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Time Reversal in Polarization Phenomena of Nuclear Interactions

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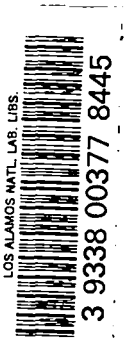
by

P. W. Keaton, Jr.



TIME REVERSAL IN POLARIZATION PHENOMENA OF NUCLEAR INTERACTIONS

by

P. W. Keaton, Jr.

ABSTRACT

The concept of "time reversal invariance" is reviewed along with some basic properties of the time reversal operator, T. Applications to specific nuclear physics problems are given which demonstrate methods of manipulating T in a calculation. Using the density matrix formalism, a relation is derived between polarization transfer (triple scattering) experiments that describe reactions and inverse reactions with initially polarized particles. The fundamental relationships are illustrated by examples. Relativistic effects and photon polarizations are not discussed.

I. INTRODUCTION

The formal properties of a time reversal operator were introduced in 1932 by Wigner[1] (see also Ref. 2). Discussions of various aspects of the time reversal operation in nuclear physics have been given by Goldberger and Watson,[3] Messiah,[4] Rodberg and Thaler,[5] and Bohr and Mottelson.[6] In an effort to make this report self-contained, the first part (which borrows heavily from the above mentioned literature) introduces basic concepts of the time reversal operation. The last part of this report (Sec. IV and Appendix B) contains derivations with a different emphasis on operator technique than other treatments of this subject. The list of references is not intended to be exhaustive. However, of particular relevance to the material presented here is work by Csonka and Moravcsik.[7]

Recent articles reviewing the present status of the time reversal invariance question have been published.[8,9]

II. A REVIEW OF THE TIME REVERSAL OPERATOR AND SOME BASIC PROPERTIES

A. The Time Reversal Operation

The time reversal operator, T, takes a state Ψ into a state $\overline{\Psi}$ in which all velocities, including "spinning" particles, are reversed. (Perhaps "velocity reversal" would have been a better name than "time reversal.") A system is said to be invariant under time reversal if the following series of operations result in returning that system to its original form: Take an isolated system at $t = 0$ and allow it to change according to the physical laws which govern it for a time t_o. At time t_o reverse the directions of all velocities. Allow the system to change according to the physical laws which govern it from time t_o until time $2t_o$. At time $2t_o$ reverse the directions of all velocities. If the system is now in exactly the same form as it was at $t = 0$, the physical laws which govern it are said to be invariant under time reversal.

The sequence is:

$$\begin{pmatrix}\text{time}\\ \text{reversal}\end{pmatrix} \times \begin{pmatrix}\text{time}\\ \text{displacement}\\ \text{by } t_o\end{pmatrix} \times \begin{pmatrix}\text{time}\\ \text{reversal}\end{pmatrix} \times \begin{pmatrix}\text{displacement}\\ \text{by } t_o\end{pmatrix} . \qquad (1)$$

If the system is invariant under this operation, it is then equal to the identity operator. Since H (the Hamiltonian) is the generator of time displacements, this invariance can be expressed as*

$$T\, e^{-iHt_o}\, T\, e^{-iHt_o} = 1 . \qquad (2)$$

Assuming the system to be invariant under time reversal, the sequence of operations in Eq. (1) are set equal to the identity operator. Applying inverse operators from the right, this leads to a second expression of a system which is invariant under time reversal:

$$\begin{pmatrix}\text{time}\\ \text{inversion}\end{pmatrix} \times \begin{pmatrix}\text{time}\\ \text{displacement}\\ \text{by } t_o\end{pmatrix} = \begin{pmatrix}\text{time}\\ \text{displacement}\\ \text{by } -t_o\end{pmatrix} \times \begin{pmatrix}\text{time}\\ \text{inversion}\end{pmatrix} , \qquad (3)$$

or the same type of operations on Eq. (2) leads to

$$T\, e^{-iHt_o} = e^{iHt_o}\, T . \qquad (4)$$

One observes that the concept of velocity reversal has led to time reversal. The operator e^{+iHt_o} on the right-hand side of Eq. (4) is an instruction to allow time to run backwards for a time t_o. It should be kept in mind that the fundamental sequence is expressed in Eqs. (1) and (2), where time always runs forward.

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*It will always be assumed here that $\hbar = 1$.

B. Antilinear Property of T

If a system at t = 0 is described by $\Psi(\vec{r},0)$, the test of Eq. (4) would require (t, of course, commutes with H)

$$T\, e^{-itH}\, \Psi(\vec{r},0) = e^{iHt}\, T\, \Psi(\vec{r},0) \ , \tag{5}$$

but (see Appendix A)

$$T\, e^{-itH}\, \Psi(\vec{r},0) = e^{-T(it)T^{-1}THT^{-1}}\, T\, \Psi(\vec{r},0) \ . \tag{6}$$

Now Eq. (4) in Eqs. (5) and (6) requires that either T be linear,

$$T(it)T^{-1} = it \qquad \text{and} \qquad THT^{-1} = -H \ , \tag{7}$$

or that T be antilinear,*

$$T(it)T^{-1} = -it \qquad \text{and} \qquad THT^{-1} = H \ . \tag{8}$$

We can choose between them by noticing that, from the point of view of time reversal, there are two important classes of physical quantities. The position coordinates, the total energy, and the kinetic energy belong to the first class which is either unrelated to time or contains an even power of the time variable. The velocity, linear and angular momentum, and components of spin in a given direction belong to the second class and contain odd powers of the time variable.

Returning to Eqs. (7) and (8), the Hamiltonian, H, of a system which is invariant under time reversal cannot contain a mixture of class I and class II, because the "either...or" relation would be destroyed. The conclusion is that since H contains some variables from the first class, it must contain

---

*An antilinear operator, A, is one with the property

$$A(\lambda_1\Psi_1 + \lambda_2\Psi_2) = \lambda_1^*(A\Psi_1) + \lambda_2^*(A\Psi_2) \ ,$$

where λ_1 and λ_2 are scalars.

only variables from the first class. That is, necessary and sufficient conditions that an isolated system be invariant under the time reversal operations expressed in Eq. (4) are that

$$THT^{-1} = H \ , \tag{9}$$

and that T be an antilinear operator.

C. Applications of T to the Schrödinger Equation

The time-dependent Schrödinger equation for the state $\Psi(\vec{r},t)$ of the Hamiltonian, H, is

$$i \frac{\partial}{\partial t} \Psi(\vec{r},t) = H \Psi(\vec{r},t) \ . \tag{10}$$

Operating from the left with the time reversal operator T, Eq. (10) becomes

$$T(i \frac{\partial}{\partial t}) T^{-1} T \Psi(\vec{r},t) = THT^{-1} T \Psi(\vec{r},t) \ . \tag{11}$$

Assuming that the system is invariant under time reversal, Eq. (9) concludes that the correct set of properties of T are expressed in Eq. (8), namely that $THT^{-1} = H$, and T is antilinear. Using Eq. (8) in Eq. (11),

$$-i \frac{\partial}{\partial t} [T \Psi(\vec{r},t)] = H[T \Psi(\vec{r},t)] \ . \tag{12}$$

Equation (12) is not the Schrödinger equation (S-E) because of the minus sign. However, the S-E can be regained by replacing t with -t.

$$i \frac{\partial}{\partial t} [T \Psi(\vec{r},-t)] = H[T \Psi(\vec{r},-t)] \ . \tag{13}$$

Equation (13) defines a new state $\overline{\Psi}$, where

$$\overline{\Psi}(\vec{r},t) = T \Psi(\vec{r},-t) \ . \tag{14}$$

Referring to Eqs. (10) and (13), one can make the statement: If $\Psi(\vec{r},t)$ is a solution of the S-E, then $\overline{\Psi}(\vec{r},t) = T \Psi(\vec{r},-t)$ is also a solution of the S-E. A reversal of motion is connected with the transformation $t \rightarrow -t$ through

the fact that observables which are odd in time change their sign and observables which are even in time do not.

D. The Complex Conjugate Operator, K

The complex conjugate operator, K, is simply an instruction to take the complex conjugate of any number or function to the right of it. The K operation is clearly antilinear since by its definition

$$K(\lambda_1\psi_1 + \lambda_2\psi_2) = \lambda_1^*(K\psi_1) + \lambda_2^*(K\psi_2) \ . \tag{15}$$

Other important properties of K are

$$K^2 = 1 \ , \qquad K = K^{-1} \ , \qquad K^* = KKK^{-1} = KKK = K \ . \tag{16}$$

The definition for the adjoint of a linear operator, B, is

$$(B\psi,\phi) = (\psi,B^\dagger\phi) \ . \tag{17}$$

The definition for the adjoint of an antilinear operator, A, is

$$(A\psi,\phi) = (\psi,A^\dagger\phi)^* \ . \tag{18}$$

It will serve as a useful example to calculate $K^\dagger$.

$$(K\psi,\phi) = (\psi^*,\phi) = (\psi^*,K\phi^*) \ . \tag{19}$$

Since $K^* = K$ from Eq. (16), we find that Eq. (19) becomes

$$(K\psi,\phi) = (\psi,K\phi)^* = (\psi,K^\dagger\phi)^* \ , \tag{20}$$

where the last equality is an application of the definition, Eq. (18). Combining Eqs. (16) and (20), we find

$$K = K^\dagger = K^{-1} \ . \tag{21}$$

An operator, A, which is antilinear, and which also satisfies $A^\dagger A = 1$, is said to be antiunitary. Clearly, K is an antiunitary operator, with the important properties summarized below.

$$K(\lambda_1\Psi_1 + \lambda_2\Psi_2) = \lambda_1^*(K\Psi_1) + \lambda_2^*(K\Psi_2)$$

$$(K\Psi,\phi) = (\Psi,K^\dagger\phi)^* \tag{22}$$

$$K^2 = 1 , \quad K = K^{-1} , \quad K^* = K , \quad K = K^\dagger , \quad K^\dagger K = 1 .$$

E. The Unitary Operator, U

It has been demonstrated above that $THT^{-1} = H$ leads to the original form of the Schrödinger equation for isolated systems which are invariant under time reversal. Here a different approach will be taken. Operating from the left on the S-E with the complex conjugate operator, K, Eq. (10) becomes

$$-i \frac{\partial}{\partial t} \Psi^*(\vec{r},t) = H^* \Psi^*(\vec{r},t) . \tag{23}$$

If a unitary operator, U, can be found such that $UH^*U^{-1} = H$, Eq. (23) could be written

$$+i \frac{\partial}{\partial t} U \Psi^*(\vec{r},-t) = HU \Psi^*(\vec{r},-t) , \tag{24}$$

where the transformation $t \to -t$ was also required to regain the S-E. The mathematical rigor necessary to prove the existence and uniqueness of U (cf. Ref. 4) is inappropriate in this treatment. We simply proceed with the assumption that a unique unitary operator exists which satisfies Eq. (24). Rewriting Eq. (24) with the use of K,

$$i \frac{\partial}{\partial t} [UK \Psi(\vec{r},-t)] = H[UK \Psi(\vec{r},-t)] . \tag{25}$$

Comparison of Eq. (25) with Eq. (13) shows that the time reversal operator, T, can be written

$$T = UK \qquad \text{or} \qquad U = TK , \tag{26}$$

where

$$U^\dagger U = 1 .$$

It follows from Eqs. (26) and (22) that

$$T^\dagger T = (UK)^\dagger (UK) = K^\dagger U^\dagger UK = K^\dagger K = 1 \ . \tag{27}$$

Therefore, since T is antilinear, it is also antiunitary. Equation (26) is the consequence of a more general theorem that every antiunitary operator can be written as the product of a unitary operator and the complex conjugate operator, K.

