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A Study of Truncation Errors in Eulerian Hydrodymics

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A Study of Truncation Errors in Eulerian Hydrodynamics

by

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ABSTRACT

A numerical and analytical study is given of the effects of truncation errors for various differencing schemes in Eulerian hydrodynamics.

Space truncation errors are studied for a conventional second order time scheme. Correlations can be made between the analytical theory and the numerical calculations.

CONTENTS

TABLES

INTRODUCTION

We desire to give some idea of the effects of truncation errors resulting from the finite differencing of the hydrodynamic equations in a fixed Eulerian mesh of one space dimension. The Eulerian form of the hydrodynamic equations will be used, as in a previous report,[1] in conservative differential and difference form. A fixed mesh in plane coordinates is assumed. Each cell of the mesh has a length of δx. The cells are numbered from left ($i = 1$) to right ($i = iM$). Given the mesh at some time and with proper boundary conditions, the usual problem is to carry the values of the quantities stored for each mesh point forward in time explicitly to a small time δt later. This numerical method is for compressible flow in the presence of shocks. The IBM Electronic Data Processing Machine, Type 7090, was used for the numerical computations.

The conservative differential form of the hydrodynamic equations in one dimension is given by:

$$\frac{\partial \rho}{\partial t} = - \frac{\partial(\rho v)}{\partial x} \tag{1}$$

$$\frac{\partial(\rho v)}{\partial t} = - \frac{\partial(p + \rho v^2)}{\partial x} \tag{2}$$

$$\frac{\partial(\rho E)}{\partial t} = - \frac{\partial(pv + \rho vE)}{\partial x} \tag{3}$$

where ρ is the density, v is the material velocity, p is the pressure, and E is the total energy per unit mass. E is the sum of the internal energy per unit mass ($\mathcal{E}$) and the kinetic energy per unit mass, i.e., $E = \mathcal{E} + \frac{1}{2} v^2$. The extension to two dimensions is indicated elsewhere.[1] A polytropic gas is assumed (with $\gamma = 5/3$) in the numerical calculations. Thus $p = (\gamma - 1)\rho\mathcal{E}$ and $c^2 = \gamma(\gamma - 1)\mathcal{E}$, where c is the sound speed.

Values of ρ_i, v_i, and $\mathcal{E}_i$ are stored and will be considered as the values of ρ, v, and $\mathcal{E}$, respectively, at the center of the ith cell of the mesh. Strictly speaking, the product of ρ_i and the volume of the ith cell is the mass in the ith cell. The products of v_i and $\mathcal{E}_i$ with this mass are the momentum and energy, respectively, in the ith cell. The conservative difference forms of equations (1), (2), and (3) are given by

$$\frac{\rho_i^{n+1} - \rho_i^n}{\delta t} = - \frac{\overline{(\rho v)}_{i+\frac{1}{2}} - \overline{(\rho v)}_{i-\frac{1}{2}}}{\delta x} \tag{4}$$

$$\frac{\rho_i^{n+1} v_i^{n+1} - \rho_i^n v_i^n}{\delta t} = - \frac{\overline{(p + \rho v^2)}_{i+\frac{1}{2}} - \overline{(p + \rho v^2)}_{i-\frac{1}{2}}}{\delta x} \tag{5}$$

$$\frac{\rho_i^{n+1} E_i^{n+1} - \rho_i^n E_i^n}{\delta t} = - \frac{\overline{(\rho v + \rho vE)}_{i+\frac{1}{2}} - \overline{(pv + \rho vE)}_{i-\frac{1}{2}}}{\delta x} \tag{6}$$

where if n indicates some time t, and $n + 1$ indicates a time $t + \delta t$ (one time step later). The method of calculating the barred quantities on the right side of these equations and the resulting truncation errors will be the subject of this report. For conservation they must be unique, i.e., for example, the number calculated for $\overline{(\rho v)}_{i+\frac{1}{2}}$ in the ith cell is to be used in the $(i + 1)^{th}$ cell for $\overline{(\rho v)}_{(i+1)-\frac{1}{2}}$. Or, if the left side of the first cell is the left boundary and the right side of the iMth cell is the left boundary of the region being studied, then by Eq. (4),

$$\sum_{i=1}^{iM} \left(\frac{\rho_i^{n+1} - \rho_i^n}{\delta t}\right) \delta x = \overline{(\rho v)}_{1-\frac{1}{2}} - \overline{(\rho v)}_{iM+\frac{1}{2}}$$

i.e., the change in mass of the region each time step is given by the mass flow in and out of the boundaries in that time step. Similarly, Eqs. (5) and (6) must be made to conserve momentum and energy, respectively. This report will not discuss the stability of the various differencing schemes, which may be found elsewhere.[2]

TRUNCATION ERRORS AND DIFFERENCING SCHEMES

Since the equations are all similar, we will show only the expansion of Eqs. (1) and (4), i.e., for advancing the density in time. The same procedure is used for Eqs. (2) and (3). Thus,

$$\rho_i^{n+1} = \rho_i^n + \delta t \left(\frac{\partial \rho}{\partial t}\right)_i^n + \frac{\delta t^2}{2} \left(\frac{\partial^2 \rho}{\partial t^2}\right)_i^n + \frac{\delta t^3}{6} \left(\frac{\partial^3 \rho}{\partial t^3}\right)_i^n + \cdots \tag{7}$$

For an explicit calculation we must evaluate the coefficients of δt, δt^2, δt^3, etc. from the values of ρ, v, and $\mathcal{E}$ which exist at the various mesh points at time step n. We will truncate (7) after second order (no δt^3 term). The coefficient of δt is given by Eq. (1), i.e., $\frac{\partial \rho}{\partial t} = - \frac{\partial (\rho v)}{\partial x}$. The coefficient of δt^2 is given by Eqs. (1) and (2), i.e.,

$$\frac{\partial^2 \rho}{\partial t} = - \frac{\partial}{\partial t} \left[\frac{\partial(\rho v)}{\partial x}\right] = - \frac{\partial}{\partial x}\left[\frac{\partial(\rho v)}{\partial t}\right] = + \frac{\partial^2}{\partial x^2}(p + \rho v^2)$$

Thus, we must calculate the coefficients of δt and δt^2 by the use of space derivatives. These space derivatives will be calculated by finite differencing schemes from the values of ρ, v, and $\mathcal{E}$ at time step n which are given at the various mesh centers. Any finite difference scheme for the space derivatives will be accurate only to some order in δx, i.e., will have truncation errors.

