily approximately. (Methods to do this were discussed in sections 2, and 3 of this chapter) Finally there is the approximation used in obtaining the expression F_{12} in the form (4.11) or what is the same thing, the step leading from (4.4) to (4.5) in the calculation of Q , in which all the tamper functions have been assumed to be nearly of the same form. We shall discuss this approximation at some further length now, but unfortunately will not develop a simple way of decreasing the error made by it.

The expression 4.5 for the kernel Q is not in general correct

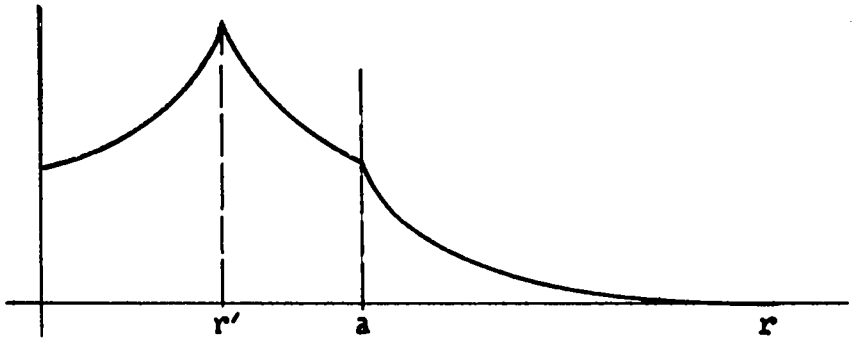
because, as has been pointed out, the transition effects will be different for the different normal modes and the step which leads from equation 4.4 to equation 4.5 will therefore be incorrect. The difficulty is equally obvious from the physical point of view that led to an interpretation of equation (4.8). Here one is now unable to say that the distribution of absorptions in the tamper depends only in size on the point in the core from which the neutrons are released. We are therefore unable to write $Q_1(r' \rightarrow r'')$ in the factored form (4.8) where each factor is a function only of one variable.

Having pointed out the difficulties, we shall now see, that there is certainly good reason to believe that the expression (4.10) for the kernel F_{12} is a useful approximation even when diffusion theory is not valid. Although in general there will be transition effects at the boundary, these effects will be qualitatively similar for the various normal modes, in that the neutron density in the tamper will always rise above the asymptotic solution close to the core-tamper interface. This positive discrepancy will always decay rapidly away from the interface but the characteristic length of this decay, while always considerably less than a mean free path in the tamper, is less for the higher normal modes than for the lower. This is so because a core density which oscillates rapidly compared with a mean free path sends positive and negative neutrons out normally to the interface in about equal numbers. This cancellation does not occur for neutrons emerging at an angle so large that the essential contribution to the emerging neutrons (within one mean free path from the interface, measured along a chord at the emerging angle) comes from the

positive part of the density closest to the interface. This emphasis on large angles in the emerging distribution for the higher modes means that the first collisions necessary to establish the asymptotic tamper solution will be made closer to the interface.

In view, however, of the qualitative similarity of the transition effects for the various normal modes, it is reasonable to believe that the combination of tamper integrals occurring in the expression 4.3 for F_{12} when used for 4.4 will be nearly independent of the indices k and j which run over all normal modes for velocities one and two. We would then obtain the expression (4.10) as an approximation to F_{12} .

We can also consider this approximation from the physical point of view that led to an interpretation of Equation (4.8). Consider a source of neutrons in the form of a spherical shell at radius r' in the core. The resulting density of absorptions can be represented by a curve like the following:



The curve in the tamper will be asymptotically an exponential but will have a transition at the interface. In the core the curve will consist of increasing and decreasing exponentials arranged to be finite at the center and have a discontinuity of slope at r' .

There will be transition effects at the source radius and at the interface. The region near the interface will not be affected by the position of the source provided the source is far enough away, i.e., more than a small fraction of a mean free path from the interface.

The shape of the curve in the tamper will therefore be independent of r' except when r' is very near to the core radius in the sense previously mentioned. Equation (4.8) will therefore be correct, except when r' is near the core radius, if $T(r)$ and $T_2(r)$ is the correct tamper solutions for a source inside an absorbing core.

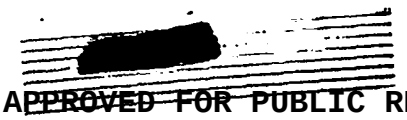
Section 5 Extension to three or more velocity groups.

The methods which we have developed so far in this chapter are sufficient to solve two-velocity problems with sufficient accuracy for practical purposes. The representation of a problem with a continuum of velocities by a reduction to one with two velocities is, however, rather crude. We should therefore like to develop methods for handling problems with three or more neutron velocities without any drastic increase in the amount of labor required. We shall also not be able to obtain arbitrary accuracy for three-velocity problems but for practical purposes we can do sufficiently well.

The approach previously used will be useful but some extension will be necessary. We can again write the neutron densities in the core at any velocity as a series of the normal modes appropriate to that velocity. We can then, in principle, write a system of simultaneous equations between the co-efficients of these normal modes to describe the balance between neutron production and neutron absorption at every velocity.

To do this we must, of course, calculate the distribution of absorptions in the tamper from each normal mode and calculate by the methods previously evolved the distribution of absorptions in the core of singly inelastically scattered neutrons returning from the absorptions in the tamper. If we then make the assumption that only the fundamental modes at each velocity are important we would then have a scheme like the one which we used for two velocity problems. One difficulty remains, that of calculating the number of absorptions of neutrons of velocity three in the core which have been doubly inelastically scattered from velocity one in the tamper. In order to be as brief as possible, we shall content ourselves here to deriving as simply as possible expressions to take into account this double inelastic scattering. We shall not attempt to exhibit the errors made in the formulas, although this can be done, because it leads to very complex expressions, and nothing essentially new is learned. The errors are of exactly the same kind (but somewhat larger, by accumulation) as those made in the discussion of the single inelastic scattering. They are due to neglect of transition effects to various degrees.

The fundamental equations (1.2) when generalized to include inelastic scattering and limited to a system in which the neutrons have just three possible velocities, become



$$A_1 = \frac{1}{\nu_{10}} G_1$$

$$A_2 = \frac{1}{\nu_{20}} G_2 + \frac{\sigma_{i12}^*}{\sigma_{1a}^*}\left(1 - \frac{1}{\nu_{10}}\right) p_{12} G_1$$

$$A_3 = \frac{1}{\nu_{30}} G_3 + \frac{\sigma_{i13}^*}{\sigma_{1a}^*}\left(1 - \frac{1}{\nu_{10}}\right) p_{13} G_1 + \frac{\sigma_{i23}^*}{\sigma_{2a}^*}\left(1 - \frac{1}{\nu_{20}}\right) p_{23} G_2$$
$$+ \frac{\sigma_{i12}^*}{\sigma_{1a}^*}\left(1 - \frac{1}{\nu_{10}}\right) p_{123} G_1 \qquad (5.1)$$

Here the quantities are defined exactly analagously to those of equations (1.2) for two groups, except to simplify the writing we have made the abbreviations

$$G_1 = S(1\to1)A_1 + S(2\to1)A_2 + S(3\to1)A_3$$

$$G_2 = S(1\to2)A_1 + S(2\to2)A_2 + S(3\to2)A_3$$

$$G_3 = S(1\to3)A_1 + S(2\to3)A_2 + S(3\to3)A_3 \qquad (5.2)$$

so that A_1, A_2, A_3 represent the number of neutrons absorbed in the core at each velocity, and G_1, G_2, G_3 have the significance of the number at each velocity which are generated in the core. The first two equations of (5.11) are just as in (1.2), the third represents the absorptions of group-three neutrons. The first term represents those generated at velocity 3 in the core which are returned to the core.

The second term represents neutrons which are generated at velocity one, are inelastically scattered in the tamper with cross section σ_{i13} directly from velocity one to three, and which are absorbed at velocity three in the core. The form is obvious and it is clear that p_{13} is just analagous to the p_{12} of previous sections but is calculated using constants of regions one and three, rather than one and two as for p_{12} . The third term represents those generated in the core at velocity two and which thereafter are scattered inelastically to three in the tamper and subsequently absorbed in the core. This leaves out of account neutrons which start at velocity one in the core, are inelastically scattered to region two in the tamper, and then before returning to the core are inelastically scattered again from two to three, eventually being absorbed in the core at velocity three. It is clear that this effect is accounted for by the last term if:

p_{123} is defined as the average probability that a neutron liberated at velocity two in the tamper (by inelastic scattering from velocity one) is inelastically scattered again to velocity three, and is absorbed in the core at that velocity three.

These equations (5.1) which are rewritten in simplest form in Chapter IV, are the equations which may be used to solve three velocity problems with practical accuracy. Their use, however, requires a method of estimating p_{123} .

We can find this by an analogue of the method used in section 3 to find p_{12} . Clearly p_{123} (like p_{12}) depends on conditions at velocity one only through the shape $T_1(r)$ of the source of neutrons at one. We can take an imaginary problem with the constants of regions two and three as they actually are, and

change region one to a region x which has the constants of two, but an inelastic cross section so adjusted that T_x is close to T_1 (exactly the same value of σ^*_{ix2} in fact as is required, as for p_{123} by equation (3.6). Then the quantity may serve as a good approximation to p_{123} . The p_{x23} can be found since we know the ν (say ν_c) required to make the whole system critical as it is just a two velocity problem with velocities two and three which can be solved by methods already worked out. The p_{x23} is adjusted so that (5.1) will give the right answer in this case. There is no loss in generality in determining the p_{x23} in simply letting group-three neutrons only regenerate group x and this simplifies the values of $S(i \to j)$. They all vanish except $S(3 \to x) = \nu_c$. Of course σ^*_{ix3} is to be taken equal to σ^*_{i23} as is every cross section for region x and 2 except that σ^*_{xa} and σ^*_{2a} differ by the extra σ^*_{ix2} in σ^*_{xa} . Equations (5.2) give $G_1 = \nu_c A_3$, $G_2 = 0$ and $G_3 = 0$ so that the equations (5.1) reduce to

$$A_1 = \frac{1}{\nu_{10}} \nu_c A_3$$

$$A_2 = 0 + \frac{\sigma^*_{ix2}}{\sigma^*_{xa}} \left(1 - \frac{1}{\nu_x}\right) p_{x2} \nu_c A_3$$

$$A_3 = 0 + \frac{\sigma^*_{i23}}{\sigma^*_{xa}} \left(1 - \frac{1}{\nu_x}\right) p_{x3} \nu_c A_3 + 0 + \frac{\sigma^*_{ix2}}{\sigma^*_{xa}} \left(1 - \frac{1}{\nu_x}\right) p_{x23} \nu_c A_3 \qquad (5.3)$$

Only the third of these is required and this determines p_{x23} immediately in terms of ν_c (divide by A_3). We may next determine by the two group problem with two and three. We may observe that this is just what is obtained if the neutrons from three feed two directly but do not feed x. That is for this case all S vanish except the single $S(3 \to 2) = \nu_c$. Hence $G_1 = G_3 = 0$, $G_2 = \nu_c A_3$, and the equations (5.1) become,

$$A_1 = 0$$

$$A_2 = \frac{1}{\nu_{20}} \nu_c A_3 + 0$$

$$A_3 = 0 + 0 + \frac{\sigma_{t23}^*}{\sigma_{2a}^*} \left(1 - \frac{1}{\nu_{20}}\right) p_{23} \nu_c A_3 + 0 \qquad (5.4)$$

Thus, again only the third equation is needed and it yields

$$\frac{1}{\nu_c} = \frac{\sigma_{t23}^*}{\sigma_{2a}^*} \left(1 - \frac{1}{\nu_{20}}\right) p_{23} \qquad (5.5)$$

when this is substituted into the third of the equations (5.3) there results an expression from which p_{x23} may be determined:

$$\frac{\sigma_{t23}^*}{\sigma_{2a}^*} \left(1 - \frac{1}{\nu_{20}}\right) p_{23} = \frac{\sigma_{t23}^*}{\sigma_{xa}^*} \left(1 - \frac{1}{\nu_x}\right) p_{x3} + \frac{\sigma_{tx2}^*}{\sigma_{xa}^*} \left(1 - \frac{1}{\nu_x}\right) p_{x23} \qquad (5.6)$$

(This expression may also be seen directly by careful reasoning). We now choose σ_{tx2}^* as in equation (3.4), and define all the g quantities and $\nu_x = \nu_{12}$ exactly as in this case.

Then we can take p_{x23} as p_{123} . We can also approximate p_{x3} by p_{13} since the former is the transfer from T_x to three and the latter is that from T_1 to three, and T_x and T_1 are nearly equal. Hence we finally obtain a useful formula for p_{123} :

$$p_{123} = \frac{p^*_{123}}{g_2 \sigma_2^*} \cdot \frac{1}{1-\frac{1}{\nu_{12}}} \cdot \frac{1}{g_{12}-g_2} \left\{ g_{12}\left(1-\frac{1}{\nu_2}\right)p_{23} - g_2\left(1-\frac{1}{\nu_{12}}\right)p_{13} \right\} \quad (5.7)$$

To use this p_{13} and p_{23} may be obtained by any of the methods outlined in sections 2 and 3. Unfortunately p_{123} is obtained as a difference of terms, and this makes the accuracy of p_{123} less than that of quantities like p_{12} . It is probably true that the best results are obtained if p_{23} and p_{13} in (5.7) are both calculated by the same method, and presumably it is best to use the method of section 3 (equation (3.5)). This is because the derivation of (3.5) is like that of (5.7) and some thought indicates that if this method is used the errors in p_{23} and p_{13} after substitution in (5.7) tend to offset each other rather than accumulate.

There are, in analogy to the case of p_{12} , many other forms that p_{123} can be written. For example, by eliminating ν_{12} by equation (3.5), and expressing it in terms of p_{12} :

$$p_{123} = \frac{\sigma^*_{123}}{g_2 \sigma_2^*} \left\{ \frac{g_2}{g_{12}-g_2}(p_{23}-p_{13}) + (1-p_{12})\,p_{23} \right\} \quad (5.8)$$

(This can also be derived directly by other methods). There is no point to this (it is identical to 5.7) if the p's are calculated by (3.5), which is the recommended proceedure, but if another formula (say 2.4) is used for the p's the relation (5.8) may be

easier to use.

Again, we might have made the analysis of p_{123} in another way, in that we could have put the constants of group two similar to group three and thus obtained a new form. This simply reverses the role of groups one and three. The easiest way of seeing what will result is to find p_{123} (obtained by formula 5.7) (simply by interchanging the constants of group three with those of group one) and then noting as in the case of p_{12}, equation (3.16), it is possible to demonstrate a reciprocity relation:

$$\frac{\sigma_{1a}}{\sigma_{1a}^*}\left(1-\frac{1}{\nu_1}\right)\frac{\sigma_{2a}^*}{\sigma_{i23}^*}p_{123}=\frac{\sigma_{3a}}{\sigma_{3a}^*}\left(1-\frac{1}{\nu_3}\right)\frac{\sigma_{2a}^*}{\sigma_{i21}^*}p_{123} \qquad (5.9)$$

so that p_{123} is known when p_{321} is found in this manner. If this method is applied to 5.8 there results

$$p_{123}=\frac{\sigma_{1a}^*}{\sigma_{1a}}\cdot\frac{\sigma_{3a}}{\sigma_{3a}^*}\cdot\frac{\sigma_{i2a}}{\sigma_{2a}^*}\cdot\frac{\left(1-\frac{1}{\nu_3}\right)}{\left(1-\frac{1}{\nu_1}\right)}\left\{\frac{g_2}{g_{32}-g_2}\left(p_{21}-p_{31}\right)+\left(1-p_{32}\right)p_{21}\right\} \qquad (5.10)$$

and of course other combinations are possible, using equations (3.5) and (3.15). In calculating p_{32} just imagine the inelastic scattering to be reversed and equal to σ_{i32}^* (but do not change σ_{2a}^* and σ_{3a}^*) so that the neutrons go from three to two instead of vice-versa. Then g_{32} is the g needed in region two to reproduce the absorption in region three.

[redacted]

Section 6 Extension to systems with finite tampers.