F. The Value of T^2

Because $T^2 \ \psi(\vec{r},t)$ is also a solution to the S-E, it follows that $T^2 \ \psi(\vec{r},t) = C \ \psi(\vec{r},t)$ where C is a constant. However, since $T^\dagger T = 1$ from Eq. (27), it follows that

$$\begin{aligned} (\psi,\psi) &= (\psi, T^\dagger T\psi) = (T\psi, T\psi)^* \\ &= (T\psi, T^\dagger TT\psi)^* = (T^2\psi, T^2\psi) \\ &= |C|^2 \ (\psi,\psi) \ , \end{aligned} \tag{28}$$

and we find $|C| = 1$. Consider the identity

$$T^3 \ \psi(\vec{r},-t) - T^3 \ \psi(\vec{r},-t) = 0 \ . \tag{29}$$

Equation (29) can be written

$$\begin{aligned} T^2(T\psi) - T(T^2\psi) &= (CT - TC)\psi \\ &= [C-C^*] \ (T\psi) = 0 \end{aligned}$$

$$C-C^* = 0 \qquad \text{and} \qquad |C| = 1 \qquad \text{imply} \qquad C = \pm 1 \ .$$

Therefore, the double application of the time reversal operator results only in either a change in sign or no change at all. For a wave function, ψ, we have

$$T^2\psi = \pm\psi \ . \tag{30}$$

It will turn out that systems with integral spin have $T^2 = +1$ and systems with half integral spin have $T^2 = -1$ (see Eq. (63)).

G. Transformation of an Operator Under T

We now proceed to find how an operator, $\underset{\sim}{Q}$, transforms under time reversal.* Although no time-dependent operators will be used, it will cause no undue difficulty to include this class. The expectation value of $\underset{\sim}{Q}(t)$ is defined as

$$\langle \underset{\sim}{Q}(t) \rangle = (\Psi(\vec{r},t),\ \underset{\sim}{Q}(t)\ \Psi(\vec{r},t)) \ . \tag{31}$$

The expectation value for $\underset{\sim}{Q}(t)$ in the time-reversed system $\overline{\Psi}(\vec{r},t) = T\ \Psi(\vec{r},-t)$ is likewise defined as

$$\langle \overline{\underset{\sim}{Q}}(t) \rangle = (\overline{\Psi}(\vec{r},t),\ \overline{\underset{\sim}{Q}}(t)\ \overline{\Psi}(\vec{r},t)) \ . \tag{32}$$

A relation can be found between $\underset{\sim}{Q}$ and $\overline{\underset{\sim}{Q}}$ by writing

$$\begin{aligned} \langle \overline{\underset{\sim}{Q}}(t) \rangle &= (T\ \Psi(\vec{r},-t),\ \overline{\underset{\sim}{Q}}(t)\ T\ \Psi(\vec{r},-t)) \\ &= (\Psi(\vec{r},-t),\ (T^{\dagger}\ \overline{\underset{\sim}{Q}}(t)\ T)\ \Psi(\vec{r},-t))* \\ &= ((T^{\dagger}\ \overline{\underset{\sim}{Q}}(t)\ T)\ \Psi(\vec{r},-t),\ \Psi(\vec{r},-t)) \\ &= (\Psi(\vec{r},-t),\ (T^{\dagger}\ \overline{\underset{\sim}{Q}}^{\dagger}(t)\ T)\ \Psi(\vec{r},-t)) \ . \end{aligned} \tag{33}$$

One notices that a relation between Eqs. (31) and (32) will make sense only if $t \to -t$. Since $\langle \underset{\sim}{Q}(t) \rangle$ is measured in a system with time running "forward" and $\langle \overline{\underset{\sim}{Q}}(t) \rangle$ is measured in a system with time running "backward," we require that a system which is invariant under time reversal obey the relationship

$$\langle \underset{\sim}{Q}(t) \rangle \equiv \langle \overline{\underset{\sim}{Q}}(-t) \rangle \ . \tag{34}$$

Substituting from Eqs. (31) and (33), Eq. (34) requires that

$$(\Psi(\vec{r},t),\ \underset{\sim}{Q}(t)\ \Psi(\vec{r},t)) = (\Psi(\vec{r},t),\ [T^{\dagger}\ \overline{\underset{\sim}{Q}}^{\dagger}(-t)\ T]\ \Psi(\vec{r},t)) \ . \tag{35}$$

---

*With the exception of T, U, K, H, and ρ (density matrix), all operators will be written with wavy lines under them. The expectation value for an operator, $\underset{\sim}{Q}$, will be written interchangeably as Q and $\langle \underset{\sim}{Q} \rangle$.

That is

$$\underset{\sim}{Q}(t) = T^{\dagger}\, \overline{\underset{\sim}{Q}}^{\dagger}(-t)\, T \tag{36}$$

or

$$\overline{\underset{\sim}{Q}}(t) = T\, \underset{\sim}{Q}^{\dagger}(-t)\, T^{\dagger} , \tag{37}$$

which are the rules for transforming an operator from a system to its time reversed system, and vice versa.

Equation (34) preserves the usual meaning of coordinate transformations which is, for the case of no explicit time dependence,

$$(\Psi, Q\phi) \equiv (\overline{\Psi}, \overline{Q\phi}) . \tag{38}$$

It is not uncommon to find the requirement $\langle Q \rangle = \langle \overline{Q} \rangle^*$ which preserves the usual meaning of operator transformation

$$\overline{Q} = TQT^{-1} , \tag{39}$$

rather than the somewhat more awkward Eq. (37). However, this results in the awkward coordinate transformation property

$$(\Psi, Q\phi) = (\overline{\Psi}, \overline{Q\phi})^* .$$

The reason both Eqs. (38) and (39) cannot be simultaneously preserved is, of course, due to the antiunitarity of T.

III. EXAMPLES OF TIME REVERSAL CALCULATIONS

A. Classical Mechanics

As is often pointed out, $\vec{F} = m \frac{d^2\vec{r}}{dt^2}$ is an equation even in time and, therefore, the solar system could run "backward" as well as "forward." This serves as a good example to apply time reversal.

Figure 1 pictures the orbit of a planet under the influence of a gravitational force. The coordinate system is arbitrarily chosen with the origin at the center of the orbit, the z-axis along $\vec{L} = \vec{r} \times \vec{p}$, the x-axis along $\vec{r}$,

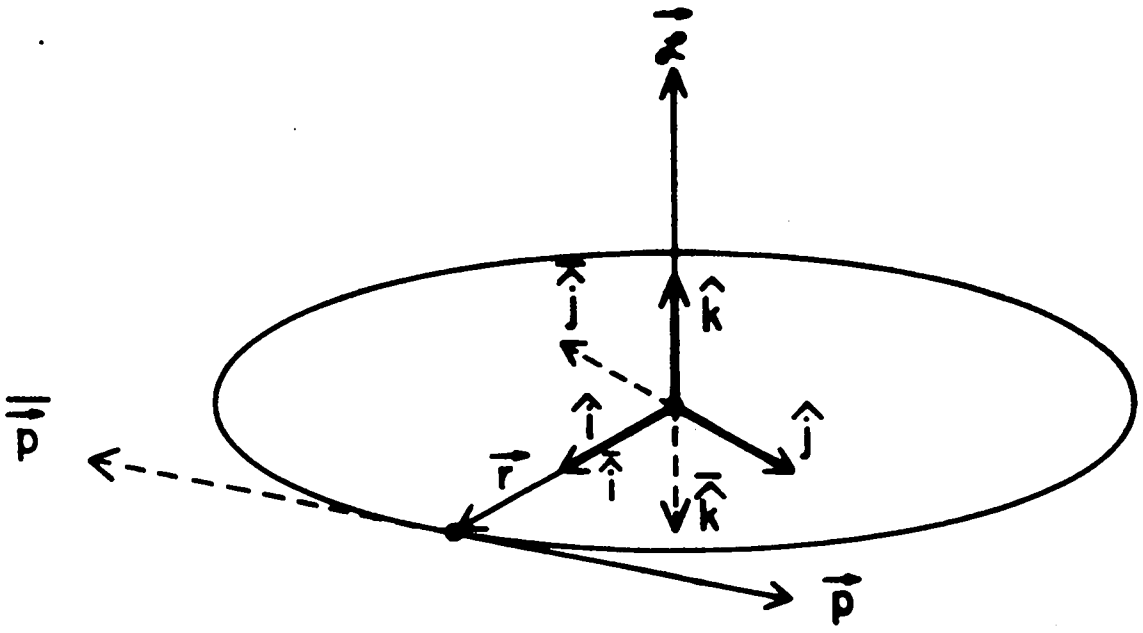


Fig. 1. The orbit of a planet under the influence of a central force. The dotted arrows indicate the effect of reversing the motion of the planet.

and the y-axis chosen to make the coordinate system right-handed. The unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ are along the x-, y-, z-axes, respectively. Because of the way in which the coordinate system is set up, $\vec{L}\cdot\hat{k} = L$. To see how $\vec{L}$ changes under time reversal, one need only note that T reverses all motions. Therefore, $\vec{p} \to -\vec{p}$ but $\vec{r} \to +\vec{r}$, since the position coordinate contains no time or motion. Therefore, we say (from Eq. (37))

$$\begin{aligned} \overline{\vec{r}} &= T\,\vec{r}^{\dagger}\,T^{\dagger} = \vec{r} \\ \overline{\vec{p}} &= T\,\vec{p}^{\dagger}\,T^{\dagger} = -\vec{p} \\ \overline{\vec{L}} &= T(\vec{r}\times\vec{p})^{\dagger}\,T^{\dagger} = -\vec{L}\ , \end{aligned} \tag{40}$$

where, of course, writing $\vec{r}^{\dagger}$, $\vec{p}^{\dagger}$, etc., is merely a formality at this point because in classical mechanics these quantities are all real. In fact, in every case in Eq. (40), the second step was a formality since the T was not used to calculate anything; the answers which we required were simply placed on the right-hand side.

The equation $\overline{\vec{L}} = -\vec{L}$ is a statement of what would happen to $\vec{L}$ if all motions were reversed--but in reference to the original coordinate system, i.e., $\overline{\vec{L}} = -L\hat{k}$. If we had "lived" in a time-reversed system in the first place, we would have set up the coordinate system with the z-axis along $\overline{\vec{r}}\times\overline{\vec{p}}$ and labeled that direction $\overline{\hat{k}}$, etc. Observe that since $\overline{\hat{k}} = -\hat{k}$, one finds

$$\overline{\vec{L}}\cdot\overline{\hat{k}} = (-\vec{L})\cdot(-\hat{k}) = \vec{L}\cdot\hat{K} \equiv L\ . \tag{41}$$

One cannot, then, observe the orbit of his planet to find which system he lives in. This, of course, is simply another way of saying that the system is invariant under the time reversal operations.

B. Quantum Mechanical Observables

A complication arises in calculating the effect of T in quantum mechanics, because one often works in different representations. As a result, the effect

of U and K, in the equation T = UK, may be different for each representation. A good example is the momentum, $\vec{p}$. In a coordinate representation,

$$\bar{\vec{p}} = T \vec{p}^{\dagger} T^{\dagger} .$$

But since $\vec{p}$ is an Hermitian operator, $\vec{p} = \vec{p}^{\dagger}$, so that

$$\bar{\vec{p}} = UK \vec{p} K^{\dagger} U^{\dagger} .$$

Now $\vec{p} = -i\vec{\nabla}$, and the complex conjugate operator, K, is sufficient to give the answer; i.e.,

$$\bar{\vec{p}} = UK(-i\vec{\nabla}) K^{\dagger} U^{\dagger} = U(i\vec{\nabla}) U^{\dagger} = i\vec{\nabla} = -\vec{p} .$$

That is, it was not necessary for U to affect $\vec{p}$. On the other hand, in the momentum representation $\vec{p}$ is a real vector. In that case,

$$\bar{\vec{p}} = (U \vec{p} U)_{\text{momentum representation}} ,$$

and one must conjure up an operator U that will take $\vec{p}$ into $-\vec{p}$ (such as a 180° rotation of the coordinate system about an axis perpendicular to $\vec{p}$). One cannot say unambiguously whether an operator is "real" or "imaginary." Although it was important to show this point explicitly, it will not cause further confusion in this report.