We will use the following methods of space differencing. Let $\frac{\partial f}{\partial x}$ be a function of x which we desire to approximate by some scheme of finite differencing, using the values of f at various mesh points; we want

$$\frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\delta x}$$

to approach $\left(\frac{\partial f}{\partial x}\right)_i$ within some order (of truncation) in δx. In this report, of course, f may be ρ, v, $\mathcal{E}$, p, etc. or some combination of these dependent variables. For differencing we will assume a three point scheme such that

$$f_{i+\frac{1}{2}} \begin{cases} = \dfrac{\xi_1 f_i + \xi_2 f_{i+1} + \xi_3 f_{i-1}}{\xi_0} & \text{if } V_T > 0 \\ = 0 & \text{if } V_T = 0 \\ = \dfrac{\xi_1 f_{i+1} + \xi_2 f_i + \xi_3 f_{i+2}}{\xi_0} & \text{if } V_T < 0 \end{cases} \qquad (8)$$

where $\xi_0 = \xi_1 + \xi_2 + \xi_3$, $V_T = v_i + v_{i+1}$, and the set (ξ_1, ξ_2, ξ_3) are constants. The only exception to Eq. (8) is that for $V_T = 0$ and $f = p$, then

$$f_{i+\frac{1}{2}} = \frac{p_i + p_{i+1}}{2}$$

V_T is a number to test the direction of flow. Since, in a three point scheme, one must have two mesh values on one side of the boundary between two cells and one mesh value on the other side of this interface, it is evident that V_T is used to select two mesh values on the side of the interface from which the hydrodynamic flow is approaching this interface.

Let

$$f_i' \equiv \left(\frac{\partial f}{\partial x}\right)_i, \quad f_i'' \equiv \left(\frac{\partial^2 f}{\partial x^2}\right)_i, \quad \text{etc.}$$

and, for $V_T > 0$, expand from Eq. (8) as

$$f_{i+\frac{1}{2}} = \frac{1}{\xi_0}\left[\xi_1 f_i + \xi_2\left(f_i + \delta x\, f_i' + \frac{\delta x^2}{2} f_i'' + \frac{\delta x^3}{6} f_i''' + \cdots\right) + \xi_3\left(f_i - \delta x\, f_i' + \frac{\delta x^2}{2} f_i'' - \frac{\Delta x^3}{6} f_i''' + \cdots\right)\right]$$

or, doing the same for $f_{i-\frac{1}{2}}(V_T > 0)$, we have

$$\begin{aligned} f_{i+\frac{1}{2}} &= f_i + \delta x f_i' \left(\frac{\xi_2 - \xi_3}{\xi_0}\right) + \frac{\delta x^2}{2} f_i'' \left(\frac{\xi_2 + \xi_3}{\xi_0}\right) + \frac{\delta x^3}{6} f_i'''\left(\frac{\xi_2 - \xi_3}{\xi_0}\right) + \cdots \\ f_{i-\frac{1}{2}} &= f_i + \delta x f_i' \left(\frac{-\xi_1 - 2\xi_3}{\xi_0}\right) + \frac{\delta x^2}{2} f_i'' \left(\frac{\xi_1 + 4\xi_3}{\xi_0}\right) + \frac{\delta x^3}{6} f_i'''\left(\frac{-\xi_1 - 8\xi_3}{\xi_0}\right) + \cdots \end{aligned} \tag{9}$$

For $V_T < 0$, we have

$$f_{i+\frac{1}{2}} = f_i + \delta x f_i' \left(\frac{\xi_1 + 2\xi_3}{\xi_0}\right) + \frac{\delta x^2}{2} f_i'' \left(\frac{\xi_1 + 4\xi_3}{\xi_0}\right) + \frac{\delta x^3}{6} f_i''' \left(\frac{\xi_1 + 8\xi_3}{\xi_0}\right) + \cdots$$

$$f_{i-\frac{1}{2}} = f_i + \delta x f_i' \left(\frac{-\xi_2 + \xi_3}{\xi_0}\right) + \frac{\delta x^2}{2} f_i'' \left(\frac{\xi_2 + \xi_3}{\xi_0}\right) + \frac{\delta x^3}{6} f_i''' \left(\frac{-\xi_2 + \xi_3}{\xi_0}\right) + \cdots \tag{10}$$

Equations (9) and (10) give

$$\frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\delta x} = f_i' + \frac{\delta x}{2} f_i'' \left(\frac{-\xi_1 + \xi_2 - 3\xi_3}{\xi_0}\right) + \frac{\delta x^2}{6} f_i''' \left(\frac{\xi_1 + \xi_2 + 7\xi_3}{\xi_0}\right) + \delta x^3 \left(\quad \right) + \cdots \qquad V_T > 0$$

$$\frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\delta x} = f_i' + \frac{\delta x}{2} f_i'' \left(\frac{\xi_1 - \xi_2 + 3\xi_3}{\xi_0}\right) + \frac{\delta x^2}{6} f_i''' \left(\frac{\xi_1 + \xi_2 + 7\xi_3}{\xi_0}\right) + \delta x^3 \left(\quad \right) + \cdots \qquad V_T < 0 \tag{11}$$

Thus, for our three point scheme for the calculation of f on an interface ($f_{i+\frac{1}{2}}$), the selection of the three constants (ξ_1, ξ_2, ξ_3) will determine a differencing scheme and determine the truncation errors. The terms in Eq. (11) containing δx, δx^2, δx^3, etc. may all be considered to be errors in

$$\left(\frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\delta x}\right)$$

as an approximation to f_i'.

We will consider in this report the types of differencing schemes given in Table 1.

Table 1. Type Differencing

Type	ξ_1	ξ_2	ξ_3	ξ_0
I	1	1	0	2
II	1	0	0	1
III	6	3	-1	8
IV	4	1	-1	4
V	7	4	1	12
VI	3	0	-1	2
VII	5	2	-1	6

Type I gives a linear differencing scheme. Type II indicates that the value on the boundary is to be taken as the value from the near cell from which the flow is approaching the interface. Type III results from a quadratic fit of the three selected mesh values which is evaluated at the interface. Type IV is the result of an extrapolation from the nearest mesh value behind the flow to the interface, using the slope given by the two outside cells. Type V is the result given by a linear least squares fit with three data points. Type VI uses the two mesh values behind the flow for a linear extrapolation to the interface. Type VII does not seem to have a simple geometrical interpolation. However, its value as a differencing scheme to reduce the truncation errors will be shown below. Though there does not seem to be any profitable reason for doing so, Type VII may be considered as a linear combination of Type I and Type VI, i.e., for $V_T > 0$,

$$f_{i+\frac{1}{2}} = \frac{2\left(\frac{3}{2} f_i - \frac{1}{2} f_{i-1}\right) + 4\left(\frac{f_i + f_{i+1}}{2}\right)}{6}$$

is the same as Type VII.

