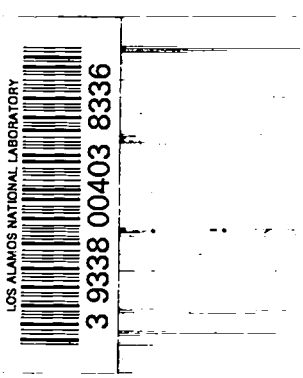
C.3

LA-3266

LOS ALAMOS SCIENTIFIC LABORATORY
LOS ALAMOS [redacted] of the [redacted] NEW MEXICO
University of California

The Interpretation Of Nuclear Excitation Functions At High Energy: Ericson Fluctuations



UNITED STATES
ATOMIC ENERGY COMMISSION
CONTRACT W-7405-ENG. 36

This report expresses the opinions of the author or authors and does not necessarily reflect the opinions or views of the Los Alamos Scientific Laboratory.

Printed in USA. Price $ 3.00. Available from the

Clearinghouse for Federal Scientific
and Technical Information,
National Bureau of Standards,
U. S. Department of Commerce,
Springfield, Virginia

LA-3266
UC-34, PHYSICS
TID-4500 (39th Ed.)

LOS ALAMOS SCIENTIFIC LABORATORY
LOS ALAMOS of the NEW MEXICO
University of California

Report written: March 15, 1965
Report distributed: May 27, 1965

The Interpretation Of Nuclear Excitation Functions At High Energy: Ericson Fluctuations

by

William R. Gibbs

Abstract

The interpretation of the nuclear reaction excitation function by statistical methods has been reviewed and some new results regarding errors and corrections have been presented. Special emphasis is placed on the effects of finite sample size on the amount of direct reaction cross section extracted. Also treated are the effects of counting statistics, generalized frequency distribution functions, and the effect of finite sample size on frequency distribution functions.

Table of Contents

I. Introduction

At very low excitation energy the nuclear reaction cross section displays resonances as a function of energy. These resonances are generally well spaced, and their properties can be studied individually by their angular characteristics. As the bombarding energy (and hence the excitation energy) is increased, D, the average spacing of these resonances, is found to decrease while Γ_0, the average width, increases. As a consequence the resonances are soon overlapping. We wish to consider here the case in which the overlapping is extreme,† i.e., $\Gamma_0/D \gg 1$.

Since individual resonances are no longer discernible in this region we must ask questions about the behavior of the cross section as a whole. This requires the use of a new technique, usually referred to as fluctuation theory or the study of Ericson fluctuations.[2,3] This technique results from the merging of the basic concepts of reaction theory with those of statistics.

In section II the results needed from reaction theory are derived, while section III covers the statistical approach to the subject. In

---

†Moldauer[1] has investigated the consequences of the violation of this condition.

all of the following sections both concepts are used as needed.

Section IV covers the finite sample size corrections to the results obtained. It is shown that these corrections are related to the Heisenberg uncertainty principle. The corrections due to counting statistics are given in section V.

Parts of section III are generalized in section VI so that frequency distribution functions, as well as the amount of direct reaction, may be calculated at any angle.

The question of resolution is dealt with in section VII, and it is shown how widths much smaller than the inherent resolution of the experiment can be measured.

II. Basic Concepts from Reaction Theory

As is commonly done, we shall divide reaction mechanisms into two classes: direct and compound. By these terms we mean short time reactions and long time reactions, respectively. The amplitude may then be written

$$A = A_S + A_L$$

where A_S is the short time reaction amplitude and A_L is the long time reaction amplitude. Since the energy average of the cross section[†] is

$$\bar{\eta} = \overline{|A_S|^2} + \overline{|A_L|^2} + 2\,\mathrm{Re}\overline{(A_S^* A_L)} \qquad (1$$

[†] Note that η instead of σ is being used to denote differential cross section to avoid confusion with the notation of statistics.

and we know that the short time amplitude is very nearly constant in energy we may imply that $\bar{A}_L = 0$. The argument goes as follows. Large energy intervals correspond to short time intervals for the wave packet interacting with the nucleus, and thus the long and short modes cannot interfere and their amplitudes add incoherently. Thus, the third term in eq. 1 is zero.

Let us consider for the moment only A_L. The form usually taken for A_L is[1,4]

$$A_L = \sum_\lambda \frac{a_\lambda}{E - E_\lambda + \frac{i\Gamma_\lambda}{2}} \quad . \tag{2}$$

The approximation may be made that Γ_λ is independent of $\lambda (\Gamma_\lambda \equiv \Gamma_0)$. The a_λ's are taken to be statistically independent complex numbers† with zero mean. From these two basic assumptions we may compute most of the quantities of interest. For example, we may compute the average cross section over an energy interval S as follows.

$$\bar{\eta} = \frac{1}{S}\int_0^S dE \sum_{\lambda\mu} \frac{a_\lambda a_\mu^*}{\left(E - E_\lambda + \frac{i\Gamma_0}{2}\right)\left(E - E_\mu - \frac{i\Gamma_0}{2}\right)}$$

$$= \frac{2\pi i}{S} \sum_{\lambda\mu} \frac{a_\lambda a_\mu^*}{\left(E_\mu - E_\lambda + i\Gamma_0\right)} \qquad (S \gg \Gamma_0)$$

†This is meant to imply that $\mathrm{Im}(a_\lambda)$ is independent of $\mathrm{Re}(a_\lambda)$ as well as that a_λ is independent of $a_{\lambda'}$.

$$= \frac{2\pi i}{S} \sum_{\lambda\mu} \frac{a_\lambda a_\mu^* \left(E_\mu - E_\lambda - i\Gamma_0\right)}{\left(E_\mu - E_\lambda\right)^2 + \Gamma_0^2}$$

$$= \frac{2\pi\Gamma_0}{S} \sum_{\lambda} \frac{|a_\lambda|^2}{\Gamma_0^2} + \frac{2\pi i}{S} \sum_{\lambda \neq \mu} \frac{a_\lambda a_\mu^* \left(E_\mu - E_\lambda - i\Gamma_0\right)}{\left(E_\mu - E_\lambda\right)^2 + \Gamma_0^2} \quad .$$

Since the second term has zero average, we will drop it, but it may have large fluctuations. The value of the first term is

$$\frac{2\pi \overline{|a|^2}}{D\Gamma_0} \quad ,$$

while a rough estimate of the r.m.s. fluctuation of the second term is

$$\frac{2\pi \overline{|a|^2}}{S\Gamma_0} \sqrt{\frac{2\Gamma_0}{D} \frac{S}{D}} \quad .$$

Thus, the first term should be expected to give a good estimate of the average cross section only if $\sqrt{\frac{2\Gamma_0}{S}} \ll 1$. It is this condition that is most often violated, and the effects of this violation form the subject matter of section IV. Since we have set up a statistical model for the reaction amplitude in the region S, we might consider the same model extended to an infinite energy range although, of course, the actual nucleus does not have this property. The relation between the infinite energy average and the finite average is treated in section IV.

We will adopt the notational convenience that a bar denotes the

average over an infinite energy region (e.g., $\bar{\eta}$ is the model expectation of the average cross section) while brackets denote the experimental average over some (implied) energy interval S (e.g., $\langle\eta\rangle$ is the measured average cross section). We may combine the processes by breaking up the infinite region into pieces of size S and then averaging some function of the local average (e.g., $\overline{\langle\eta\rangle^2}$). The process implied by the bar is an ensemble average.

With this notation, then, we may write

$$\bar{\eta} = \frac{2\pi\overline{|a|^2}}{\Gamma_0 D} \quad . \tag{3}$$

With the same assumptions we may also obtain the value of the autocorrelation function defined below,

$$R(\epsilon) \equiv \frac{\overline{\eta(E)\eta(E+\epsilon)} - \bar{\eta}^2}{\bar{\eta}^2} = \frac{1}{1 + \frac{\epsilon^2}{\Gamma_0^2}} \equiv h\left(\frac{\epsilon}{\Gamma_0}\right) . \tag{4}$$

If we assume that the excitation function was measured in very small energy steps, then the experimental average is expressed by

$$\langle\eta\rangle = \frac{1}{S}\int_0^S \eta(E)dE \quad , \tag{5}$$

and since the ensemble average operation is linear

$$\overline{\langle\eta\rangle} = \bar{\eta} \quad . \tag{6}$$

We may also compute $\overline{\langle\eta\rangle^2}$ by the following means[5]

$$\overline{\langle\eta\rangle^2} = \overline{\left[\frac{1}{S}\int_0^S \eta(E)dE\right]^2}$$

$$= \frac{1}{S^2}\int_0^S dE \int_0^S dE' \; \overline{\eta(E)\eta(E')}$$

$$= \frac{1}{S^2}\int_0^S dE \int_{-E}^{S-E} dx \; \overline{\eta(E)\eta(E+x)}$$

$$= \frac{\bar{\eta}^2}{S^2}\int_0^S dE \int_{-E}^{S-E} dx \left(1 + \frac{1}{1 + \frac{x^2}{\Gamma_0^2}}\right) \quad \text{(from eq. 4)}$$

$$= \bar{\eta}^2 + \frac{\bar{\eta}^2\Gamma_0^2}{S^2}\int_0^S dE \int_{-E}^{S-E} \frac{dx}{\Gamma_0^2 + x^2}$$

$$= \bar{\eta}^2 + \frac{2\bar{\eta}^2 \tan^{-1} T}{T} - \frac{\bar{\eta}^2}{T^2}\,\text{Log}\,(1 + T^2) \qquad \left(T \equiv \frac{S}{\Gamma_0}\right) .$$

From this we may see

$$\text{Var}\,(\langle\eta\rangle) \equiv \overline{\langle\eta\rangle^2} - \overline{\langle\eta\rangle}^2$$

$$= \bar{\eta}^2\left[\frac{2\tan^{-1} T}{T} - \frac{1}{T^2}\,\text{Log}\,(1 + T^2)\right] . \qquad (7$$

This gives us an estimate of the error we may be making in a measurement of the average cross section.

All of the preceding applies to all spinless particles and nuclei. If we have a particle of spin s incident upon a target of spin I leading to two final particles with spins I' and s', then Ericson[2] has shown that, in certain limits, the autocorrelation function, as defined by eq. 4, becomes

$$R(\epsilon) \equiv \frac{\overline{\eta(E)\eta(E+\epsilon)} - \bar{\eta}^2}{\bar{\eta}^2} = \frac{1}{N} h\left(\frac{\epsilon}{\Gamma_0}\right) , \tag{8}$$

where

$$N = \frac{(2s+1)(2I+1)(2s'+1)(2I'+1)}{2} . \tag{9}$$

One of the limits needed was that N be large, so we may expect this to be only asymptotically correct. The meaning of this result will become clear in the next section.

Up to this point we have been treating only the long time component. If we have both modes contributing we may compute the autocorrelation function as follows.

$$A = A_S + A_L ,$$

$$\eta = \eta_S + \eta_L + 2\mathrm{Re}(A_S^* A_L) ,$$

$$\eta^2 = \eta_S^2 + \eta_L^2 + 2\eta_S\eta_L + (2\mathrm{Re}\, A_S^* A_L)^2 + 4\eta_S\, \mathrm{Re}(A_S^* A_L) + 4\eta_L\, \mathrm{Re}(A_S^* A_L) ,$$

$$\overline{\eta^2} = \eta_S^2 + \overline{\eta_L^2} + 2\eta_S\bar{\eta}_L + \overline{(2\ \mathrm{Re}A_S^* A_1)^2} \quad .$$

As we have already noted, $\overline{\eta_L^2} = 2\bar{\eta}_L^2$. For the last term we have:[†]

$$\overline{(2\ \mathrm{Re}A_S^* A_L)^2} = 4\ \overline{(A_S^+ A_L^+ + A_S^- A_L^-)^2}$$

$$= 4\left(A_S^{+2}\overline{A_L^{+2}} + A_S^{-2}\overline{A_L^{-2}} + 2\,A_S^+ A_S^-\ \overline{A_L^+ A_L^-}\right) \quad .$$

Since

$$\overline{A_L^{+2}} = \overline{A_L^{-2}} \text{ and } \overline{A_L^+ A_L^-} = 0 \ ^{\ddagger} \quad , \qquad (10$$

$$\overline{(2\ \mathrm{Re}A_S^* A_L)^2} = 2\eta_S\bar{\eta}_L \quad ,$$

so that

$$\overline{\eta^2} = \eta_S^2 + 2\bar{\eta}_L^2 + 4\eta_S\bar{\eta}_L \quad .$$

Since

$$\bar{\eta} = \eta_S + \bar{\eta}_L \quad ,$$

[†] The notation (+) and (-) is used for real and imaginary parts of a quantity (e.g., $A_S = A_S^+ + iA_S^-$).

[‡] These follow from the assumption that $\overline{a_\lambda^{+2}} = \overline{a_\lambda^{-2}}$ and $\overline{a_\lambda^+ a_\lambda^-} = 0$ which were partially implied above.

if we define

$$y \equiv \frac{\eta_s}{\bar{\eta}} \quad ,$$

we find

$$\frac{\overline{\eta^2} - \bar{\eta}^2}{\bar{\eta}^2} \equiv R(0) = (1 - y^2) \quad . \tag{11}$$

So the general case may be written as

$$R(\epsilon) = \frac{(1 - y^2)}{N} \; h\left(\frac{\epsilon}{\Gamma_0}\right) \quad . \tag{12}$$

Thus, it is possible to determine the long time lifetime $\left(\frac{\hbar}{\Gamma_0}\right)$ and the relative amounts of short and long time reaction cross sections.

One might note here that it would seem that higher moments of η might be useful. The third and fourth moments can be obtained by the same methods used in this section, but, with much less trouble, the general formula will be developed in the next section.

It might also be noted that Γ_0 is a natural unit of energy so the convention that all energies are measured in units of Γ_0 could well be adopted. Such quantities will be referred to as specific (e.g., specific resolution) but will be denoted by the same symbol.

III. The Statistical Approach

Since at any given energy the quantity $A_L^{\pm}$ is made up of a large number of independent additive contributions ($\Gamma_0/D \gg 1$), we are led immediately to the central limit theorem of statistics. Let us state and prove this theorem.