We now wish to develop a method for treating the case of a homogeneous, spherical core surrounded by a homogeneous, spherical tamper with finite outer radius. In the first place to apply our methods at all, we must know the solutions (critical ν , etc.) for one-velocity problems with finite tampers. If the core and tamper are of equal mean free path the method of the extrapolated end point described by Frankel, Nelson, and Goldberg in LA-53 and LA-258 can be used to obtain very accurate results. When the free-paths are unequal the approximate methods of Feynman (LA -608) or, of Serber (LA - 234) can be used.

We shall consider only the case of two neutron velocities. Not only will the critical ν values be different but the fundamental equations (1.2) will have to be altered in case the tamper is finite. The term in these equations,

$$\frac{\sigma_{t12}^*}{\sigma_{1a}^*}\left(1-\frac{1}{\nu_{10}}\right) p_{12}\left[S(1\to 1)A_1 + S(2\to 1)A_2\right] \qquad (6.1)$$

must be changed in form, if p_{12} is still defined as the number of core absorptions at velocity two per neutron liberated in the tamper. Of course, the formula for p_{12} will be altered, but beside that the term (6.1) in equations (1.2) must be replaced by:

$$\frac{\sigma_{t12}^*}{\sigma_{1a}^*}\frac{\left(1-\frac{1}{\nu_{10}}\right)}{1+L_1} p_{12}\left[S(1\to 1)A_1 + S(2\to 1)A_2\right] \qquad (6.2)$$

[redacted]

[redacted]

This term is to represent the number of neutrons liberated in the core at velocity one which are inelastically scattered (with cross-section σ_{i12}^{*}) in the tamper and are eventually absorbed in the core. It can be understood in the following way. Of the neutrons liberated in the core, which are $S(1\rightarrow 1)A_1+S(2\rightarrow 1)A_2$ in number, a fraction $\frac{1}{\nu_{10}}$ are returned to the core. Hence $1-\frac{1}{\nu_{10}}$ are not returned to the core. One of two things may have happened to these neutrons. They either have been absorbed in the tamper, or they have leaked out of the outside of the tamper. Suppose we define: L_1
is the number of neutrons (of velocity one) which leak out of the tamper for each neutron which is absorbed in the tamper.

Then it is clear that
$\frac{1-\frac{1}{\nu_{10}}}{1+L_1}\left[S(1\rightarrow 1)A_1+S(2\rightarrow 1)A_2\right]$
is the number of neutrons of velocity one which are absorbed in the tamper. Of those absorbed the fraction, $\frac{\sigma_{i12}^{*}}{\sigma_{1a}^{*}}$
are inelastically scattered, and these have the chance p_{12}
of finding their way back to the core at velocity two, by definition. Thus the form (6.2) is explained.

For finite tamper problems then we have two quantities to calculate L_1 and p_{12} . We should first point out that p_{12} cannot generally be calculated by a method of imaginary inelastic scattering analogous to that used in Section 3. This is because it is not possible to reproduce closely the function $T_1(r)$ by adjusting an absorption cross section in region two to make a function $T_x = T_{12}$.

[redacted]

[redacted]

This is because, although the asymptotic exponential rates h_1 and h_2 may be made to coincide, the boundary conditions at the outer edge are different, and thus the ratio of rising to falling exponentials cannot generally be correctly matched at the same time. Put another way, the extrapolated end points will be unequal. For that reason we can find p_{12} only by the analogue of the method of section 2 in which we use explicit functions for T_1 and T_2 and neglect transition effects. This we proceed to do, finding first L_1 and then p_{12}.

Consider the neutrons which are lost from group one in the tamper. The absorptions of neutrons of group one are distributed approximately as

$$c \cdot \sigma_{1a}^* \frac{\sinh h_1 (b + x_1 - r)}{r}$$

where c is a constant h_1 is the decay constant of the asymptotic solution in the tamper and x_1 is the extrapolated end point at the outside of the tamper. This distribution neglects the complicated transition effects at the core-tamper interface and at the outside of the tamper. We have defined a quantity L_1 as the number of neutrons of velocity one which leak out of the tamper per neutron absorbed. This quantity can be approximately calculated in the following way. With the distribution just given, we integrate over the tamper to obtain the total number of absorptions and calculate the neutron current leaving the outer edge of the tamper. The current corresponding to an asymptotic neutron flux density N is:

$$\vec{J} = -\frac{g\sigma^*}{h^2} \nabla N \qquad (6.3)$$

[redacted]

This is easily shown by consideration of the integral equation and is seen to approach the diffusion theory expression for small g, which is correct.

The expression (6.3) will give the current wherever the asymptotic density is correct. We make the approximation that the current at the surface is correctly given by (6.3). This is valid for $g = 0$, where the current is independent of radius, but becomes worse as g is increased, because of the difference between the absorption in the transition effect at the outside surface and the absorption in the asymptotic density, which we have used to replace the transition effect.

We now obtain for L_1, by dividing the above calculated current by the total absorptions:

$$L_1 = \frac{h_1 b \cosh h_1 x_1 + \sinh h_1 x_1}{h_1 a \cosh h_1 (b + x_1 - a) + \sinh h_1 (b + x_1 - a) - h_1 b \cosh h_1 x_1 - \sinh h_1 x_1} \quad (6.4)$$

where a and b are the core radius and the outside tamper radius, respectively. * Note: * Frankel and Goldberg (LA-258) describe methods whereby L_1 can be calculated exactly (excluding transition effects at the core-tamper interface). That is the approximate formula (20) for $\bar{J}$ is corrected at the outside edge of the tamper. The result is that the combination $h_1 b \cosh h_1 x_1 + \sinh h_1 x_1$ which appears in the numerator, and last half of the demominator, and which arises from (20), is to be replaced by (if we call $\sigma_1 = 1$)

$$h_1 \sqrt{\frac{h_1^2 - g_1}{2g_1(1-h_1^2)}} \left(b + \frac{1}{\pi}\int_0^1 T_c\, ds\right)$$

where T_c is a certain expression depending on g given in LA-258. These expressions would be identical if the square root could be replaced by the $\cosh h_1 x_1$ and if $\frac{1}{\pi}$ times the integral could be replaced by $\tanh \frac{(h_1 x_1)}{h_1}$

That this is nearly possible can be seen from the following table.

g	$\sqrt{\frac{h^2-g}{2g(1-h^2)}}$	$\cosh(hx)$	$\frac{1}{\pi}\int_0^1 T_c\, ds$	$\tanh \frac{h_1 x_1}{h_1}$
0.0	1.00	1.00	.71	.71
.2	1.23	1.21	.75	.79
.4	1.72	1.63	.80	.87
.6	3.26	2.94	.87	.95

Since g is rarely as large as 0.4 we can use the simpler form given in the text. Actually if g is as large as 0.4 the entire formula is inaccurate for another reason. For such a large g transition effects at the core tamper interface are considerable and limit the accuracy of (20).

To find p_{12} we shall use an approximate representation of the tamper densities by asymptotic solutions of the integral equation. We define a kernel $Q_2(\vec{x} \to \vec{x}')$ as the flux density of neutrons of velocity two at a point $\vec{x}'$ arising from a source of one neutron per second at $\vec{x}$. We call the tamper solution for group one $T_1(\vec{x})$ and approximate it by the asymptotic solution so that:

$$T_1(\vec{x}) = \frac{\sinh h_1(b + x_1 - r)}{r} \qquad (6.5)$$

To the accuracy which we used in deriving the previous expressions for P_{12}, we have for p_{12} :

$$P_{12} = \sigma_{2a} \frac{16\pi^2 \int_a^b r^2 dr \int_0^a r'^2 dr' T_1(r) Q_2(r \to r')}{4\pi \int_a^b r^2 dr\, T_1(r)} \tag{6.6}$$

As before, we replace $Q_2(r-r')$ by $Q_2(r' \to r)$, because of the fact that it is a symmetric one-velocity diffusion kernel. We then note that

$$4\pi \int_0^a r'^2 dr' Q_2'(r' \to r)$$

is just the flux density in the tamper of neutrons of velocity two, arising from a uniform source spread over the core, of unit strength per unit volume. Of the neutrons from such a source, $\frac{4\pi a^3}{3}$ in number per second, a fraction equal approximately to $\frac{1}{\nu_2}$ are absorbed in the core, a fraction $1 - \frac{1}{\nu_2}$ are lost in the tamper, and of these a fraction $\frac{1}{1+L_2}$ (L_2 is defined by replacing 1 by 2 in the definition of L_1) are actually absorbed in the tamper. We can then write the desired density in the tamper as:

$$\frac{1-\frac{1}{\nu_2}}{1+L_2} \cdot \frac{4\pi a^3}{3} \cdot \frac{T_2(r)}{\sigma_{2a}^* \, 4\pi \int_a^b r^2 dr\, T_2(r)} \tag{6.7}$$

where $T_2(r)$ is to be approximated by:

$$T_2(r) = \frac{\sinh h_2 (b + x_2 - r)}{r} \tag{6.8}$$

If we now do the indicated integrations, we obtain an explicit expression for :

$$P_{12}=\frac{a^3}{\sigma}\frac{\sigma_{2a}}{\sigma_{2a}^*}\left(1-\frac{1}{\nu_2}\right)\frac{h_1^2\,h_2^2}{[S_1+h_1 a c_1-\Sigma_1-h_1 b K_1][S_2+h_2 a c_2]}$$

$$\times\left\{\frac{(S_1C_2-\Sigma_1K_2)+(C_1S_2-K_1\Sigma_2}{h_1+h_2}-\frac{(S_1C_2-\Sigma K_2)-(C_1S_2-K_1\Sigma_2)}{h_1-h_2}\right\}$$

(6.9)

The following abbreviations have been used:

$S_1=\sinh h_1(b-a+x_1)$ $\quad$ $\Sigma_1=\sinh h_1 x_1$

$C_1=\cosh h_1(b-a+x_1)$ $\quad$ $K_1=\cosh h_1 x_1$

$S_2=\sinh h_2(b-a+x_2)$ $\quad$ $\Sigma_2=\sinh h_2 x_2$

$C_2=\cosh h_2(b-a+x_2)$ $\quad$ $K_2=\cosh h_2 x_2$

It can easily be shown that as b is allowed to become very large, this expression for P_{12} approaches the expression given by equation (2.4) of Chapter III for an infinite tamper. This is to be expected since the two derivations are completely parallel except for the thickness of the tamper. The formula as it stands is complicated, and some experience in its use would probably permit one to find simpler approximations to it, but we have not done a great deal of work with finite tampers as yet.

In appendix VII we give a numerical check of the procedure which we have developed for two group finite tamper problems. No actual calculations of this sort are available, because of their difficulty when done by more conventional methods. We therefore use the method to solve a one-velocity problem which has been artificially split into a two-velocity problem by the addition of an inelastic cross-section in the tamper. The problem can then be solved exactly as a one-velocity problem and we thus obtain a reasonable check of the method.

To extend the method to problems with three or more velocities would require the study of quantities analagous to p_{123} for the finite tamper. This is difficult to work out for the general case, and entirely satisfactory formulae for this case have not been worked out.

If the total mean free path is nearly the same in all energy groups then the methods of section 2 and section 3 can be applied to obtain an appropriate formula for the analogue of p_{123}

Chapter IV Special Methods for the Solution of Many Velocity Problems.

We propose in this chapter to describe specific procedures, based on the developments of the preceding chapters, for the solution of a useful variety of problems. We shall show how to calculate the multiplication rates of systems composed of a homogeneous spherical core and a homogeneous spherical tamper with arbitrary outside radius.

We shall try to devise suitable tests of the accuracy of the methods given and shall indicate how the methods may be extended to include more general problems.

Section 1 Reduction of multiplying systems to equivalent critical problems.

We first state and prove the following useful theorem. Any system in which the neutron density depends on time through a factor $e^{\alpha t}$ can be thought of as a critical system (density constant in time) in which a velocity - dependent capture cross-section equal to $\frac{\alpha}{v}$ is added to the system. The proof is elementary. Consider the Boltzmann equation describing the system. If $n(\vec{x}, \vec{v}, t)$ is the complete density function, we have:

$$\frac{\delta n}{\delta t} + \vec{v} \cdot \vec{\nabla} n + v\sigma n = S \qquad (1.1)$$

Here S is the source of neutrons being restored to $\vec{x}$ and $\vec{v}$ and is explicitly independent of time, being the result of some linear integral operation on n. We write $n = n_0(\vec{x}, \vec{v})\, e^{\alpha t}$ and obtain:

$$\alpha n_0 + \vec{v} \vec{\nabla} n_0 + v\sigma n_0 = S_0$$

$$\text{or } \vec{v} \vec{\nabla} n_0 + v\left(\sigma + \frac{\alpha}{v}\right) n_0 = S_0 \qquad (1.2)$$

Here S_0 is the result of the same operation on n_0 that gave S when operating on n. From equation (1.2) we then see that the system is equivalent to a time independent one with total cross-section $\sigma + \frac{\alpha}{v}$ and no additional source involving α, so that the $\frac{\alpha}{v}$ is essentially a capture cross section.

We shall then treat only critical systems, it being assumed that all multiplying or decaying systems have been reduced to critical by the use of the above theorem.

Section 2 Methods for problems with no inelastic scattering in the tamper.

Let us begin by outlining the method which seems most useful for the solution of problems in which the tamper does not scatter inelastically. We shall use the first lower approximation developed in Chapter II. We consider a system composed of a homogeneous, spherical core with radius a and a homogeneous, spherical tamper with outside radius b. The core is characterized by a total cross section $\sigma_t(v)$, a fission cross section $\sigma_f(v)$, a capture cross section $\sigma_c(v)$, an inelastic cross section $\sigma_i(v)$, a fission spectrum $\chi(v)$ normalized to unit integral, an inelastic spectrum $\psi(v' \to v)$ $(=0 \text{ for } v > v')$ normalized to unit integral over v, a total cross section $\sigma_t^*(v)$ in the tamper, and a capture cross section $\sigma_c^*(v)$ in the tamper. The value of ν for fission is given and it is desired to find the multiplication rate of the neutron density. We guess a multiplication rate α and add $\frac{\alpha}{v}$ to the total and capture cross sections in core and tamper. We then wish to find whether the resulting system is critical with the given value of ν. We do this by finding the value of ν required for criticality. If the critical ν is higher than the actual ν we have guessed too high a value for α and must adjust the next guess accordingly.

Usually two trials will give a useful value for the multiplication rate.

In order to find the critical value of ν we write equation (2.3) of Chapter II:

$$A(v)=\frac{1}{\nu_0(v)}\int_0^\infty dv' S(v'\to v)A(v') \tag{2.1}$$

We can write $S(v'\to v)$ explicitly in terms of the cross sections (σ and σ_c are understood to have the $\frac{\alpha}{v}$ absorption added). We define an absorption cross section $\sigma_a(v)=\sigma_i(v)+\sigma_c(v)+\sigma_f(v)$ and write $S(v'\to v)$ as:

$$S(v'\to v)=\nu x(v)\frac{\sigma_f(v')}{\sigma_a(v')}+\psi(v'\to v)\frac{\sigma_i(v')}{\sigma_a(v')} \tag{2.2}$$

Equation (1.3) becomes, remembering that inelastic scattering must decrease the energy of a neutron:

$$A(v)=\frac{\nu x(v)}{\nu_0(v)}\int_0^\infty dv'\frac{\sigma_f(v')}{\sigma_a(v')}A(v')+\frac{1}{\nu_0(v)}\int_v^\infty dv'\psi(v'\to v)\frac{\sigma_i(v')}{\sigma_a(v)}A(v') \tag{2.3}$$

For convenience, we choose the size of $A(v)$ so that one neutron emerges from fission per second. That is, we place:

$$\nu\int_0^\infty dv\,\frac{\sigma_f(v)}{\sigma_a(v)}A(v)=1 \tag{2.4}$$

With this normalization, equation (5) becomes:

$$A(v) = \frac{\chi(v)}{\nu_0(v)} + \frac{1}{\nu_0(v)} \int_V^\infty dv' \psi(v' \to v) \frac{\sigma_i(v')}{\sigma_a(v')} A(v') \qquad (2.5)$$

All the functions in equation (2.5) are known except for $A(v)$ and $\nu_0(v)$. Since $\nu_0(v)$ is just the ν required for criticality in a certain known one-velocity problem (Chapter II), we can calculate it by the methods suggested in Chapter I so that only $A(v)$ remains to be found. Equation (2.5) can then be integrated numerically step-by-step. Suppose v_0 is the highest energy of the fission spectrum. Then $A(v_0)$ is just $\frac{\chi(v_0)}{\nu_0(v_0)}$. We start $A(v)$ with this initial value and find it for successively lower valuesof v by using the values already found to do the integral on the right-hand side of equation (2.5). We finally obtain the complete function $A(v)$, on the assumption of the normalization (2.4). If the function $A(v)$ is then substituted in the normalization condition and the integral performed, we obtain the value of ν required for criticality.