We can now return to the coefficients in Eq. (11) which determine the truncation errors. Table 2 illuminates the effect of the ξ's on the truncation errors.

Table 2. Truncation Errors

		δx Coefficients		δx^2 Coefficient
Type	(ξ_1,ξ_2,ξ_3)	$\left(\frac{-\xi_1 + \xi_2 - 3\xi_3}{\xi_0}\right)$	$\left(\frac{+\xi_1 - \xi_2 + 3\xi_3}{\xi_0}\right)$	$\left(\frac{\xi_1 + \xi_2 + 7\xi_3}{\xi_0}\right)$
I	(1,1,0)	0	0	1
II	(1,0,0)	-1	+1	1
III	(6,3,-1)	0	0	1/4
IV	(4,1,-1)	0	0	-1/2
V	(7,4,1)	-1/2	+1/2	3/2
VI	(3,0,-1)	0	0	-2
VII	(5,2,-1)	0	0	0

The fact that Types II and V do not give an accuracy even to order δx may be noted. In fact, both give the effect of a diffusion term being added to the differential equations, with Type II giving twice as much diffusion as that given by Type V. This fact and the effect of the errors in the δx^2 terms will be evident in the numerical examples given later in this paper.

Let us again return to Eq. (7). The coefficient of δt^2 will have a term involving the second space derivative. Differencing this space derivative in an ordinary conserving manner, i.e., like

$$\frac{\frac{p_{i+1} - p_i}{\delta x} - \frac{p_i - p_{i-1}}{\delta x}}{\delta x} = \frac{p_{i+1} + p_{i-1} - 2p_i}{\delta x^2} = p_i'' + \frac{2}{4!} p_i'''' \delta x^2 + \cdots \tag{12}$$

will give accuracy to order δx. This term is then accurate to order $\delta t^2 \delta x$, i.e., third order. In this report, all the numerical examples used a linear (first order) differencing scheme for the calculation of the coefficient of the term which is second order in time. The various types of differencing schemes listed above were used for the calculation of the coefficient of the δt term in Eq. (7). Thus, Types II and V give accuracy only to order δt, while Types I, III, IV, and VI give accuracy to order $\delta t \delta x$. Type VII may give accuracy to order as high as $\delta t \delta x^2$ for this term, depending on the considerations discussed in the next paragraph.

Because the right side of our hydrodynamic equations are nonlinear in the dependent variables, there are several ways these terms may be grouped for differencing. We will discuss two methods of grouping and designate them as Group A and Group B. By Group A we will mean that f of Eq. (8) is to be replaced by ρv when used to calculate the coefficient of δt in Eq. (7), i.e.,

$$f_{i+\frac{1}{2}} = (\rho v)_{i+\frac{1}{2}} = (\xi_1 \rho_i v_i + \xi_2 \rho_{i+1} v_{i+1} + \xi_3 \rho_{i-1} v_{i-1})/\xi_0$$

for $V_T > 0$, etc. Group A will thus result in the truncation errors indicated in Table II being valid for

$$\frac{(\rho v)_{i+\frac{1}{2}} - (\rho v)_{i-\frac{1}{2}}}{\delta x}$$

as an approximation to

$$\left[\frac{\partial(\rho v)}{\partial x}\right]_i$$

Similarly, Group A for the first order time terms means that f of Eq. (8) is replaced by $p + \rho v^2$ and **f is replaced by pv** $+ \rho vE$ in Eqs. (5) and (6), respectively. In other words, Group A means that the terms on the right hand side of the hydrodynamic equations are fitted as a group. By Group B we will mean that each dependent variable on the right hand side of the hydrodynamic equations (for coefficient of δt terms) will use Eq. (8) individually, i.e.,

$$(\rho v)_{i+\frac{1}{2}} = \rho_{i+\frac{1}{2}} v_{i+\frac{1}{2}} = (\xi_1 \rho_i + \xi_2 \rho_{i+1} + \xi_3 \rho_{i-1})(\xi_1 v_i + \xi_2 v_{i+1} + \xi_3 v_{i-1})/\xi_0^2$$

for $V_T > 0$, etc. Thus, by Group B one can expand and obtain (for $V_T > 0$)

$$\begin{aligned}\frac{\rho_{i+\frac{1}{2}} v_{i+\frac{1}{2}} - \rho_{i-\frac{1}{2}} v_{i-\frac{1}{2}}}{\delta x} &= \rho_i v_i' + \rho_i' v_i + \frac{\delta x}{2}\left\{\left[\rho_i v_i'' + v_i \rho_i''\right]\left[\frac{-\xi_1 + \xi_2 - 3\xi_3}{\xi_0}\right]\right. \\ &\left. + 2\rho_i' v_i' \left[\left(\frac{\xi_2 - \xi_3}{\xi_0}\right)^2 - \left(\frac{\xi_1 + 2\xi_3}{\xi_0}\right)\right]\right\} + \frac{\delta x^2}{6}\left\{\left[\rho_i v_i'' + v_i \rho_i''\right]\right. \\ &\times \left[\frac{\xi_1 - \xi_2 + 7\xi_3}{\xi_0}\right] + 3\left[\rho_i' v_i'' + \rho_i'' v_i'\right]\left.\left[\frac{(\xi_2 - \xi_2)(\xi_2 + \xi_2) + (\xi_1 + 2\xi_3)(\xi_1 + 4\xi_3)}{\xi_0^2}\right]\right\} \\ &+ \cdots\end{aligned} \qquad (13)$$

Using Type VII (5, 2, -1) differencing, one sees that the last term in the coefficient of δx^2 does not vanish. Another instructive example of Group B is also given by considering spherical geometry. Considering the divergence in one dimension

$$\left[\frac{\partial(r^2 f)}{r^2 \partial r} = \frac{\partial f}{\partial r} + \frac{2f}{r}\right]$$

with $r_{i+\frac{1}{2}} = r_i \pm \delta r/2$ and $V_T > 0$, we find

$$\begin{aligned}\frac{r^2_{i+\frac{1}{2}} f_{i+\frac{1}{2}} - r^2_{i-\frac{1}{2}} f_{i-\frac{1}{2}}}{r_i^2\, \delta r} &= f'_i + \frac{2f_i}{r_i} + \frac{\delta r}{2}\left(f''_i + \frac{2f'_i}{r_i}\right)\left(\frac{-\xi_1 + \xi_2 - 3\xi_3}{\xi_0}\right) \\ &+ \frac{\delta r^2}{6}\left[f'''_i\left(\frac{\xi_1 + \xi_2 + 7\xi_3}{\xi_0}\right) + \frac{3f''_i}{r_i}\left(\frac{\xi_1 + \xi_2 + 5\xi_3}{\xi_0}\right) + \frac{3}{2}\frac{f'_i}{r_i^2}\right] + \cdots\end{aligned} \tag{14}$$