Central Limit Theorem: Let $\{x_i\}$ be a set of m numbers drawn from a population with some distribution characterized by the distribution function f(x). Further, let all of the moments of x be finite. Now consider the quantity z defined by

$$z = \frac{1}{m} \sum_{i=1}^{m} x_i \quad .$$

Then

i) $\bar{z} = \bar{x}$,

ii) $\sigma_z^2 \equiv \text{Var}\,(z) \equiv \overline{z^2} - \bar{z}^2 = \frac{1}{m} \text{Var}\,(x)$,

iii) In the limit of large m, z is normally distributed, i.e., the distribution function of z, g(z), is given by

$$g(z) = \frac{1}{\sqrt{2\pi}\,\sigma_z} e^{-(z-\bar{z})^2/2\sigma_z^2} \left[1 + \mathcal{O}\left(\frac{1}{m^2}\right)\right] \qquad (13$$

independent of what form is taken for f(z).

Proof:

i) $$\bar{z} = \overline{\frac{1}{m}\sum_{i=1}^{m} x_i} = \frac{1}{m}\sum_{i=1}^{m} \bar{x}_i$$

$$= \bar{x}$$

ii) $$\overline{z^2} = \frac{1}{m^2}\sum_{i,j}^{m} \overline{x_i x_j}$$

$$= \frac{1}{m^2}\sum_{i=1}^{m} \overline{x_i^2} + \frac{1}{m^2}\sum_{i\neq j}^{m} \overline{x_i x_j}$$

$$= \frac{1}{m}\overline{x^2} + \frac{1}{m^2}(m)(m-1)\,\bar{x}^2$$

$$= \bar{x}^2 + \frac{\overline{x^2} - \bar{x}^2}{m}$$

$$\overline{z^2} - \bar{z}^2 = \frac{1}{m}\,\mathrm{Var}\,(x)$$

iii) To prove this part let us define the characteristic function as the Fourier transform of a distribution function.

$$\varphi(t) = \int_{-\infty}^{\infty} e^{ixt} f(x)\,dx$$

Note

$$\frac{d^p}{dt^p}\varphi(t)\Big|_{t=0} = i^p \int_{-\infty}^{\infty} x^p f(x)dx = i^p\overline{x^p} \quad .$$

Now let

$$\varphi_m(t) = \int_{-\infty}^{\infty} e^{izt} g(z)dz$$

$$= \sum_{j=0}^{\infty} \frac{t^j i^j \overline{z^j}}{j!} \quad .$$

But we may write the moments of z in a different manner

$$\overline{z^j} = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_m \left(\frac{1}{m}\sum_{k=1}^{m} x_k\right)^j f(x_1)f(x_2)\cdots f(x_m)$$

$$= (-i)^j \frac{d^j}{dt^j} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_m \, e^{\frac{i}{m}\sum_{k=1}^{m} x_k t} f(x_1)f(x_2)\cdots f(x_m)\Big|_{t=0}$$

$$= (-i)^j \frac{d^j}{dt^j}\left[\varphi\left(\frac{t}{m}\right)\right]^m\Big|_{t=0} \quad ,$$

so

$$\varphi_m(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left\{\frac{d^j}{dt^j}\left[\varphi\left(\frac{t}{m}\right)\right]^m\Big|_{t=0}\right\} = \varphi^m\left(\frac{t}{m}\right)$$

$$= \varphi^m\left(\frac{t}{m}\right) \quad .$$

For large m

$$\varphi\left(\frac{t}{m}\right) \equiv \int_{-\infty}^{\infty} e^{\frac{ixt}{m}} f(x)dx$$

$$= 1 + \frac{it\bar{x}}{m} - \frac{t^2\overline{x^2}}{2m^2} - \cdots$$

if all of the moments of x are finite, so that

$$\varphi_m(t) = \left(1 + \frac{it\bar{x}}{m} - \frac{t^2\overline{x^2}}{2m^2} + \cdots\right)^m = e^{m \log\left(1 + \frac{it\bar{x}}{m} - \frac{t^2\overline{x^2}}{2m^2} + \cdots\right)}$$

$$= e^{m\left(\frac{it\bar{x}}{m} - \frac{t^2\overline{x^2}}{2m^2} + \frac{t^2\bar{x}^2}{2m^2} + \cdots\right)}$$

$$= e^{it\bar{x} - \frac{t^2\sigma_x^2}{2m}\left[1 + \mathcal{O}\left(\frac{1}{m^2}\right)\right]}$$

or

$$\varphi_m(t) = e^{it\bar{z} - \frac{t^2\sigma_z^2}{2}\left[1 + \mathcal{O}\left(\frac{1}{m^2}\right)\right]} \qquad \text{as } m \to \infty \quad .$$

If we now note that

$$\int_{-\infty}^{\infty} e^{izt}\left(\frac{1}{\sqrt{2\pi}\,\sigma_z}\, e^{-\frac{(z-\bar{z})^2}{2\sigma_z^2}}\right) dz = e^{it\bar{z} - \frac{t^2\sigma_z^2}{2}} \quad ,$$

we may conclude

$$g(z) \to \frac{1}{\sqrt{2\pi}\,\sigma_z}\, e^{-\frac{(z-\bar{z})^2}{2\sigma_z^2}} \left[1 + \mathcal{O}\left(\frac{1}{m^2}\right)\right] \qquad \text{as } m \to \infty$$

Q. E. D.

Note that i) and ii) hold independent of the value of m, while iii) is valid for large m only.

Since the form of eq. 2 is valid for each amplitude corresponding to a different M substate, we may say that for each set of spin projections (s_z, I_z, s_z', I_z') there is an amplitude z_i (where i denotes a collection of four numbers chosen from the set $\{s_z, I_z, s_z', I_z'\}$) such that the cross section is given by

$$\eta = \sum_{i=1}^{(2s+1)(2I+1)(2s'+1)(2I'+1)} |z_i|^2 \quad . \qquad (14$$

The z_i, in accordance with eq. 10 and the ideas just presented, will be taken to have the following properties

i) $z_i = z_i^+ + iz_i^-$, $\quad \mathrm{Var}\,(z_i^+) = \mathrm{Var}\,(z_i^-) = \sigma_i^2$

$$\overline{z_i^+} = \mu_i^+ \quad , \qquad \overline{z_i^-} = \mu_i^-$$

$$\overline{z_i^+ z_j^-} = \mu_i^+ \mu_j^-$$

$$\overline{z_i^+ z_j^+} = \mu_i^+ \mu_j^+ \quad , \quad \overline{z_i^- z_j^-} = \mu_i^- \mu_j^- \quad , \quad i \neq j$$

ii) z_i^+ and z_i^- are normally distributed.

Since changing the signs of all spin projections does not change the cross section, not all contributions to the sum (14) are independent. By counting, the number of independent contributions may be seen to be

$$N = \frac{(2s + 1)(2I + 1)(2s' + 1)(2I' + 1) + 1}{2} \quad \text{(all spins integral)} \tag{15a}$$

$$N = \frac{(2s + 1)(2I + 1)(2s' + 1)(2I' + 1)}{2} \quad \text{(otherwise)} \quad . \tag{15b}$$

Let us rewrite (14) so that it contains only independent real numbers. We find that†

$$\eta = \sum_{i=1}^{2N} y_i^2 \tag{16}$$

† For the case of all spins integral the equation presented is not quite correct since the terms corresponding to $s_z = I_z = s_z' = I_z' = 0$ should have a factor of 1/2 multiplying them. This could be absorbed into the mean and variance of this term, however.

where the y_i are simply the $z_i^{\pm}$ relabeled and multiplied by $\sqrt{2}$. Consider now the case of $\mu_i^+ = \mu_i^- = 0$ and $\sigma_i^2 \equiv \sigma^2$. This means we have only long time reactions, and the average cross section through each of the M substates is the same. This yields a χ^2 distribution for η/σ^2 with 2N degrees of freedom.

$$f\left(\frac{\eta}{\sigma^2}\right) = \frac{1}{2^N(N-1)!}\left(\frac{\eta}{\sigma^2}\right)^{N-1} e^{-\frac{\eta}{2\sigma^2}} \tag{17}$$

or, since $\bar{\eta} = 2N\sigma^2$,

$$f(\eta) = \frac{N^N}{\bar{\eta}(N-1)!}\left(\frac{\eta}{\bar{\eta}}\right)^{N-1} e^{-N\frac{\eta}{\bar{\eta}}} , \tag{18}$$

which is a gamma distribution.

From the distribution function we may obtain a formula for all of the moments of η

$$\overline{\eta^m} = \frac{1}{N^m}\frac{(N-1+m)!}{(N-1)!}\bar{\eta}^m . \tag{19}$$

For example

$$\overline{\eta^2} = \frac{N(N+1)}{N^2}\bar{\eta}^2$$

$$= \bar{\eta}^2 + \frac{1}{N}\bar{\eta}^2$$

so that

$$\frac{\overline{\eta^2} - \bar{\eta}^2}{\bar{\eta}^2} = \frac{1}{N}$$

as in eq. 8.

IV. Sample Size Effects

Now let us suppose that an excitation function has been measured over an energy span S, and an autocorrelation function analysis gives an experimental determination of Γ_0, Γ. We would now like to know the number of independent points that have been measured. We may proceed by assuming that n independent points have been measured and solve for n by requiring that the results of sections II and III be consistent.

If only n points contain all of the information, we need use only these points to compute the quantities of interest. For example

$$\langle \eta \rangle = \frac{1}{n} \sum_{i=1}^{n} \eta_i \quad .$$

From the Central Limit Theorem ii)

$$\text{Var } \langle \eta \rangle = \frac{1}{n} \text{ Var } (\eta) = \frac{\bar{\eta}^2}{n} \quad . \tag{20}$$

Combining (20) and (7) we have

$$n = \frac{T^2}{2T \tan^{-1} T - \text{Log } (1 + T^2)} \tag{21}$$

$$\rightarrow \frac{T}{\pi} \qquad \text{as } T \rightarrow \infty \quad .$$

Similar considerations using Var $\langle \eta\eta' \rangle$, where η and η' are two independent cross sections, yield

$$n = \frac{3T^2}{5T \tan^{-1} T - 2 \text{ Log } (1 + T^2)} \rightarrow \frac{T}{\left(\frac{5}{6}\pi\right)} \quad . \tag{22}$$

Similarly

$$\text{Var } (\langle \eta \rangle^2) \qquad \text{yields} \qquad n \rightarrow \frac{T}{\pi} \tag{23}$$

and

$$\text{Var } (\langle \eta^2 \rangle) \qquad \text{yields} \qquad n \rightarrow \frac{T}{\left(\frac{9}{10}\pi\right)} \quad . \tag{24}$$

The n's resulting from eq. (22) and eq. (21) are plotted in Figure 1 along with T/π. Since the average cross section appears in every expression, the result expressed by eq. (21) would seem to be the best estimate of n to use. It may be observed from Figure 1 that in the region considered a good approximation to eq. (21) is

$$n = \frac{T}{\pi} + 1 \quad . \tag{25}$$

We now wish to consider the effect of a finite n on the value and uncertainty of measured quantities.

A. Sample Size Effects on R(0)

The effect of a finite sample size on R(0) has been investigated by

Monte Carlo procedures. The quantity R(0) was computed many ($\geq 5{,}000$) times from the expressions

$$R(0) = \frac{\langle \eta^2 \rangle - \langle \eta \rangle^2}{\langle \eta \rangle^2} \tag{26}$$

$$\langle \eta^2 \rangle = \frac{1}{n} \sum_{i=1}^{n} \eta_i^2 \quad , \qquad \langle \eta \rangle = \frac{1}{n} \sum_{i=1}^{n} \eta_i$$

where the η_i were chosen from a χ^2 distribution with 2N degrees of freedom. In this fashion $N\bar{R}(0)$ and Var (NR(0)) were obtained. $N\bar{R}(0)$ is plotted for N = 1, 2, 3 in Figures 2, 3, and 4. The solid curves are reasonable fits to order $1/n^2$. It may be noted that since $R(0) \equiv 0$ for $n = 1$ and $N(0) \to 1$ as $n \to \infty$ there is only one free parameter in the fit. All three cases were found to be fit by

$$N\bar{R}(0) = \frac{(n - 1)(4n - 4 + N)}{4n^2} \quad . \tag{27}$$

In Figures 5, 6, and 7 the variances are plotted for N = 1, 2, and 3. No such formula as appears in eq. (27) seems to be available, but it is not as necessary in this case since this is an estimate of the error involved and is slowly changing with N. The approximate values given by Hall[6] are plotted on the same graphs.

If we allow the possibility of some direct reaction we have in general

$$y = f_1(N,n,R)$$

and

$$\sigma_R^2 = f_2(N,n,R) \quad .$$

These functions have been computed numerically for N = 1 and are plotted in Figures 8 - 23. Figures 8 - 15 show y as a function of R for various values of n. It may be noted that the interaction between y and n is small, i.e., the curve may be reasonably represented by

$$y = \left[1 - \frac{4n^2R}{(n - 1)(4n - 3)}\right]^{1/2}$$

which leads one to believe that the formula

$$y = \left[1 - \frac{4n^2NR}{(n - 1)(4n - 4 + N)}\right]^{1/2} \qquad (28$$

may be a good approximation, at least for N = 1, 2, and 3.

The quantity σ_R^2 is plotted as function of R in Figures 16 - 23. Thus, one may obtain the uncertainty in R, and by applying this to the abscissa of the appropriate figure of y vs R obtain the error in y without recourse to the propagation of error formula. Since we have previously noted that σ_R^2 is a slowly varying function of N it is reasonable to use NR(0) for the abscissa of both curves.

We may note that the results show that there is an apparent increase of fast component because of a finite energy range. This is actually

related to the uncertainty principle, as we may see below.

If we prepare the nucleus in a long-lived state (or states) with width Γ_0, the fraction which will have decayed after time t is

$$f = 1 - e^{-\frac{\Gamma_0 t}{\hbar}} \approx \frac{\Gamma_0 t}{\hbar} \quad .$$

Referring to the arguments given in section II, we consider anything as fast which is coherent with the "instantaneous" component. This last is occurring during the time it takes for the incident particle to cross the nucleus

$$t_I = \frac{2R}{v} = \frac{2MR}{\hbar K} = \frac{\hbar KR}{E} \quad .$$

It is this instantaneous component that we usually associate with direct processes and the part that decays with width Γ_0 that we associate with compound processes. We see that there is a mixing of a little compound process into the "fast." This fraction is $\Gamma_0 KR/E$ which usually is so small as to be negligible. However, this assumes perfect time definition at the nucleus. If there is some finite energy average (ΔE) involved then there will be an uncertainty in the time of arrival of the bombarding particle during which the decaying state can interfere with the instantaneous process so that the fraction of compound coherent with direct is $\Gamma_0/\Delta E$.