To assure that total angular momentum, $\vec{J} = \vec{L} + \vec{S}$, transform in a consistent manner, we require that intrinsic spin transform in the same manner as does orbital angular momentum. Therefore,

$$\begin{aligned} \bar{\vec{L}} &= T \vec{L}^{\dagger} T^{\dagger} = -\vec{L} \\ \bar{\vec{S}} &= T \vec{S}^{\dagger} T^{\dagger} = -\vec{S} \\ \bar{\vec{J}} &= T \vec{J}^{\dagger} T^{\dagger} = -\vec{J} . \end{aligned} \tag{42}$$

Specific operators for transforming angular momentum will serve as a useful example. Consider first the case of spin 1/2 (using the Pauli spin matrices).

$$\underset{\sim}{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \qquad \underset{\sim}{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \qquad \underset{\sim}{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{43}$$

$$\vec{\underset{\sim}{\sigma}}^{\dagger} \equiv \vec{\underset{\sim}{\sigma}} .$$

Notice that K already takes $\underset{\sim}{\sigma}_y \to -\underset{\sim}{\sigma}_y$ since $\underset{\sim}{\sigma}_y$ is complex. The only problem left is to find a U such that

$$U \underset{\sim}{\sigma}_x U^{\dagger} = -\underset{\sim}{\sigma}_x$$

$$U \underset{\sim}{\sigma}_z U^{\dagger} = -\underset{\sim}{\sigma}_z . \tag{44}$$

By direct computation, one can demonstrate that $U = i\sigma_y$ has the required properties. That is,

$$T \vec{\underset{\sim}{\sigma}}^{\dagger} T^{-1} = (i\underset{\sim}{\sigma}_y K) \vec{\underset{\sim}{\sigma}} (K\, i\underset{\sim}{\sigma}_y) = -\vec{\underset{\sim}{\sigma}} . \tag{45}$$

This can be generalized. Angular momentum matrix elements $\langle j'm'|\underset{\sim}{J}_x|jm\rangle$ and $\langle j'm'|\underset{\sim}{J}_z|jm\rangle$ are real numbers. The matrix elements $\langle j'm'|\underset{\sim}{J}_y|jm\rangle$ are pure imaginary numbers. One can therefore use K to take the matrix $\underset{\sim}{J}_y$ into $-\underset{\sim}{J}_y$. The proper effect on $\underset{\sim}{J}_x$ and $\underset{\sim}{J}_z$ can be obtained by a 180° rotation about the y-axis. This is easily stated in terms of Euler angles and "D" matrices (cf. Rose[10]), which rotate the coordinate system.

$$D^j_{m'm}(\alpha,\beta,\gamma) = (jm'|e^{-i\alpha\underset{\sim}{J}_z} e^{-i\beta\underset{\sim}{J}_y} e^{-i\gamma\underset{\sim}{J}_z}|jm)$$

$$= e^{-im'\alpha}(jm'|e^{-i\beta\underset{\sim}{J}_y}|jm)\, e^{-im\gamma}$$

$$= e^{-im'\alpha}\, d^j_{m'm}(\beta)\, e^{-im\gamma} . \tag{46}$$

In particular, a rotation about the y-axis is

$$U_{m'm} = D^j_{m'm}(0,\pi,0) = (jm'|e^{-i\pi\underset{\sim}{J}_y}|jm) = d^j_{m'm}(\pi) . \tag{47}$$

For the special case of $j = 1/2$,

$$U = \underset{\sim}{d}^{1/2}(\pi) = i\underset{\sim}{\sigma}_y , \tag{48}$$

which is consistent with Eq. (45). For a general angular momentum, $\vec{J}$, the time reversal operator, T, can be written

$$T = UK = \underset{\sim}{d}^{j}(\pi)\ K\ . \tag{49}$$

C. Exchange of States

The overlap between two wave function Ψ_1 and Ψ_2 can be transformed in a consistent manner.

$$\begin{aligned}
[\Psi_2(t),\Psi_1(t)] &= [\Psi_2^*(t),\Psi_1^*(t)]^* \\
&= [K\Psi_2(t),U^\dagger UK\Psi_1(t)]^* \\
&= [T\Psi_2(t),T\Psi_1(t)]^* \\
&= [\overline{\Psi}_2(-t),\overline{\Psi}_1(-t)]^* \\
&= [\overline{\Psi}_1(-t),\overline{\Psi}_2(-t)]\ .
\end{aligned} \tag{50}$$

The effect of going from a system to the time-reversed system is to take the initial state $\Psi_1(t)$ into the final state $\overline{\Psi}_1(-t)$ of the time-reversed system, and the final state $\Psi_2(t)$ into the initial state $\overline{\Psi}_2(-t)$.

D. The Phase of Time Reversal States in Angular Momentum

All systems which are treated will be assumed to be invariant under the time reversal operations. That is, only systems with $THT^{-1} = H$ will be considered. No further reference will be made to explicit time dependence. Henceforth, only stationary states and time-independent operators will be considered.

The need to calculate the phase of time-reversed states is perhaps best illustrated by a plane wave. Consider a wave function

$$\chi_k = e^{i\vec{k}\cdot\vec{r}}\ . \tag{51}$$

The prescription for calculating $\overline{\chi}_\alpha = T\chi_\alpha$ has been

$$T\chi_k = UK\chi_k = e^{UK(i\vec{k}\cdot\vec{r})K^\dagger U^\dagger} = e^{-iU(\vec{k}\cdot\vec{r})U^\dagger} = e^{-i\vec{k}\cdot\vec{r}} , \qquad (52)$$

where the complex conjugate operator, K, effectively performed the function $\vec{k} \to -\vec{k}$ ($\vec{k}$, of course, is momentum) and U had no effect in this case. One can, instead, replace each operator in χ_k by the known time-reversed form, i.e., $\overline{\vec{k}} = -\vec{k}$, etc. Although this is adequate to get the form of $\overline{\chi}_k$, there may be a phase, η_k, which multiplies the answer. If this method is used, the wave function is referred to here as $\chi_{\overline{k}}$. Thus, we write

$$\overline{\chi}_k = \eta_k \chi_{\overline{k}} . \qquad (53)$$

In the case of the plane wave above, we know that $\overline{\vec{k}} = -\vec{k}$; therefore,

$$\overline{\chi}_k = \eta_k \chi_{\overline{k}} = \eta_k e^{i\overline{\vec{k}}\cdot\overline{\vec{r}}} = \eta_k e^{-i\vec{k}\cdot\vec{r}} . \qquad (54)$$

Comparing Eqs. (52) and (54), we see that for this case $\eta_k = +1$. Of course, the plane wave was merely used here to make a point. That $\eta_k = +1$ is the result of a more general property of the coordinate representation, utilizing the fact that $T|\vec{r}\rangle = |\vec{r}\rangle$. For example,

$$\chi_k(\vec{r}) = \langle\vec{r}|k\rangle$$

$$\overline{\chi_k(\vec{r})} = \langle T\vec{r}|Tk\rangle$$

$$\chi_{\overline{k}}(\vec{r}) = \langle\vec{r}|Tk\rangle .$$

But since

$$\langle\vec{r}|Tk\rangle = \langle T\vec{r}|Tk\rangle = \langle\vec{r}|T^\dagger Tk\rangle^* = \langle\vec{r}|k\rangle^* ,$$

we find

$$\chi_{\overline{k}}(\vec{r}) = \overline{\chi}_k(\vec{r}) = \chi_k^*(\vec{r}) .$$

Before calculating the phase of general angular momentum, it will be helpful to first calculate the phase for orbital angular momentum in a physical problem. The spatial representation of an energy eigenstate of a scattered

particle in a spinless system can be written

$$\Psi_{k\ell m}(\vec{r}) = N \frac{F_\ell(kr)}{kr} [i^\ell Y_{\ell m}(\theta,\phi)] \tag{55}$$

$$\overline{\Psi} = \Psi^* = N \frac{F_\ell(kr)}{kr} [(-i)^\ell Y^*_{\ell m}(\theta,\phi)] .$$

But $Y^*_{\ell m} = (-1)^m Y_{\ell -m}$, and

$$\overline{\Psi}_{k,\ell,m} = \eta_{k,\ell,m} \Psi_{\overline{k},\overline{\ell},\overline{m}} = (-1)^{\ell+m} \Psi_{k,\ell,-m} . \tag{56}$$

Therefore,

$$\eta_{k,\ell,m} = \eta_{\ell,m} = (-1)^{\ell+m} . \tag{57}$$

Generalizing this to total angular momentum, we first note that

$$\begin{aligned} TJ_z T^{-1} &= -J_z \\ TJ_\pm T^{-1} &= -J_\mp , \end{aligned} \tag{58}$$

where $J_\pm = J_x \pm iJ_y$ are the "step-up" and "step-down" angular momentum operators.[10] Starting with

$$J_z \Psi^M_J = M \Psi^M_J , \tag{59}$$

and applying T from the left,

$$J_z[T\Psi^M_J] = -M[T\Psi^M_J] , \tag{60}$$

which shows that $T\Psi^M_J$ is also an eigenstate of J_z. It is related to Ψ^{-M}_J by a constant

$$T\Psi^M_J = \eta^{-M}_J \Psi^{-M}_J , \tag{61}$$

where $|\eta^{-M}_J| = 1$ because $T^2\Psi^M_J = \pm\Psi^M_J$ (cf. Eq. (30)). The calculation of $TJ_-\Psi^M_J$ will yield the necessary phase relation by first calculating $T[J_-\Psi^M_J]$, and then calculating $[TJ_-]\Psi^M_J = [-J_+T]\Psi^M_J = -J_+[T\Psi^M_J]$.

$$TJ_-\psi_J^M = T(\text{constant})\psi_J^{M-1} = (\text{constant})\eta_J^{-M+1}\,\psi_J^{-M+1} \quad , \tag{62}$$

where (constant) = [(J+M)(J-M+1)] and is just the (real) matrix element for J_-.[10] Next,

$$TJ^-\psi_J^M = -J_+T\psi_J^M = -\eta_J^{-M}\,J_+\psi_J^{-M}$$

$$= -\eta_J^{-M}(\text{constant})\psi_J^{-M+1} \quad . \tag{63}$$

Comparing Eq. (62) and Eq. (63), the relation must be

$$\eta_J^{-M} = -\eta_J^{-M+1}$$

or

$$\eta_J^M = \eta_J(-1)^M \quad .$$

One can choose η_J arbitrarily so long as $(-1)^M$ is evident. To be consistent with the orbital angular momentum, $i^\ell\,Y_\ell^m$, one must choose $\eta_J = (-1)^J$, and

$$T\psi_J^M = (-1)^{J+M}\,\psi_J^{-M}$$

$$\eta_J^M = (-1)^{J+M} \quad . \tag{64}$$

We note that

$$T^2\psi_J^M = \eta_J^M\,\eta_J^{-M}\,\psi_J^M = (-1)^{2J}\,\psi_J^M \quad . \tag{65}$$

As stated after Eq. (30), if J is half-integer, $T^2 = -1$, and if J is integer, $T^2 = +1$.

E. Effects of T Transformations on S Matrix Elements

The incoming and outgoing waves of a nuclear reaction can be written operationally as

$$\psi_i^{(+)} = \Omega^{(+)}\,\phi$$

$$\psi_f^{(-)} = \Omega^{(-)}\,\phi \quad , \tag{66}$$

where

$$\Omega^{(\pm)} = \left[1 + \frac{1}{E-H\pm i\epsilon}\right] \quad ;$$

then

$$K\Omega^{(\pm)}K^{-1} = \Omega^{(\mp)}$$

and

$$T\Psi_i^{(+)} = \eta_f \Psi_f^{(-)}$$

$$T\Psi_f^{(-)} = \eta_i \Psi_i^{(+)} \quad . \tag{67}$$

An S matrix element becomes

$$S_{fi} = (\Psi_f^{(-)}, \Psi_i^{(+)}) = (T\Psi_i^{(+)}, T\Psi_f^{(-)})$$

$$= \eta_i^* \eta_f (\Psi_{\bar{i}}^{(-)}, \Psi_{\bar{f}}^{(+)}) \quad . \tag{68}$$

Since $|\eta| = 1$, one finds

$$|S_{fi}|^2 = |S_{\bar{i}\bar{f}}|^2 \quad . \tag{69}$$

This is a relation between the cross section for a reaction and the cross section for the inverse reaction. An incoming wave has all its motions reversed, making it an outgoing wave, and vice versa. This is the analogue of the statement in classical mechanics that if time is reversed, all systems retrace their paths (see Fig. 2).