The last term in the coefficient of δr^2 cannot be made to vanish by any selection of ξ_1, ξ_2, and ξ_3. The advantage to using the Group B method is that, since $\rho_{i+\frac{1}{2}}$, $v_{i+\frac{1}{2}}$, and $\mathcal{E}_{i+\frac{1}{2}}$ are calculated once for each cell and used to calculate all the other terms which are combinations of them, calculation time is saved. However, the numerical examples given later in this report will indicate that the resulting truncation errors are not desirable.

For a stagnation region one may note that, if the flow is into a cell on both interfaces ($V_T < 0$ for $f_{i+\frac{1}{2}}$ and $V_T > 0$ for $f_{i-\frac{1}{2}}$),

$$\frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\delta x} = f'_i\left(\frac{2\xi_1 + 4\xi_3}{\xi_0}\right) + \frac{\delta x^2}{6} f'''_i\left(\frac{2\xi_1 + 16\xi_3}{\xi_0}\right) + \cdots \tag{15}$$

and, if the flow is out of a cell on both interfaces ($V_T > 0$ for $f_{i+\frac{1}{2}}$

and $V_T < 0$ for $f_{i-\frac{1}{2}}$),

$$\frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\delta x} = 2\,\frac{\xi_2 - \xi_3}{\xi_0}\left(f_i' + \frac{\delta x^2}{6} f_i'''\right) + \cdots \tag{16}$$

The result is that the truncation errors are essentially increased by one order for all the schemes (except Type I) considered in this report.

To summarize the methods discussed above and used in this report, we note that, for Eq. (4),

$$\overline{(\rho v)}_{i+\frac{1}{2}} = (\rho v)^n_{i+\frac{1}{2}} - \frac{\delta t}{2\delta x}\left[(p + \rho v^2)^n_{i+1} - (p + \rho v^2)^n_i\right] \tag{17}$$

The first term on the right is calculated by Group A and Group B for the various types of differencing. The second term on the right is the second order time term which is used in this report.

NUMERICAL RESULTS

The numerical solution to a steady shock was used to study the effects of truncation errors. In front of the shock all the cells contained initial values of $\rho_0 = 1.0$, $v_0 = 0$, and $\mathcal{E}_0 = 0$. With $\gamma = 5/3$, the theoretical values behind the shock are $\rho_s = 4.0$, $v_s = 1.0$, and $\mathcal{E}_s = 0.5$. The shock was assumed to enter the mesh on the left boundary, and its progress to the right was calculated as a function of time. The left interface of the first cell and all mesh values to the left of this cell, i.e., the left boundary conditions, were assumed to have the values of ρ_s, v_s, and $\mathcal{E}_s$. The theoretical shock velocity is $4/3\ v_s$. The theoretical position of the shock and values of ρ, v, and $\mathcal{E}$ are plotted as straight lines (connecting dots) in the numerical examples below.

Table 3 gives the "key" to the problems.

Table 3. Key to Problems

Problem Number	Type Differencing	δx	Group
1	I (1,1,0)	1	A
101	I (1,1,0)	1	B
2	II (1,0,0)	1	A
102	II (1,0,0)	1	B
3	III (6,3,-1)	1	A
103	III (6,3,-1)	1	B
4	IV (4,1,-1)	1	A
104	IV (4,1,-1)	1	B
5	V (7,4,1)	1	A
105	V (7,4,1)	1	B
6	VI (3,0,-1)	1	A
106	VI (3,0,-1)	1	B
7	VII (5,2,-1)	1	A
107	VII (5,2,-1)	1	B
14 (3 Plots)	I (1,1,0)	1/4	A
17 (3 Plots)	VII (5,2,-1)	1/4	A

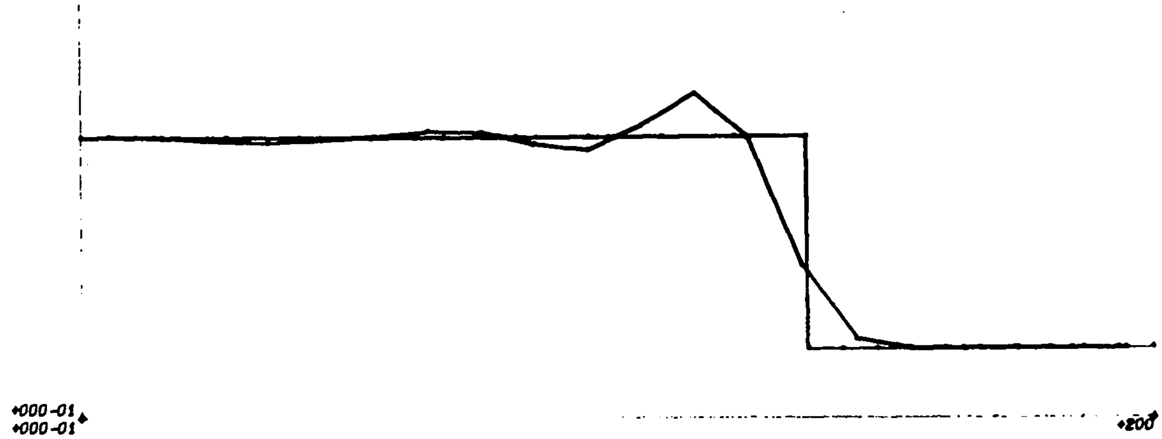
A constant value of $\delta t = 0.3\ \delta x/v_s$ was used for each problem. Values of δt from $0.4\ \delta x/v_s$ to $0.05\ \delta x/v_s$ made essentially no difference in the numerical results for the better differencing schemes. However, Problem 6 is given as an example to illustrate the difficulty of using a constant δx for all time, i.e., one should use a different

$$\delta t = 0.3 \frac{\delta x}{|v_{max}|}$$

for each cycle, where $|v_{max}|$ is the velocity of maximum magnitude which exists in the mesh at the existing time. Problem 6 is shown at a time when it has just become unstable. If δt had been calculated for each cycle in Problem 6 as indicated above, then the problem would have remained stable, but large oscillations would be evident because of the inaccuracies of the differencing scheme, i.e., due to truncation errors. Space runs from $x = 0$ to $x = 20.0$ for all problems shown.