Because of the nature of the extraction of the fast fraction it is

not clear what the equivalent energy averaging interval is. Note that the ideas just presented give only an upper limit for this fraction since we compute only the amount coherent with the fast mode, and hence the part with non-zero mean. This coherent part is not required to be constant.

B. Sample Size Effects on Γ

If one investigates the autocorrelation function for $\epsilon \gg \Gamma_0$ it may be observed that there are fluctuations which depend on a finite sample for their existence. A calculation shows that the period of these fluctuations is appreciably greater than Γ ($\sim 2\Gamma$). It is natural to assume that these are the same fluctuations which cause the uncertainty at $\epsilon = 0$. With this reasoning we will write (for $y = 0$)

$$NR(\epsilon) = \frac{1}{1 + \frac{\epsilon^2}{\Gamma_0^2}} + B(\epsilon) \quad . \tag{29}$$

Since $B(\epsilon)$ is not as rapidly varying as the other term we will consider it to be independent of ϵ. Because

$$NR(0) = 1 + B(0)$$

we know the mean and variance of $B(0)$. If we determine the value of Γ by looking at the autocorrelation function near $\epsilon = 0$ we may say

$$\frac{d}{d\epsilon^2}\left[\frac{\overline{NR(\epsilon)}}{\overline{NR(0)}}\right]\Bigg|_{\epsilon=0} = -\frac{4n^2}{\Gamma_0^2(n-1)(4n-4+N)} = -\frac{1}{\Gamma^2}$$

which leads to

$$\Gamma = \Gamma_0 \sqrt{\frac{(n-1)(4n-4+N)}{4n^2}} \qquad (30$$

for the expected value of Γ with a finite sample. Also since

$$\Gamma^2 = \Gamma_0^2 (1 + B)$$

the error in Γ is given by

$$\sigma_\Gamma^2 = \left(\frac{\Gamma_0^2}{2\Gamma}\right)^2 \sigma_{NR}^2 \quad . \qquad (31$$

C. The Effect of Finite Sample Size on Frequency Distribution Functions

The number of entries in the sample taken has an effect on the form of the distribution function one should expect to measure. This is owing to the fact that since an absolute scale is difficult to establish for the cross section, the measured values must be normalized to have mean value one before they can be compared with the theoretical prediction.

To see how this can affect the form of the function, consider the extreme case of a sample size of 1. This means we select a single number, z, normalize it so that it has value 1, and then plot it. The result is that we see a distribution function which is

$$f(z) = \delta(z - 1)$$

regardless of the initial distribution.

If we draw a sample of size 2 we may see by similar arguments that the resulting distribution will always be symmetric about 1.

The general problem would seem to be a difficult one. However, it is easy enough to prepare a machine code which will transform any arbitrary distribution function to its finite sample size analogue by means of Monte Carlo techniques. The difficulty with this method is the computing time required for sufficient accuracy.

For the distribution function given by eq. 18 the general transformation may be made analytically. The result is

$$f(z) = \frac{(Nn - 1)!}{n^N(N - 1)!(Nn - N - 1)!} z^{N-1} \left(1 - \frac{z}{n}\right)^{Nn-N-1}$$

where $z = \eta/\langle\eta\rangle$. This equation is derived in the Appendix.

V. The Effect of Counting Statistics

Since, in practice, the cross sections are measured by a counting process and there are random fluctuations associated with this process, one might worry that these fluctuations would strongly affect the apparent values obtained. Let us consider this effect on the quantities of interest. We may write, in general, for counting data

$$h = h_0 + \sqrt{h_0}\ z \qquad (32$$

where h is the number of counts actually obtained, h_0 is the true number of counts corresponding to the actual cross section, and z is a random

variable with mean zero and variance 1. If there is the same amount of integrated beam on the target for each measurement, there is a single constant, α, which relates the cross section to the number of counts.

$$\eta = \alpha n = \alpha n_0 + \alpha\sqrt{n_0}\ z$$

$$= \eta_0 + \alpha^{1/2}\ \eta_0^{1/2}\ z$$

where η_0 is the true cross section. We may now calculate R(0) to be

$$R(0) = \frac{\overline{\eta^2} - \bar{\eta}^2}{\bar{\eta}^2} = \frac{\overline{\eta_0^2} + \alpha\bar{\eta}_0 - \bar{\eta}_0^2}{\bar{\eta}_0^2}$$

$$= R_0(0) + \frac{\alpha}{\bar{\eta}_0} = R_0(0) + \frac{1}{\bar{n}} \quad . \tag{33a}$$

$\bar{n}$ is the average number of counts taken in the set of data being analyzed. If a total of m data points are used in the analysis, the variance of R(0) due to the present cause is

$$\sigma_R^2 \approx \frac{1}{m\bar{n}} \quad . \tag{33b}$$

Both of the results above are valid only in the limit of large m.

For $R(\epsilon)$ ($\epsilon > 0$) there is no correction ($m \to \infty$) since no cross section is multiplied by itself. It we compute Γ by comparison with a Lorentzian form, we have

$$\frac{R(\epsilon)}{R(0)} = \frac{R_0(0)}{\left(1 + \frac{\epsilon^2}{\Gamma_0^2}\right)\left(R_0(0) + \frac{1}{\bar{n}}\right)} = \frac{1}{1 + \frac{\epsilon^2}{\Gamma^2}}$$

$$= \frac{R(0) - \frac{1}{\bar{n}}}{R(0)\left(1 + \frac{\epsilon^2}{\Gamma_0^2}\right)} \quad .$$

This leads to a correction given by

$$\Gamma_0^2 = \Gamma^2 \left[\frac{\bar{n}R(0)}{\bar{n}R(0) - 1 - \frac{\Gamma^2}{\epsilon^2}}\right] \qquad (34$$

which depends on the value of ϵ chosen for the comparison.

VI. Generalized Frequency Distributions

In the general case the restriction to all equal variances for the long time reaction to different M substates is rather poor. Let us now investigate the consequences of relaxing the assumption.

If we write as before

$$\eta = \sum_{i=1}^{2N} y_i^2 \quad ,$$

assume

$$\sigma_i^{+2} = \sigma_i^{-2} = \frac{1}{2\alpha_i} \quad , \tag{35}$$

and define

$$x_i = \mu_i^{+2} + \mu_i^{-2} \quad , \tag{36}$$

then the frequency distribution function is given by

$$f(\eta) = \mathfrak{L}^{-1} \left\{ \prod_{i=1}^{N} \alpha_i \, (\alpha_i + t)^{-1} \, e^{-\frac{t x_i \alpha_i}{\alpha_i + t}} \right\} \quad . \tag{37}$$

The quantity in the braces is the moment generating function with argument -t so that

$$\frac{d^m}{dt^m} \left\{ \prod_{i=1}^{N} \alpha_i \, (\alpha_i - t)^{-1} \, e^{\frac{t x_i \alpha_i}{\alpha_i - t}} \right\} \Bigg|_{t=0} = \overline{\eta^m} \quad . \tag{38}$$

Note that $1/\alpha_i$ is the average long time cross section to the i^{th} M substate and as such may be calculated by some sort of Hauser-Feshbach expression (modified to correct for the interference with direct reaction). x_i is the short time reaction cross section to the i^{th} M substate and might be calculated by means of a direct reaction calculation (again possibly modified).

Equation 37 cannot be inverted for the general case, but there are some interesting special cases. For only one M substate (N = 1)

$$f(\eta) = \alpha e^{-\alpha(\eta+x)} I_0 (2\sqrt{x}\,\alpha\sqrt{\eta}) \quad . \tag{39}$$

Since this is the inverse Laplace transform of one factor in the braces of eq. 37, any given case may be built up by convolution (numerically, if necessary).

For $x_i \equiv 0$, eq. 37 can be solved in general to give

$$f(\eta) = \prod_{i=1}^{N} \alpha_i \sum_{\substack{\text{cyclic} \\ \text{permu-} \\ \text{tations}}} \frac{e^{-\alpha_1 \eta}}{(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1)\cdots(\alpha_N - \alpha_1)} \quad . \tag{40}$$

If all $\alpha_i \equiv \alpha$ and we define $X = \sum_{i=1}^{N} x_i$, then eq. 37 gives

$$f(\eta) = \alpha\left(\frac{\eta}{X}\right)^{\frac{N-1}{2}} e^{-\alpha(X+\eta)} I_{N-1} (2\alpha\sqrt{X}\sqrt{\eta}) \quad . \tag{41}$$

This is a generalization of eq. 39.

We may now use eq. 38 to investigate the behavior of R(0) in this more general case. For the first two moments we obtain

$$\bar{\eta} = \sum_{i=1}^{N} \left(\frac{1}{\alpha_i} + x_i\right) \tag{42}$$

$$\overline{\eta^2} = \bar{\eta}^2 + \sum_{i=1}^{N} \left(\frac{1}{\alpha_i^2} + 2\frac{x_i}{\alpha_i}\right) \quad . \tag{43}$$

Now define

$$Z = \sum_{i=1}^{N} \frac{1}{\alpha_i} \quad ,$$

$$\beta_i = \frac{1}{\alpha_i Z} \quad , \qquad \gamma_i = \frac{x_i}{X} \quad , \tag{44}$$

and

$$y = \frac{X}{\bar{\eta}} \quad .$$

With these definitions we may write

$$R(0) = (1 - y)^2 \sum_{i=1}^{N} \beta_i^2 + 2y(1 - y) \sum_{i=1}^{N} \beta_i \gamma_i$$

$$= (1 - y)^2 a + 2y(1 - y)b \quad . \tag{45}$$

The quantity $1/a$ is defined as N_{eff}, the effective number of M substates. To see this significance we note that the relative cross section for a reaction proceeding through a given M substate is determined mostly by angular momentum considerations and only in second order by the reaction mechanism. If this is true we may set $a = b$ (since $\beta_i = \gamma_i$) and thus

$$R(0) = a(1 - y^2) \quad .$$

If calculations of a and b can be made, more exact values of y may be obtained from the data. The first-order correction is given by

$$y' = y + (1 - y)\Delta \tag{46}$$

where $\Delta = \frac{b}{a} - 1$, y is the value extracted by use of N_{eff}, and y' is the corrected value of y.

VII. The Effect of Resolution

Let us assume the measurement has been made with a resolution described by a square resolution function with full width ρ. In this case the cross section measured will be $\tilde{\eta}(E)$ instead of the true value $\eta(E)$ where

$$\tilde{\eta}(E) = \frac{1}{\rho}\int_{E - \frac{\rho}{2}}^{E + \frac{\rho}{2}} \eta(E')dE' \quad . \tag{47}$$

Using Γ_0 as the energy unit we have

$$\overline{\tilde{\eta}(E)\tilde{\eta}(E + \epsilon)} = \frac{1}{\rho^2}\int_{E - \frac{\rho}{2}}^{E + \frac{\rho}{2}} dE' \int_{E + \epsilon - \frac{\rho}{2}}^{E + \epsilon + \frac{\rho}{2}} dE'' \, \overline{\eta(E')\eta(E'')}$$

$$= \frac{1}{\rho^2} \int_{E - \frac{\rho}{2}}^{E + \frac{\rho}{2}} dE' \int_{E - E' + \epsilon - \frac{\rho}{2}}^{E - E' + \epsilon + \frac{\rho}{2}} dx \, \overline{\eta(E')\eta(E' + x)}$$

$$= \bar{\eta}^2 + \frac{\bar{\eta}^2}{\rho^2} \int_{E - \frac{\rho}{2}}^{E + \frac{\rho}{2}} dE' \int_{E - E' + \epsilon - \frac{\rho}{2}}^{E - E' + \epsilon + \frac{\rho}{2}} \frac{dx}{1 + x^2}$$

$$= \bar{\eta}^2 + \frac{\bar{\eta}^2}{\rho^2} \int_{E - \frac{\rho}{2}}^{E + \frac{\rho}{2}} dE' \left[\tan^{-1}(E - E' + \epsilon + \frac{\rho}{2}) - \tan^{-1}(E - E' + \epsilon - \frac{\rho}{2}) \right]$$

$$= \bar{\eta}^2 + \frac{\bar{\eta}^2}{\rho^2} \Big\{ (\epsilon + \rho) \tan^{-1}(\epsilon + \rho) - \frac{1}{2} \operatorname{Log} \left[1 + (\epsilon + \rho)^2 \right]$$

$$+ (\epsilon - \rho) \tan^{-1}(\epsilon - \rho) - \frac{1}{2} \operatorname{Log} \left[1 + (\epsilon + \rho)^2 \right]$$

$$- 2\epsilon \tan^{-1} \epsilon + \operatorname{Log} (1 + \epsilon^2) \Big\}$$

and

$$R(\epsilon) = \frac{1}{\rho^2} \Big\{ (\epsilon + \rho) \tan^{-1}(\epsilon + \rho) + (\epsilon - \rho) \tan^{-1}(\epsilon - \rho)$$

$$- \frac{1}{2} \operatorname{Log} \left[1 + (\epsilon + \rho)^2 \right] - \frac{1}{2} \operatorname{Log} \left[1 + (\epsilon - \rho)^2 \right] \qquad (48$$

$$- 2\epsilon \tan^{-1} \epsilon + \operatorname{Log} (1 + \epsilon^2) \Big\}$$

For $\epsilon = 0$ we have

$$R(0) = \frac{2\rho \tan^{-1} \rho - \text{Log}\,(1 + \rho^2)}{\rho^2} \quad . \tag{49}$$

Figure 24 shows R(0) as a function of ρ. Since this curve is a function only of the specific resolution, any units may be used as long as the same units are used for the resolution and width.

If Γ is measured by the slope of the autocorrelation function at the origin ($\epsilon \ll 1$)

$$\Gamma^2 = \left[\frac{2\rho \tan^{-1} \rho - \text{Log}\,(1 + \rho^2)}{\rho^2}\right] (1 + \rho^2)\, \Gamma_0^2 \quad . \tag{50}$$

For arbitrary ϵ eq. 48 must be solved numerically to obtain the correction function for Γ.