More explicitly, if the initial and final states each contain two spins S_1, S_2 and S_1', S_2', respectively, time reversal invariance requires

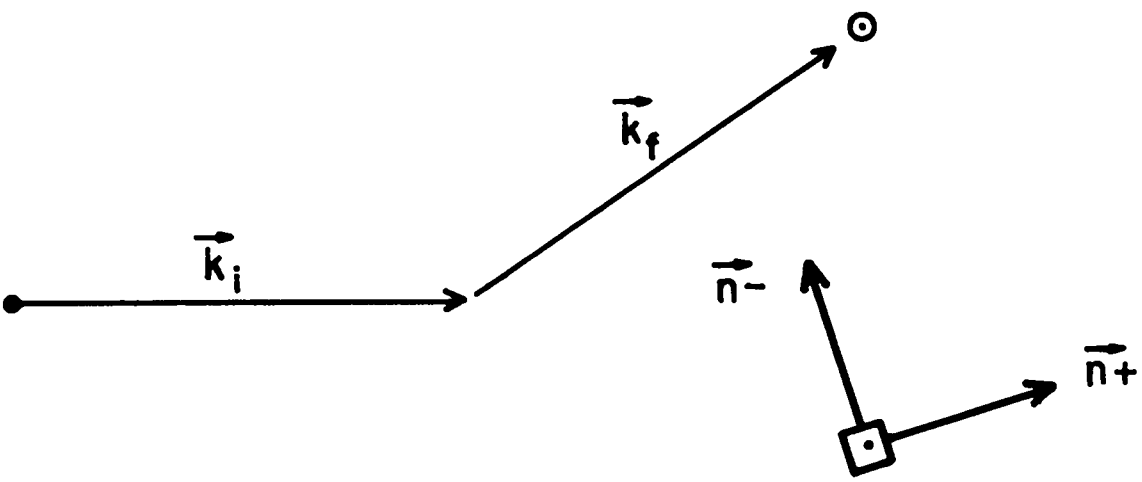
$$\langle \vec{k}_f; S_1', S_2', m_1', m_2'; \beta_f | S | \vec{k}_i; S_1, S_2, m_1, m_2; \beta_i \rangle$$

$$= (-1)^{S_1 + m_1 + S_2 + m_2 + S_1' + m_1' + S_2' + m_2'}$$

$$\times \langle -\vec{k}_i, S_1, S_2, -m_1, -m_2; \beta_i | S | -\vec{k}_f; S_1', S_2', -m_1', -m_2'; \beta_f \rangle \quad , \tag{70}$$

where $\vec{k}_i$ and $\vec{k}_f$ are the initial and final momenta, and β_i and β_f are quantum numbers which are not affected by time reversal.

Finally, if the S matrix elements are written in total angular momentum representation, conservation of angular momentum requires that



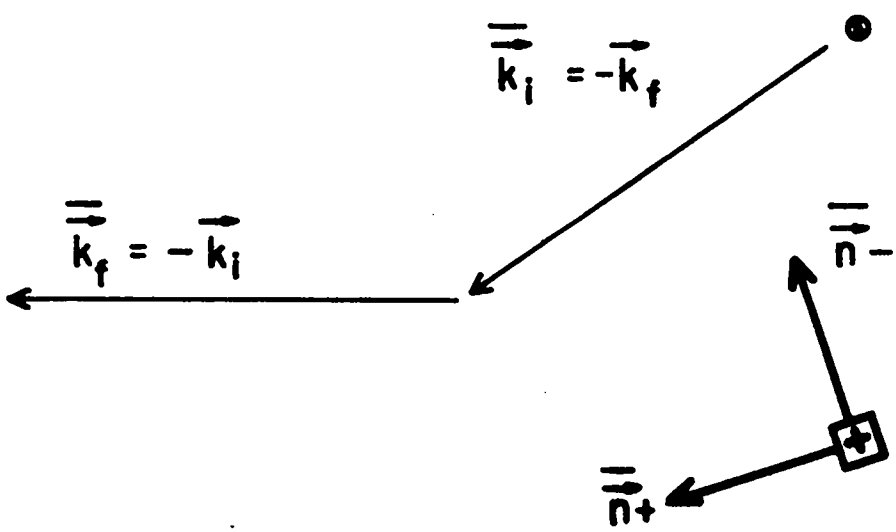


Fig. 2. Top: The momentum vectors of a nuclear reaction producing a polarized particle out of the paper and indicated by ⊙. Bottom: The effect of reversing all motions, including the "spinning" of the particle. Spin pointing into the paper is indicated by ⊗.

$$\langle J_f, M_f, \beta_f | S | J_i, M_i, \beta_i \rangle = \langle \beta_f | S_{J_i} | \beta_i \rangle \, \delta_{J_i, J_f} \, \delta_{M_i, M_f} \, . \tag{71}$$

Time reversal invariance implies

$$\langle J_f, M_f, \beta_f | S | J_i, M_i, \beta_i \rangle = (-1)^{J_f + M_f + J_i + M_i} \langle J_i, -M_i, \beta_i | S | J_f, -M_f, \beta_f \rangle \, . \tag{72}$$

Combining these two equations,

$$\langle \beta_f | S_J | \beta_i \rangle = \langle \beta_i | S_J | \beta_f \rangle \, . \tag{73}$$

That is, the S matrix is symmetric in the total angular momentum representation as a consequence of time reversal invariance.

IV. TIME REVERSAL INVARIANCE IN NUCLEAR REACTIONS

We consider here only nuclear reactions which have two particles in the entrance channel (C) and two particles in the exit channel (C'). This can be written symbolically as

$$A_C + B_C \rightarrow A_{C'} + B_{C'} \, . \tag{74}$$

A. Density Matrix

A statistical description is appropriate for scattering experiments. The beam and target consist of an incoherent mixture of various spin orientations. This suggests that we expect the beam and target to constitute a statistical ensemble of systems, each consisting of a pair of interacting particles. Let there be N such systems and $\Psi(\alpha)$, ($\alpha = 1, 2, \ldots N$) be a description of each. The mean value of a physical operator $\underset{\sim}{Q}$ is

$$\langle \underset{\sim}{Q} \rangle = \frac{1}{N} \sum_{\alpha=1}^{N} \langle \Psi(\alpha) | \underset{\sim}{Q} | \Psi(\alpha) \rangle \, . \tag{75}$$

Expressing the $\Psi(\alpha)$ in an orthonormal basis,

$$\Psi(\alpha) = \sum_j a_j(\alpha) | j \rangle \, ,$$

where

$$\sum_j |a_j(\alpha)|^2 = 1 \ .$$

Now expressing $\langle \underset{\sim}{Q} \rangle$ again,

$$\langle \underset{\sim}{Q} \rangle = \frac{1}{N} \sum_{\alpha=1}^{N} \sum_{j,k} a_k^*(\alpha) \langle k | \underset{\sim}{Q} | j \rangle a_j(\alpha) \ . \tag{76}$$

We define the Hermitian matrix, called the density matrix, ρ, as

$$\langle j | \rho | k \rangle = \sum_{\alpha=1}^{N} \frac{a_j(\alpha) a_k^*(\alpha)}{N} \ ,$$

so that

$$\langle \underset{\sim}{Q} \rangle = \sum_{j,k} \langle j | \rho | k \rangle \langle k | \underset{\sim}{Q} | j \rangle \ ,$$

or

$$\langle \underset{\sim}{Q} \rangle = \mathrm{Tr}(\rho \underset{\sim}{Q}) \ . \tag{77}$$

Clearly also

$$\mathrm{Tr}\rho = 1 \ ,$$

because

$$\sum_j |a_j(\alpha)|^2 = 1 \ .$$

B. Density Matrix After a Collision

The definitions of the previous section allow us to calculate the "final" density matrix, ρ_f, after two particles undergo a reaction. We write the initial wave

$$\phi_i = e^{ikz} \vec{\chi} \ ,$$

where $\vec{\chi}$ is an eigenvector of the initial spins, expanded in a particular basis, $\vec{\chi} = \sum_j a_j |j\rangle$, where $|j\rangle$ are the base vectors. The scattered function can be written as

$$\psi^{(+)}(r) \underset{r\to\infty}{\to} e^{ikz} \vec{\chi} + \frac{e^{ikz}}{r} \underset{\sim}{M}(\theta,\phi) \vec{\chi} \ . \tag{78}$$

Initially, the expectation value of an operator $\underset{\sim}{Q}$ is, as before,

$$\langle \underset{\sim}{Q} \rangle_i = \frac{\langle \chi | \underset{\sim}{Q} | \chi \rangle}{\langle \chi | \chi \rangle} = \frac{\sum_{j,k} a_k^* \langle k | \underset{\sim}{Q} | j \rangle a_j}{\sum_j |a_j|^2}$$

$$= \frac{\sum_{j,k} a_j a_k^* \langle k | \underset{\sim}{Q} | j \rangle}{\sum_j |a_j|^2}$$

$$= \sum_{j,k} \rho_{jk}^{(i)} Q_{Kj} = \mathrm{Tr}(\rho_i \underset{\sim}{Q}) \ . \qquad (79)$$

After the reaction, the expectation value of the same operator is (suppressing the α index for the present)

$$\langle \underset{\sim}{Q} \rangle_f = \frac{\langle \vec{\chi}' | \underset{\sim}{Q} | \vec{\chi}' \rangle}{\langle \vec{\chi}' | \vec{\chi}' \rangle} = \frac{\langle \underset{\sim}{M} \vec{\chi} | \underset{\sim}{Q} | \underset{\sim}{M} \vec{\chi} \rangle}{\langle \underset{\sim}{M} \vec{\chi} | \underset{\sim}{M} \vec{\chi} \rangle}$$

$$= \frac{\sum_{jk} \sum_{\ell m} a_k^* M_{\ell k}^* \langle \ell | \underset{\sim}{Q} | m \rangle M_{mj} a_j}{\sum_{j,k,\ell} a_k^* M_{\ell k}^* M_{\ell j} a_j} \ .$$

That is,

$$\langle \underset{\sim}{Q} \rangle_f = \frac{\sum_{\ell m} \left(\sum_{jk} M_{mj} a_j a_k^* M_{k\ell}^* \right) \langle \ell | \underset{\sim}{Q} | m \rangle}{\sum_{j,k,\ell} M_{\ell j} a_j a_k^* M_{k\ell}^*}$$

$$= \frac{\sum_{\ell m} (\underset{\sim}{M} \rho_i \underset{\sim}{M}^\dagger)_{m\ell} \langle \ell | \underset{\sim}{Q} | m \rangle}{\sum_{\ell} (\underset{\sim}{M} \rho_i \underset{\sim}{M}^\dagger)_{\ell\ell}}$$

We can write

$$\langle \underset{\sim}{Q} \rangle_f = \frac{\mathrm{Tr}[\underset{\sim}{M}\rho_i \underset{\sim}{M}^\dagger \underset{\sim}{Q}]}{\mathrm{Tr}[\underset{\sim}{M}\rho_i \underset{\sim}{M}^\dagger]} \ . \tag{80}$$

And the final density matrix is defined as

$$\rho_f = \frac{\underset{\sim}{M}\rho_i \underset{\sim}{M}^\dagger}{\mathrm{Tr}[\underset{\sim}{M}\rho_i \underset{\sim}{M}^\dagger]} \ , \tag{81}$$

where the above calculation shows that

$$\langle \underset{\sim}{Q} \rangle_f = \mathrm{Tr}[\rho_f \underset{\sim}{Q}] \ . \tag{82}$$

Before going on, it will be shown that $\mathrm{Tr}(\underset{\sim}{M}\rho_i \underset{\sim}{M}^\dagger)$ is a cross section. This fact will often be used to "label" this quantity. Assume that a basis has been chosen which diagonalizes the initial density matrix. In that case,

$$\begin{aligned} \mathrm{Tr}[\underset{\sim}{M}\rho_i \underset{\sim}{M}^\dagger] &= \sum_j \Big(\sum_m M_{jm} \rho_{mm}^{(i)} \tilde{M}^*_{mj} \Big) \\ &= \sum_j \sum_m |M_{jm} a_m|^2 \\ &= \sum_j \sigma_j = \sigma(\alpha) \ , \end{aligned} \tag{83}$$

where the initial state has been taken as $\chi = \sum_m a_m |m\rangle$ and $\sigma_j(\alpha)$ is the cross section for scattering to the state $|j\rangle$. The index α has been replaced to point out that this is the scattering for a particular pair, α, of particles. The average cross section for the ensemble is then

$$\sigma(\theta,\phi) = \frac{1}{N} \sum_{\alpha=1}^{N} \sigma(\alpha) = \mathrm{Tr}[\underset{\sim}{M}\rho_i \underset{\sim}{M}^\dagger] \ . \tag{84}$$

C. Density Matrix Relations for Observables

The initial beam polarization can be expanded in terms of $\underset{\sim}{\omega^\ell}(S_1)$, a complete and orthogonal set of matrices spanning the space of S_1, the incident

particle spin. The superscript, ℓ, is an index taking values 0, 1,..., $2S_1$. Likewise, the initial target polarization can be expanded in terms of $\underset{\sim}{\omega}^m(S_2)$, a complete and orthogonal set of matrices spanning the space of S_2, the target spin. The superscript, m, takes integral values from 0 to $2S_2$. It will be convenient to always take $\underset{\sim}{\omega}^0(S_1) = \underset{\sim}{1}_1$ and $\underset{\sim}{\omega}^0(S_2) = \underset{\sim}{1}_2$, where $\underset{\sim}{1}_1$ and $\underset{\sim}{1}_2$ are the identity operators in S_1 and S_2 spaces, respectively.