PROBLEM RESULTS

The density versus x plots for Problems 1-7 and 101-107 and the density, velocity, and internal energy versus x plots for Problems 14 and 17 are shown on the following pages.



DENSITY VS X

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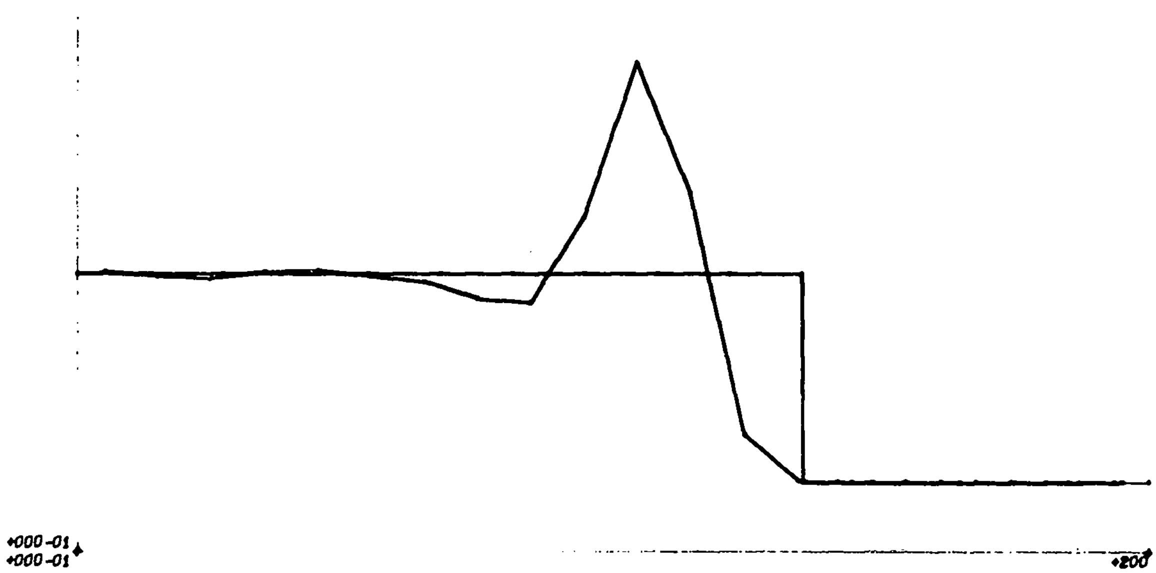

DENSITY VS X

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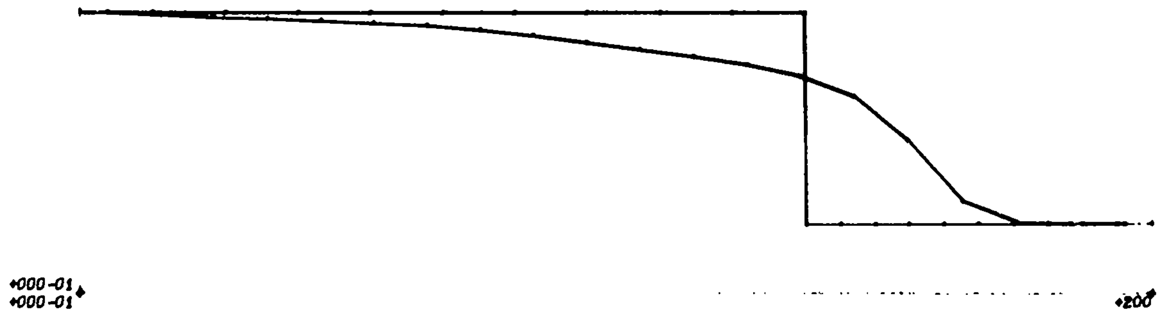

DENSITY VS X

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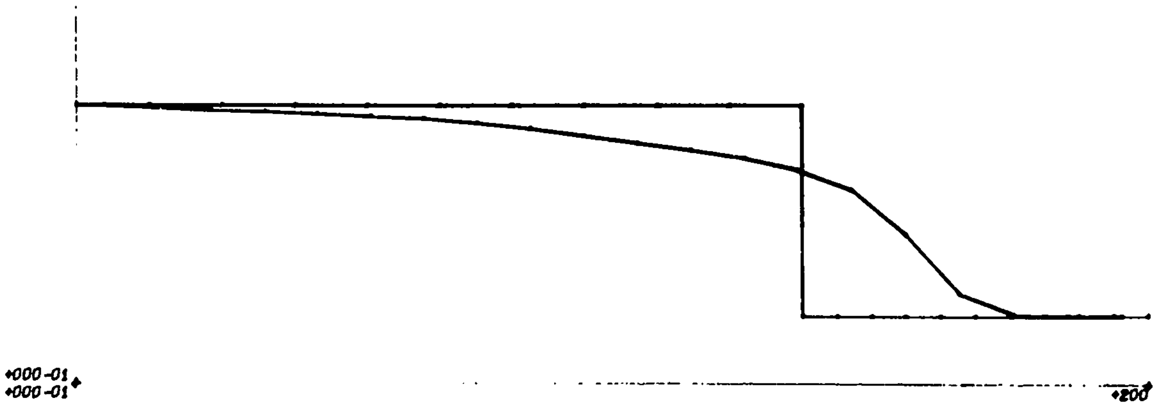