This procedure is a useful one in cases where one desires a detailed picture of the spectrum of neutrons present inside the system or escaping from it. It may be expected to be very accurate for the untamped case, in which case the assumption of no inelastic scattering in the tamper is exact and in addition the first upper approximation will be extremely good because of the similarity in shape of the untamped eigenfunctions at all velocities. Its accuracy may be less but will still be very good in cases where the properties of the tamper do not vary wildly with velocity.

Section III Special simplifying assumption for inelastic scattering.

There are some cases with a continium of velocities in which the integral equation can be solved exactly, and a great deal of numerical work can be avoided. For example, it may be sufficiently accurate to represent the inelastic scattering cross sections in a special form such that the integration of equation (2.5) is much simplified. Suppose we could write:

$$\sigma_i(v')\,\phi(v' \to v) = \sum_{n=1}^{N} \sigma_n(v')\,\chi_n(v) \qquad (3.1)$$

This is just the assumption that the inelastic scattering behaves as though there existed various varieties of fission, each with its own cross section as a function of velocity $\sigma_n(v')$, and associated with each kind of fission a different normalized spectrum $\chi_n(v)$; and with a ν of 1. In order to represent real inelastic scattering $\sigma_n(v)$ should be taken different from zero only for values of v <u>higher</u> than those for which $\chi_n(v)$ is different from zero.

If we now write equation (2.5) we have:

$$A(v) = \frac{\chi(v)}{\nu_0(v)} + \frac{1}{\nu_0(v)} \sum_{n=1}^{N} \chi_n(v) \int_0^{\infty} \frac{\sigma_n(v')}{\sigma_a(v')} A(v')\,dv' \qquad (3.2)$$

This can be reduced to a set of N linear inhomogeneous algebraic equations for N unknowns. We define:

$$A_n = \int_0^{\infty} \frac{\sigma_n(v)}{\sigma_a(v)} A(v)\,dv \qquad (3.3)$$

and obtain:

$$A(v) = \frac{x(v)}{\nu_0(v)} + \sum_{n=1}^{N} A_n \frac{x_n(v)}{\nu_0(v)} \tag{3.4}$$

which gives the function $A(v)$ in terms of the known functions $x(v)$, $x_n(v)$ and $\nu(v)$ and the unknown coefficients A_n . These coefficients can be determined by multiplying equation (3.4) by $\frac{\sigma_n(v)}{\sigma_a(v)}$ and integrating over all v . We obtain:

$$A_m = \int_0^\infty \frac{\sigma_m(v)}{\sigma_a(v)} \frac{x(v)}{\nu_0(v)} dv + \sum_{n=1}^{N} A_n \int_0^\infty \frac{\sigma_m(v)}{\sigma_a(v)} \frac{x_n(v)}{\nu_0(v)} dv \tag{3.5}$$

This set of linear algebraic equations are easily solved (if N is not large) and we then have $A(v)$ from (3.4). We can finally find the ν for criticality by performing the integral indicated in equation (2.4).

It should be finally pointed out that the integral equation can also be solved if the inelastic scattering can be represented by another form. Suppose $\phi(v' \to v)$ is of a form analagous to that arising from collisions of neutrons with protons, equation (2.5) can be integrated as a first order differential equation. That is, suppose we can write, with sufficient accuracy

$$\phi(v' \to v) = \phi_1(v')\phi_2(v) \qquad v' > v \tag{3.6}$$

$$= 0 \qquad v' < v$$

(for neutrons on protons this can be done exactly, $\phi_1 = \frac{1}{(v')^2}$, $\phi_2 = 2v$. We then have for equation (2.5)

$$A(v)\frac{x(v)}{\nu_0(v)} + \frac{\phi_2(v)}{\nu_0(v)}\int_v^\infty dv' \frac{\sigma_i(v')}{\sigma_a(v')}\phi_1(v')A(v')$$

$$\text{or } \frac{\nu_0(v)}{\phi_2(v)}A(v) = \frac{x(v)}{\phi_2(v)} + \int_0^\infty dv' \frac{\sigma_i(v')}{\sigma_a(v')}\phi_1(v')A(v') \qquad (3.7)$$

This can be differentiated with respect to v to obtain:

$$\frac{d}{dv}\left[\frac{\nu_0(v)}{\phi_2(v)}A(v)\right] + \frac{\sigma_i(v)}{\sigma_a(v)}\phi_1(v)A(v)\frac{d}{dv}\left[\frac{x(v)}{\phi_2(v)}\right] \qquad (3.8)$$

This can immediately be integrated to give $A(v)$ as an indefinit integral on v . In general the integral cannot be performed analytically, but good accuracy can be obtained by the use of Simpson's Rule, or some equivalent method. The application of this method to the treatment of the hydride problem has been given by Ehrlich in LA-508.

Section 4 The method of velocity groups

A somewhat less cumbersome procedure, which is not exact but will give good accuracy in critical mass calculations (but not so detailed an energy spectrum), is the so called group approximation. We assume that neutrons have only certain discrete energies and average the cross sections in such a way that reasonable results are obtained. This approximation can be put on a rigorous basis if there is no inelastic scattering in the tamper. We proceed to do this by one possible method.

In equation (2.5) we write $A(v) = \sigma_a(v)\, N(v)$, where $N(v)$ is the flux density at velocity v integrated over the core. Equation (2.5) becomes, if we multiply by $\nu_0(v)$ and remember equation (5.11) of Chapter I (for ν_0 in terms of f, where f is the number of excess neutrons required per collision for criticality):

$$\left[\sigma_a(v) + f(v)\sigma(v)\right] N(v) = \nu\, x(v) \int_0^\infty dv' \sigma_i(v')\, N(v') + \int_v^\infty dv' \psi(v' \to v)\, \sigma_i(v')\, N(v') \qquad (4.1)$$

In equation (4.1), we interpret the left-hand side as the number of neutrons lost from velocity v per second and the right hand side as the number added to velocity v per second. Equation (4.1) of course says that these must be equal for criticality. The first term on the left, $\sigma_a N$, is the number lost by absorption, whereas $f\sigma N$ is the number lost by leakage into the tamper. We shall think of $f\sigma$ as an effective cross section for loss of leakage.

We now define n energy regions or groups, as we will call them, which do not overlap but collectively include the complete range of energy. Let us call u_i and v_i the lower and upper limits respectively of i th group, and understand that the first group has the lowest energy. We then integrate equation (4.1) over the i th group and obtain:

$$\int_{u_i}^{v_i} \sigma_a(v) N(v) dv + \int_{u_i}^{v_i} f(v) \sigma_f(v) N(v) dv =$$

$$\nu \int_{u_i}^{v_i} \chi(v) dv \int_0^{\infty} \sigma_f f(v') N(v') dv'$$

$$+ \int_{u_i}^{v_i} dv \int_v^{\infty} dv' \psi(v' \to v) \sigma_i(v') N(v') \quad (4.2)$$

We define:

$$\int_{u_i}^{v_i} N(v) dv = N_i$$

$$\frac{\int_{u_i}^{v_i} \sigma_a(v) N(v) dv}{\int_{u_i}^{v_i} N(v) dv} = \overline{\sigma_a}^i$$

$$\frac{\int_{u_i}^{v_i} f(v) \sigma_f(v) N(v) dv}{\int_{u_i}^{v_i} N(v) dv} = \overline{f\sigma}^i \quad (4.3)$$

$$\int_{u_i}^{v_i} \chi(v) dv = \chi_i$$

$$\frac{\int_{u_i}^{v_i} \sigma_f(v) N(v) dv}{\int_{u_i}^{v_i} N(v) dv} = \overline{\sigma_f}^i$$

[redacted]

Equation (4.2) can then be written, reversing the order of integration in the term involving the inelastic cross section:

$$\overline{\sigma_a}^i N_i + \overline{f\sigma}^i N_i = \nu\, x_i \sum_{j=1}^{n} \overline{\sigma_f}^j N_j$$

$$+ \int_{v_i}^{\infty} dv' \sigma_i(v') N(v') \int_{u_i}^{v_i} dv\, \psi(v' \to v)$$

$$+ \int_{u_i}^{v_i} dv' \sigma_i(v') N(v') \int_{u_i}^{v'} dv\, \psi(v' \to v) \qquad (4.4)$$

We further define

$$\frac{\int_{u_j}^{v_j} dv' \sigma_i(v') N(v') \int_{u_i}^{v_i} dv\, \psi(v' \to v)}{\int_{u_j}^{v_j} dv\, N(v)} = \overline{\sigma_i}^{ji}$$

(4.5)

The quantity $\overline{\sigma_i}^{ji}$ is essentially an average inelastic scattering cross section from the j th group to the i th group. It is different from zero only if $j \geq i$. Its values for $i = j$ represent an addition to the elastic cross sections arising from inelastic scattering which fails to remove a neutron from the group in which it started. In the definition (4.5), the v_i in the integral over v could be replaced by v', since $\psi(v' \to v) = 0$ for $v > v'$.

Equation (4.4) can now be written:

$$\overline{\sigma_a}^i N_i + \overline{f\sigma}^i N_i = \nu\, x_i \sum_{j=1}^{n} \overline{\sigma_f}^j N_j + \sum_{j=1}^{n} \overline{\sigma_i}^{ji} N_j$$

(4.6)

If the average cross sections occurring in equation (4.6) have been obtained by integrating over the actual spectrum, the set of N_i and the ν obtained will be identical with their correct values. The approximation which will be used is that of using an approximate spectrum (guessed or calculated on some simplified assumptions) to obtain the average cross sections. The solution of the set of equations (4.6) will then yield an approximate value of ν and the approximate number of neutrons in each group. If a reasonable number of groups are used, so that the cross sections do not vary widely within any group, the approximation to the average cross sections will be excellent and we can therefore expect to obtain accurate results for ν and the spectrum.

The procedure which we have given contains specific recipes for averaging the absorption cross sections, the fission spectrum, and the inelastic spectrum, all in the core. It is implied that the average velocity of a group is given by:

$$\frac{1}{\overline{v}^i} = \frac{\int_{u_i}^{v_i} \frac{1}{v} N(v)\,dv}{\int_{u_i}^{v_i} N(v)\,dv} \qquad (4.7)$$

This is so, because the velocity enters only in the form of an absorption cross section $\frac{\alpha}{v}$

No very direct procedure is implied for averaging the total core cross section, the total tamper cross section, or the tamper capture cross section. These quantities enter only in the calculation of $\overline{f\sigma}^i$ and all that can be said precisely is that $f\sigma$ should be averaged in the same way as an absorption cross section.

In practice it is usually adequate to calculate f_{σ_i} as the f_σ for a one-velocity problem whose cross sections in both core and tamper are reasonable averages over the i' th group of the actual cross sections.

It is clear that the procedure which we have outlined for solving the integral equation (2.1) will work equally well for the set of simultaneous equations (4.6), if all integrations are replaced by appropriate summations. The use of the group approximation makes much more practicable the application of the sequence of upper and lower approximations described in Chapter II. This remains difficult, however, because the one-velocity eigenfunctions are not sufficiently well known. Upper and lower limits can be put on the critical radius and the critical ν by the use of a perturbation scheme like that which led to equation (II-9) of Appendix II. These procedures are sufficiently illustrated by their application to some two-group critical calculations in Appendix IV.

Section 5 Methods including inelastic scattering in the tamper.

We now propose to apply the methods of Chapters II and III to the calculation of critical problems in which inelastic scattering is present in the tamper. We do this first for the case of an infinite tamper. We discuss the case of three neutron velocity groups, since this fully illustrates the general method.

As in Chapter III, we define a total absorption cross section in the tamper as the sum of inelastic and capture cross sections (assuming no fission in the tamper). Neutrons leaving the core make inelastic collisions in the tamper and these collisions then form a source, in the tamper, of lower energy neutrons.

The diffusion of these neutrons back into the core must then be treated by the use of the quantities P_{12}, P_{13}, P_{23}, P_{123} defined in equations (2.4), (3.5), (5.7), or (5.8) of Chapter III.

Using a notation similar to that of equation (4.6), we write:

$$\sigma_{1a} N_1 + f_1 \sigma_1 N_1 = \nu \chi_1 \left(\sigma_{1f} N_1 + \sigma_{2f} N_2 + \sigma_{3f} N_3\right) \tag{5.1}$$

On the left-hand side is the total number of neutrons removed per second from group one in the core by absorption and leakage. The right hand side is the total number entering group one per second (by fission). The term $f\sigma$, which gives the leakage, includes the loss by inelastic scattering in the tamper, since f_1 is assumed to be calculated with inelastic scattering in the tamper treated as capture.

We can write a similar equation for the second velocity group:

$$\sigma_{2a} N_2 + f_2 \sigma_2 N_2 = \nu \chi_2 \left(\sigma_{1f} N_1 + \sigma_{2f} N_2 + \sigma_{3f} N_3\right) + \sigma_{i12} N_1 + f_1 \sigma_1 \frac{\sigma_{i12}^*}{\sigma_{1a}^*} P_{12} N_1 \tag{5.2}$$

The last term on the right is just the total number of absorptions per second taking place in the core of neutrons diffusing back into the core after having been degraded to velocity two by an inelastic collision in the tamper.

This is easily seen if it is remembered that $f_1 \sigma_1 N_1$ is the total number of neutrons of velocity one which are absorbed in the tamper per second, that a fraction $\frac{\sigma_{i12}^*}{\sigma_{1a}^*}$ of these are degraded to velocity two, and that a fraction P_{12} of these (Chapter III) are absorbed in the core. Equation (5.2) is then best understood by transposing this term to the left. The left-hand side is then just the total number of neutrons per second removed from velocity two in the core by absorption and leakage, exclusive of those which were returned from a degradation in the tamper. The right hand side, to which it must be equal, is just the total number of neutrons of velocity two produced per second in the core.

A similar equation for group three follows immediately:

$$\sigma_{3a} N_3 + f_3 \sigma_3 N_3 = \nu \times_3 (\sigma_{1f} N_1 + \sigma_{2f} N_2 + \sigma_{3f} N_3)$$

$$+ \sigma_{i3} N_1 + \sigma_{i23} N_2$$

$$+ f_1 \sigma_1 \frac{\sigma_{i13}^*}{\sigma_{1a}^*} P_{13} N_1 + f_2 \sigma_2 \frac{\sigma_{i23}^*}{\sigma_{2a}^*} P_{23} N_2$$

$$+ f_1 \sigma_1 \frac{\sigma_{i12}^*}{\sigma_{1a}^*} P_{123} N_1 \qquad (5.3)$$

The only term in equation (5.3) requiring further explanation is the last one, which must be added in to take care of the neutrons which are twice degraded in the tamper before reentering the core. The equations (5.1), (5.2) and (5.3) are exactly equivalent to (5.1) and (5.2) of Chapter III, simply rewritten in a form more convenient for practical calculation.

If only two groups are used, just omit equation (5.3) and set $N_3 = 0$. In this case, if the tamper is finite, replace P_{12} by $\frac{P_{12}}{1 + L_1}$.