Orthogonal matrices are defined as

$$\mathrm{Tr}[\underset{\sim}{\omega}^{\ell\dagger}(s)\ \underset{\sim}{\omega}^m(S)] = (2S+1)\ \delta_{\ell m} \ . \tag{85}$$

Since $\underset{\sim}{\omega}^0 = \underset{\sim}{1}$, it follows that all other matrices in the orthogonal set must be traceless.

A direct product of the beam set of matrices, $\underset{\sim}{\omega}^\ell(S_1)$, and the target set of matrices, $\underset{\sim}{\omega}^m(S_2)$, forms a complete orthogonal set of matrices, $\underset{\sim}{\Omega}_\mu$, for the entrance channel C.

$$\underset{\sim}{\Omega}_\mu = [\underset{\sim}{\omega}^\ell(S_1)] \times [\underset{\sim}{\omega}^m(S_2)] \ . \tag{86}$$

Similarly, one can form such a set for the exit channel, C',

$$\underset{\sim}{\Omega}_{\mu'} = [\underset{\sim}{\omega}^{\ell'}(S_1')] \times [\underset{\sim}{\omega}^{m'}(S_2')] \ , \tag{87}$$

where $\ell' = 0, 1, \ldots, 2S_1'$ and $m' = 0, 1, \ldots, 2S_2'$. (A prime on the subscript of Ω distinguishes a complete set in C from a complete set in C'.)

To summarize, a complete orthogonal set of matrices can be formed to describe the entrance channel, C, with the properties.

$$\mathrm{Tr}[\underset{\sim}{\Omega}_\mu^\dagger\ \underset{\sim}{\Omega}_\nu] = (2S_1+1)(2S_2+1)\ \delta_{\mu\nu} \ ; \quad \underset{\sim}{\Omega}_o = \underset{\sim}{1}_C \ . \tag{88}$$

A complete orthogonal set of matrices can be found to describe the exit channel, C', with the properties

$$\mathrm{Tr}[\underset{\sim}{\Omega}_{\mu'}^\dagger\ \underset{\sim}{\Omega}_{\nu'}] = (2S_1'+1)(2S_2'+1)\ \delta_{\mu'\nu'} \ ; \quad \underset{\sim}{\Omega}_{o'} = \underset{\sim}{1}_{C'} \ .$$

Since the $\underset{\sim}{\Omega}$ matrices span the spin space,[11] the initial density matrix* can be expanded as

$$\rho_i = \frac{1}{(2S_1+1)(2S_2+1)} \sum_\mu \Omega_\mu^i \underset{\sim}{\Omega}_\mu \ , \tag{89}$$

where $\Omega_\mu^i = \langle \underset{\sim}{\Omega}_\mu \rangle^i = \mathrm{Tr}[\rho_i \underset{\sim}{\Omega}_\mu]$ is the component of Ω_μ which is contained in the entrance channel (beam and/or target). It is convenient to take these matrices as complete and orthogonal. One can use a set that is simply "at least complete" (more likely overcomplete and not orthogonal) if the set $\underset{\sim}{B}$ is defined so that the expansion of ρ_i becomes

$$\rho_i = \sum_k b_k B_k \ ; \qquad b_k = \langle \underset{\sim}{B}_k \rangle \ .$$

The second relationship can be satisfied without orthogonality.[11] From Eq. (81), the final density matrix is

$$\rho_f = \frac{\underset{\sim}{M} \rho_i \underset{\sim}{M}^\dagger}{\mathrm{Tr}[\underset{\sim}{M} \rho_i \underset{\sim}{M}^\dagger]} \ . \tag{90}$$

Some definitions will be helpful:

$$\begin{aligned} I_o &\equiv \frac{1}{2S+1} \mathrm{Tr}[\underset{\sim}{M}\underset{\sim}{M}^\dagger] \\ \bar{I}_o &= \frac{1}{2S'+1} \mathrm{Tr}[\underset{\sim}{\bar{M}}\underset{\sim}{\bar{M}}^\dagger] \\ I(\theta,\phi) &\equiv \mathrm{Tr}[\underset{\sim}{M} \rho_i \underset{\sim}{M}^\dagger] \\ \bar{I}(\theta,\phi) &= \mathrm{Tr}[\underset{\sim}{\bar{M}} \rho_i \underset{\sim}{\bar{M}}^\dagger] \ . \end{aligned} \tag{91}$$

where I_o is the differential cross section for an unpolarized entrance channel

---

*For these calculations, we will assume that the Ω matrices are Hermitian, so that $\Omega^\dagger = \Omega$; a treatment of non-Hermitian matrices is given in Appendix B.

($\rho_i = \frac{1}{2S+1} 1_C$) and $I(\theta,\phi)$ is a differential cross section for general polarization in the entrance channel. The bars refer, of course, to the time-reversed reaction (or inverse reaction). Since the expectation value of $\underset{\sim}{\Omega}_{\mu'}$ can be written

$$\Omega^f_{\nu'} = \mathrm{Tr}[\rho_f \, \underset{\sim}{\Omega}_{\nu'}] \ , \tag{92}$$

we substitute from Eqs. (90) and (91) for ρ_f in Eq. (92), and write different forms of the very important relation:

$$\mathrm{Tr}[\underset{\sim}{M}\rho_i \underset{\sim}{M}^\dagger] \ \Omega^f_{\nu'} = \mathrm{Tr}[\underset{\sim}{M}\rho_i \underset{\sim}{M}^\dagger \ \underset{\sim}{\Omega}_{\nu'}]$$

$$\mathrm{Tr}[\underset{\sim}{M}\rho_i \underset{\sim}{M}^\dagger] \ \Omega^f_{\nu'} = \frac{1}{2S+1} \sum_\mu \Omega^i_\mu \ \mathrm{Tr}[\underset{\sim}{M}\underset{\sim}{\Omega}_\mu \underset{\sim}{M}^\dagger \ \underset{\sim}{\Omega}_{\nu'}] \tag{93}$$

or

$$\boxed{I(\theta,\phi) \cdot \Omega^f_{\nu'} = \frac{1}{2S+1} \sum_\mu \Omega^i_\mu \ \mathrm{Tr}[\underset{\sim}{M}\underset{\sim}{\Omega}_\mu \underset{\sim}{M}^\dagger \ \underset{\sim}{\Omega}_{\nu'}]} \ . \tag{94}$$

D. Calculations of $\overline{M}$ and $\overline{\Omega}$

Before $\overline{M}$ and $\overline{\Omega}$ are calculated, a recalculation of the general operator $\underset{\sim}{\overline{Q}}$ will be made to emphasize that the earlier results of Eqs. (36) and (37) are the same, even though the matrix operator $\underset{\sim}{Q}$ may not be square (we are specifically concerned with the scattering matrix, M, for a reaction involving arbitrary entrance and exit spins). Considering only stationary states, time reversal invariance can be written as

$$(\Psi_{C'}, \underset{\sim}{Q}\Psi_C) = (\overline{\Psi}_C, \underset{\sim}{\overline{Q}} \, \overline{\Psi}_{C'}) \ . \tag{95}$$

Using the notation $\overline{\Psi}_C = T_C \Psi_C$ and $\overline{\Psi}_{C'} = T_{C'} \Psi_{C'}$, the transformation from the time-reversed system becomes

$$
\begin{aligned}
(\bar{\Psi}_C, \underset{\sim}{\bar{\Omega}} \bar{\Psi}_{C'}) &= (T_C \Psi_C, \underset{\sim}{\bar{\Omega}} T_{C'} \Psi_{C'}) \\
&= (\Psi_C, T_C^{\dagger} \underset{\sim}{\bar{\Omega}} T_{C'} \Psi_{C'})^* \\
&= (T_C^{\dagger} \underset{\sim}{\bar{\Omega}} T_{C'} \Psi_{C'}, \Psi_C) \\
&= (\Psi_{C'}, T_{C'}^{\dagger} \underset{\sim}{\bar{\Omega}}^{\dagger} T_C \Psi_C) \ . \qquad (96)
\end{aligned}
$$

Comparing Eq. (95) and Eq. (96), the relation is seen to be

$$
\begin{aligned}
\underset{\sim}{\Omega} &= T_{C'}^{\dagger} \underset{\sim}{\bar{\Omega}}^{\dagger} T_C \\
\underset{\sim}{\bar{\Omega}} &= T_C \underset{\sim}{\Omega}^{\dagger} T_{C'}^{\dagger} \ . \qquad (97)
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\underset{\sim}{\bar{M}} &= T_C \underset{\sim}{M}^{\dagger} T_{C'}^{\dagger} \quad ; \quad \underset{\sim}{M}^{\dagger} = T_C^{\dagger} \underset{\sim}{\bar{M}} T_{C'} \\
\underset{\sim}{\bar{M}}^{\dagger} &= T_{C'} \underset{\sim}{M} T_C^{\dagger} \quad ; \quad \underset{\sim}{M} = T_{C'}^{\dagger} \underset{\sim}{\bar{M}}^{\dagger} T_C \ . \qquad (98)
\end{aligned}
$$

The expectation values of the $\underset{\sim}{\Omega}$ matrices are measured, of course, either in the entrance channel C or the exit channel C'. Therefore,

$$
\begin{aligned}
\underset{\sim}{\bar{\Omega}}_{\mu} &= T_C \underset{\sim}{\Omega}_{\mu} T_C^{\dagger} \\
\underset{\sim}{\bar{\Omega}}_{\nu'} &= T_{C'} \underset{\sim}{\Omega}_{\nu'} T_{C'}^{\dagger} \ . \qquad (99)
\end{aligned}
$$

The results of Eqs. (98) and (99) will be used in the next section.

E. Polarization Transfer Coefficients

The type of polarization measurements with which we will be concerned are the polarizations of $A_{C'}$, $B_{C'}$, or both, after a reaction in which A_C (or B_C) was initially polarized. This can be written symbolically as

$$
\vec{A}_C + B_C \rightarrow \vec{A}_{C'} + B_{C'} \ . \qquad (100)
$$

We seek a relation between the parameters which describes the reaction in Eq. (100) and those which describe the polarization of A_C (and/or B_C) in the inverse reaction with $A_{C'}$ (and/or $B_{C'}$) initially polarized. This can be

written symbolically as

$$\vec{A}_{C'} + B_{C'} \quad \vec{A}_C + B_C \ . \tag{101}$$

The expression for the final polarization, $\Omega^f_{\nu'} = \langle \underset{\sim}{\Omega}_{\nu'} \rangle^f$, in Eq. (100) when the initial particles have polarizations described by $\Omega^i_\mu = \langle \underset{\sim}{\Omega}_\mu \rangle^i$ are given in Eq. (94) as

$$I(\theta,\phi) \cdot \Omega^f_{\nu'} = \frac{1}{2S+1} \sum_\mu \Omega^i_\mu \, \mathrm{Tr}[\underset{\sim}{M} \underset{\sim}{\Omega}_\mu \underset{\sim}{M}^\dagger \underset{\sim}{\Omega}_{\nu'}] \ . \tag{102}$$