DENSITY VS X

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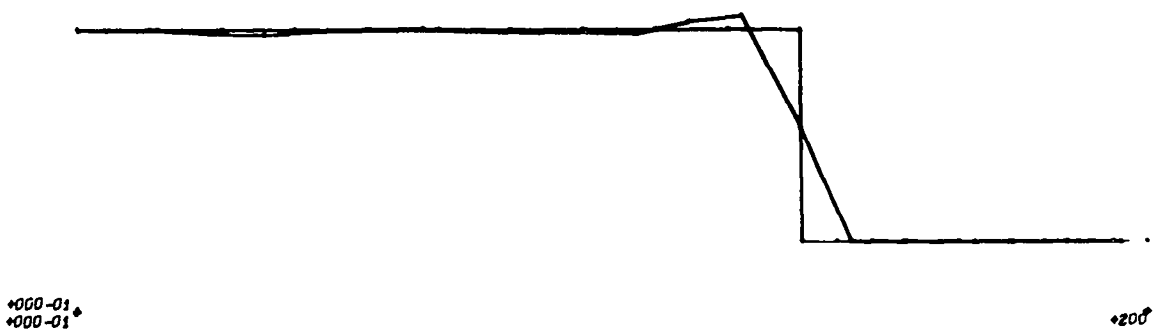
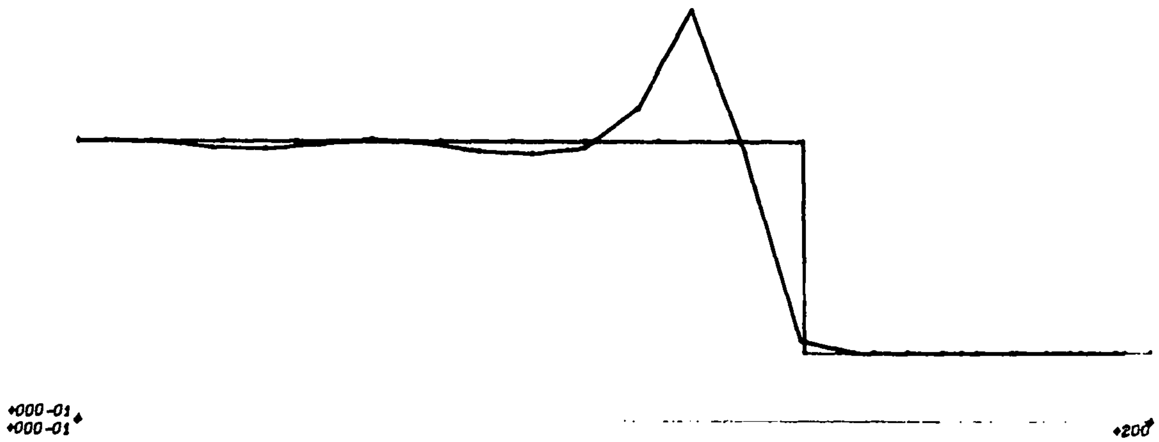


DENSITY VS X

FRCB NO = 3 DX= 1.00E 00 IM= 20 TIME= 1.020000E 01

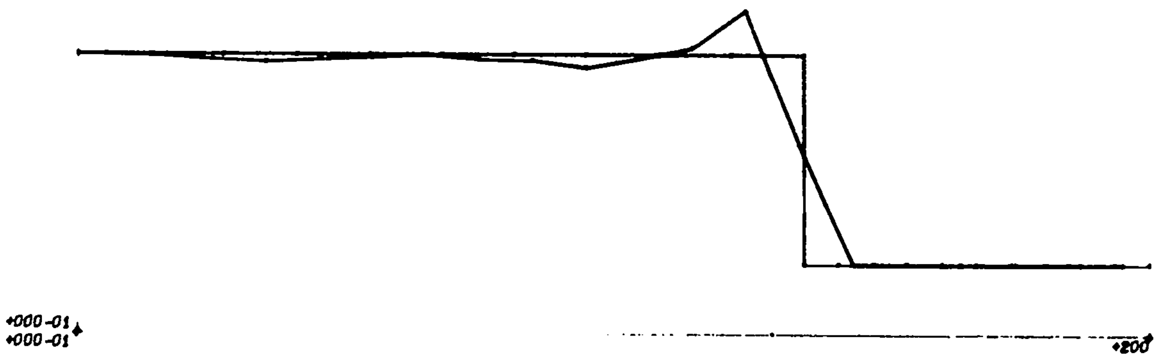


DENSITY VS X

FRCB NO = 103 DX= 1.00E 00 IM= 20 TIME= 1.020000E 01

DENSITY VS X

FRCB NO = 4 DX= 1.00E 00 IM= 20 TIME= 1.020000E 01



DENSITY VS X

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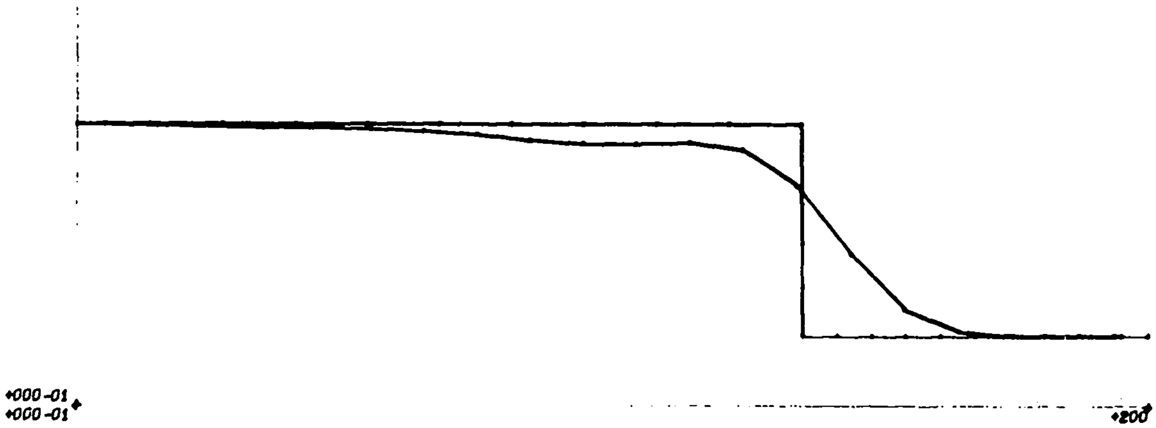

DENSITY VS X

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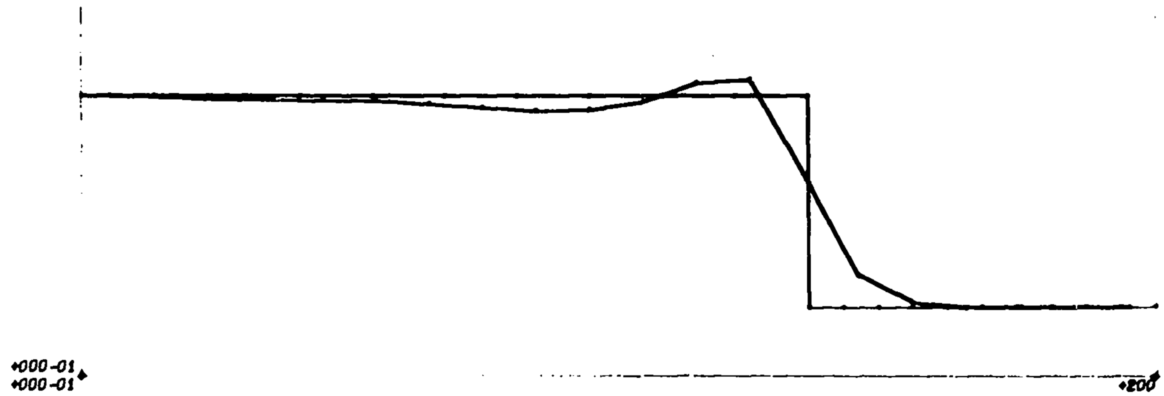