The method of solution of these three equations is just that described for the equations (4.6). That is, the total number of neutrons produced by fission per second is normalized to unity (for convenience) and the equations solved step by step for N_1, N_2, N_3. The value of ν is then found from:

$$\frac{1}{\nu} = (\sigma_{1f} N_1 + \sigma_{2f} N_2 + \sigma_{3f} N_3) \tag{5.4}$$

Although we have made a considerable effort to provide a reasonably vigorous foundation for these equations, it is clear that their best justification must come from actual numerical comparisons with more trustworthy calculations. It is impractical to do an actual calculation (except with the methods here given) to an accuracy greater than that given by a two-group calculation with the so-called P_3 spherical harmonic approximation to the solution of the Boltzmann equation. Many comparisons of this type have been made, with excellent results, but for our purposes it is sufficient to make the comparison with two group diffusion theory. This is done in Appendix V.

It is also profitable to make various sorts of internal comparisons of the methods here developed. One questionable point is that of the accuracy attained by the use of the quantities P_{12}, P_{123} etc. We can shed some light on this by performing a typical three-group critical calculation in two ways, first by the use of the definitions (3.5) and (5.8) (Chapter III) for P_{12} and P_{123},

and second by the use of the definitions (3.15) and (5.7). These two sets of definitions were derived in widely different ways and were seen to be most useful under different sets of conditions. The discrepancy between the two approximate calculations can then be taken as a reliable estimate of the absolute error introduced by our approximations. A comparison of this sort is also made in Appendix VI.

It is difficult, in the few selected examples given in these appendices to convey the impression of confidence in the accuracy of these formulas which one obtains when actually working with them and comparing them time and time again with surprising success to the results of much more difficult but more exact calculations.

It should finally be pointed out in a little more detail why the group approximation (or something similar) is essential to the treatment of inelastic scattering in the tamper. Suppose we had a tamper in which many inelastic collisons could be made by a single neutron. We would then have to use quantities like p_{12}, p_{123}, p_{1234} etc. These quantities (except for the first two) seem to be exceedingly difficult to calculate with any degree of accuracy, so that a restriction to a small number of velocities is necessary.

There do exist problems however with a continuum of velocities which can be treated by our methods. This can be done if the inelastic scattering function in the tamper is so restricted that a neutron once inelastically scattered can make no more inelastic collisions, or possibly only one more. We illustrate with the case where no more than one inelastic collision in the tamper is

permitted. We write the integral equation analagous to the simultaneous equations which we have been using:

$$A(v) = \frac{1}{\nu_0(v)} \int_0^\infty dv' S(v' \to v) A(v')$$

$$+ \int_0^\infty dv' \int_0^\infty dv'' A(v') S(v' \to v'') \left[1 - \frac{1}{\nu_0(v'')}\right] \frac{\sigma_i^*(v'')}{\sigma_a^*(v'')} \phi^*(v'' \to v) p(v''; v) \tag{5.5}$$

Here the second term on the right gives the number of absorptions taking place in the core by reason of single inelastic scattering in the tamper. The function $p(v'';v)$ would be written in our previous notation as $p_{v''v}$ and is just the return to the core of neutrons of velocity v released in the tamper distribution characteristic of velocity v''. Equation (5.5) will not in general describe the system correctly. It <u>will</u> however do so if $\phi^*(v'' \to v)$ and $\sigma_i^*(v)$ are nowhere different from zero for the same value of v. In this case, a neutron once degraded cannot be again inelastically scattered and equation (5.5) is correct, at least to the extent that our previous approximations are valid.

The equation (5.5) can in principle be solved, in particular by a numerical step by step procedure, but the work will in general be prohibitive. By the use of the continuous analogue of p_{123}, a similar equation can be written in the case where a neutron can be degraded not more than twice in succession in the tamper. This equation is naturally very complicated and about equally useless.

Section 6 Other problems amenable to the methods described herein.

In actual practice it will usually be necessary to treat systems which are less idealized than the ones which we have considered so far. A partial list of useful extensions of the foregoing formalism might include methods for treating:

1. Systems departing from spherical symmetry.
2. Systems containing holes, or other inhomogeneities.
3. Subcritical systems containing a steady source.

We shall outline how several of these generalizations may be made. Consider first the case of a non-spherical system, such as a homogeneous cubical core imbedded in an infinite homogeneous tamper. If the cubical core problem were solved in general for neutrons of a single velocity, the extension to many velocities would be simple because none of our basic results thus far depend essentially on the spherical symmetry of the core. They do depend on the assumption that the fundamental modes at the various neutron velocities do not differ radically in shape in the core. This assumption should be equally valid for cores of any not too irregular shape, so that to treat the cubical core, we have only to obtain the eigenvalues for the appropriate one-velocity problems (with cubical cores). This is not easy, but the difficulty does not lie in our many-velocity formalism.

Consider now an otherwise spherical, homogeneous system with an infinite tamper having a small hole at its center. Such a system is encountered when one changes the degree of criticality of a system by removing material from the center. The consequent change in the value of ν required for criticality can be

obtained by the consistent application of the formalism thus far developed with the specification that all one-velocity eigenvalues are to be calculated for systems with holes in the center. The reasoning here is exactly the same as that just used for the non-spherical case, but it is now possible to carry through the calculation with ease. This is so because the eigenvalue for a spherical one-velocity system with a hole in the center is easily calculated by a simple perturbation scheme involving only the eigenvalue for the system without the hole, the volume of the hole and the ratio of the neutron density at the center to the average neutron density in the core.

If the hole is not at the center, the perturbation will also involve the angular distribution of neutron velocities at the hole, and if the hole is in the tamper, the procedure becomes meaningless. More general inhomogeneities in density can presumably be well treated by the same method if they occur in the core. Here, the difficulty in obtaining good one-velocity eigenvalues becomes very severe. Again, variations in tamper density render the method less valid, because the tamper densities in the one-velocity problems cannot be made to coincide so closely by changing the absorptions.

If the inhomogeneity consists of variations of composition, so that the system is not divided into a core and a tamper each with homogeneous composition, the method breaks down. A clumsy perturbation treatment can still be made, if the composition is nearly uniform, but the main advantages of the method are lost.

Another important type of problem is that of a subcritical system which multiplies a steady neutron source. The application of the formalism to a problem of this type is sufficiently illustrated by consideration of an untamped active sphere with a source at the center. Here one uses one of the series of upper and lower approximations described in Chapter II, with the given source introduced into the original integral equation. A set of inhomogeneous equations will be obtained which can be solved to obtain an approximation to the number of fissions occurring per second.

Better results will be obtained by the use of the first collision source, rather than the point source at the center. Neither the first upper nor lower approximations will be adequate, because of the bad shape of the source, but the second lower approximation will usually give good accuracy.

This problem has been very carefully studied from the point of view of the methods developed herein by Ashkin and by these and other methods by Serber. Ashkin was able to treat the multiplication of an untamped sphere with a continuum of velocities. For practical problems with metal spheres, however, the method of velocity groups was sufficient. For small spheres Serber's method of successive collisions was also able to deal with the complete velocity spectrum. The matters are discussed in detail in Serber's LAMS - 253.

APPENDIX I. Diffusion theory check of assumptions for first lower approximation.

We wish to obtain some numerical confirmation of the basic assumption underlying the U-1 and L-1 approximations. We consider a two-velocity problem in the diffusion approximation, with the following constants:

$$\sigma_1 = \sigma_1^* = 4 \qquad \sigma_{1a} = 1 \qquad \sigma_{1c}^* = 0 \qquad a = 0.48690$$

$$\sigma_2 = \sigma_2^* = 8 \qquad \sigma_{2a} = 2 \qquad \sigma_{2c}^* = 0.11626 \tag{1}$$

The quantity a is the radius of the core.

The fundamental modes at the two velocities have the shapes, in the core:

$$U_1(r) = \frac{\sin k_1 r}{r}$$

$$U_2(r) = \frac{\sin k_2 r}{r} \tag{2}$$

where k_1 and k_2 are to be determined. The corresponding tamper densities are $\frac{e^{-h_1 r}}{r}$ and $\frac{e^{-h_2 r}}{r}$. The quantities h_1 and h_2 can be determined from the cross sections in the usual way, and we then require that the logarithmic derivatives of the densities be continuous at $r = a$. The problem has been so adjusted that:

$$k_1 a = \frac{\pi}{2}$$

$$k_2 a = \frac{5\pi}{8} \tag{3}$$

We can then find the value of ν which would have to be associated with the core absorption cross section at each velocity to maintain these fundamental modes at critical. These turn out to be:

$$\nu_1 = 1.86731$$
$$\nu_2 = 1.33880 \qquad (4)$$

We then calculate, both exactly and with the approximations of Chapter II, the density of absorptions of neutrons of velocity two from a source with the shape of u_1 . We have:

$$-\frac{1}{3\sigma_2}\nabla^2(nv)_2 + \sigma_{2a}(nv)_2 = \frac{\sin k_1 r}{r} \qquad (5)$$

This yields for $(nv)_2$:

$$r(nv)_2 = \frac{3\sigma_2}{k_1^2 + 3\sigma_2\sigma_{2a}}\sin k_1 r + A \sinh\sqrt{3\sigma_2\sigma_{2a}}\, r \qquad (6)$$

Where A must be adjusted to satisfy the boundary condition at $r=a$. Equating the logarithmic derivative of (6) at $r=a$ to $-h_2$, we obtain:

$$A = \frac{3\sigma_2}{k_1^2 + 3\sigma_2\sigma_{2a}} \frac{k_1 \cos k_1 a + h_2 \sin k_1 a}{\sqrt{3\sigma_2\sigma_{2a}}\cos h\sqrt{3\sigma_2\sigma_{2a}}\, a + h_2 \sin h\sqrt{3\sigma_2\sigma_{2a}}\, a} \qquad (7)$$

The density of absorption can then be calculated as $(nv)_2\sigma_{2a}$.

With the L-1 approximation of Chapter II, this density of absorptions will have the shape $\frac{\sin k_2 r}{r}$ and a size such

that its volume integral is just $\frac{1}{\nu_2}$ times the volume integral of the source. This yields:

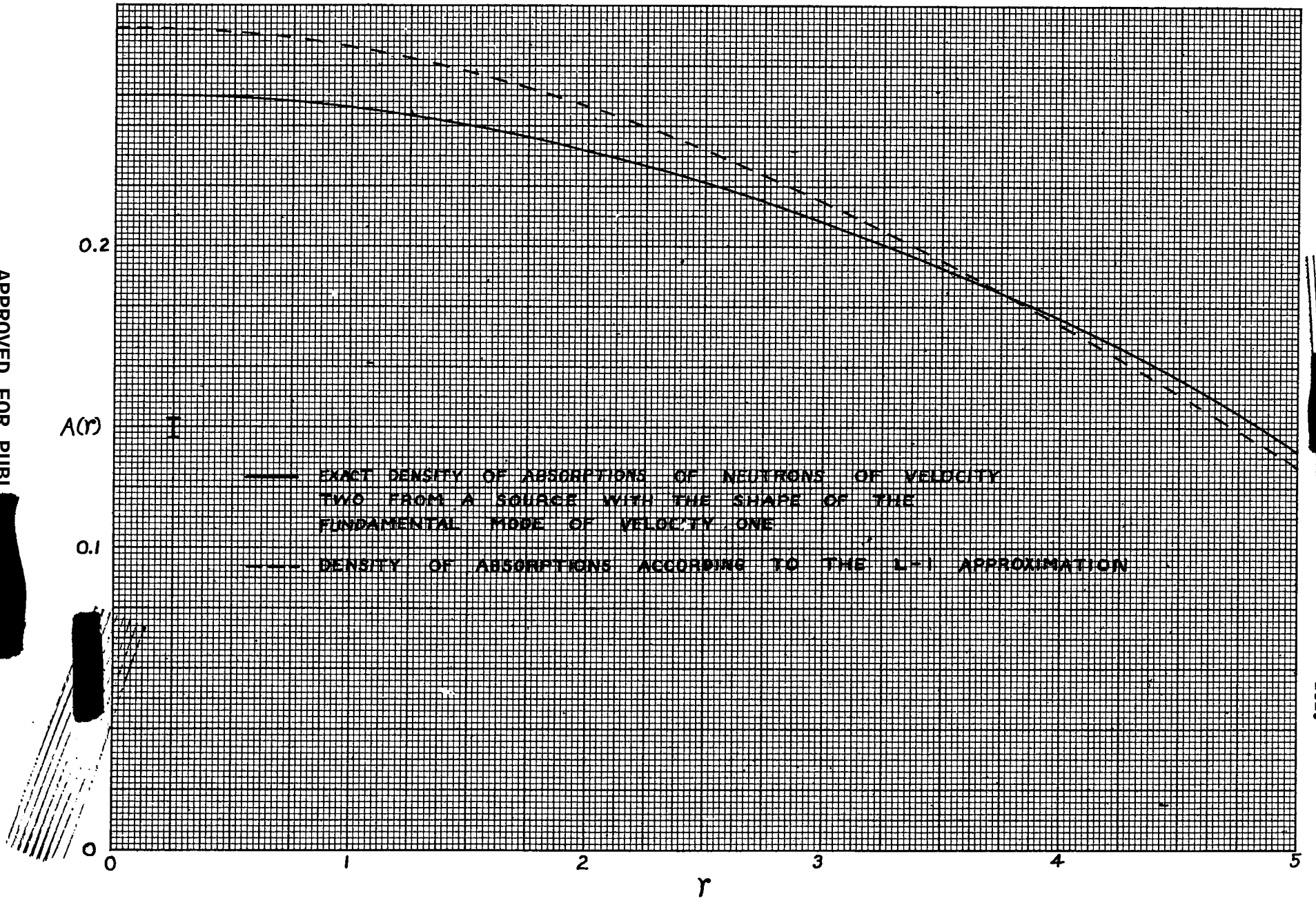
$$(n v)_2 \sigma_{2a} = \frac{1}{\nu_2} \frac{\int_0^a r\,dr \sin k_1 r \cdot \frac{\sin k_2 r}{r}}{\int_0^a r\,dr \sin k_2 r} \qquad (8)$$

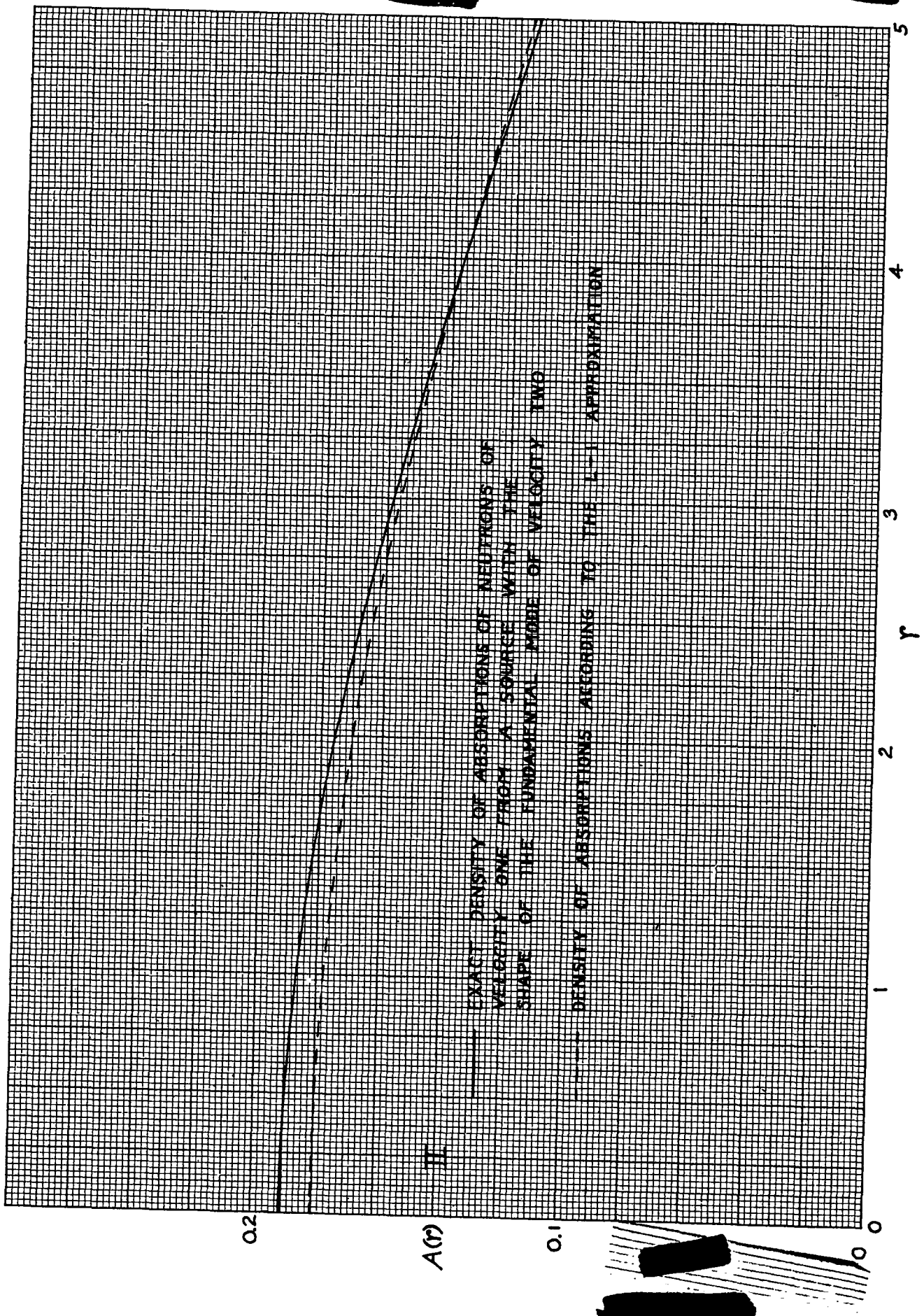
The exact expression is compared with the approximate expression in the first of the graphs immediately following. An exactly similar calculation can be made of the absorptions of neutrons of velocity one arising from a source $\frac{\sin k_2 r}{r}$. The results are plotted in the second of the succeeding graphs.