For $\epsilon/\rho \ll 1$ (but no restriction on the absolute magnitude of ϵ) we obtain

$$R(\epsilon) = \frac{1}{\frac{\rho}{\pi} + 1} - \frac{\epsilon^2}{\rho^2}\left(\frac{1}{\frac{\epsilon}{\pi} + 1} - \frac{1}{1 + \rho^2}\right) \tag{51}$$

and, further, if $\epsilon \gg \pi$

$$R(\epsilon) = \frac{\pi}{\rho}\left(1 - \frac{\epsilon}{\rho}\right) \quad . \tag{52}$$

In absolute units this becomes

$$R(\epsilon) = \frac{\pi\Gamma_0}{\rho}\left(1 - \frac{\epsilon}{\rho}\right) \tag{53}$$

so that measurements of very small widths may be made along with a simultaneous experimental determination of the resolution, i.e.,

$$\Gamma_0 = \frac{R(0)\rho}{\pi} \tag{54}$$

while ρ may be extracted from

$$\frac{R(\epsilon)}{R(0)} = 1 - \frac{\epsilon}{\rho} \quad . \tag{55}$$

One may note that $R(0)$ as low as 0.03 might be easily measured so that widths of the order of 0.01 ρ could be deduced.

All of the above is for $N = 1$. If $N > 1$, simply replace

$$R(0) \rightarrow NR(0)$$

to obtain the more general result.

VIII. Appendix: Some Analytic Results

A. Finite Sample Transformation of a Gamma Distribution

Suppose a sample of size n, $(x_1, x_2, \cdots x_n)$, is drawn from a population with a gamma distribution of mean 1. Thus,

$$f(x) = \frac{N^N}{(N-1)!}\, x^{N-1}\, e^{-Nx} \quad .$$

We wish to inquire as to the distribution of z where

$$z = \frac{x_1}{\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)}$$

since this is the process carried out in normalizing the experimental cross sections to mean 1. By the same technique used in proving the Central Limit Theorem but using the Laplace transform in place of the Fourier transform, we may write

$$\mathcal{L}\left(f(z)\right) = \frac{N^{nN}}{\left[(N-1)!\right]^n}\int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_n \exp\left(-\frac{nx_1 t}{\sum_{i=1}^{n} x_i}\right) \tag{A1}$$

$$x_1^{N-1}\, x_2^{N-1} \cdots x_n^{N-1}\, e^{-N\sum_{i=1}^{n} x_i} .$$

We now make the transformation of variables

$$z = \frac{nx_1}{\sum_{i=1}^{n} x_i}, \quad y_2 = x_2,\ y_3 = x_3, \cdots,\ y_n = x_n \quad ;$$

so that

$$x_1 = \frac{z\sum_{i=2}^{n} y_i}{n - z}, \quad x_2 = y_2,\ x_3 = y_3, \cdots,\ x_n = y_n \quad .$$

The Jacobian of this transformation is given by

$$J = \frac{n \sum_{i=2}^{n} y_i}{(n - z)^2} \quad . \tag{A2}$$

With this change of variable, eq. A1 becomes

$$\mathcal{L}\left(f(z)\right) = \frac{N^{nN}}{\left[(N - 1)!\right]^n} \int_0^n dz \, \frac{n \, e^{-zt}}{(n - z)^2} \int_0^\infty dx_2 \cdots \int_0^\infty dx_n$$

$$\left(\sum_{i=2}^{n} x_i \right) \left(\frac{z \sum_{i=2}^{n} x_i}{n - z} \right)^{N-1} x_2^{N-1} \cdots x_n^{N-1} \, e^{-\frac{N}{1 - \frac{z}{n}} \sum_{i=2}^{n} x_i} \quad . \tag{A3}$$

From A3 it is easy to see that

$$f(z) = 0 \qquad \text{for } z > n$$

and for $0 < z < n$

$$f(z) = \frac{N^{nN}}{\left[(N - 1)!\right]^n} \, \frac{z^{N-1}}{(n - z)^{N+1}}$$

$$\int_0^\infty dx_2 \cdots \int_0^\infty dx_n \left(\sum_{i=2}^{n} x_i \right)^N x_2^{N-1} \cdots x_n^{N-1} \, e^{-\frac{N}{1 - \frac{z}{n}} \sum_{i=2}^{n} x_i} \tag{A4}$$

Thus, we are led to consider integrals of the following type.

$$\int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_m \left(\sum_{i=1}^m x_i\right)^N x_1^{N-1} \cdots x_m^{N-1} e^{-a\sum_{i=1}^m x_i}$$

$$= \sum_{\substack{n_1, n_2 \cdots n_m \\ \left(\sum_{j=1}^m n_j = N\right)}} \frac{N!}{n_1! n_2! n_3! \cdots n_m!} \int_0^\infty dx_1 \int_0^\infty dx_2 \cdots \int_0^\infty dx_m x_1^{n_1+N-1} \cdots x_m^{n_m+N-1} e^{-a\sum_{i=1}^m x_i}$$

$$= \sum_{\substack{n_1, n_2 \cdots n_m \\ \left(\sum_{j=1}^m n_j = N\right)}} \frac{N!}{n_1! n_2! n_3! \cdots n_m!} \left[\frac{(n_1 + N - 1)!(n_2 + N - 1)! \cdots (n_m + N - 1)!}{a^{n_1+N} \; a^{n_2+N} \; a^{n_3+N} \cdots a^{n_m+N}}\right]$$

$$= \frac{1}{a^{N(m+1)}} \sum_{\substack{n_1, n_2 \cdots n_m \\ \left(\sum_{j=1}^m n_j = N\right)}} \frac{N!(n_1 + N - 1)!(n_2 + N - 1)! \cdots (n_m + N - 1)!}{n_1! \; n_2! \; n_3! \cdots n_m!} \quad .$$

Since the sum is only a constant we may drop it and find the normalization at the end. This gives

$$f(z) \propto \frac{z^{N-1}}{(n - z)^{N+1}} \left(1 - \frac{z}{n}\right)^{Nn}$$

$$\propto z^{N-1} \left(1 - \frac{z}{n}\right)^{Nn-N-1} \quad .$$

This is easily integrated to obtain the normalization. The final result is

$$f(z) = \frac{(Nn - 1)!}{n^N (n - 1)!(Nn - N - 1)!} z^{N-1} \left(1 - \frac{z}{n}\right)^{Nn-N-1} . \qquad \text{(A5}$$

B. Analytic Determination of $N\bar{R}$ for n = 2

Suppose we measure two values, x and y, which come from a gamma distribution (eq. 18). We may express R(0) as

$$R = \frac{\frac{1}{2}(x^2 + y^2) - \frac{1}{4}(x + y)^2}{\frac{1}{4}(x + y)^2}$$

$$= \left(\frac{x - y}{x + y}\right)^2 . \qquad \text{(A6}$$

Thus, we have

$$\mathcal{L}\Big(f(R)\Big) = \frac{N^{2N}}{\left[(N - 1)!\right]^2} \int_0^\infty dx \int_0^\infty dy \; e^{-\left(\frac{x-y}{x+y}\right)^2 t} x^{N-1} y^{N-1} \; e^{-N(x+y)} .$$

Now make the change of variables

$$p = \left(\frac{x - y}{x + y}\right) , \qquad z = y ,$$

then

$$x = z\left(\frac{1 + p}{1 - p}\right) , \qquad y = z ,$$

and

$$|J| = \frac{2z}{(1 - p)^2} .$$

Thus, we have

$$\mathcal{L}\left(f(R)\right) = \frac{2\,N^{2N}}{\left[(N-1)!\right]^2}\int_{-1}^{1} dp\,\frac{e^{-p^2t}}{(1-p)^2}\int_{0}^{\infty} dz\; z^{2N-1}\left(\frac{1+p}{1-p}\right)^{N-1} e^{-\frac{2N}{1-p}z}$$

$$= \frac{2\,N^{2N}(2N-1)!}{\left[(N-1)!\right]^2(2N)^{2N}}\int_{-1}^{1} dp\,\frac{e^{-p^2t}}{(1-p)^2}\left(\frac{1+p}{1-p}\right)^{N-1}(1-p)^{2N}$$

$$= \frac{2(2N)!}{\left[(N-1)!\right]^2\,2N\,2^{2N}}\int_{-1}^{1}(1-p^2)^{N-1}\,e^{-p^2t}\,dp \qquad \text{(A7}$$

$$= \frac{2\,N(2N)!}{(N!)^2\,2^{2N}}\int_{0}^{1}(1-p^2)^{N-1}\,e^{-p^2t}\,dp$$

$$= \frac{N\,(2N)!}{(N!)^2\,2^{2N}}\int_{0}^{1}(1-R)^{N-1}\,\frac{e^{-Rt}}{\sqrt{R}}\,dR \quad ,$$

from which it is clear that

$$f(R) = \frac{N\,(2N)!}{(N!)^2\,2^{2N}}\,\frac{(1-R)^{N-1}}{\sqrt{R}} \qquad 0 < R < 1 \qquad \text{(A8}$$

$$= 0 \qquad R > 1 \text{ or } R < 0 \quad .$$

Meanwhile we get from eq. A7

$$N\bar{R} = \frac{2N^2\,(2N)!}{(N!)^2\,2^{2N}} \int_0^1 p^2\,(1 - p^2)^{N-1}\,dp \quad . \qquad \text{(A9}$$

We may then compare these results with eq. 27 for the case of n = 2. Note that this is a very stringent test of eq. 27 since it is exact at n = 1 or n = ∞ and was fit to the calculations at n = 4. It is expected to be the greatest in error around n = 2.

N	$N\bar{R}$	$\frac{4 + N}{16}$
1	0.3333	0.3125
2	0.4000	0.3750
3	0.4285	0.4375
4	0.4444	0.5000

We may see that eq. 27 gives a good representation to within about 10% even for the case of n = 2.

Figure Captions

Figure 1. The effective number of independent observations (n) as a function of the energy range covered (T). The lower (solid) curve would be suitable for autocorrelation functions, while the upper curve should be used in the comparison of two independent cross sections. The dotted curve (T/π) is given for comparison. Energy is measured in units of Γ_0.

Figures 2, 3, and 4. The expected value of R(0) as a function of the sample size (n). The generated input data were characterized by a χ^2 distribution with 2N degrees of freedom.

Figures 5, 6, and 7. The variance of R(0) as a function of sample size. The generated input data were characterized by a χ^2 distribution with 2N degrees of freedom.

Figures 8 - 15. The fraction of direct reaction as a function of R(0) for n = 2, 4, 6, 8, 10, 12, 15, and 20. N = 1 for all of these curves.

Figures 16 - 23. The variance of R(0) plotted as a function of R(0) for n = 2, 4, 6, 8, 10, 12, 15, and 20. N = 1 for all of these curves.

Figure 24. R(0) as a function of resolution for several values of Γ_0. Arbitrary units may be used for the resolution as long as Γ_0 is measured in the same units.

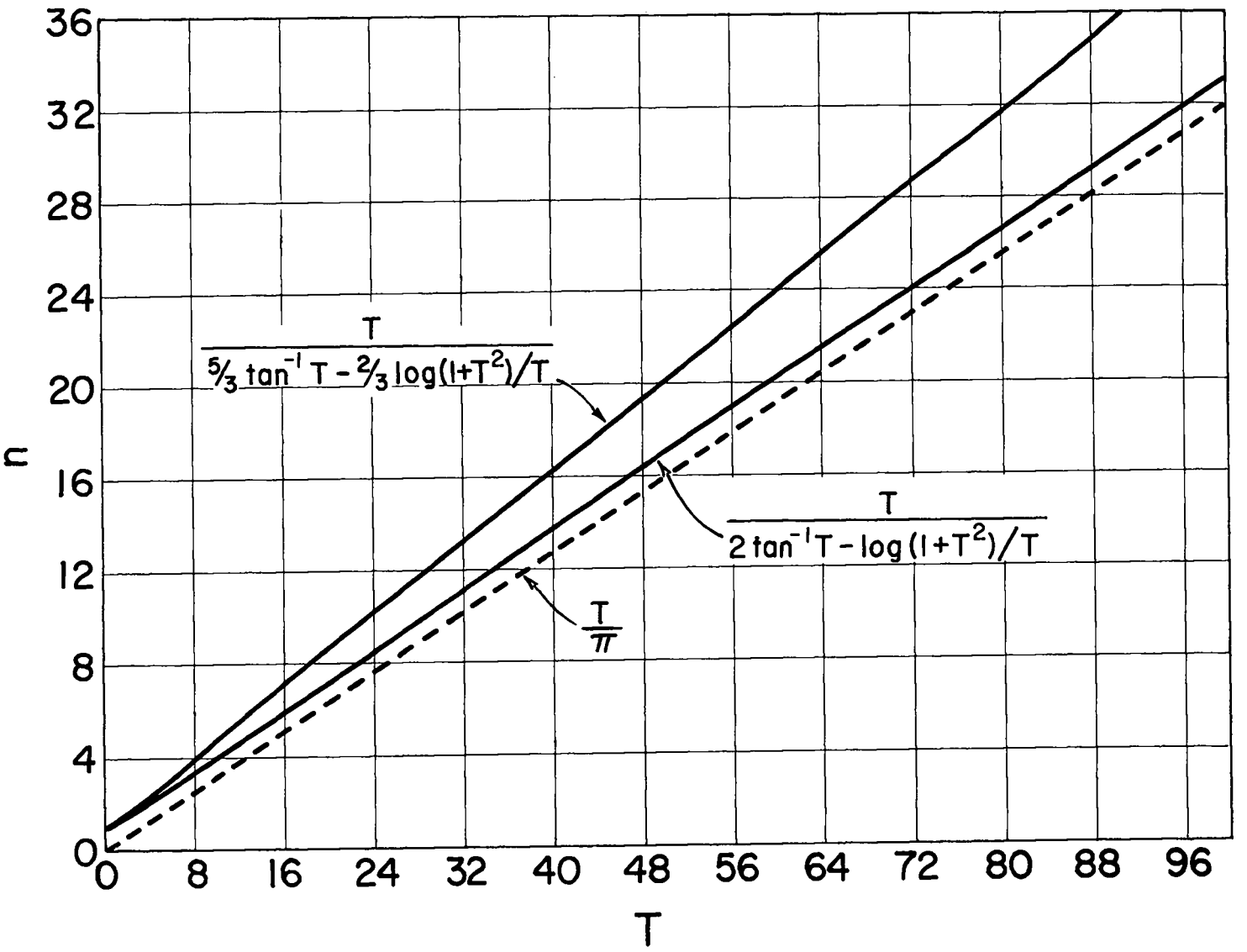

36
32
28
24
20
16
12
8
4
0
C
0
8
16
24
32
40
48
56
64
72
80
88
96
T
$\frac{T}{5/3 \tan^{-1} T - 2/3 \log(1+T^2)/T}$
$\frac{T}{2 \tan^{-1} T - \log(1+T^2)/T}$
$\frac{T}{\pi}$

Figure 1.