The expression for the final polarization, $\Omega^f_\mu = \langle \Omega_\mu \rangle^f$, from the inverse reaction in Eq. (101) may be given by

$$\overline{I}(\theta,\phi) \ \Omega^f_\mu = \frac{1}{2S'+1} \sum_{\nu'} \Omega^i_{\nu'} \, \mathrm{Tr}[\underset{\sim}{\overline{M}} \underset{\sim}{\Omega}_{\nu'} \underset{\sim}{\overline{M}}^\dagger \underset{\sim}{\Omega}_\mu] \ ,$$

where the bars indicate values for the inverse reaction. Just as the coordinate system is not changed by T, the base matrices are not changed from $\underset{\sim}{\Omega}$ to $\underset{\sim}{\overline{\Omega}}$. The quantity Ω^f_μ is the polarization of A_C in the inverse reaction of Eq. (101), but referred to the coordinate system of the reaction in Eq. (100) (see Fig. 2). However, it will be more convenient to work with the quantity $\overline{\Omega}^f_\mu$, and at the end of a calculation relate this to Ω^f_μ as follows:

$$\overline{\Omega}^f_\mu = \langle \overline{\Omega}_\mu \rangle^f = \langle \Omega_\mu(-\vec{S}) \rangle^f = (-1)^r \, \Omega^f_\mu \ .$$

In the last equation, the fact that the Ω_μ are made up of products of spatial components of spin operators is used to determine r. For example, in the spherical tensors of Appendix A, r = k, and we have

$$T_{kq}(\vec{S}) = (-1)^k \, T_{kq}(-\vec{S}) \ .$$

Therefore, the inverse reaction in Eq. (101) can be described by

$$\bar{I}(\theta,\phi)\ \bar{\Omega}^{f}_{\mu} = \frac{1}{2S'+1} \sum_{\nu'} \bar{\Omega}^{i}_{\nu'}\ \mathrm{Tr}[\underline{\underline{M\Omega}}_{\nu}, \underline{\underline{M}}^{\dagger}\bar{\Omega}_{\mu}] \ , \tag{103}$$

where $\bar{\Omega}^{f}_{\mu} = \langle \underline{\underline{\bar{\Omega}}}_{\mu} \rangle^{f} = \langle T\underline{\underline{\Omega}}_{\mu} T^{\dagger} \rangle^{f}$ and $\bar{I}(\theta,\phi)$ is the differential cross section for the inverse reactions. We can now show that the two trace formulae in Eqs. (102) and (103) are equivalent. Using Eqs. (98) and (99),

$$\mathrm{Tr}[\underline{\underline{M\Omega}}_{\nu}, \underline{\underline{M}}^{\dagger}\bar{\Omega}_{\mu}] = \mathrm{Tr}[T_{C}\underline{\underline{M}}^{\dagger}T^{\dagger}_{C'} \cdot T_{C'}\underline{\underline{\Omega}}_{\nu}, T^{\dagger}_{C'} \cdot T_{C}\underline{\underline{M}}T^{\dagger}_{C} \cdot T_{C}\underline{\underline{\Omega}}_{\mu}T^{\dagger}_{C}]$$

$$= \mathrm{Tr}[\underline{\underline{M}}^{\dagger}\underline{\underline{\Omega}}_{\nu}, \underline{\underline{M\Omega}}_{\mu}]$$

$$= \mathrm{Tr}[\underline{\underline{M\Omega}}_{\mu}\underline{\underline{M}}^{\dagger}\underline{\underline{\Omega}}_{\nu'}] \ , \tag{104}$$

because $T^{\dagger}T = 1$, and $\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$ for any matrices A and B. For brevity, we can define a polarization transfer coefficient, $\Lambda^{\nu'}_{\mu}$, where

$$\Lambda^{\nu'}_{\mu} = \mathrm{Tr}[M\Omega_{\mu}M^{\dagger}\Omega_{\nu'}]$$

$$\Lambda^{\bar{\mu}}_{\bar{\nu}'} = \mathrm{Tr}[\overline{M\Omega}_{\nu}, \overline{M^{\dagger}\Omega}_{\mu}] \ , \tag{105}$$

and Eq. (104) is simply a statement that for nuclear reactions which obey time reversal invariance,

$$\Lambda^{\nu'}_{\mu} \equiv \Lambda^{\bar{\mu}}_{\bar{\nu}'} \ , \tag{106}$$

(Ref. 12 defines polarization transfer coefficients for spherical tensors).

Substituting Eqs. (105) and (106) in Eqs. (102) and (103),

$$I(\theta,\phi) \cdot \Omega^f_{\nu'} = \frac{1}{2S+1} \sum_\mu \Omega^i_\mu \Lambda^{\nu'}_\mu$$

$$\bar{I}(\theta,\phi) \cdot \bar{\Omega}^f_\mu = \frac{1}{2S'+1} \sum_{\nu'} \bar{\Omega}^i_{\nu'} \Lambda^{\nu'}_\mu \qquad (107)$$

$$\Lambda^{\nu'}_\mu = \mathrm{Tr}[M\Omega_\mu M^\dagger \Omega_{\nu'}] \ .$$

Consider the following examples of Eq. (107) for arbitrary reactions given in Eqs. (100) and (101).

(1) Cross Section: $\nu' = 0$, $\mu = 0$. In this case, Eq. (107) becomes

$$I_o = \frac{1}{2S+1} \Lambda^0_o = \frac{1}{2S+1} \mathrm{Tr}[\underset{\sim}{M}\underset{\sim}{M}^\dagger]$$

$$\bar{I}_o = \frac{1}{2S'+1} \Lambda^0_o \ ,$$

where $\Omega^i_o \equiv 1$, $\Omega^f_o \equiv 1$. Therefore,

$$(2S+1)\, I_o = (2S'+1)\, \bar{I}_o \ . \qquad (108)$$

(2) Polarization: $\nu' = 0$, $\mu \neq 0$. Here the polarization in the inverse reaction with an unpolarized beam (and/or target) is sought. Then Eq. (107) becomes

$$(2S'+1)\, \bar{I}_o \cdot \bar{\Omega}^f_\mu = \Lambda^0_\mu = \mathrm{Tr}[\underset{\sim}{M}\underset{\sim}{\Omega}_\mu \underset{\sim}{M}^\dagger] \ , \qquad (109)$$

where $\bar{\Omega}^f_\mu = \langle T_C \underset{\sim}{\Omega}_\mu T_C^\dagger \rangle^f$. It is the role of T_C in this case to replace the entrance channel spin, $\vec{S}$, by $-\vec{S}$. Writing $\underset{\sim}{\Omega}_\mu$ explicitly as a function of $\vec{S}$, this becomes

$$\bar{\Omega}^f_\mu = \langle \underset{\sim}{\Omega}_\mu(-\vec{S}) \rangle^f \ . \qquad (110)$$

Using Eqs. (108) and (110) in Eq. (109), we find

$$(2S+1)\, I_o \, \langle \underset{\sim}{\Omega}_\mu(-\vec{S}) \rangle^f = \Lambda^0_\mu \ . \qquad (111)$$

(3) Cross Section: $\nu' = 0$, $\mu \neq 0$. An initially polarized beam (or target) has components Ω_μ^i. Equation (107) becomes

$$(2S+1)\ I(\theta,\phi) = \sum_\mu \Omega_\mu^i\ A_\mu^0 \ . \tag{112}$$

Substituting Eq. (111) into Eq. (112), we have the very important relation

$$I(\theta,\phi) = I_o \sum_\mu \Omega_\mu^i\ \overline{\Omega}_\mu^f$$

$$= I_o \sum_\mu \Omega_\mu^i\ \langle \Omega_\mu(-\vec{S}) \rangle^f \ ,$$

or (see Appendix B)

$$\boxed{I(\theta,\phi) = I_o \sum_\mu \langle \Omega_\mu(\vec{S}) \rangle^i\ \langle \Omega_\mu(-\vec{S}) \rangle^f} \ , \tag{113}$$

where Ω_μ^i and $\overline{\Omega}_\mu^f$ are, of course, to be determined in the same coordinate system.

(4) Polarization Transfer: $\nu' \neq 0$, $\mu \neq 0$. Multiplying the first equation in Eq. (107) by $\sum_{\nu'} \overline{\Omega}_{\nu'}^i$, and the second equation in Eq. (105) by $\sum_\mu \Omega_\mu^i$, we find

$$(2S+1)\ I(\theta,\phi) \sum_{\nu'} \overline{\Omega}_{\nu'}^i\ \Omega_{\nu'}^f = (2S'+1)\ \overline{I}(\theta,\phi) \sum_\mu \Omega_\mu^i\ \overline{\Omega}_\mu^f \ .$$

Eliminating S and S' with Eq. (108), this becomes

$$\boxed{\frac{I(\theta,\phi)}{I_o} \sum_\mu \Omega_\mu^i\ \overline{\Omega}_\mu^f = \frac{\overline{I}(\theta,\phi)}{\overline{I}_o} \sum_{\nu'} \overline{\Omega}_{\nu'}^i\ \Omega_{\nu'}^f} \ , \tag{114}$$

which is a relation between the polarization transfer values Ω_ν^f, and the polarization transfer values $\overline{\Omega}_\mu^f$ of the inverse reaction.

As a simple example of Eq. (114), consider the $T(\vec{d},\vec{n})^4He$ reaction. We wish to find a relation between the neutron polarization transfer along the

$\vec{k}_i \times \vec{k}_f$ axis (y-axis) when the deuteron beam is also polarized along that axis, and the deuteron polarization transfer along the $\vec{k}_i \times \vec{k}_f$ axis when the neutron beam is polarized along the same axis (i.e., $^4\text{He}(\vec{n},\vec{d})\text{T}$). We will designate $p_y^y = \langle \underset{\sim}{\sigma_y} \rangle^y$ for the neutron polarization transfer. The superscript y indicates that $\langle \underset{\sim}{\sigma_y} \rangle$ is to be measured with an incident polarized deuteron beam, $\langle \underset{\sim}{S_y} \rangle^i$. Likewise, the deuteron polarization transfer will be designated $\bar{P}_y^y = \langle \underset{\sim}{\bar{S}_y} \rangle^y$. In this notation, p_y^0 and P_y^0 are the usual polarization values obtained by initially unpolarized beams. Substituting Eq. (113) for $I(\theta,\phi)$ and similarly for $\bar{I}(\theta,\phi)$, Eq. (114) becomes*

$$[1 + \tfrac{3}{2} P_y^i \bar{P}_y^0][1 + \tfrac{3}{2} P_y^i \bar{P}_y^y] = [1 + \bar{p}_y^i p_y^0][1 + \bar{p}_y^i p_y^y]$$

$$\tfrac{3}{2} P_y^i[\bar{P}_y^0 + \bar{P}_y^y + P_y^i \bar{P}_y^0 \bar{P}_y^y] = \bar{p}_y^i[p_y^0 + p_y^y + \bar{p}_y^i p_y^0 p_y^y] \ . \qquad (115)$$

At zero degrees, both $\bar{P}_y^0$ and p_y^0 must be zero. In that case,

$$\tfrac{3}{2} P_y^i P_y^y = p_y^i p_y^y \ ; \qquad \text{Lim}(\vec{k}_i \times \vec{k}_f) \to 0 \ ; \qquad |\vec{k}_i|, |\vec{k}_f| \neq 0 \ , \qquad (116)$$

where Eq. (116) has also been expressed with $\bar{P}_y^y = -P_y^y$, $\bar{p}_y^i = -p_y^i$. If the measurement p_y^y is divided by the incident deuteron beam polarization $\frac{3}{2} P_y^i$, and likewise P_y^y, Eq. (116) states

$$\frac{P_y^y}{(p_y^i)} = \frac{p_y^y}{(\frac{3}{2} P_y^i)} \ ; \qquad \text{Lim}(\vec{k}_i \times \vec{k}_f) \to 0 \ ; \qquad |\vec{k}_i|, |\vec{k}_f| \neq 0 \ , \qquad (117)$$

This is but an example of the type of information which is expected to be obtainable in the near future.[13]

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*The need for the "3/2" factor is clear from Appendix A.

V. TIME REVERSAL INVARIANCE IN ELASTIC SCATTERING

The fact that elastic scattering leaves the particles in the same final state as their initial state places restrictions on the scattering matrix $M(\vec{k}_f, \vec{k}_i, \vec{S}_1, \vec{S}_2)$, where $\vec{k}_i$ and $\vec{k}_f$ are the initial and final momenta, respectively, and where $\vec{S}_1$ and $\vec{S}_2$ are the beam and target spin, respectively. In particular, since one cannot distinguish between the reaction and the inverse reaction (if the interaction is invariant under time reversal), the M matrix must satisfy

$$M(\vec{k}_f, \vec{k}_i, \vec{S}_1, \vec{S}_2) = M(\overline{\vec{k}}_f, \overline{\vec{k}}_i, \overline{\vec{S}}_1, \overline{\vec{S}}_2)$$
$$= M(-\vec{k}_i, -\vec{k}_f, -\vec{S}_1, -\vec{S}_2) \ . \qquad (118)$$

Some of the consequences of this restriction will be discussed.