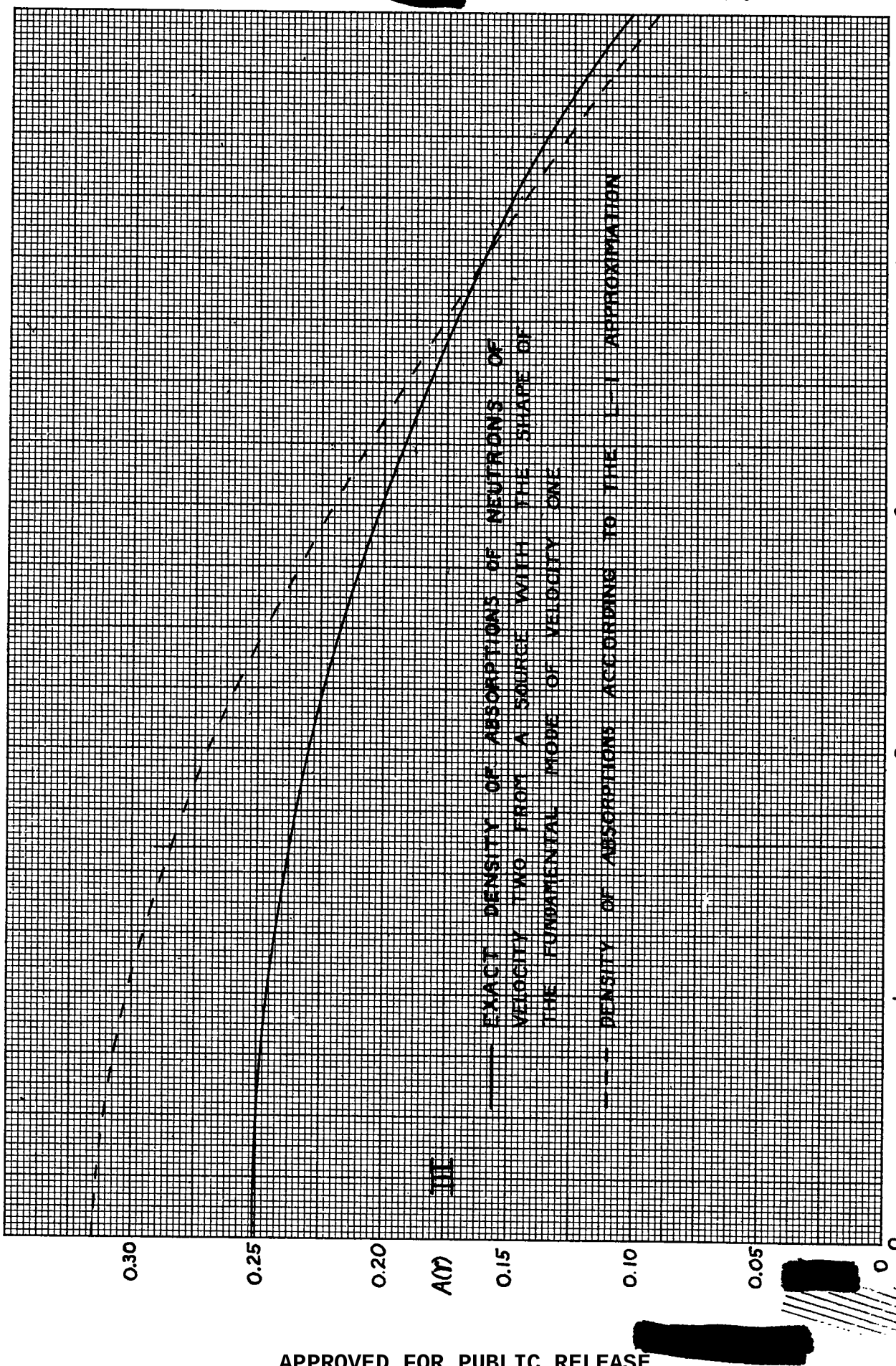
A more stringent test of our approximations would be obtained in a case where the values of $k_1 a$ and $k_2 a$ differed more widely. This has been done for a similar case with $k_1 a = \frac{\pi}{2}$ and $k_2 a = \frac{3\pi}{4}$ and the results plotted in the third and fourth graphs following. It can be seen from these plots that our approximation is a reasonable one.

It is of interest to consider the number of neutrons absorbed per neutron emitted from the source. This number has been approximated by $\frac{1}{\nu}$ and can be calculated exactly by multiplying equation (6) by σ_{2a}, integrating over the core, and dividing the result by the integral of the source over the core. We label the four cases previously considered in the order given and list the values of $\frac{1}{\nu}$ and the correct absorption ratio:

Case	$\frac{1}{\nu}$	Correct Ratio
I	0.7469	0.7395
II	0.5355	0.5398
III	0.7020	0.6773
IV	0.5698	0.5547







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APPENDIX II
The sign of the errors of the upper and lower approximations

We wish to show that the critical ν and critical radius given by the first lower approximation (equation (2.3), Chapter II) will always be less than those given by the first upper approximation (equation 3.2), Chapter II). To show this we fix the radius and the value of ν and inquire by what factor would the neutron production per absorption need to be increased to make the system critical. That is, we write instead of equations (2.3) and (3.2) the following:

$$\lambda_L A_L(v) = \frac{1}{\nu_0(v)} \int dv' S(v' \to v) A_L(v') \tag{1}$$

$$\lambda_U A_U(v) = \frac{1}{\nu_0(v)} \int dv' S(v' \to v) M_0(v, v') A_U(v') \tag{2}$$

where $$M_0(v, v') = \frac{\int d\vec{x}\, \psi_0(\vec{x}, v) \psi_0(\vec{x}, v')}{\sqrt{\int d\vec{x}\, \psi_0^2(\vec{x}, v) \int d\vec{x}\, \psi_0^2(\vec{x}, v')}}$$

Here λ_L and λ_U are the quantities by which $S(v' \to v)$ must be divided in the first lower and first upper approximations respectively in order to make the system critical. If λ_L is unity, for example, the system is just critical by the first lower approximation. If λ_L is greater than unity, the system produces too

many neutrons and is therefore supercritical. We proceed to show that λ_L is always equal to or greater than λ_U. A given system will therefore be more active in the first lower approximation, and hence the critical radius and critical ν will be less in this approximation.

We note that $M_0(v,v')$ will be equal to or less than unity by the Schwarz inequality. We therefore write:

$$M_0(v,v') = 1 - N(v,v') \tag{3}$$

where N is a positive function which will be small compared with unity if the eigenfunctions $\psi_0(\vec{x},v)$ are reasonable. We imagine that equation (1) has been solved and the eigenvalue λ_L obtained. We then obtain λ_U by a first-order perturbation calculation. We write:

$$\lambda_U = \lambda_L + \Delta\lambda$$

$$A_U = A_L + \Delta A \tag{4}$$

Equation (2) becomes, if we neglect obviously second order terms:

$$\lambda_L A_L + \lambda_L \Delta A + \Delta\lambda A_L = \frac{1}{\nu_0(v)} \int dv' S(v' \to v) \left[A_L(v') - N A_L(v') + \Delta A(v') \right] \tag{5}$$

We use equation (1) to eliminate two terms and obtain:

$$\lambda_L \Delta A + \Delta\lambda A_L = \frac{1}{\nu_0(v)} \int dv' S(v' \to v) \Delta A(v') - \frac{1}{\nu_0(v)} \int dv' S(v' \to v) N A_L(v') \tag{6}$$

There is an integral operator adjoint to that in equation (1) whose eigenvalue is also λ_L. The corresponding eigenfunction satisfies:

$$\lambda_L \bar{A}_L(v') = \int dv' S(v' \to v) \frac{1}{\nu_0(v)} \bar{A}_L(v) \tag{7}$$

We multiply equation (6) by $\bar{A}_L$ and integrate over V. We obtain:

$$\lambda_L \int dv \bar{A}_L(v) \Delta A(v) + \Delta\lambda \int dv \bar{A}_L(v) A_L(v)$$
$$= \int dv \int dv' \frac{1}{\nu_0(v)} S(v' \to v) \bar{A}_L(v) \Delta A(v')$$
$$- \int dv \int dv' \frac{1}{\nu_0(v)} S(v' \to v) N(v,v') \bar{A}_L(v) A_L(v') \tag{8}$$

Use of equation (7) yields:

$$\lambda_L \int dv \bar{A}_L(v) \Delta A(v) + \Delta\lambda \int dv \bar{A}_L(v) A_L(v')$$
$$= \lambda_L \int dv' \bar{A}_L(v') \Delta A(v')$$
$$- \int dv \int dv' \frac{1}{\nu_0(v)} S(v' \to v) N(v,v') \bar{A}_L(v) A_L(v') ,$$

or, with some obvious manipulations:

$$\Delta\lambda = - \frac{\int dv \int dv' \frac{1}{\nu_0(v)} S(v' \to v) N(v,v') \bar{A}_L(v) A_L(v')}{\int dv \bar{A}_L(v) A_L(v)} \tag{9}$$

We now show that $\bar{A}_L(v)$ and $A_L(v)$ do not change sign and hence that $\Delta\lambda$ is essentially negative. A rigorous proof of this assertion seems difficult but a simple physical argument can be given. If we have a system which according to equation (1) is not critical, then it is intuitively clear that it could be made critical by dividing $S(v' \to v)$ by a real positive number λ_L. If the system has thus been made critical, we can normalize $A_L(v)$ so that one neutron, for example, emerges from fissions per second.

Since all processes contemplated in equation (1), except for fission, degrade the energy of neutrons, we can then use equation (1) to solve for $A_L(v)$ by a step-by-step method. A positive number of neutrons will be present at the highest energies because a positive number of neutrons are emerging from fission. Positive numbers of neutrons will then be present at all lower energies by reason of inelastic scattering from above and positive numbers of neutrons emerging into the lower energies from fission. With this normalization it must then be true that $A_L(v)$ is positive for all v .

To show that $\bar{A}_L(v)$ is also positive definite we use the same argument. The equation (7) can be thought as giving the neutron spectrum for a problem in which inelastic scattering takes neutrons from low energies to high and the fission spectrum and cross section are interchanged. We then normalize $\bar{A}_L(v)$ so that one neutron emerges from this inverted fission process per second and solve step-by-step starting at the low energy end. $\bar{A}_L(v)$ must then be positive also. It then follows from equation (9) and the arguments just made that $\Delta\lambda$ is negative and the assertion with which we started is therefore true.

We should now like to demonstrate that λ_L and λ_U lie on opposite sides of the correct value. We again use a first-order perturbation treatment, which becomes somewhat more difficult because of the necessity of taking into account the space variation of the kernel as well as the eigenfunctions. We consider three kernels:

[redacted]

$$P_L(\vec{x}' \to \vec{x}, v) = \frac{1}{\nu_0(v)} \sum_{n=0}^{\infty} \frac{\psi_n(\vec{x}, v)\,\psi_n(\vec{x}', v)}{\int d\vec{x}\,\psi_n^2(\vec{x}, v)} = \frac{1}{\nu_0(v)} \delta(\vec{x} - \vec{x}')$$

$$P(\vec{x}' \to \vec{x}, v) = \sum_{n=0}^{\infty} \frac{1}{\nu_n(v)} \frac{\psi_n(\vec{x}, v)\,\psi_n(\vec{x}', v)}{\int d\vec{x}\,\psi_n^2(\vec{x}, v)} \tag{10}$$

$$P_U(\vec{x}' \to \vec{x}, v) = \frac{1}{\nu_0(v)} \frac{\psi_0(\vec{x}, v)\,\psi_0(\vec{x}', v)}{\int d\vec{x}\,\psi_0^2(\vec{x}, v)}$$

We write the integral equation (15) of Chapter II as:

$$\lambda A(\vec{x}, v) = \int d\vec{x}' \int dv' P(\vec{x}' \to \vec{x}, v)\, S(v' \to v)\, A(\vec{x}', v') \tag{11}$$

Here λ is again the number by which $S(v' \to v)$ would have to be divided to make the given system critical. If the three kernels (10) are inserted in equation (11), three different values of the eigenvalue λ are determined. We shall denote these by λ_L, λ, and λ_U. The first measures the activity of the system according to the first lower approximation, the second gives the true activity of the system, and the third measures the activity according to the first upper approximation. We then wish to show that if the three values of λ are calculated by a first-order perturbation scheme starting from the first lower approximation, the true value of λ must lie between λ_L and λ_U.

We write the integral equation (11) as:

$$(\lambda_L - \Delta\lambda)\left[A_L(\vec{x}, v) + \Delta A(\vec{x}, v)\right]$$

$$= \int d\vec{x}' \int dv' S(v' \to v)\left[P_L(\vec{x}' \to \vec{x}, v) - \Delta P(\vec{x}' \to \vec{x}, v)\right]\left[A_L(\vec{x}, v) + \Delta A(\vec{x}, v)\right] \tag{12}$$

[redacted]

The function $A_L(\vec{x},v)$ satisfies the following equation:

$$\lambda_L A_L(\vec{x},v) = \int dv' S(v' \to v) \frac{1}{\Sigma_0(v)} A_L(\vec{x},v') \tag{13}$$

In finding λ_L we could ignore the space dependence of $A_L(\vec{x},v)$ because of the highly degenerate form of equation (13). If, however, we wish to base a perturbation calculation on $A_L(\vec{x},v)$, we must be more careful. We write:

$$A_L(\vec{x},v) = A_L(v) f(\vec{x}) \tag{14}$$

Here $f(x)$ is an arbitrary function of position to be determined later and $A_L(v)$ is just the solution of equation (1). We rewrite equation (12), neglecting second-order terms and making the cancellations, possible by virtue of equation (13):

$$\lambda_L \Delta A(\vec{x},v) - \Delta\lambda A_L(v) f(\vec{x})$$
$$= \int dv' S(v' \to v) \frac{1}{\Sigma_0(v)} \Delta A(\vec{x},v) - \int d\vec{x}' \int dv' S(v' \to v) \Delta P(\vec{x}' \to \vec{x}, v) A_L(v) f(\vec{x}') \tag{15}$$

We multiply through by the function $\bar{A}_L(v)$ satisfying the integral equation (7) adjoint to equation (1), and integrate over velocity. Two terms in equation (15) then cancel by virtue of equation (7) and we obtain:

$$\Delta\lambda f(\vec{x}) \int dv \bar{A}_L(v) A_L(v)$$
$$= \int d\vec{x}' \int dv' \int dv S(v' \to v) \bar{A}_L(v) A_L(v') \Delta P(\vec{x}' \to \vec{x}, v) f(\vec{x}')$$
$$\text{or } \Delta\lambda f(\vec{x}) = \int d\vec{x}' \frac{\int dv' \int dv S(v' \to v) \bar{A}_L(v) A_L(v') \Delta P(\vec{x}' \to \vec{x}, v)}{\int dv \bar{A}_L(v) A_L(v)} f(\vec{x}') \tag{16}$$

Equation (16) is an integral equation for the unknown function $f(\vec{x})$. The quantity $\Delta\lambda$ enters as the eigenvalue of an integral operator which is symmetric since

$$\Delta P(\vec{x}' \to \vec{x}, v) = \Delta P_L(\vec{x} \to \vec{x}', v)$$

Consider the three kernels (10). We define two kernels ΔP :

$$\Delta P(\vec{x}' \to \vec{x}, v) = P_L(\vec{x}' \to \vec{x}, v) - P(\vec{x}' \to \vec{x}, v)$$

$$\Delta P_U(\vec{x}' \to \vec{x}, v) = P_L(\vec{x}' \to \vec{x}, v) - P_U(\vec{x}' \to x, v) \qquad (17)$$

We see from (10), remembering that the ν_n form an increasing sequence, that ΔP and ΔP_u are each kernels with all positive eigenvalues. Furthermore, ΔP_u is obtained from ΔP by adding a kernel with all positive eigenvalues. If $\Delta\lambda$ corresponds to ΔP and $\Delta\lambda_u$ to ΔP_u , we see that $\Delta\lambda$ and $\Delta\lambda_U$ are the highest eigenvalues of symmetric kernels which are sums of kernels with positive eigenvalues (remembering that $\bar{A}_L(v)$ and $A_L(v)$ can be chosen positive definite). Furthermore $\Delta\lambda_U$ is the highest eigenvalue of a kernel which is obtained from that for $\Delta\lambda$ by the addition of kernels with all positive eigenvalues. It therefore follows, from a lemma to be proved, that $\Delta\lambda$ and $\Delta\lambda_U$ are positive and that $\Delta\lambda_U$ is greater than $\Delta\lambda$, so that $\lambda_L > \lambda > \lambda_U$ as we wished to show.