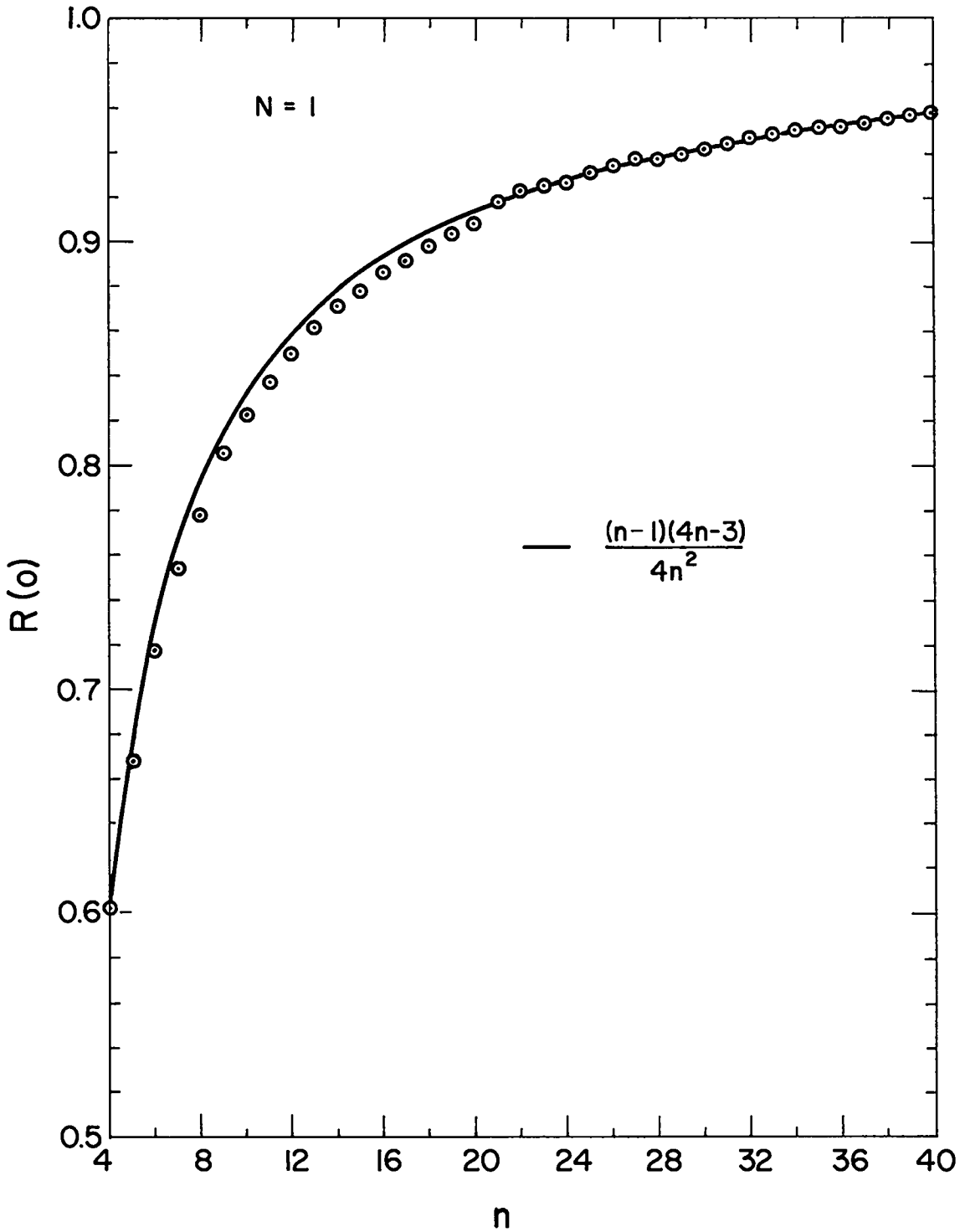


Figure 2.

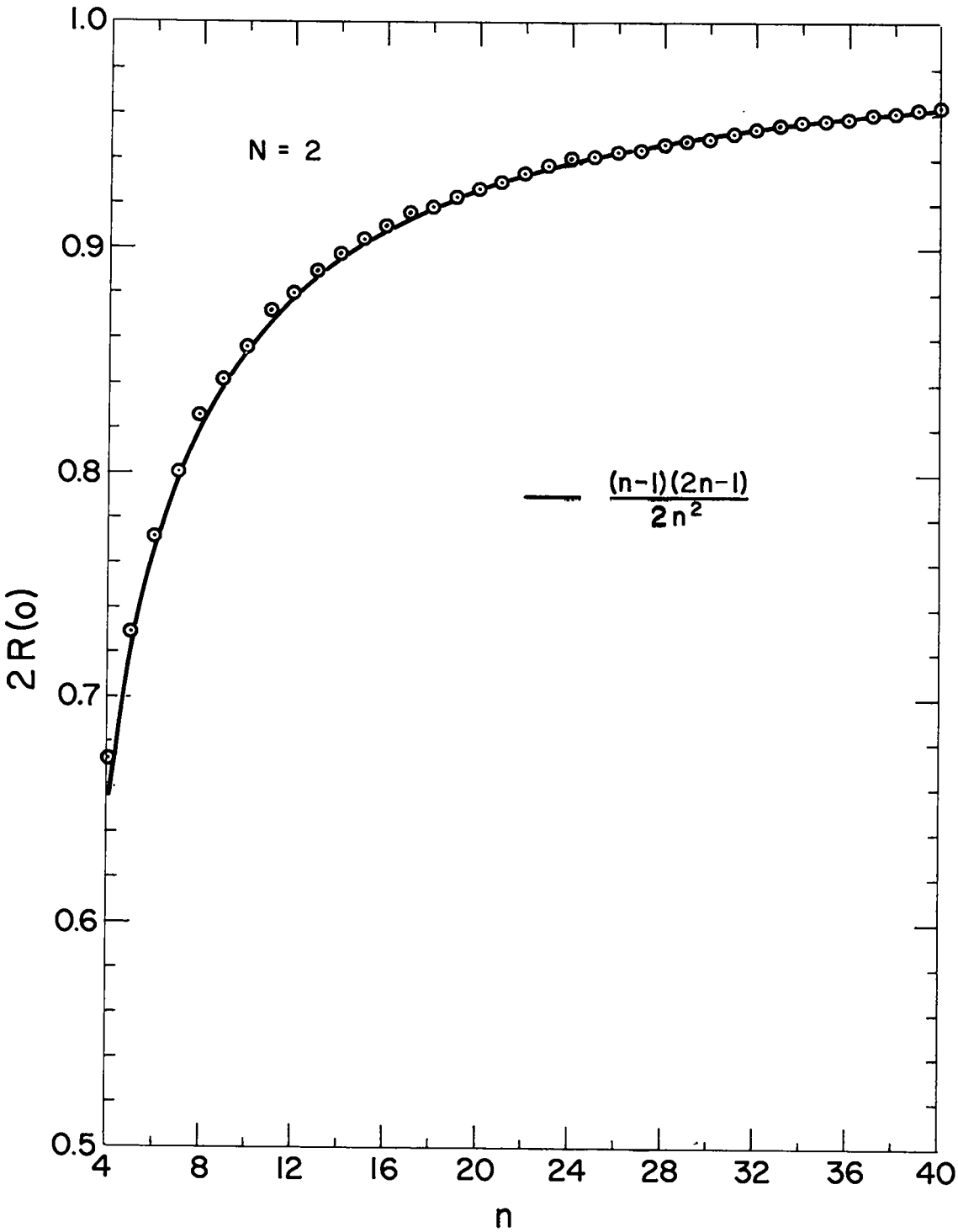


Figure 3.

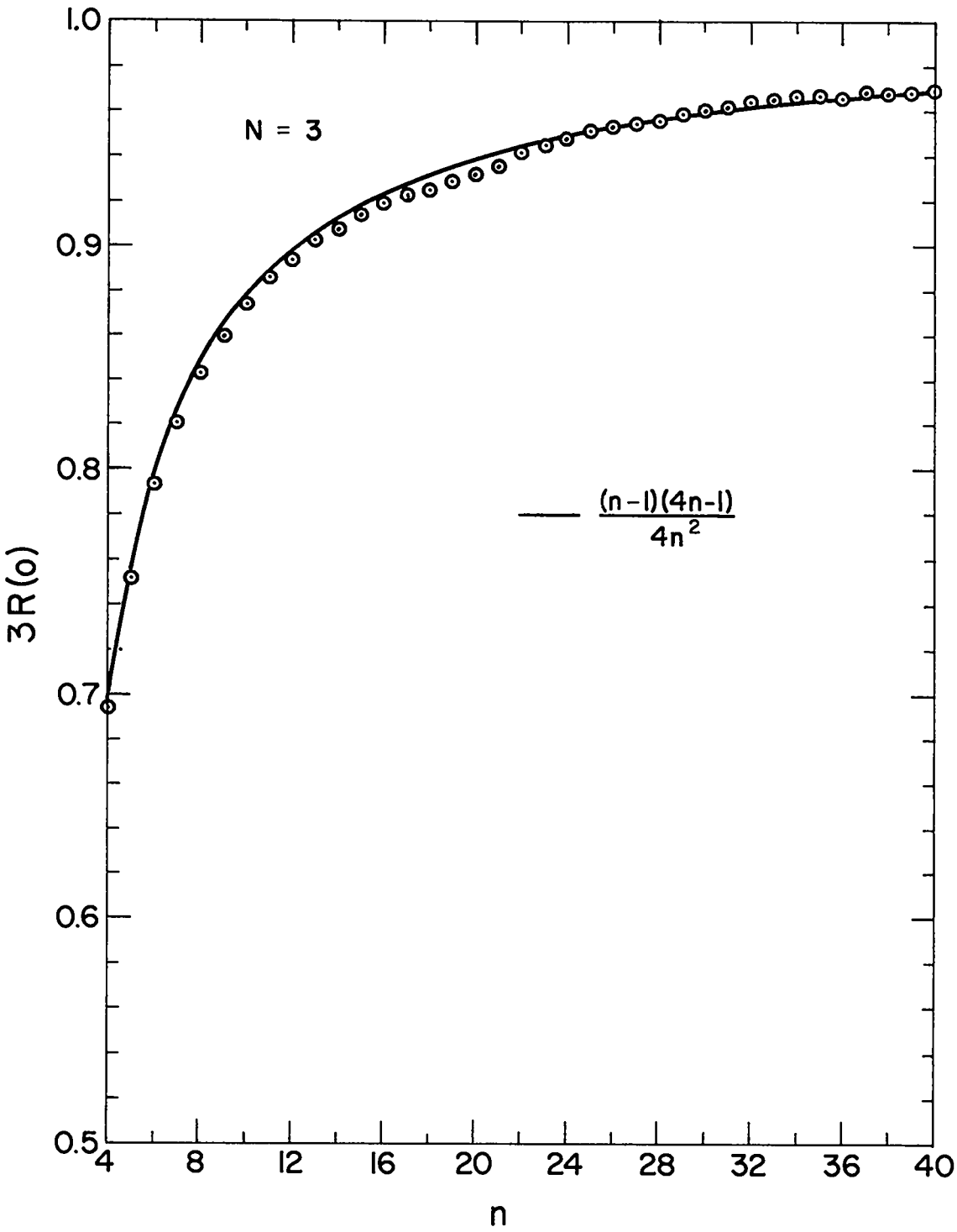


Figure 4.

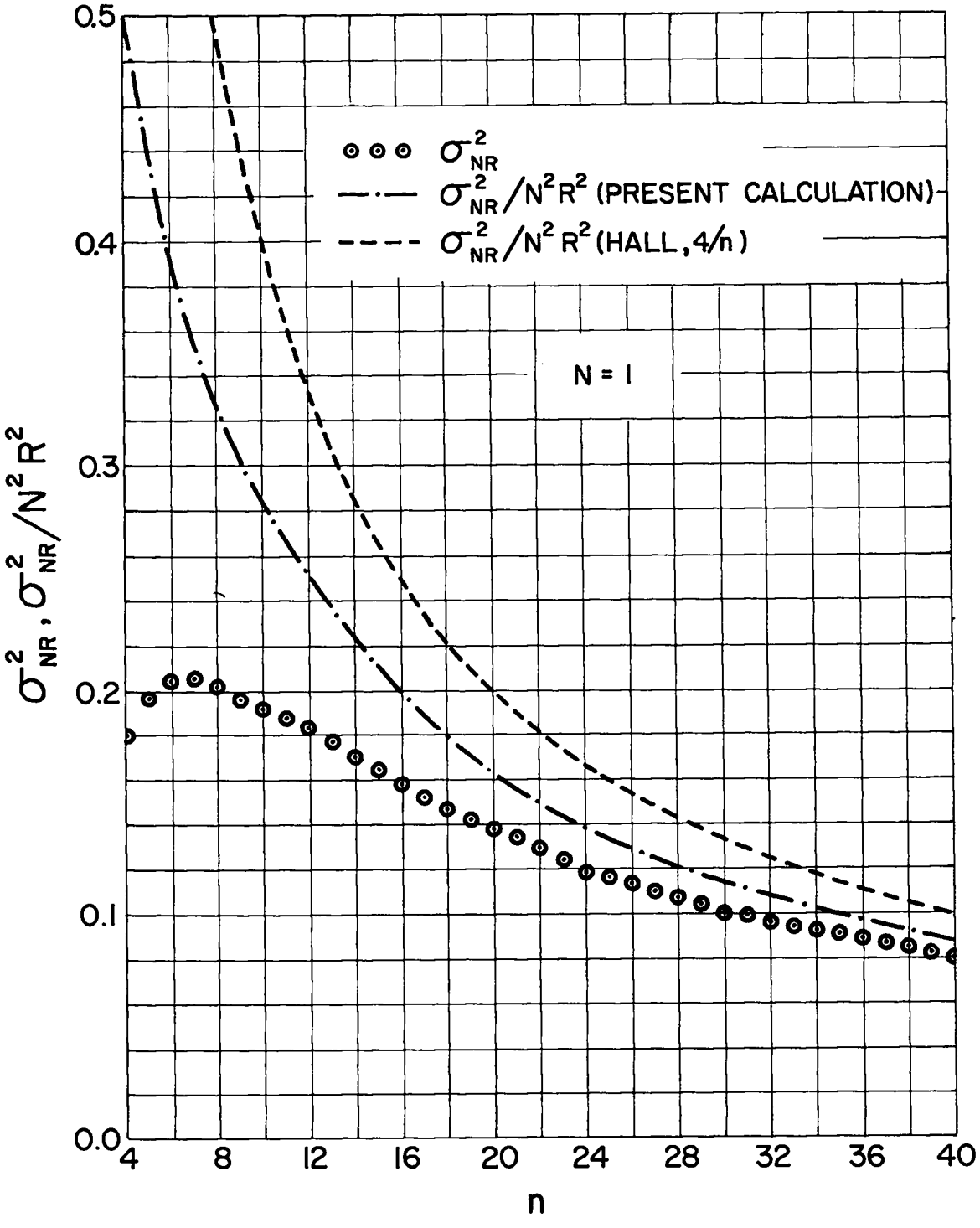

0.5
0.4
0.3
0.2
0.1
0.0
4
8
12
16
20
24
28
32
36
40
n
σ^2_{NR}
σ^2_{NR}/N^2R^2 (PRESENT CALCULATION)
σ^2_{NR}/N^2R^2 (HALL, 4/n)
N = 1
σ^2_{NR}, σ^2_{NR}/N^2R^2

Figure 5.