DENSITY VS X

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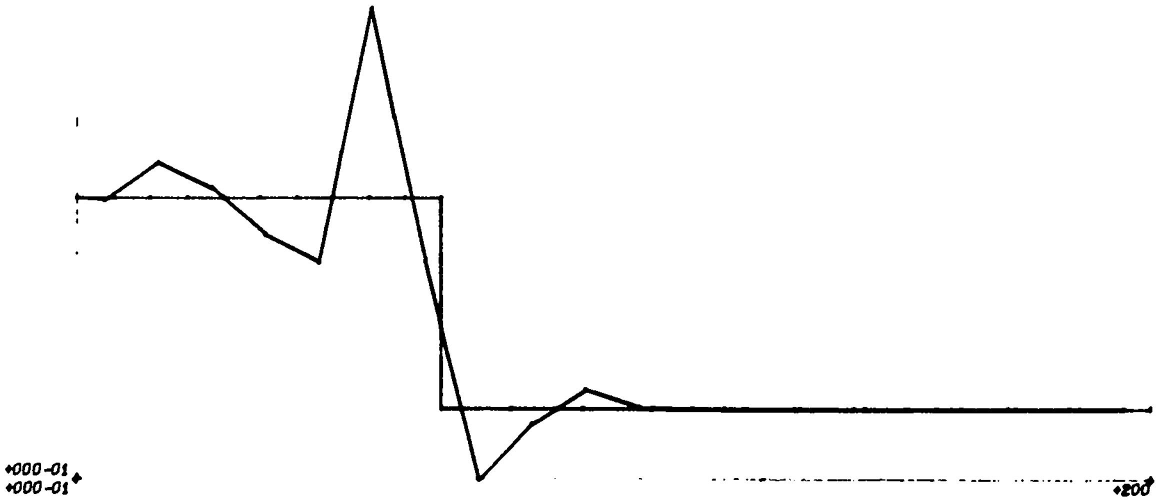
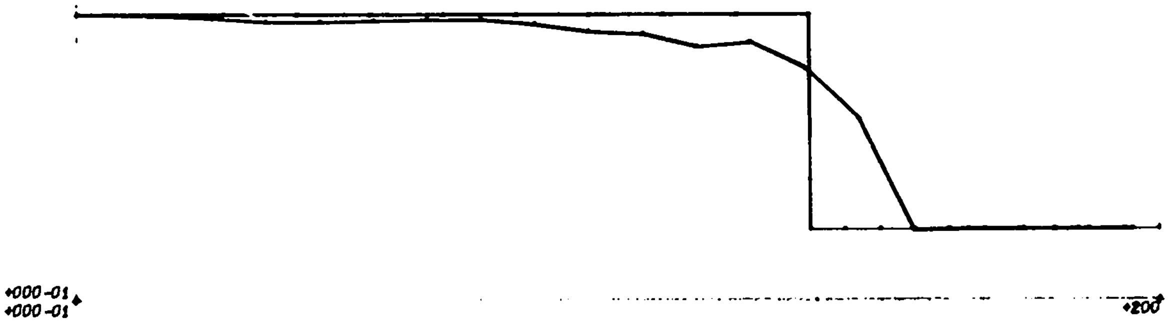