We now prove the required lemma. Let L and M be two symmetric kernels, each with all positive eigenvalues forming a decreasing sequence from a highest eigenvalue. Let the highest eigenvalues of L and M be λ and μ respectively. We have:

$$\lambda = \frac{(\psi, L\psi)}{(\psi, \psi)} \qquad \mu = \frac{(x, Mx)}{(x, x)} \tag{18}$$

where ψ and x are functions so chosen that the expression equated to λ and μ attain their maximum possible values. The bracket notation of course indicates an inner product. Consider the kernel $S = L + M$. Its eigenvalues are all positive since any eigenvalue is a possible value of:

$$\frac{(\phi, S\phi)}{\phi, \phi} = \frac{(\phi, L\phi)}{(\phi, \phi)} + \frac{(\phi, M\phi)}{\phi, \phi} \tag{19}$$

and each term in (19) is essentially positive as can be seen by expanding ϕ in eigenfunctions of L or M for the two terms respectively. The highest eigenvalue of S, σ, will be the absolute maximum of the expression (19). If we let $\phi = \psi$, for example, we obtain:

$$\sigma \geq \lambda + \frac{(\psi, M\psi)}{(\psi, \psi)} \tag{20}$$

We have just argued that:

$$\frac{(\psi, M\psi)}{(\psi, \psi)} \geq 0 \tag{21}$$

so that:

$$\sigma \geq \lambda \tag{22}$$

or by reversing the argument:

$$\sigma \geq \mu \tag{23}$$

We have therefore proved the required result, that the highest eigenvalue of the sum of two kernels with positive eigenvalues (each with a highest eigenvalue) is greater than the highest eigenvalue of either kernel separately. Actually, of course, the proofs lack rigor because they are based on perturbation arguments to first order, without a demonstration that the higher order terms are small.

APPENDIX III. Results of first approximations for several special problems

We now propose to apply the proceedures of Appendix II to an actual many velocity problem. We first assume the validity of diffusion theory for the calculation of all eigenvalues. In spite of the shortcomings of diffusion theory, we can still expect a meaningful check, since our approximate methods will be applied to a many velocity diffusion equation, which we shall also solve exactly.

We define a problem as follows:

$$\sigma(v) = \sigma^*(v) = \frac{1}{v} \qquad \chi(v) = 1 \quad \text{if } 0<v<1$$

$$\sigma_f(v) = \sigma_c^*(v) = \frac{K^2}{3v} \qquad = 0 \quad \text{if } 1<v \tag{1}$$

$$\sigma_c(v) = \sigma_{in}(v) = 0$$

We then write the diffusion equation:

$$-\frac{1}{3\sigma_{tr}(v)}\nabla^2 N(\vec{x},v)+\sigma_f(v)N(\vec{x},v)=\nu\chi(v)\int_0^\infty \sigma_f(v')N(\vec{x},v')dv'$$

$$-\frac{1}{3\sigma_{tr}^*(v)}\nabla^2 N(\vec{x},v)+\sigma_c^*(v)N(\vec{x},v)=0 \qquad 0<v<a$$

$$a<v \qquad (2)$$

Here $N(\vec{x},v)$ is the flux density at $\vec{x}$ per unit range of velocity and ν is the total number of neutrons per fission required for criticality. We write:

$$F(\vec{x})=\int_0^\infty \sigma_f(v)N(\vec{x},v)\,dv \qquad 0<v<a$$

$$=0 \qquad a<v \qquad (3)$$

for the density of fissions, and insert the given cross sections:

$$\nabla^2 N(\vec{x},v)-\frac{K^2}{v^2}N(\vec{x},v)=-3\nu\frac{\chi(v)}{v}F(\vec{x}) \qquad (4)$$

This equation can be formally solved for N in terms of F by the use of the Green's function for the differential operator on the left:

$$N(\vec{x},v)=\frac{3\nu\chi(v)}{4\pi v}\int d\vec{x}'\,\frac{e^{-\frac{K}{v}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}F(\vec{x}') \qquad (5)$$

We then multiply through by $\sigma_f(v)$ and integrate over v to obtain a space integral equation for $F(\vec{x})$:

[redacted]

$$F(\vec{x}) = \frac{\nu}{4\pi} K^2 \int d\vec{x}' \frac{F(\vec{x}')}{|\vec{x}-\vec{x}'|} \int_0^1 \frac{dv}{v^2} e^{-\frac{K}{v}|\vec{x}-\vec{x}'|}$$

$$\text{or} \quad F(\vec{x}) = \frac{\nu}{4\pi} K^2 \int d\vec{x}' \frac{F(\vec{x}')}{|\vec{x}-\vec{x}'|} \frac{e^{-K|\vec{x}-\vec{x}'|}}{K|\vec{x}-\vec{x}'|} \tag{6}$$

We rewrite this:

$$F(\vec{x}) = \frac{\nu}{4\pi} K \int d\vec{x}' \frac{e^{-K|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|^2} F(\vec{x}') \tag{7}$$

We then see that our many-velocity diffusion theory is equivalent to a one-velocity integral theory problem. The density of fissions becomes a neutron density, ν becomes the $1+f$ needed for criticality, and K becomes the total cross section in the equivalent one-velocity problem. The integral is to be taken only over the core, so that the one-velocity problem is that of an untamped sphere of radius a .

We take $K=1$ and $a=1$. The usual extrapolated end-point method then gives $\nu = 1.9878$.

We now propose to obtain ν by the $U-1$ and $L-1$ approximations. We first use the $L-1$ method and then obtain the approximate answer in the $U-1$ approximation by the use of the perturbation scheme developed in this appendix.

Equation (1.8) of Chapter II gives for ν calculated by the L-1 method:

$$\frac{1}{\nu_L} = \int_0^\infty \frac{x(v)}{\nu(v)} dv \int_0^1 \frac{dv}{\nu(v)} \tag{8}$$

Here $\nu(v)$ is the value of ν required to make a one-velocity system critical with the constants of velocity v . Writing $k(v)$ for the wave-number of the fundamental mode at velocity v , we must match the logarithmic derivatives of $\sin k$ and $e^{-\frac{k}{v}r}$ at $r=a$. This gives:

$$k \cot ka = -\frac{K}{v}$$
$$\text{or } \frac{\tan k}{k} = -v \tag{9}$$

We find $\nu(v)$ by remembering that in the one-velocity diffusion theory:

$$(\nu-1)\frac{\sigma_f}{\sigma} = f = \frac{k^2}{3\sigma^2} \quad \text{or } \nu = 1+\frac{\sigma}{\sigma_f} f$$
$$= 1+\frac{k^2}{3\sigma\sigma_f} = 1+\frac{k^2}{K^2} v^2 = 1+k^2v^2 \tag{10}$$

Combining (8) and (10), and taking k from equation (9), we have:

$$\frac{1}{\nu_2} = \int_0^1 \frac{dv}{1+v^2k^2(v)} \tag{11}$$

This integral can be performed analytically and yields $\nu_L = 1.9790$, which is slightly below the exact answer previously obtained. This is of course in the direction expected from the theorem proved in the first part of this appendix. It is easily estimated that this discrepancy in the values of ν amounts to a difference of 0.6% in the critical radius or 1.8% in the mass.

The difference between the true value of ν and ν_L arises from the fact that the one-velocity eigenfunctions do not have the same shape.

The spread in shapes can be indicated by giving the values of ka for $v = 0$ and $v = 1$. For $v = 0$, $ka = \pi$ and for $v = 1$, $ka = 0.65\pi$

We now wish to apply the methods developed in the first part of this appendix to find an upper limit for ν. We have

$$S(v' \rightarrow v) = \nu x(v) \tag{12}$$

and $$\lambda_L A_L(v) = \frac{1}{\nu(v)} \int_0^\infty dv' \nu x(v) A_L(v')$$

so that $$A_L(v) = \frac{\nu_L x(v)}{\nu(v)}$$

We give ν its correct value and then have $\frac{\nu}{\lambda_L} = \nu_L$. The equation adjoint to (12) is:

$$\lambda_L \bar{A}_L(v) = \int_0^\infty dv' \frac{\nu x(v')}{\nu(v')} \bar{A}_L(v') \tag{13}$$

so that $\bar{A}_L(v)$ = a constant = 1, say.

We have $\nu_U = \frac{\nu}{\lambda_U} = \frac{\nu}{\lambda_L + \Delta\lambda}$, where from equation (9) of this appendix, $\Delta\lambda$ is given (approximately) by:

$$\Delta\lambda = \frac{-\int dv \int dv' \frac{1}{\nu(v)} \nu x(v) N(v, v') \frac{\nu_L x(v')}{\nu(v')}}{\int dv \frac{\nu_L x(v)}{\nu(v)}}$$

$$= -\nu\nu_L \int_0^1 dv \int_0^1 dv' \frac{N(v, v')}{\nu(v)\nu(v')} \tag{14}$$

where $N(v,v')$ is defined by equation (3). The indicated double integral can be done numerically and we obtain $\Delta\lambda=0.0161$ and hence $\nu_\infty = 2.0113$. This is too high as expected.

APPENDIX IV. Numerical test of the convergence of higher approximations

We propose to solve a two-velocity problem with no inelastic scattering in the tamper, in the diffusion limit. For simplicity we consider the same problem discussed in Appendix I. We specify the cross-section and fission spectrum a little more closely:

σ_1	= 4	σ_2	= 8	
σ_{112}	= 1	σ_{2c}	= 0	
σ_{1c}	= 0	σ_{2f}	= 2	a = 0.48690
σ_{1f}	= 0	χ_2	= 0	
χ_1	= 1	σ_{2*}	= 8	
σ_{1*}	= 4	σ_{2c*}	= 0.11626	
σ_{1c*}	= 0			

The radius of the tamper is infinite.

A pair of coupled diffusion equations for the neutron densities can be written and solved in a straightforward but lengthy fashion. When this is done, it is found that if ν were equal to 2.509 the system would be critical.

We shall now apply to this problem the first four of the sequency of successive approximations described at the end of Chapter II. The fundamental integral equations are:

$$A_1(\vec{x}) = \nu \int_{CORE} d\vec{x}' P_1(\vec{x}' \to \vec{x}) A_2(\vec{x}')$$

$$A_2(\vec{x}) = \int_{CORE} d\vec{x}' P_2(\vec{x}' \to \vec{x}) A_1(\vec{x}') \qquad (1)$$

[redacted] [redacted]

These are just the equation (15) of Chapter II modified to take into account the existence of only two velocities and particularized to the cross sections of the present problem. The space functions A_1 and A_2 are, respectively, the densities of inelastic collisions and of fissions. The kernels P_1 and P_2 are just the one-velocity diffusion kernels for neutrons of velocities one and two which were defined in Chapter II. Let us define u_0, u_1, u_2 , - - - - -to be the successive eigenfunctions of the kernel P_1 and $\frac{1}{\mu_0}$, $\frac{1}{\mu_1}$,- - -to be its corresponding eigenvalues. The function u_0 is then the distribution of fissions in the fundamental mode of a one-velocity problem whose constants are the constants of velocity, one with the total core absorption cross section considered as fission. The quantity μ_0 is the value of ν required to keep this one-velocity problem critical. We define a similar sequence of functions V_0, V_1, V_2 ----and a sequence of eigenvalues $\frac{1}{\nu_0}, \frac{1}{\nu_1}, \frac{1}{\nu_2}$, ---- for velocity two.

The L-1 approximation proceeds by writing:

$$P_1(\vec{x}' \to \vec{x}) = \frac{1}{\mu_0} \int (\vec{x} - \vec{x}')$$
$$P_2(\vec{x}' \to \vec{x}) = \frac{1}{\nu_0} \int (\vec{x} - \vec{x}') \qquad (2)$$

as outlined in Chapter II. The equations (1) become:

$$A_1 = \frac{\nu}{\mu_0} A_2$$
$$A_2 = \frac{1}{\nu_0} A_1 \qquad (3)$$

which yields $\nu = \mu_0 \nu_0 = 2.500$, which is certainly excellent accuracy.

The U-1 approximation assumes:

$$P_1(\vec{x}' \to \vec{x}) = \frac{1}{\mu_0} \frac{u_0(\vec{x})\, u_0(\vec{x}')}{(u_0 u_0)}$$

$$P_2(\vec{x}' \to \vec{x}) = \frac{1}{\nu_0} \frac{\nu_0(\vec{x})\, \nu_0(\vec{x}')}{(\nu_0 \nu_0)} \tag{4}$$

where $(u_0 u_0)$ is the volume integral over the core of $u_0^2(\vec{x})$. A similar notation will now prove very useful. The equations (1) become:

$$A_1 = \frac{\nu}{\mu_0} \frac{(u_0 A_2)}{(u_0 u_0)} u_0$$

$$A_2 = \frac{1}{\nu_0} \frac{(\nu_0 A_1)}{(\nu_0 \nu_0)} \nu_0 \tag{5}$$

These can be conveniently solved by multiplying the first by ν_0 and integrating and the second by u_0 and integrating. This yields:

$$(\nu_0 A_1) = \frac{\nu}{\mu_0} \frac{(u_0 \nu_0)}{(u_0 u_0)} (u_0 A_2)$$

$$(u_0 A_2) = \frac{1}{\nu_0} \frac{(u_0 \nu_0)}{(\nu_0 \nu_0)} (\nu_0 A_1) \tag{6}$$

which are simply two homogeneous linear equations for the numbers $(\nu_0 A_1)$ and $(u_0 A_2)$. We immediately obtain:

$$\nu = \mu_0 \nu_0 \frac{(u_0 u_0)(\nu_0 \nu_0)}{(u_0 \nu_0)^2} = 2.516 \tag{7}$$

which is above the correct answer and slightly closer to it than was L-1.