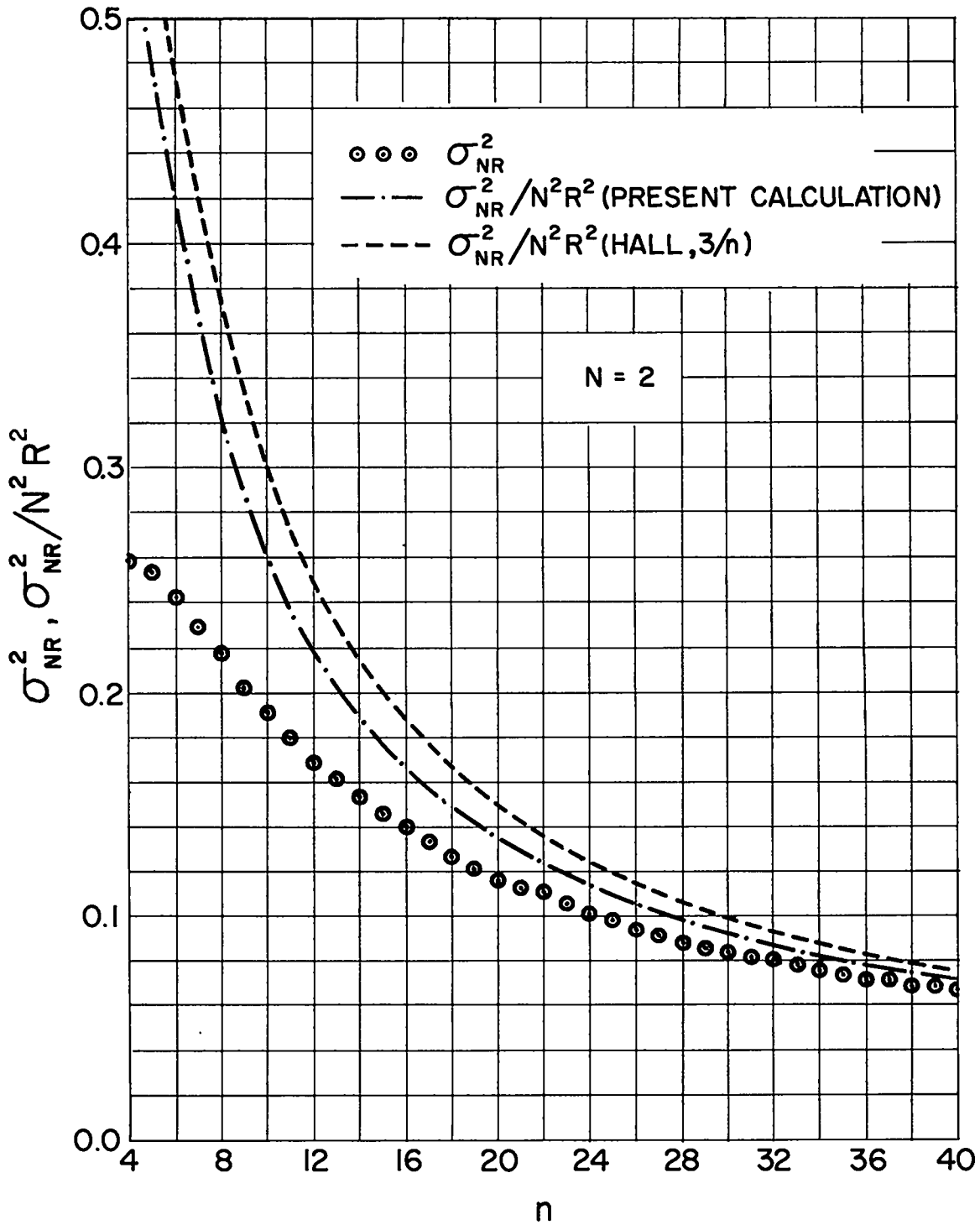

0.5
0.4
0.3
0.2
0.1
0.0
4
8
12
16
20
24
28
32
36
40
n
σ^2_{NR}, σ^2_{NR}/N^2R^2
σ^2_{NR}
σ^2_{NR}/N^2R^2 (PRESENT CALCULATION)
σ^2_{NR}/N^2R^2 (HALL, 3/n)
N = 2

Figure 6.

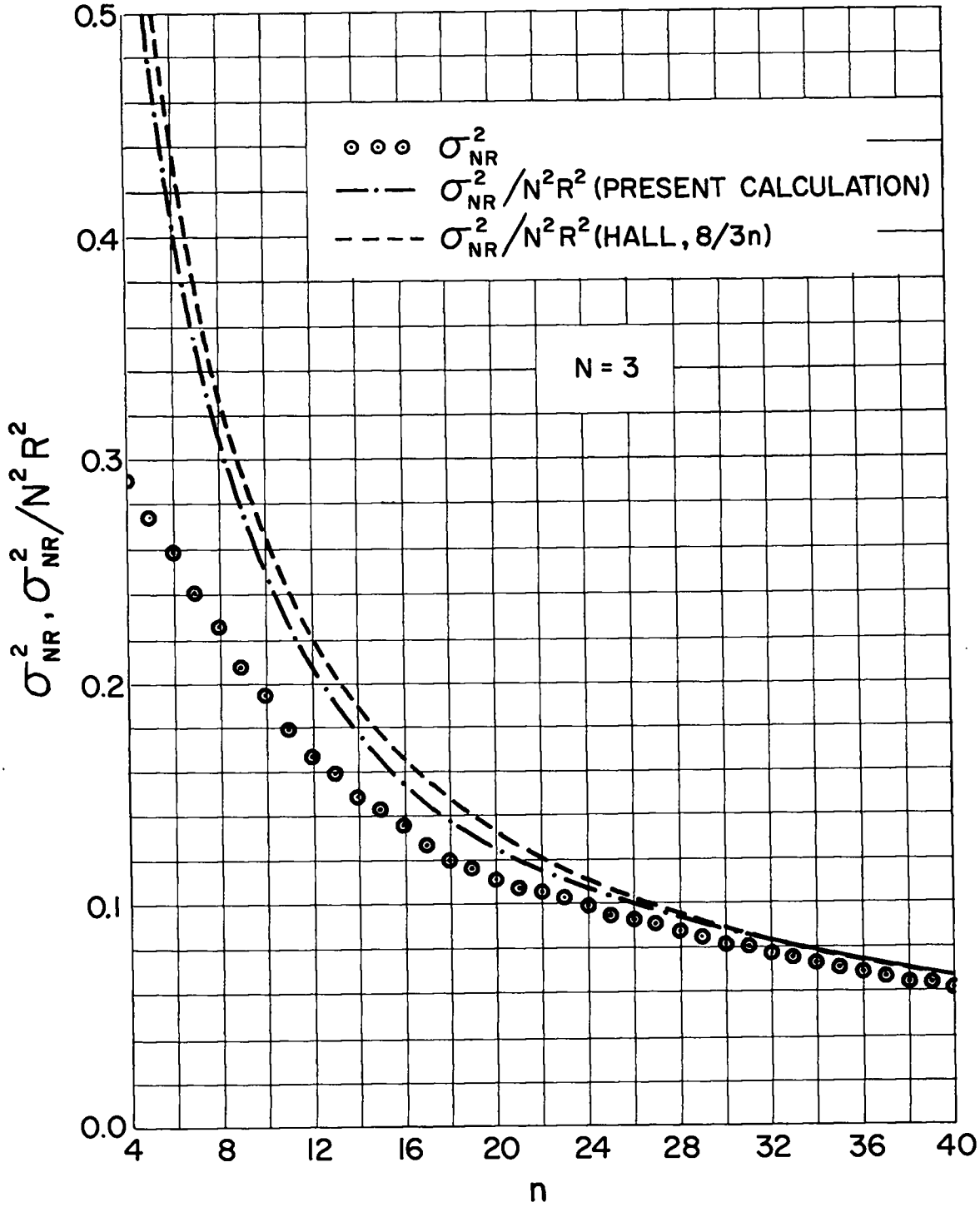

0.5
0.4
0.3
0.2
0.1
0.0
4
8
12
16
20
24
28
32
36
40
n
σ^2_{NR}, σ^2_{NR}/N^2R^2
σ^2_{NR}
σ^2_{NR}/N^2R^2 (PRESENT CALCULATION)
σ^2_{NR}/N^2R^2 (HALL, 8/3n)
N = 3

Figure 7.

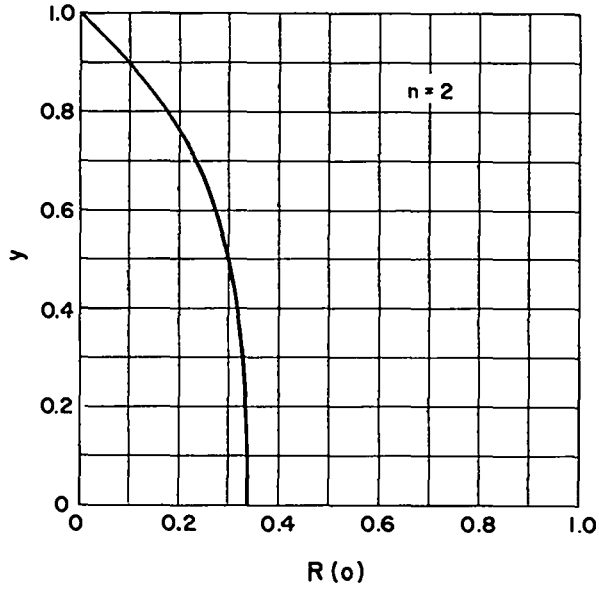


Figure 8.

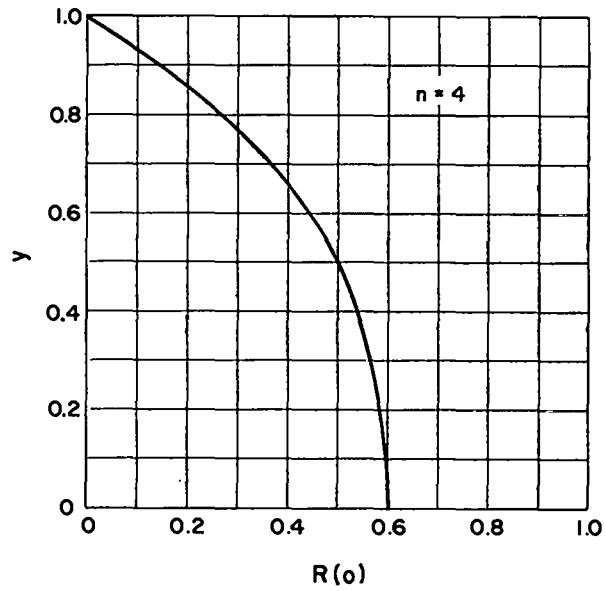


Figure 9.

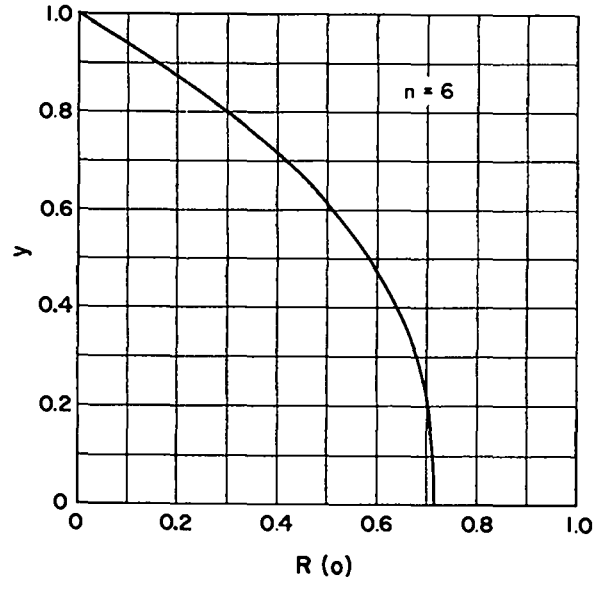


Figure 10.

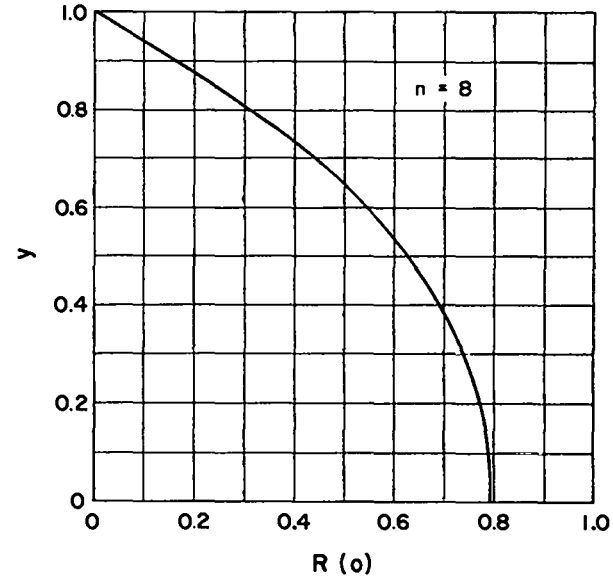


Figure 11.