A. Coordinate System

The transformation, $\overline{M}$, in Sec. IV did not include $\vec{k}_i \to -\vec{k}_f$ and $\vec{k}_f \to -\vec{k}_i$, because that would include a change of the coordinate system. Recall that Eq. (37) expressed both $\underset{\sim}{\overline{Q}}$ and $\underset{\sim}{Q}$ in the same coordinate system. Specifically, the transformation is

$$\overline{M} = \mathbb{T}M^{\dagger}(\hat{k}_f, \hat{k}_i, \vec{S}_1, \vec{S}_2)T^{\dagger} = M(\hat{k}_f, \hat{k}_i, -\vec{S}_1, -\vec{S}_2) \ . \qquad (119)$$

The meaning will become clear with a specific example. The most natural coordinate system in which to discuss time reversal invariance is the following set:

$\hat{n}$ a unit vector along $\vec{k}_i \times \vec{k}_f$ (taken here as the y-axis)

$\hat{n}_+$ a unit vector along $\vec{k}_f + \vec{k}_i$ (taken here as the z-axis)

$\hat{n}_-$ a unit vector along $\vec{k}_f - \vec{k}_i$ (taken here as the x-axis) ,

where specific x-, y-, and z-axes are assigned for convenience. Notice that as a consequence of $\overline{\vec{k}}_i = -\vec{k}_f$, $\overline{\vec{k}}_f = -\vec{k}_i$, and $\overline{\vec{S}} = -\vec{S}$, the time reversal of the coordinate system requires (see Fig. 2)

$$\begin{aligned} \bar{\hat{n}} &= -\hat{n} \qquad \text{or} \qquad \bar{\hat{n}}\cdot\bar{\vec{S}} = \hat{n}\cdot\vec{S} \\ \bar{\hat{n}}_+ &= -\hat{n}_+ \qquad\qquad \bar{\hat{n}}_+\cdot\bar{\vec{S}} = \hat{n}_+\cdot\vec{S} \qquad \text{Time reversal invariance .} \\ \bar{\hat{n}}_- &= +\hat{n}_- \qquad\qquad \bar{\hat{n}}_-\cdot\bar{\vec{S}} = -\hat{n}_-\cdot\vec{S} \end{aligned} \tag{120}$$

One can exchange one system with the other by a rotation of 180° about $\hat{n}_-$. The notation for this transformation will be R_-.

For future comparison, it is pointed out here that conservation of parity requires $[\vec{k}_i] = -\vec{k}_i$, $[\vec{k}_f] = -\vec{k}_f$, and $[\vec{S}] = \vec{S}$, where the brackets [] designate the transformed observable. In this case,

$$\begin{aligned} [\hat{n}] &= \hat{n} \qquad \text{or} \qquad [\hat{n}]\cdot[\vec{S}] = \hat{n}\cdot\vec{S} \\ [\hat{n}_+] &= -\hat{n}_+ \qquad \text{or} \qquad [\hat{n}_+]\cdot[\vec{S}] = -\hat{n}_+\cdot\vec{S} \qquad \text{Parity Conservation .} \\ [\hat{n}_-] &= -\hat{n}_- \qquad \text{or} \qquad [\hat{n}_-]\cdot[\vec{S}] = -\hat{n}_-\cdot\vec{S} \end{aligned} \tag{121}$$

B. A Spin 1/2 Particle on a Spin-0 Target

The M matrix can be expanded in terms of the complete set, σ_j.

$$M = A\,1 + \vec{B}\cdot\sigma \ ,$$

where

$$\begin{aligned} \vec{B} &= (\vec{B}\cdot\vec{n}_-)\vec{n}_- + (\vec{B}\cdot\vec{n})\vec{n} + (\vec{B}\cdot\vec{n}_+)\vec{n}_+ \\ &= B_-\vec{n}_- + B_o\vec{n} + B_+\vec{n}_+ \end{aligned} \tag{122}$$

$$\begin{aligned} \bar{M} &= TM^\dagger T^\dagger = T[A^*\,1 + \vec{B}^*\cdot\vec{\sigma}]T^\dagger \\ &= U[A\,1 + \vec{B}\cdot\vec{\sigma}^*]U \ , \end{aligned}$$

where the function of U is a rotation about the y-axis, and $\sigma_y^* = -\sigma_y$. Therefore,

$$\bar{M} = A\,1 - \vec{B}\cdot\vec{\sigma} \ . \tag{123}$$

The inverse reaction will be specified in the inverse coordinate system (a rotation, R_-, of 180° about $\hat{n}_-$). Therefore,

$$R_-\bar{M}R_-^{-1} = A\,1 - \sigma_x B_- + \sigma_y B_o + \sigma_z B_+ \ .$$

Since Eq. (118) holds for elastic scattering,

$$R_{-}\overline{M}R_{-}^{-1} = M \tag{124}$$

implies that $B_{-} = 0$. (Of course, parity conservation requires that not only is $B_{-} = 0$, but also $B_{+} = 0$.)

C. Elastic Scattering of General Spin

We again designate by Ω_{μ} a complete, orthogonal set of matrices which describe the polarization of the beam (with spin $\vec{S}_1$) and the target (with spin $\vec{S}_2$). However, for present purposes there will be no loss of generality if we consider only an unpolarized beam of spin $\vec{S}$ incident on an unpolarized target of arbitrary spin. In this case, the polarizations of the particle after the reaction and the inverse reaction are described in Eq. (107) as

$$\begin{aligned}(2S+1)\, I_o \langle\Omega_\mu\rangle^f &= \mathrm{Tr}[MM^\dagger\Omega_\mu] \\ (2S+1)\, I_o \langle\overline{\Omega}_\mu\rangle^f &= \mathrm{Tr}[\overline{MM^\dagger}\overline{\Omega}_\mu] = \mathrm{Tr}[M\Omega_\mu M^\dagger] \ .\end{aligned} \tag{125}$$

However, since Eq. (124) is a general property for elastic scattering, Eq. (125) can be written

$$\begin{aligned}\mathrm{Tr}[M\Omega_\mu M^\dagger] = \mathrm{Tr}[\overline{MM^\dagger}\overline{\Omega}_\mu] &= \mathrm{Tr}[R_{-}^{-}MR_{-}R_{-}^{-}M^\dagger R_{-}\overline{\Omega}_\mu] \\ &= \mathrm{Tr}[MM^\dagger R_{-}\overline{\Omega}_\mu R_{-}^{-1}] \ .\end{aligned} \tag{126}$$

Neglecting normalizing constants let $\Omega_\mu = S_\mu$, components of angular momentum. Then Eq. (126) states that

$$\begin{aligned}\mathrm{Tr}[MS_zM^\dagger] &= \mathrm{Tr}[MM^\dagger S_z] \\ \mathrm{Tr}[MS_yM^\dagger] &= \mathrm{Tr}[MM^\dagger S_y] \\ \mathrm{Tr}[MS_xM^\dagger] &= -\,\mathrm{Tr}[MM^\dagger S_x] \ .\end{aligned} \tag{127}$$

However, the last equation in Eq. (127) is zero because of the conservation of parity (i.e., $P_x = 0$ after scattering an unpolarized beam). Therefore,

we can write the familiar relation

$$\mathrm{Tr}[M\vec{S}M^{\dagger}] = \mathrm{Tr}[MM^{\dagger}\vec{S}] \ , \tag{128}$$

which is a consequence of time reversal invariance for a parity conserving M matrix.

VI. CONCLUSION

The general properties for the time reversal operation for nuclear reactions have been discussed. Equation (113) (and its generalization, which is Eq. (B-11) in Appendix B) was derived and gives a useful relation for the cross section of reactions containing polarized particles in the entrance channel. A relation between polarization transfer experiments was derived (Eq. (114)) and is presently being investigated. This report ends where most treatments of time reversal invariance for nuclear interactions begin--namely with elastic scattering. The restrictions of time reversal invariance in the elastic scattering process will be the subject of a future report.

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APPENDIX A

Spin-one tensors are given in terms of the symmetric, traceless, Hermitian set $\underset{\sim}{1}$, $\underset{\sim}{S}_i$, and $\underset{\sim}{S}_{ij} = \frac{1}{2}(\underset{\sim}{S}_i \underset{\sim}{S}_j + \underset{\sim}{S}_j \underset{\sim}{S}_i) - \frac{2}{3}\delta_{ij}$, where the matrix form of the components $\underset{\sim}{S}_j$ are (i,j = 1,2,3 or x,y,z).

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad S_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

However, this set is overcomplete and not orthogonal. A convenient set of orthogonal and Hermitian matrices for spin one is

$$\underset{\sim}{\Omega}_o = \underset{\sim}{1} \quad \text{(unit matrix)}$$

$$\underset{\sim}{\Omega}_1 = \sqrt{\frac{3}{2}}\,\underset{\sim}{S}_1 = \sqrt{\frac{3}{2}}\,\underset{\sim}{S}_x$$

$$\underset{\sim}{\Omega}_2 = \sqrt{\frac{3}{2}}\,\underset{\sim}{S}_2 = \sqrt{\frac{3}{2}}\,\underset{\sim}{S}_y$$

$$\underset{\sim}{\Omega}_3 = \sqrt{\frac{3}{2}}\,\underset{\sim}{S}_3 = \sqrt{\frac{3}{2}}\,\underset{\sim}{S}_z$$

$$\underset{\sim}{\Omega}_4 = \sqrt{\frac{3}{2}}\,(\underset{\sim}{S}_1 \underset{\sim}{S}_2 + \underset{\sim}{S}_2 \underset{\sim}{S}_1) = \sqrt{6}\,S_{12}$$

$$\underset{\sim}{\Omega}_5 = \sqrt{\frac{3}{2}}\,(\underset{\sim}{S}_2 \underset{\sim}{S}_3 + \underset{\sim}{S}_3 \underset{\sim}{S}_2) = \sqrt{6}\,S_{23}$$

$$\underset{\sim}{\Omega}_6 = \sqrt{\frac{3}{2}}\,(\underset{\sim}{S}_3 \underset{\sim}{S}_1 + \underset{\sim}{S}_1 \underset{\sim}{S}_3) = \sqrt{6}\,S_{31}$$

$$\underset{\sim}{\Omega}_7 = \sqrt{3}\,(\underset{\sim}{S}_1^2 - \underset{\sim}{S}_2^2) = \sqrt{\frac{3}{2}}\,(S_{11} - S_{22})$$

$$\underset{\sim}{\Omega}_8 = \sqrt{\frac{1}{2}}\,(3\underset{\sim}{S}_3^2 - 2) = \frac{3}{\sqrt{2}}\,(S_{33}) .$$

These matrices have the properties

$$\underset{\sim}{\Omega}_j^\dagger = \underset{\sim}{\Omega}_j$$

$$\mathrm{Tr}\;\underset{\sim}{\Omega}_j \underset{\sim}{\Omega}_K = 3\delta_{jK} ,$$

and $\underset{\sim}{\Omega}_7$ and $\underset{\sim}{\Omega}_8$ can be replaced by

$$\sqrt{\frac{3}{2}}\,(\underset{\sim}{S}_2^2 - \underset{\sim}{S}_3^2) \qquad \text{and} \qquad \sqrt{\frac{1}{2}}\,(3\underset{\sim}{S}_1^2 - \underset{\sim}{2}) \ ,$$

respectively, or by

$$\sqrt{\frac{3}{2}}\,(\underset{\sim}{S}_3^2 - \underset{\sim}{S}_1^2) \qquad \text{and} \qquad \sqrt{\frac{1}{2}}\,(3\underset{\sim}{S}_2^2 - \underset{\sim}{2}) \ ,$$

respectively, depending on which is more convenient for a specific problem. Another commonly used set of matrices for spin one are the non-Hermitian, orthogonal, complete set of spherical tensors, $\underset{\sim}{T}_{kq}$.