In the L-2 approximation, we write:

$$P_1(\vec{x}' \to \vec{x}) = \left(\frac{1}{\mu_0} - \frac{1}{\mu_1}\right) \frac{u_0(\vec{x}')\, u_0(\vec{x})}{(u_0\, u_0)} + \frac{1}{\mu_1} \int (\vec{x} - \vec{x}')$$

$$P_2(\vec{x}' \to \vec{x}) = \left(\frac{1}{\nu_0} - \frac{1}{\nu_1}\right) \frac{v_0(\vec{x}')\, v_0(\vec{x})}{(v_0\, v_0)} + \frac{1}{\nu_1} \int (\vec{x} - \vec{x}') \qquad (8)$$

The integral equations become:

$$A_1 = \nu \left(\frac{1}{\mu_0} - \frac{1}{\mu_1}\right) \frac{(u_0 A_2)}{(u_0 u_0)}\, u_0 + \frac{\nu}{\mu_1} A_2$$

$$A_2 = \left(\frac{1}{\nu_0} - \frac{1}{\nu_1}\right) \frac{(v_0 A_1)}{(v_0 v_0)}\, v_0 + \frac{1}{\nu_1} A_1 \qquad (9)$$

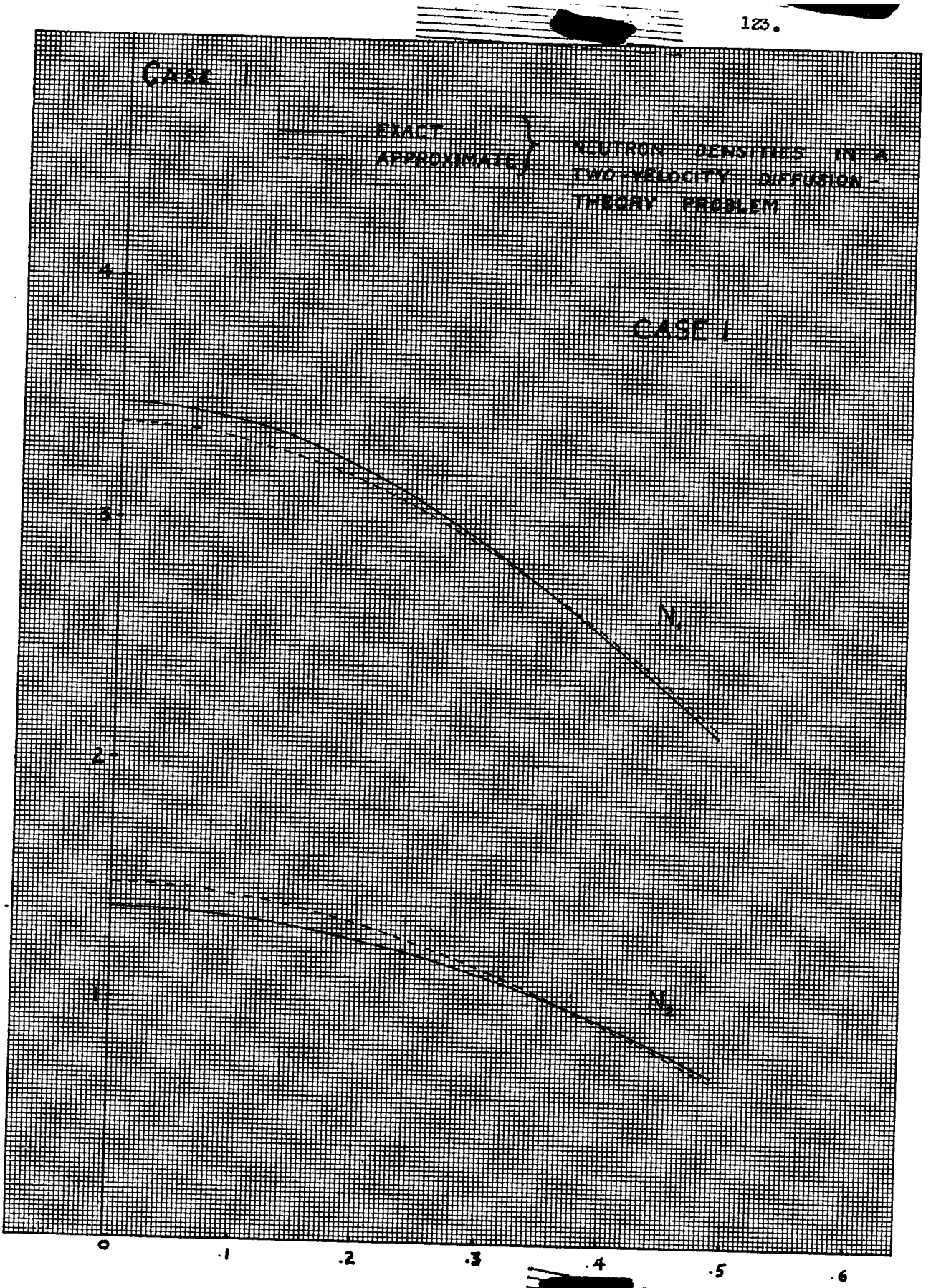
We multiply each equation by u_0 and by v_0 and integrate over the core. This yields four homogeneous linear equations for $(u_0 A_1)$, $(v_0 A_1)$, $(u_0 A_2)$, and $(v_0 A_2)$. Setting the secular determinant equal to zero yields $\nu = 2.508$, in good agreement with the correct value, but slightly lower as expected.

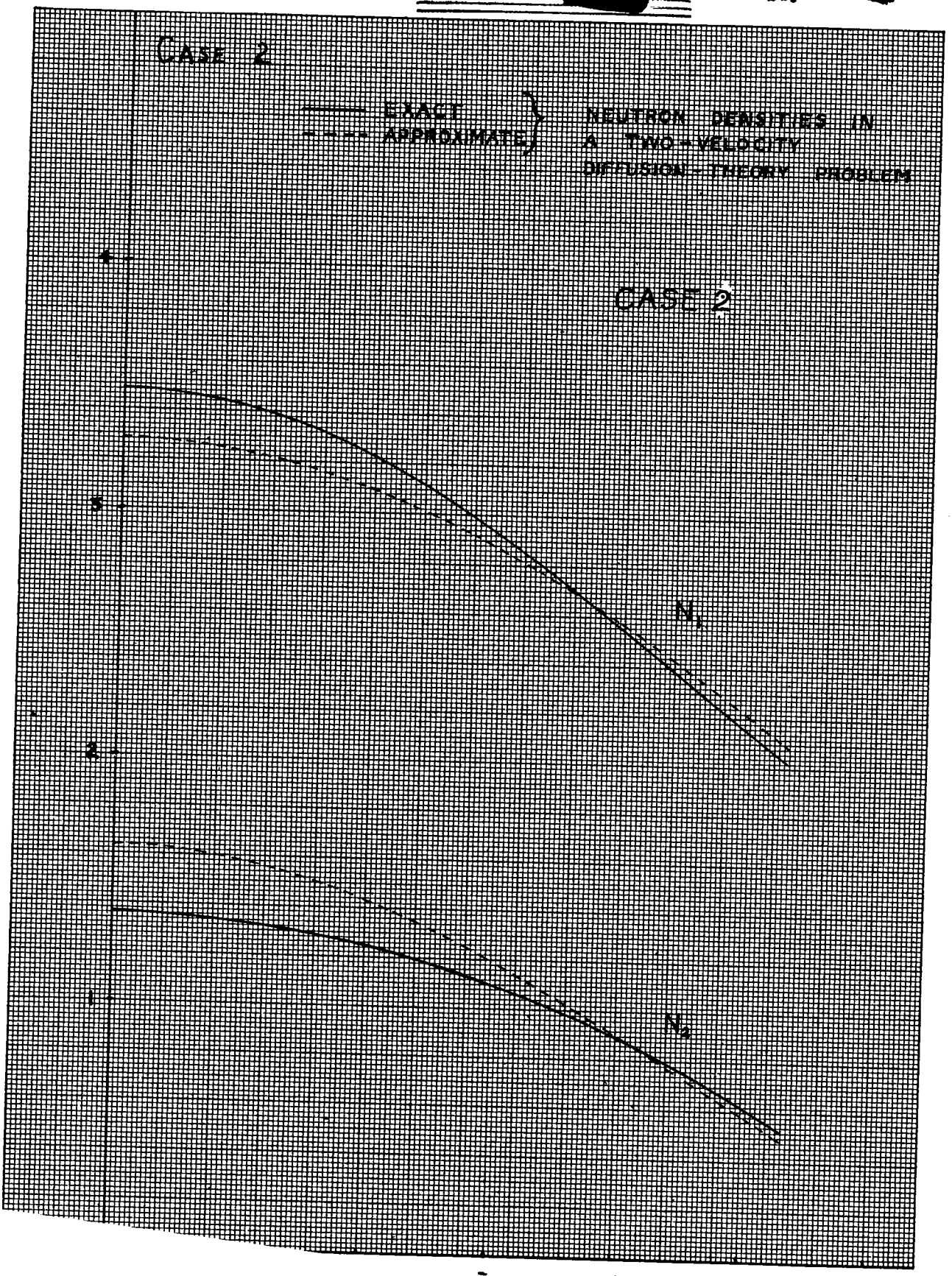
Nothing will be gained by going through the U-2 approximation in detail, but it is interesting to note that it gives $\nu = 2.509$ to that many figures. These figures are summarized in the first column of the table below.

An exactly similar set of calculations has been done for the second problem discussed in Appendix I, namely the one with different radius and tamper capture cross section. In this one the fundamental modes have phases at the outside of the core of $\frac{\pi}{2}$ and $\frac{3\pi}{4}$, so that the set of approximations should converge less rapidly than in the previous case. The results are given in the second column of the table.

	ν, First Problem	ν, Second Problem
Exact	2.509	2.548
L-1	2.500	2.500
U-1	2.516	2.596
L-2	2.508	2.543
U-2	2.509	2.552

It is finally of considerable interest to compare the neutron flux densities as calculated from the exact two-group diffusion theory and the L-1 approximation. In the first graph following are plotted two solid curves representing N_1 and N_2, the neutron flux densities at velocities one and two respectively, calculated exactly. The dotted curves are just the fundamental modes for the two velocities, with their relative sizes adjusted so that the ratio of fissions to inelastic collisions is just that given by the L-1 approximation. The first graph is for the first case discussed in this appendix and the second graph is for the second case with $ka = \frac{\pi}{2}$ and $\frac{3\pi}{4}$ for the two velocities. The agreement between solid and dotted curves is remarkable in view of the seeming crudity of the approximation.





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APPENDIX V

TWO-VELOCITY DIFFUSION THEORY WITH INELASTIC SCATTERING IN THE TAMPER

We shall attempt to investigate the validity of the approximation scheme developed in Chapter III for the case of an inelastically scattering tamper. Our procedure will parallel that of Appendix IV in that we will discuss a two-velocity diffusion theory problem, first exactly and then by means of the equations (4.1) of Chapter III and the indicated successive approximations to their solution.

We take for the constants of the problem the following:

$$\sigma_1 = \sigma_1^* = 4 \qquad \sigma_2 = \sigma_2^* = 8$$

$$\sigma_{1f} = \sigma_{1c} = \sigma_{1c}^* = 0 \qquad \sigma_{2f} = 2$$

$$\sigma_{12} = 0.2813 \qquad \sigma_{2c} = \sigma_{2c}^* = 0$$

$$\sigma_{12}^* = 0.2205$$

$$x_1 = 1 \qquad x_2 = 0 \qquad a = 0.5$$

The exact two-velocity diffusion equations plus the usual diffusion theory boundary conditions yield $\nu = 2.467$.

We shall now apply to this problem the successive approximations (developed in Section 5 of Chapter III) to the solution of the integral equations (4.1) of Chapter III. These equations, specialized to the present problem, are:

$$A_1(\vec{x}) = \nu \int_{CORE} d\vec{x}'\, P_1(\vec{x}' \to \vec{x})\, A_2(\vec{x}')$$

$$A_2(\vec{x}) = \int d\vec{x}'\, P_2(\vec{x}' \to \vec{x})\, A_1(\vec{x}') + \nu \int_{CORE} d\vec{x}'\, F_{12}(\vec{x}' \to \vec{x})\, A_2(\vec{x}') \quad (1)$$

with F_{12} given in terms of P and P_2 by (4.11) of Chapter III.

As in Appendix III we call the sequence of one-velocity ν's for velocity one μ_0, μ_1 ,--- and the corresponding eigenfunctions u_0, u_1 ,---. For velocity two we have ν_0, ν_1 ,--- and v_0, v_1 ---.

We then make the L-1 approximation of Chapter II:

$$P_1(\vec{x}' \to \vec{x}) = \frac{1}{\mu_0} \Big/ (\vec{x} - \vec{x}')$$

$$P_2(\vec{x}' \to \vec{x}) = \frac{1}{\nu_0} \Big/ (\vec{x} - \vec{x}') \qquad (2)$$

$$F_{12}(r' \to r) = \frac{\sigma_{t12}}{\sigma_{1a}^*}\left(1 - \frac{1}{\mu_0}\right) \frac{P_{12}}{4\pi \Big/_0^a r^2 dr} = \left(1 - \frac{1}{\mu_0}\right) \frac{P_{12}}{(11)}$$

where (11) is an abbreviation for the volume integral over the core of the product of the quantities inside the bracket.

The equations (1) become:

$$A_1(\vec{x}) = \frac{\nu}{\mu_0} A_2(\vec{x})$$

$$A_2(\vec{x}) = \frac{1}{\nu_0} A_1(\vec{x}) + \nu\left(1 - \frac{1}{\mu_0}\right) \frac{P_{12}}{(11)} (1A_2) \qquad (3)$$

These can be integrated over the core to obtain two algebraic equations for $(1A_1)$ and $(1A_2)$:

$$(1A_1) = \frac{\nu}{\mu_0} (1A_2)$$

$$(1A_2) = \frac{1}{\nu_0} (1A_1) + \nu\left(1 - \frac{1}{\mu_0}\right) p_{12} (1A_2) \qquad (4)$$

Inserting the values for μ_0, ν_0 , and p_{12}, we obtain $\nu = 2.500$. This is too high by 2% of $\nu - 1$, which amounts to about 4% in the critical mass.

We now make the U-1 approximation:

$$P_1(\vec{x}' \to \vec{x}) = \frac{1}{\mu_0} \frac{u_0(\vec{x}) u_0(\vec{x}')}{(u_0 u_0)}$$

$$P_2(\vec{x}' \to \vec{x}) = \frac{1}{\nu_0} \frac{v_0(\vec{x}) v_0(\vec{x}')}{(v_0 v_0)} \qquad (5)$$

$$F_{12}(\vec{x}' \to \vec{x}) = \left[1 - \frac{1}{\mu_0} \frac{(1 u_0)}{(u_0 u_0)} u_0(\vec{x}')\right]\left[1 - \frac{1}{\nu_0} \frac{(1 v_0)}{(v_0 v_0)} v_0(\vec{x})\right] \frac{p_{12}}{(1 - \frac{1}{\nu_0})(11)}$$

The equations (1) become:

$$A_1(\vec{x}) = \frac{\nu}{\mu_0} \frac{u_0(\vec{x})}{(u_0 u_0)} (u_0 A_2) \qquad (6)$$

$$A_2(\vec{x}) = \frac{1}{\nu_0} \frac{v_0(\vec{x})}{(v_0 v_0)} (v_0 A_1) \frac{\nu p_{12}}{(1 - \frac{1}{\nu_0})(11)} \left[(1 A_2) - \frac{1}{\mu_0} \frac{(1 u_0)}{(u_0 u_0)} (u_0 A_2)\right]\left[1 - \frac{1}{\nu_0} \frac{(1 v_0)}{(v_0 v_0)} v_0(\vec{x})\right]$$

These are of somewhat more complicated form than (3) but they can be reduced to three algebraic equations by multiplying the first equation by v_0 , the second by u_0 and by unity and integrating over the core. This yields the equations;

$$(v_0 A_1) = \frac{\nu}{\mu_0} \frac{(u_0 v_0)}{(u_0 u_0)} (u_0 A_2)$$

$$(u_0 A_2) \frac{1}{\nu_0} \frac{(u_0 v_0)}{(v_0 v_0)} (v_0 A_1) + \frac{\nu p_{12}}{(1 - \frac{1}{\nu_0})(11)} \left[(1 u_0) - \frac{1}{\nu_0} \frac{(1 v_0)(u_0 v_0)}{(v_0 v_0)}\right]\left[(1 A_2) - \frac{1}{\mu_0} \frac{(1 u_0)}{(u_0 u_0)} (u_0 A_2)\right]$$

$$(1 A_2) = \frac{1}{\nu_0} \frac{(1 v_0)}{(v_0 v_0)} (v_0 A_1) + \frac{\nu p_{12}}{(1 - \frac{1}{\nu_0})(11)} \left[(11) - \frac{1}{\nu_0} \frac{(1 v_0)(1 v_0)}{(v_0 v_0)}\right]\left[(1 A_2) - \frac{1}{\mu_0} \frac{(1 u_0)}{(u_0 u_0)} (u_0 A_2)\right] \qquad (7)$$

These can be solved, with the information needed for L-1 plus a knowledge of u_0 and v_0. We obtain $\nu = 2.441$. This is too low by less than 2% in $\nu - 1$.