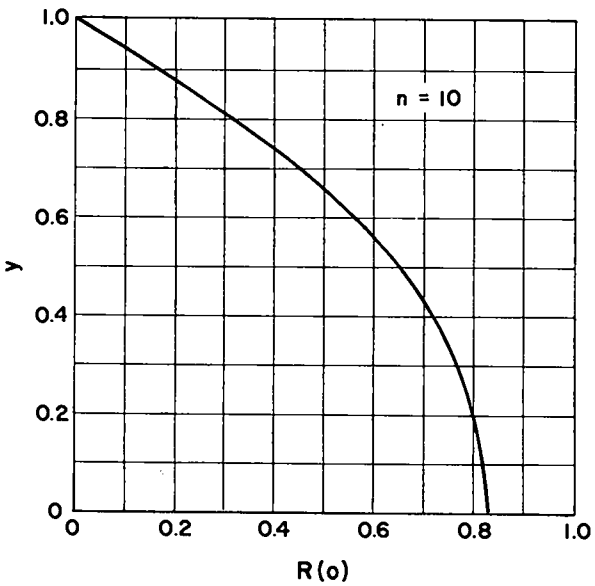


Figure 12.

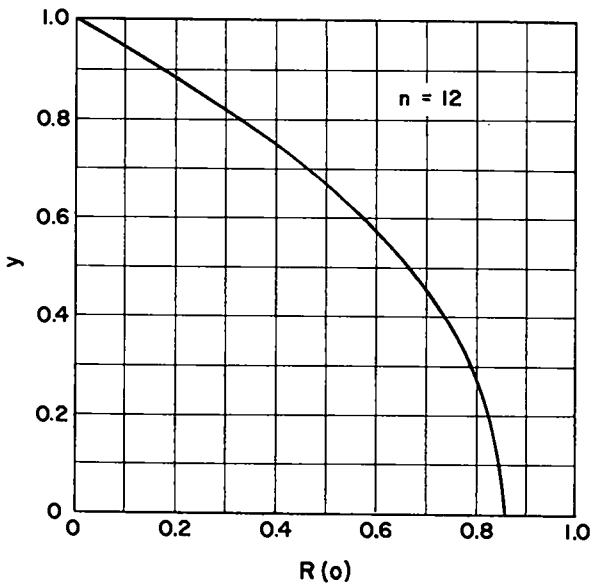


Figure 13.

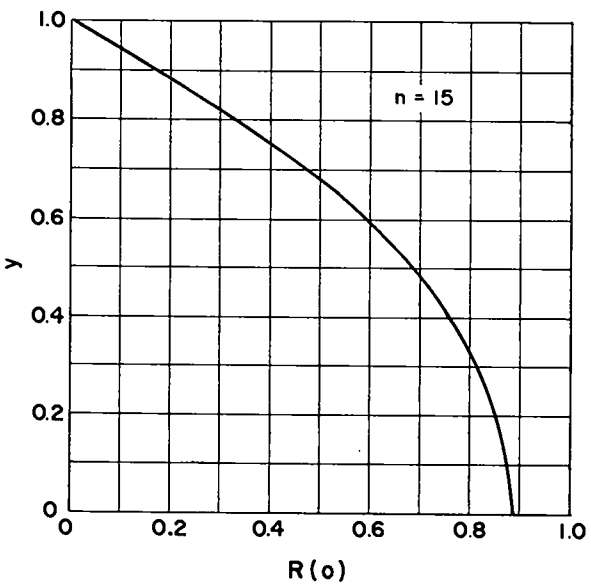


Figure 14.

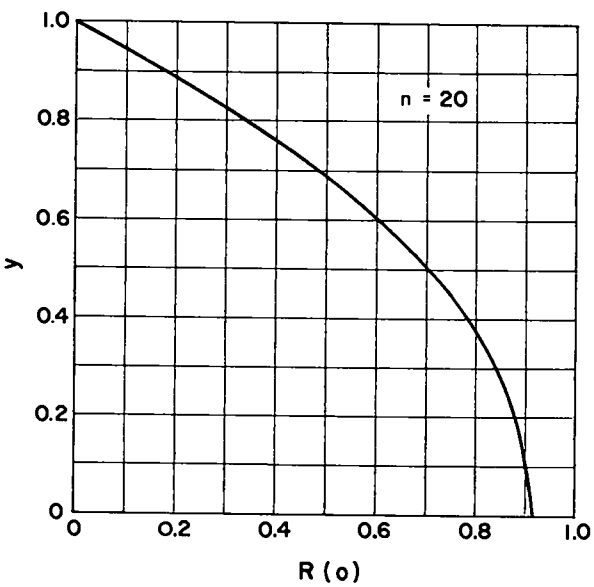


Figure 15.

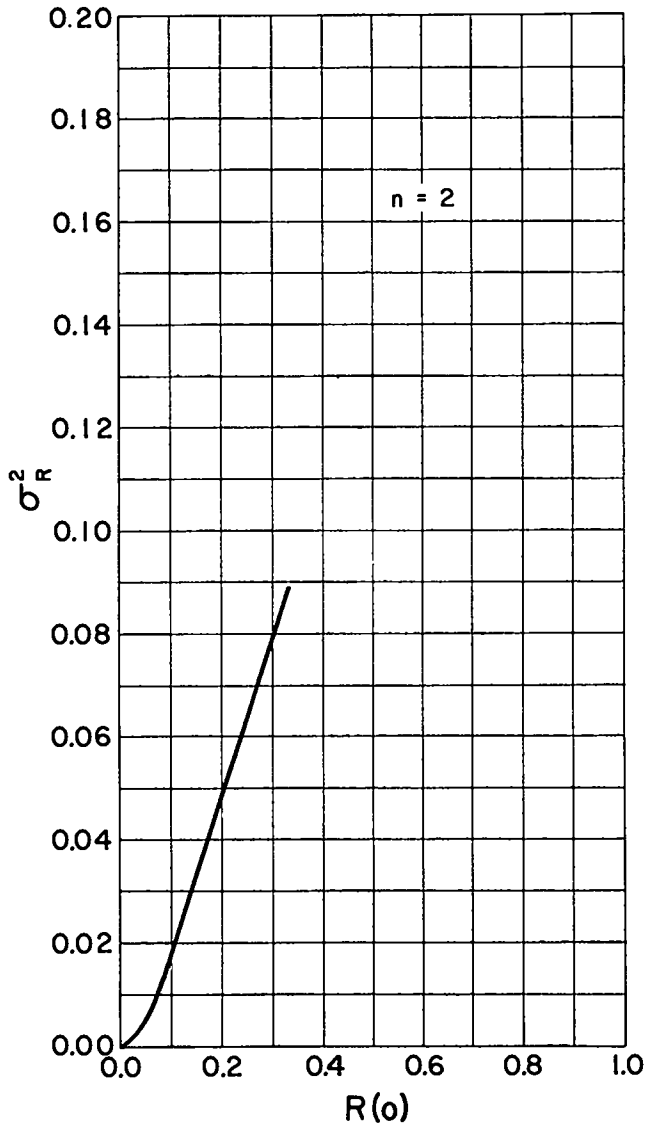


Figure 16.

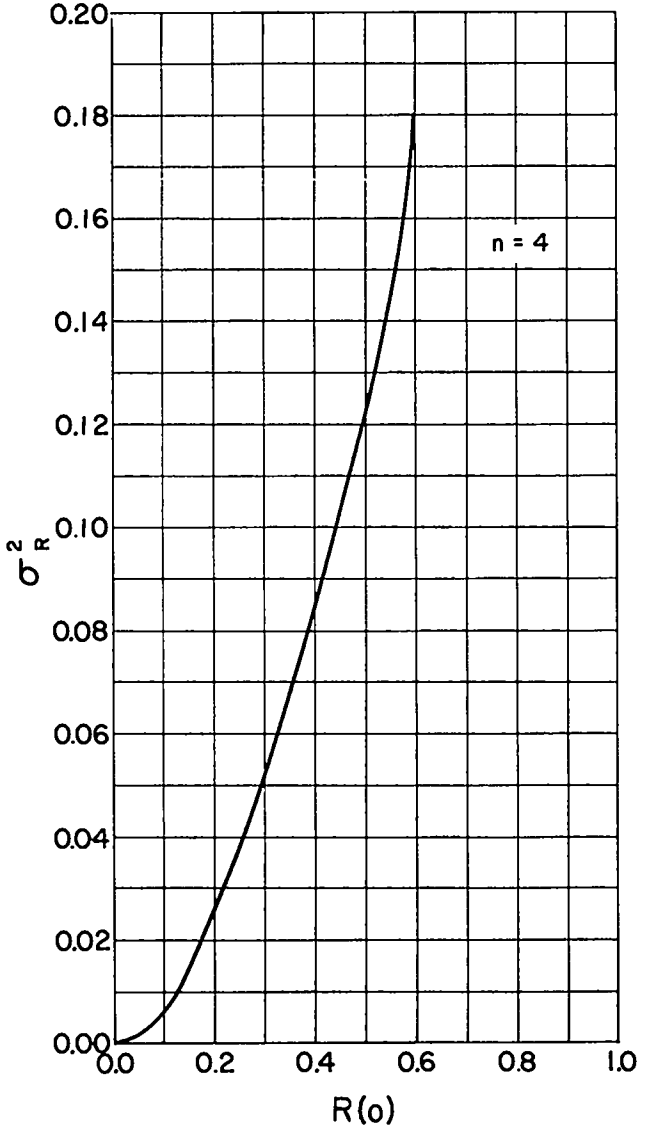


Figure 17.

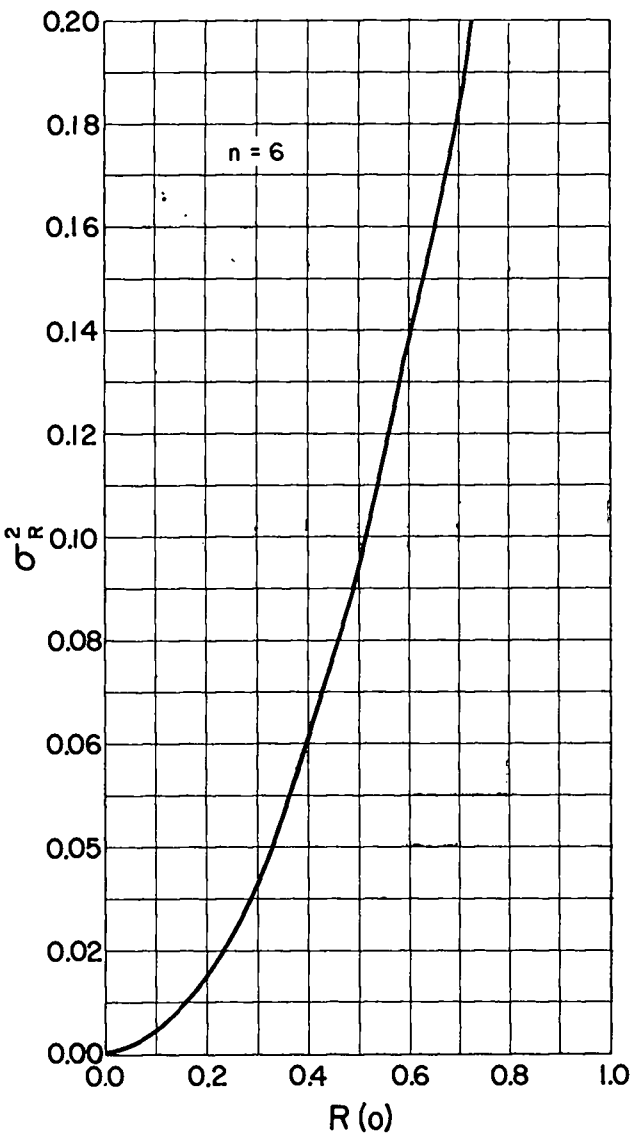


Figure 18.

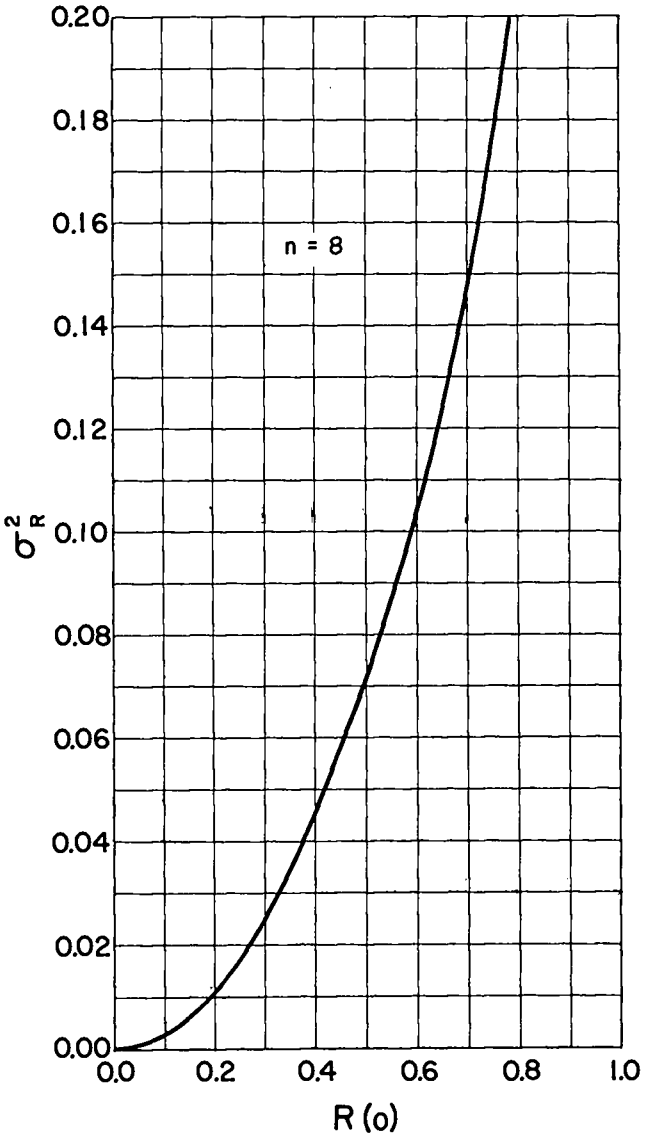


Figure 19.