$$\underset{\sim}{T}_{00} = \underset{\sim}{1}$$

$$\underset{\sim}{T}_{10} = \sqrt{\frac{3}{2}}\,\underset{\sim}{S}_z$$

$$\underset{\sim}{T}_{1\pm1} = \mp \frac{\sqrt{3}}{2}\,(\underset{\sim}{S}_x \pm i\underset{\sim}{S}_y)$$

$$\underset{\sim}{T}_{20} = \frac{1}{\sqrt{2}}\,(3\underset{\sim}{S}_3^2 - 2)$$

$$\underset{\sim}{T}_{2\pm1} = \mp \frac{\sqrt{3}}{2}\,[(\underset{\sim}{S}_x \underset{\sim}{S}_z + \underset{\sim}{S}_z \underset{\sim}{S}_x) \pm i(\underset{\sim}{S}_y \underset{\sim}{S}_z + \underset{\sim}{S}_z \underset{\sim}{S}_y)]$$

$$\underset{\sim}{T}_{2\pm2} = \frac{\sqrt{3}}{2}\,[(\underset{\sim}{S}_x^2 - \underset{\sim}{S}_y^2) \pm i(\underset{\sim}{S}_x \underset{\sim}{S}_y + \underset{\sim}{S}_y \underset{\sim}{S}_x)] \ .$$

These matrices have the properties

$$\underset{\sim}{T}^{\dagger}_{kq} = (-1)^q \underset{\sim}{T}_{k-q}$$

$$\mathrm{Tr}[\underset{\sim}{T}^{\dagger}_{kq} \underset{\sim}{T}_{k'q'}] = 3\,\delta_{kk'}\,\delta_{qq'} \ .$$

In the early part of this report we have used the exponential notation for an operator. The operator $e^{\mathscr{O}}$ is defined as

$$e^{\mathscr{O}} \equiv \left[1 + \mathscr{O} + \frac{\mathscr{O}\cdot\mathscr{O}}{2!} + \frac{\mathscr{O}\cdot\mathscr{O}\cdot\mathscr{O}}{3!} + \ldots\right] .$$

With this definition, we proceed to calculate $Ae^{\mathcal{O}}$, for any operators A and $\mathcal{O}$.

$$Ae^{\mathcal{O}} = A[1 + \mathcal{O} + \frac{\mathcal{O}\cdot\mathcal{O}}{2!} + \frac{\mathcal{O}\cdot\mathcal{O}\cdot\mathcal{O}}{3!} + \ldots] A^{-1}A$$

$$= [1 + A\mathcal{O}A^{-1} + \frac{A\mathcal{O}\cdot\mathcal{O}A^{-1}}{2!} + \frac{A\mathcal{O}\cdot\mathcal{O}\cdot\mathcal{O}A^{-1}}{3!} + \ldots] A$$

$$= [1 + A\mathcal{O}A^{-1} + \frac{A\mathcal{O}A^{-1}\cdot A\mathcal{O}A^{-1}}{2!} + \frac{A\mathcal{O}A^{-1}\cdot A\mathcal{O}A^{-1}\cdot A\mathcal{O}A^{-1}}{3!} + \ldots] A$$

$$= [1 + (A\mathcal{O}A^{-1}) + \frac{(A\mathcal{O}A^{-1})^2}{2!} + \frac{(A\mathcal{O}A^{-1})^3}{3!} + \ldots] A$$

$$= e^{A\mathcal{O}A^{-1}} \cdot A \quad .$$

We have used the last expression in Eq. (6).

APPENDIX B

One can, of course, expand a density matrix in terms of a non-Hermitian set of orthogonal matrices. Consider, for example, the spherical tensors

$$\rho_i = \frac{1}{(2S+1)} \sum_{k,q} \langle T_{kq} \rangle T^{\dagger}_{kq} \quad . \tag{B-1}$$

Since $\rho_i^{\dagger} = \rho_i$, we must have from Eq. (B-1)

$$[\langle T_{kq} \rangle T^{\dagger}_{kq} + \langle T_{k-q} \rangle T^{\dagger}_{k-q}] = [\langle T_{kq} \rangle T^{\dagger}_{kq} + \langle T_{k-q} \rangle T^{\dagger}_{k-q}]^{\dagger}$$

$$= [\langle T_{kq} \rangle^{*} T_{kq} + \langle T_{k-q} \rangle^{*} T_{k-q}] \quad . \tag{B-2}$$

Since $T^{\dagger}_{kq} = (-1)^{]} T_{k-q}$, Eq. (B-2) is satisfied by

$$\langle T_{kq} \rangle^{*} = (-1)^{q} \langle T_{k-q} \rangle \quad . \tag{B-3}$$

The spherical tensors are not Hermitian operators and, therefore, they are not observable. However, one can make up a non-Hermitian operator Ω_{μ} with observable components by writing

$$\Omega_{\mu} = \frac{1}{2} [\Omega^{\dagger}_{\mu} + \Omega_{\mu}] + \frac{i}{2} [i\Omega^{\dagger}_{\mu} - i\Omega_{\mu}]$$

$$= V_{\mu} + iW_{\mu} \quad , \tag{B-4}$$

where $V_{\mu} = V^{\dagger}_{\mu}$ and $W_{\mu} = W^{\dagger}_{\mu}$. The density matrix is written

$$\rho_i = \frac{1}{(2S+1)} \sum_{\mu} \langle \Omega_{\mu} \rangle^{i} \, \Omega^{\dagger}_{\mu} \qquad ; \qquad \mathrm{Tr} \; \Omega^{\dagger}_{\mu} \, \Omega_{\nu} = (2S+1) \, \delta_{\mu\nu}$$

$$I(\theta,\phi) = \mathrm{Tr}[M\rho_i M^{\dagger}]$$

$$= \frac{1}{(2S+1)} \sum_{\mu} \langle \Omega_{\mu} \rangle^{i} \, \mathrm{Tr}[M\Omega^{\dagger}_{\mu} M^{\dagger}] \quad . \tag{B-5}$$

Substituting Eq. (B-4) into Eq. (B-5),

$$(2S+1) \; I(\theta,\phi) = \sum_{\mu} \langle \Omega_{\mu} \rangle^{i} \, [\mathrm{Tr}(MV_{\mu}M^{\dagger}) - i\mathrm{TR}(MW_{\mu}M^{\dagger})] \quad .$$

But since $M = T_{C'}^{\dagger}\overline{M}^{\dagger}T_C$, $M^{\dagger} = T_C^{\dagger}\overline{M}T_{C'}$, from Eq. (96),

$$(2S+1)\, I(\theta,\phi) = \sum_{\mu} \langle\Omega_\mu\rangle^{i} \left[\mathrm{Tr}(\overline{M M^{\dagger}} V_\mu) - i\mathrm{TR}(\overline{M M^{\dagger}} W_\mu)\right] . \tag{B-6}$$

But $\overline{V}_\mu$ and $\overline{W}_\mu$ are observables in the inverse reaction, namely

$$\overline{I}_o \langle \overline{V}_\mu \rangle = \frac{1}{2S'+1} \mathrm{Tr}[\overline{M M^{\dagger}} V_\mu]$$

$$\overline{I}_o \langle \overline{W}_\mu \rangle = \frac{1}{2S'+1} \mathrm{Tr}[\overline{M M^{\dagger}} W_\mu] . \tag{B-7}$$

The observables V_μ and W_μ are tensors made up of components of spin. Since the effect of T here is to take $\vec{S}$ into $-\vec{S}$, we note also that

$$\overline{V}_\mu(\vec{S}) = V_\mu(-\vec{S})$$

$$\overline{W}_\mu(\vec{S}) = W_\mu(-\vec{S}) . \tag{B-8}$$

Recalling that $(2S+1)I_o = (2S'+1)\overline{I}_o$, we substitute Eqs. (B-7) and (B-4) into (B-6) to find

$$I(\theta,\phi) = I_o \sum_{\mu} \langle\Omega_\mu\rangle^{i} \langle\overline{\Omega}_\mu^{\dagger}\rangle^{f} , \tag{B-9}$$

where the superscript f denotes final polarization in the inverse reaction. Using Eqs. (B-8) and (B-4) in Eq. (B-9), we obtain

$$I(\theta,\phi) = I_o \sum_{\mu} \langle\Omega_\mu\rangle^{i} \langle\Omega_\mu^{\dagger}(-\vec{S})\rangle^{f} . \tag{B-10}$$

This is a generalization of Eq. (111). To emphasize this important result, we rewrite Eq. (B-10) in the general form. Consider the reaction

$$\vec{A}_C + \vec{B}_C \quad A_{C'} + B_{C'} ,$$

with polarized target (and/or beam) with beam spin $\vec{S}_1$ and target spin $\vec{S}_2$. We can form a complete, orthogonal set of matrices from $\vec{S}_1$ and $\vec{S}_2$ which we will label $\Omega_\mu(\vec{S}_1,\vec{S}_2)$. These matrices are required to obey the orthogonality relationship $\mathrm{Tr}[\Omega_\mu^{\dagger}\,\Omega_\nu] = (2S_1+1)(2S_2+1)\delta_{\mu\nu}$. If the entrance channel contains

polarized particles (i.e., not all $\Omega_\mu^i = 0$, $\mu > 0$, where $\Omega_\mu^i = \langle\Omega_\mu\rangle^i = \mathrm{Tr}\rho_i\Omega_\mu$, and $\Omega_o = 1$) then the cross section $I(\theta,\phi)$ is given from Eq. (B-10) as

$$\boxed{I(\theta,\phi) = I_o \sum_\mu \langle\Omega_\mu(\vec{S}_1,\vec{S}_2)\rangle^i \langle\Omega_\mu^\dagger(-\vec{S}_1,-\vec{S}_2)\rangle^f} \quad , \tag{B-11}$$

where, of course, all quantitites are expressed in the same coordinate system. To illustrate this, consider the cross section for the $T(\vec{d},n)^4He$ reaction, where the incident deuteron beam has tensor polarization which can be expressed in terms of the spherical tensors, T_{kq}. These spherical tensors have the properties

$$\begin{aligned} T_{kq} &= (-1)^q T_{k-q} \\ T_{kq}(\vec{S}) &= (-1)^k T_{kq}(-\vec{S}) \ . \end{aligned} \tag{B-12}$$

Substituting Eq. (B-12) into Eq. (B-11), we find (cf. Satchler[14])

$$I(\theta,\phi) = I_o \sum_{k,q} (-1)^{k+q} T_{kq}^i T_{k-q}^f \ , \tag{B-13}$$

where the T_{kq}^i are the expectation values of the incident deuteron beam, and the T_{kq}^f are the expectation values of the deuteron polarization in the inverse reaction (i.e., $^4He(n,\vec{d})T$), but expressed in the same coordinate system as T_{kq}^i.

REFERENCES

1. E. Wigner, Nachrichten der Göttingen Gesellshaft der Wissenschaften (Math. Physik. Klasse, 1932) p. 546.
2. E. P. Wigner, Group Theory (Academic Press, New York, 1959).
3. M. L. Goldberger and K. M. Watson, Collision Theory (John Wiley & Sons, New York, 1964).
4. A. Messiah, Quantum Mechanics (John Wiley & Sons, New York, 1966) Vol. II.
5. L. S. Rodberg and R. M. Thaler, Introduction to the Quantum Theory of Scattering (Academic Press, New York, 1967).
6. A. Bohr and B. R. Mottelson, Nuclear Structure (W. A. Benjamin, Inc., New York, 1969) Vol. I.
7. P. L. Csonka and M. J. Moravcsik, Phys. Rev. 152, 1310 (1966).
8. E. M. Henley, Am. Rev. of Nucl. Sci. 19, 367 (1969).
9. Sov. Phys. Uspekhi 11, #4 (January-February, 1969).
10. M. E. Rose, Elementary Theory of Angular Momentum (John Wiley & Sons, New York, 1957).
11. J. L. Gammel, P. W. Keaton, Jr., and G. G. Ohlsen, "Formalism for the $\vec{T}(\vec{d},\vec{n})^4$He and $^3\vec{He}(\vec{d},\vec{p})^4$He Reactions," (LASL report to be published).
12. G. R. Satchler, "The Scattering and Polarization of Spin One Particles," Oak Ridge National Laboratory Report ORNL-2861 (January 1960).
13. J. E. Simmons, W. B. Broste, G. P. Lawrence, J. L. McKibben, and G. G. Ohlsen, Bull. Am. Phys. Soc. 14, 1230 (1969).
14. G. R. Satchler, Nucl. Phys. 8, 65 (1958).