By an exactly similar procedure the L-2 approximation can be made, yielding $\nu = 2.470$; which is very close.

Possibly a better test would be in a case with more inelastic scattering in the tamper. We increase the inelastic cross section in the tamper drastically and alter several core cross sections to the following:

$$\sigma_1 = \sigma_1^* = 4 \qquad \sigma_2 = \sigma_2^* = 8$$

$$\sigma_{1f} = \sigma_{1c} = \sigma_{1c}^* = 0 \qquad \sigma_{2f} = 1 \qquad x_1 = 1 \qquad a = 0.5$$

$$\sigma_{12} = 0.05834 \qquad \sigma_2 = \sigma_{2c}^* = 0 \qquad x_2 = 0.$$

$$\sigma_{12}^* = 1.8505$$

The exact value of ν is 2.339. The L-1 value is 2.391, too high by 4% in $\nu - 1$ or about 8% in the mass. The U-1 and L-2 approximations give, respectively, $\nu = 2.319$ and $\nu = 2.341$. These are 1.5% low and a doubtful 0.1% in $\nu - 1$, respectively, so that in spite of the relatively poorer accuracy of L-1, good convergence is still obtained.

It should be noted that the so-called upper and lower approximations have apparently reversed their roles. It cannot be proved that this will always be so, or that they will always bracket the answer, but it is certain that as the inelastic scattering in the tamper approaches zero, they will become true upper and lower

approximations to ν and a. As have been pointed out in the text in Chapter III this series of approximations can only be expected to converge to the true answer in the diffusion theory limit. In integral theory there is already an error in the fundamental equation (4.1) for F_{12} of Chapter III.

APPENDIX VI.

THREE-VELOCITY PROBLEM IN INTEGRAL THEORY.

We consider a three-group problem with inelastic scattering in an infinite tamper. We choose a core radius of two-length units and cross sections as follows:

$\sigma_1 = \sigma_1^* = 1$ $\quad \sigma_2 = \sigma_2^* = 1.2$ $\quad \sigma_2 = \sigma_3^* = 1.5$

$\sigma_{1f} = 0$ $\quad \sigma_{2f} = 0.2$ $\quad \sigma_{3f} = 0.5$

$\sigma_{i12} = \sigma_{i12}^* = 0.1$ $\quad \sigma_{i23} = \sigma_{i23}^* = 0.12$

$\sigma_{i13} = \sigma_{i13}^* = 0.1$ $\quad \sigma_{2c} = 0.2$

$\sigma_{ic} = 0.2$ $\quad \sigma_{2c}^* = 0$

$\sigma_{1c}^* = 0$

$x_1 = 1.0$ $\quad x_2 = 0$ $\quad x_3 = 0$

We need numerous auxiliary quantities. First there are the total absorption cross sections in the core and tamper:

$\sigma_{1a} = 0.4$ $\quad \sigma_{2a} = 0.52$ $\quad \sigma_{3a} = 0.5$

$\sigma_{1a}^* = 0.2$ $\quad \sigma_{2a}^* = 0.12$ $\quad \sigma_{3a}^* = 0$

We then find the absorption numbers, g, in the tamper:

$g_1 = 0.2$ $\quad g_2 = 0.1$ $\quad g_3 = 0$

We will need the rates of decay, h, of the asymptotic tamper solutions given by:

$$\frac{\frac{h}{\sigma^*}}{\tanh^{-1}\frac{h}{\sigma^*}} = 1-g \qquad (1)$$

We find:

$h_1 = 0.7103 \qquad h_2 = 0.6301 \qquad h_3 = 0$

We shall also require the quantities g_{12}, g_{21}, etc. defined in Chapter III. These are given by relations of the form:

$$\frac{\frac{h_1}{\sigma_2^*}}{\tanh^{-1}\frac{h_1}{\sigma_2^*}} = 1-g_{12} \qquad (2)$$

These yield:

$g_{12} = .1303 \qquad g_{21} = .1504$

$g_{13} = .0798 \qquad g_{31} = 0$

$g_{23} = .0619 \qquad g_{32} = 0$

We must now find the quantities $f_1, f_2, f_3, f_{12}, f_{13}$ etc., which are needed for the calculation of the critical ν by the method of Chapter IV. These quantities f are the number of extra neutrons per collision required to keep various one-velocity systems critical. The relation of these auxiliary problems to the actual problem has been defined in Chapter III. We give in tabular form the constants specifying these problems:

Problem	σ^a	σ^{*a}	g	$\frac{h}{\sigma^*}$
1	2.0	2.0	0.2	0.7103
2	2.4	2.4	0.1	0.5251
3	3.0	3.0	0	0
12	2.4	2.4	0.1303	0.5919
13	3.0	3.0	0.0798	0.4735
23	3.0	3.0	0.0619	0.4201
21	2.0	2.0	0.1504	0.6301
31	2.0	2.0	0	0
32	2.4	2.4	0	0

As stated in Chapter I, there exist rapid and accurate methods for obtaining the f's in general. For the purpose at hand, the mean free paths in core and tamper were purposely set equal for each velocity, so that the more precise method of the extrapolated end point described by Frankel and Nelson in LA-53 could be applied. According to this method, we first choose the unit of length so that the total cross section is unity and introduce an auxiliary variable k, which is just the wave number of the sinusoidal asymptotic solution in the core. We then have the two relations:

$$k = \frac{\pi - \tan^{-1}\frac{k}{h}}{a - x_1}$$

$$1 + f = \frac{k}{\tan^{-1} k} \qquad (3)$$

The first of these is a diffusion theory boundary condition

applied at a distance x , inside the actual boundary. The second is just the usual relation between f and k . We can find x_1 as a function of f and g from a graph in LA-258 (By Frankel and Goldberg). The procedure is then to guess a value of k , calculate f by the second equation, look up x_1 , and calculate an improved value of k by substituting the quantities thus far found in the right-hand side of the first equation. This procedure yields the following values of f :

$f_1 = 0.3226$ $f_2 = 0.2244$ $f_3 = 0.0862$

$f_{12} = 0.2332$ $f_{21} = 0.3100$

$f_{13} = 0.1555$ $f_{31} = 0.1834$

$f_{23} = 0.1495$ $f_{32} = 0.1311$

It will finally be convenient to calculate values of ν , the total number of neutrons necessary per core absorption to keep these auxiliary systems critical. This can be done with the usual relation:

$$\nu = 1 + \frac{\sigma}{\sigma_a} f \qquad (4)$$

We obtain:

$\nu_1 = 1.8065$ $\nu_2 = 1.5178$ $\nu_3 = 1.2586$

$\nu_{12} = 1.5382$ $\nu_{21} = 1.7750$

$\nu_{13} = 1.4665$ $\nu_{31} = 1.4585$

$\nu_{23} = 1.4485$ $\nu_{32} = 1.3025$

We shall now need the quantities $p_{12}, p_{13}, p_{23}, p_{123}$. We will make several calculations of the value of ν required to make our three velocity problem critical. For the first we use the equations (3.5) and (5.8) of Chapter III for p_{12} and p_{123}. By simply substituting the numbers already found in these equations, we obtain:

$$p_{12}=0.1074 \qquad p_{13}=0.3541 \qquad p_{23}=0.3364$$
$$p_{123}=0.2419$$

By using the equations (3.15) and (5.7) of Chapter III we can get an alternative set of values for the above quantities. The vanishing of the tamper absorption in the third group now requires the taking of limits in which σ_{3a}^* approaches zero and the diffusion theory relation (exact for small absorption) between g and h is used to obtain limiting values for $\frac{g_{31}}{\sigma_{3a}^*}$ and $\frac{g_{32}}{\sigma_{3a}^*}$. These limits are:

$$\frac{g_{31}}{\sigma_{3a}^*} \longrightarrow \frac{\sigma_3^*}{\sigma_1^{*2}} = 1.5$$

$$\frac{g_{32}}{\sigma_{3a}^*} \longrightarrow \frac{\sigma_3^*}{\sigma_2^{*2}} = 1.0417$$

The following values are then easily obtained:

$$p_{12}=0.1129 \qquad p_{13}=0.3626 \qquad p_{23}=0.3394$$
$$p_{123}=0.3243$$

The first three quantities are seen to be in reasonable agreement with their previous values, but the two values of p_{123} do not agree well.

[redacted] [redacted]

(Some of this discrepancy may arise from the fact that p_{123} does not have so many significant figures as p_{12}, because of the subtractions occurring in it.)

We now use the first set of values and the method embodied in equations (5.1), (5.2), and (5.3) of Chapter IV to find the value of ν for criticality. Equation (5.1), with the specified normalization to one neutron emitted from fission per second, becomes:

$$N_1 = \frac{x_1}{\sigma_{1a} + f_1\sigma_1} = \frac{x_1}{\nu_1\sigma_{1a}} = 1.38388$$

Equation (5.2) becomes:

$$N_2 = \frac{1}{\nu_2\sigma_{2a}} \left\{ x_2 + \sigma_{i12} N_1 + f_1\sigma_1 \frac{\sigma^*_{i12}}{\sigma^*_{1a}} p_{12} N_1 \right\} = 0.20572$$

Similarly, equation (5.3) becomes:

$$N_3 = 0.5002$$

We can now calculate ν from equation (5.4) of Chapter IV. We obtain

$$\nu = 3.4336$$

By performing exactly the same calculation with the other set of values for p_{12}, p_{13}, p_{23}, p_{123} we obtain:

$$\nu = 3.2463$$

This discrepancy in the value of ν will give rise to a discrepancy in the critical mass if ν is specified. This corresponding discrepancy in mass can be estimated to be about 16%, which is large, but not uselessly so. It is interesting to compare this discrepancy with that obtained by changing the tamper inelastic

cross sections to capture cross sections. This produces such a large change in ν (to 6.1828) that it is difficult to be sure of the mass change, but it probably amounts to approximately a 300% increase. The 16% uncertainty in the mass arising from our approximate method is then seen to be negligible in comparison with the possible uncertainty introduced by the presence of inelastic scattering.

It would of course be more reasonable to treat the tamper inelastic cross section as an elastic cross section rather than a capture, if it is desired to ignore the effect of the inelastic scattering. This at least does not throw away neutrons which are actually present. This calculation is easily done and leads to a value $\nu = 4.7407$, which corresponds to a critical mass too large by about 150%. We then see that the 16% discrepancy in mass between our first two calculations is really negligible in comparison with the total effect on the mass of the inelastic scattering in the tamper. We can estimate that our approximate methods will give the effect of the tamper inelastic scattering on the critical mass to about 10%, which is excellent. The example was chosen to show up the effect of inelastic scattering to an extreme degree. Under more practical conditions the effects of inelastic scattering are not nearly so drastic.

It should be pointed out that in actual practice, the differential cross sections will not be known with complete accuracy. If they were, the reduction to average three group cross sections is too ambiguous a procedure to be considered reliable. The best available procedure is then to adjust the three group cross sections so that they are in reasonable agreement with the known

differential cross sections, and at the same time so that their use in the approximate formalism here developed will give correct results for the available experiments on assemblies which are in the neighborhood of critical. The formalism can then be considered a valid and convenient method for predicting the results of other experiments which are not too remotely related the the experiments which were used in the adjustment of the cross sections.

It is instructive at this point to check the accuracy of the expression obtained for p_{12} by the use of the asymptotic exponential approximation to the tamper densities in the one velocity problems. We use equation (2.4) of Chapter III to obtain:

$$p_{12} = 0.1077 \qquad p_{13} = 0.3618 \qquad p_{23} = 0.3438$$

We then insert these values in equation (5.8) of Chapter III to obtain:

$$p_{123} = 0.2474$$

It is seen that these values agree well with the ones previously obtained, so that the value of ν for criticality will be given well with this simpler procedure.

We see that, at least for the example used, this much simpler method is just as trustworthy as the methods previously used. It will presumably become relatively poorer as the tamper absorptions are increased, but it should always be used unless greater accuracy is clearly necessary.

Finally, it should be noticed that with the present formalism, the use of more than three groups is of doubtful advantage, inasmuch as the discrepany between the three values of p_{123} indicates that a quantity such as p_{1234} may be completely uncertain.

APPENDIX VII.

NUMERICAL TEST OF THE THEORY FOR A FINITE TAMPER.

We assume a two-velocity system with the following constants:

$\sigma_1 = \sigma_1^* = 1 \quad \sigma_{1f} = 0.3 \quad \sigma_{i12} = 0 \quad \sigma_{1c} = 0 \quad \sigma_{i12}^* = 0.2 \quad \sigma_{1c}^* = 0$

$\sigma_2 = \sigma_2^* = 1 \quad \sigma_{2f} = 0.3 \quad \sigma_{2c} = 0 \quad \sigma_{2c}^* = 0$

$x_1 = 1 \qquad x_2 = 0$

$a = 1.4 \qquad b = 3.0$

We inquire what value of ν for fission will make the system critical. The constants have been so chosen that a neutron at either velocity is equally effective in causing a fission, so that the problem really is a one-velocity problem with constants:

$\sigma = \sigma^* = 1 \qquad \sigma_f = 0.3 \qquad \sigma_c = 0 \qquad \sigma_c^* = 0$

$a = 1.4 \qquad b = 3.0$

The f for this problem can be found by the extrapolated end-point method and is equal to 0.4505. We can immediately find the ν for criticality to be 2.5017.

We now apply the method of Chapter IV for solving two-velocity problems with finite tampers. For this purpose we require that for a one-velocity problem with the constants of group one. This problem is identical with the previous one, except for the fact that $g_1 = 0.2$. This can also be solved exactly and yields

$f_1 = 0.5255$, which is equivalent to $\nu_1 = 2.7517$.

The other f required is f_2, which is just the one first given. The quantity L_1 defined in equation (64) of Chapter III is equal to 1.2486. The quantity p_{12} defined by equation (6.9) of Chapter III can be found by taking the limit of $\frac{h_2^2}{\sigma_{2a}^*}$ for small h_2 and is equal to 0.1112.

Equations (5.1) and (5.2) of Chapter IV with p_{12} replaced by $\frac{p_{12}}{(1+L_1)}$ and $N_3 = 0$ can then be solved for N_1 and N_2 and the value of ν found by the usual procedure. We obtain $\nu = 2.5324$. We have therefore made a mistake in $\nu - 1$ of about 2% by the use of our method. This corresponds to a mistake in critical mass of something like 4%, which is certainly tolerable for most purposes.

We might ask for the total effect on ν of the finiteness of the tamper. The value of ν for the same system with $b = \infty$ is 2.1230. This corresponds to something like a 60% mass decrease from the previous case, so that our approximate treatment gives the effect on the mass of the finiteness of the tamper to something like 7%. This certainly constitutes a very useful degree of accuracy.

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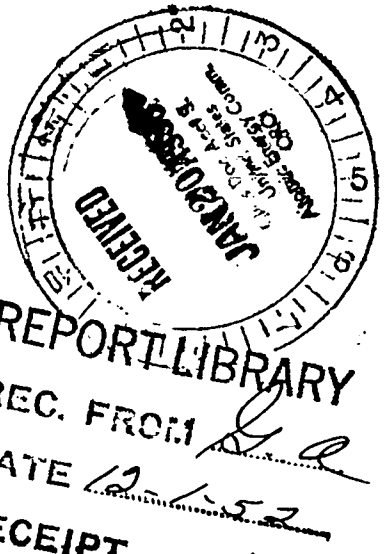

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