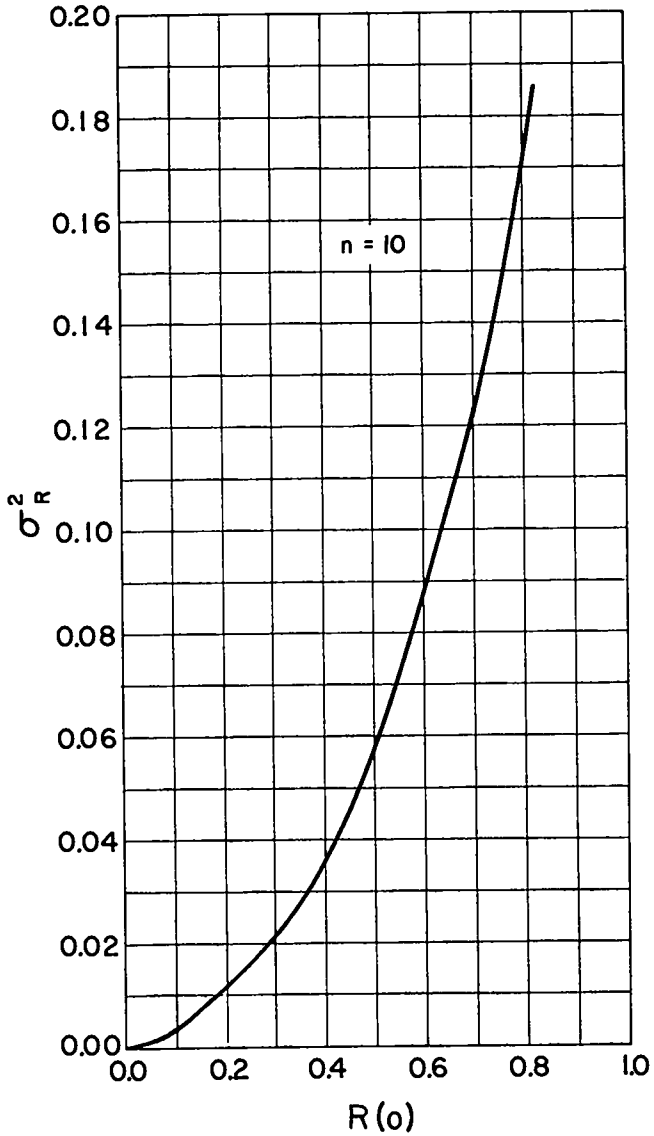


Figure 20.

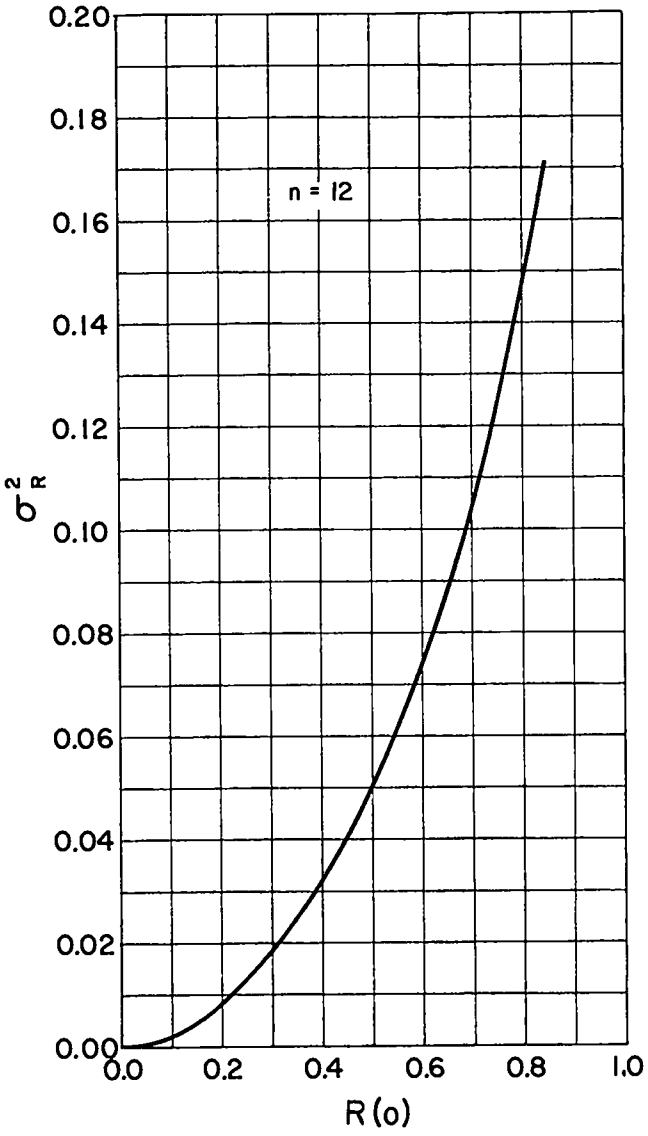


Figure 21.

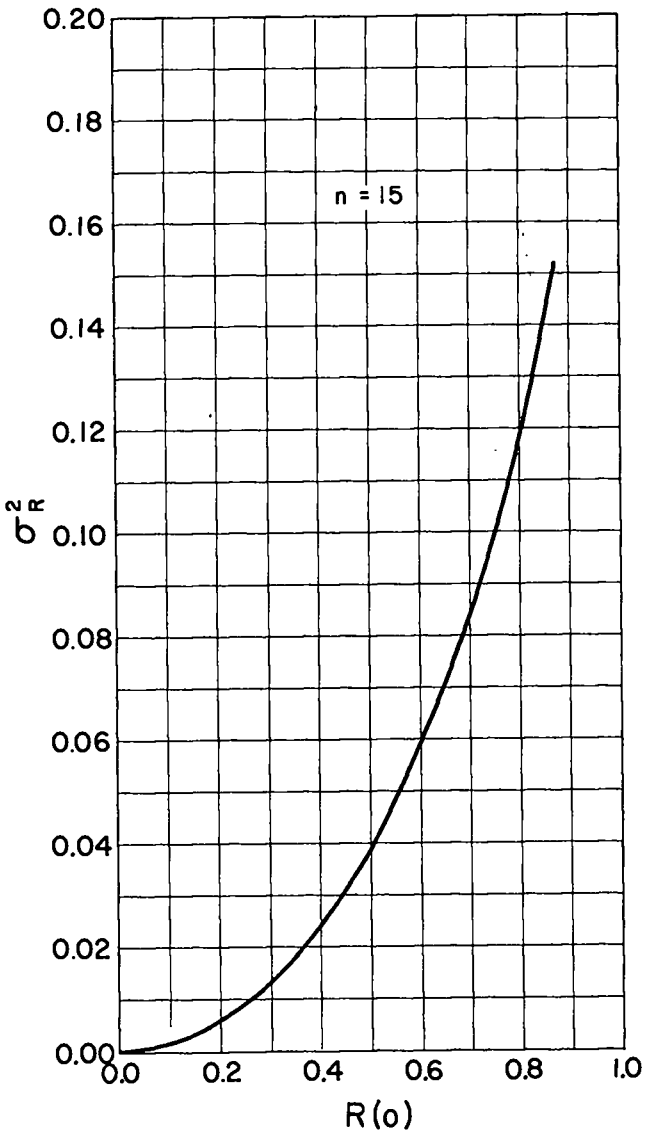


Figure 22.

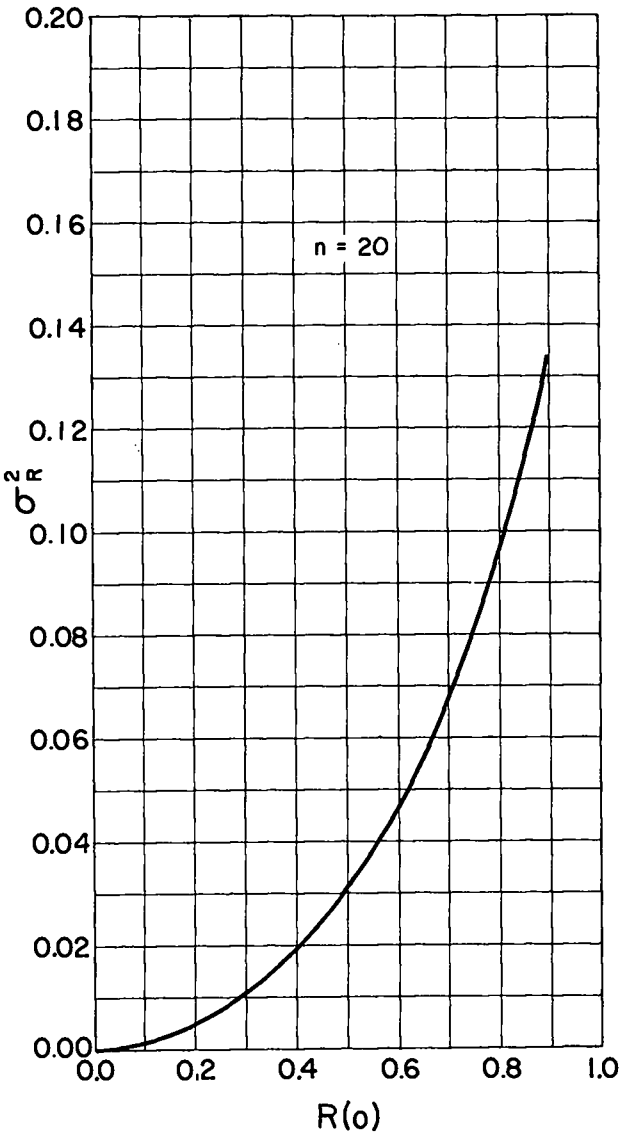


Figure 23.

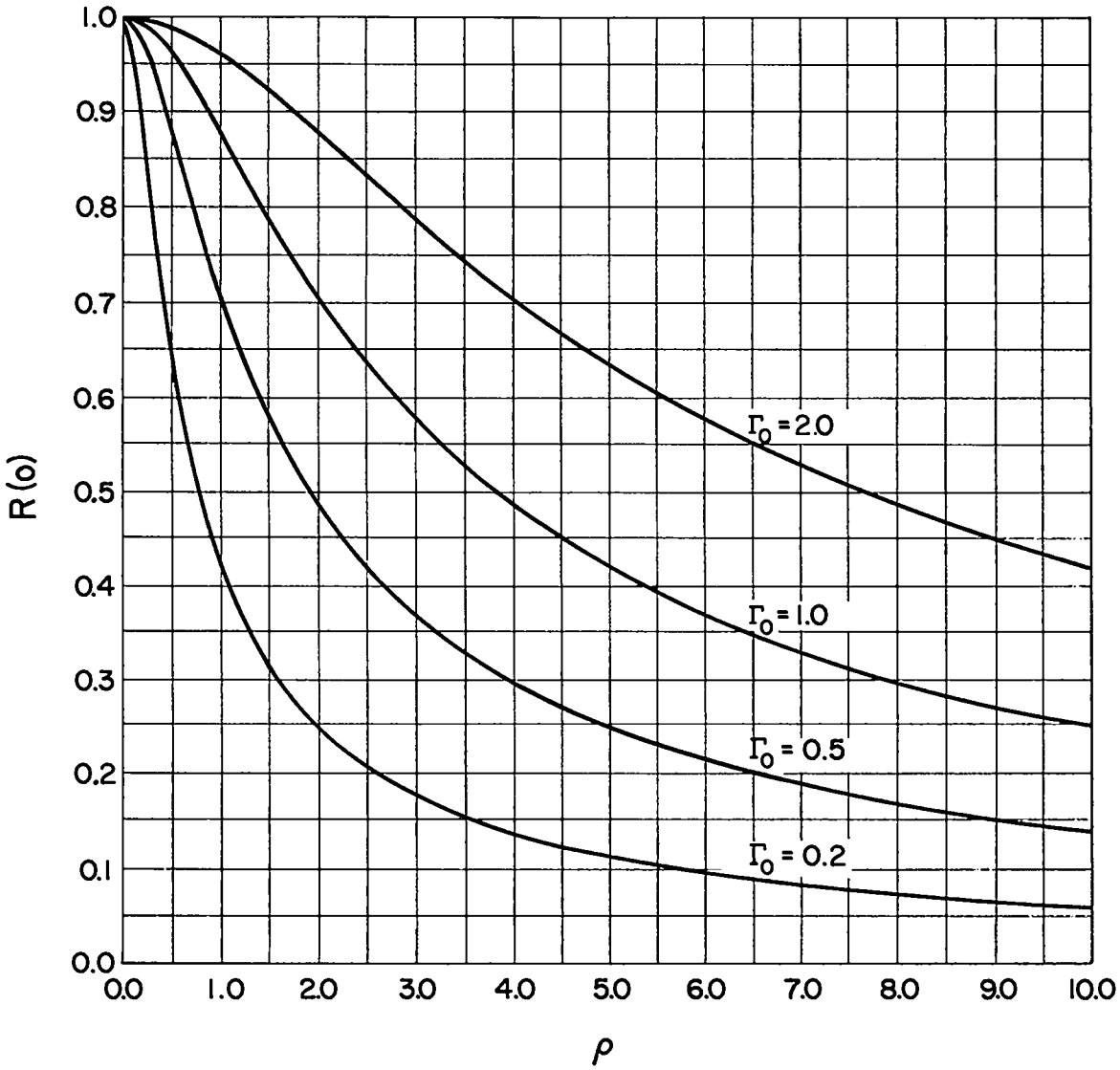

R(0)
ρ
Γ0 = 2.0
Γ0 = 1.0
Γ0 = 0.5
Γ0 = 0.2
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0

Figure 24.

References

1. P. A. Moldauer, Phys. Letters, 8, 70 (1964).

2. T. Ericson, Ann. Phys. (N. Y.), 23, 390 (1963).

   T. Ericson, Phys. Letters, 4, 258 (1963).

   T. Ericson, Phys. Rev. Letters, 5, 430 (1960).

   T. Ericson, Advan. Phys., 9, 425 (1960).

   T. Ericson, Varenna Lectures, 1961, Course 23, Academic Press, New York, 1963.

3. D. M. Brink and R. O. Stephen, Phys. Letters, 5, 77 (1963).

   Brink, Stephen, and Tanner, Nucl. Phys., 54, 577 (1964).

4. J. Humblet and L. Rosenfeld, Nucl. Phys., 26, 529 (1961).

5. M. Böhning, International Conference on Nuclear Physics, Paris, 1964. Contribution C158.

6. I. Hall, Phys. Letters, 10, 199 (1964).