DENSITY VS X

FRCB NO = 6 DX= 1.00E 00 IM= 20 TIME= 5.100000E 00

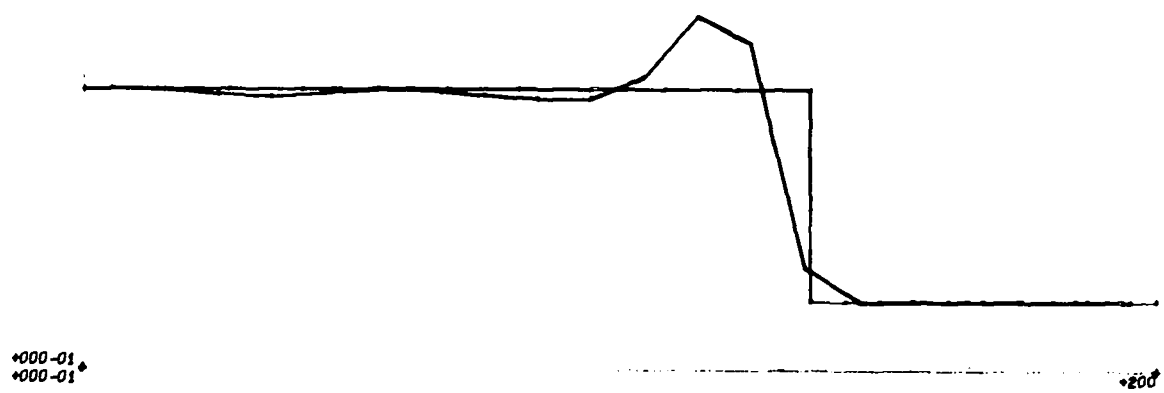


DENSITY VS X

FRCB NO = 106 DX= 1.00E 00 IM= 20 TIME= 1.020000E 01

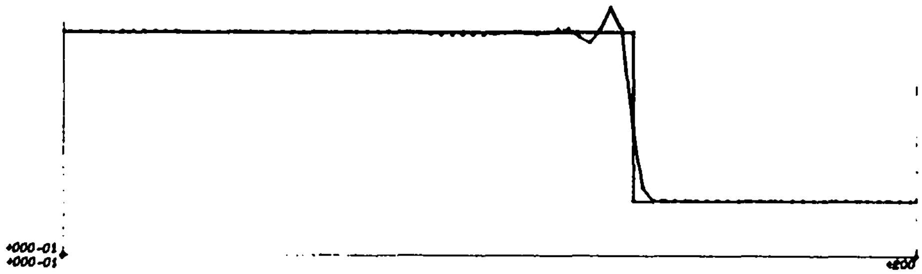
DENSITY VS X

FRCB NC = 7 DX= 1.00E 00 IM= 20 TIME= 1.020000E 01



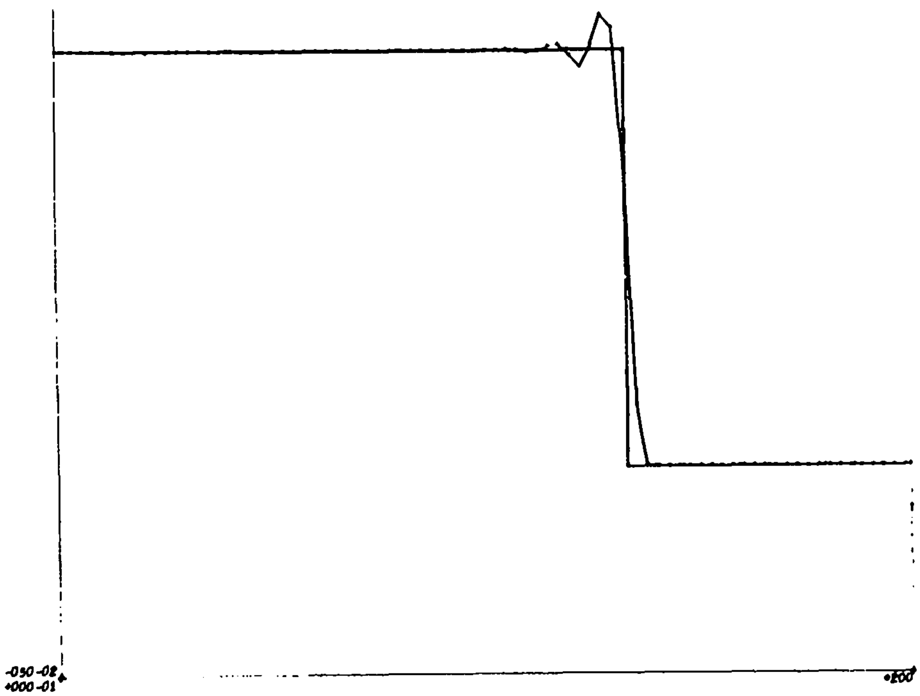
DENSITY VS X

FRCB NC = 107 DX= 1.00E 00 IM= 20 TIME= 1.020000E 01



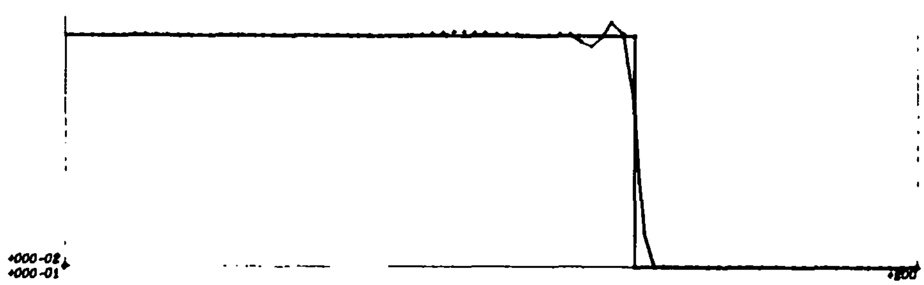
DENSITY VS X

PROB NO = 14 DX= 2.50E-01 IN= 80 TIME= 1.005000E 01



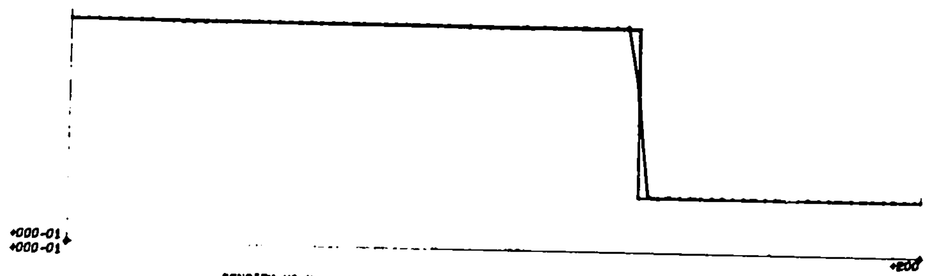
VELOCITY VS X

PROB NO = 14 DX= 2.50E-01 IN= 80 TIME= 1.005000E 01



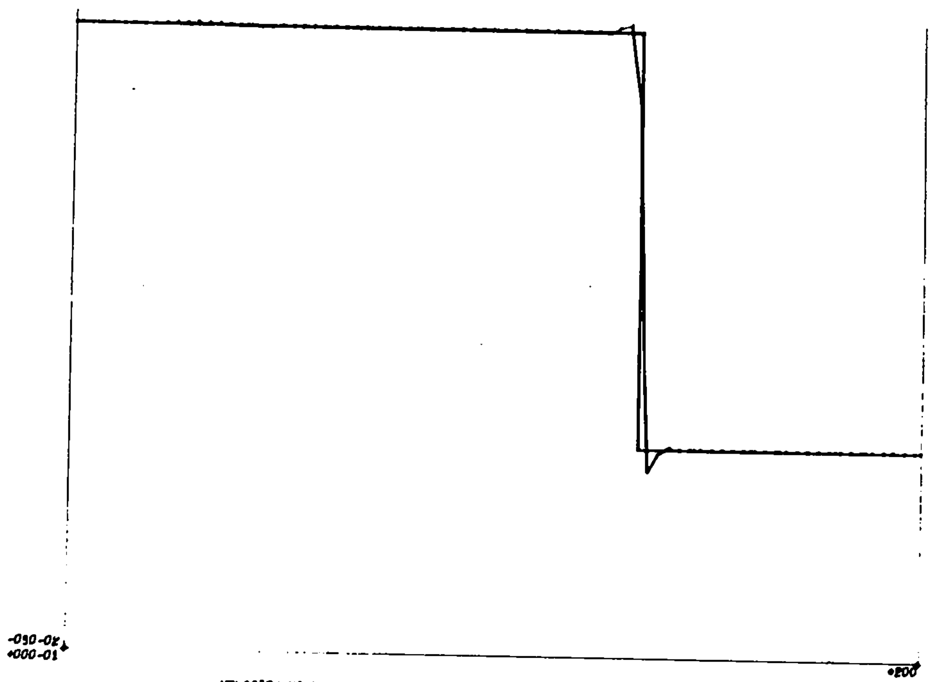
INT ENERGY VS X

PROB NO = 14 DX= 2.50E-01 IN= 80 TIME= 1.005000E 01



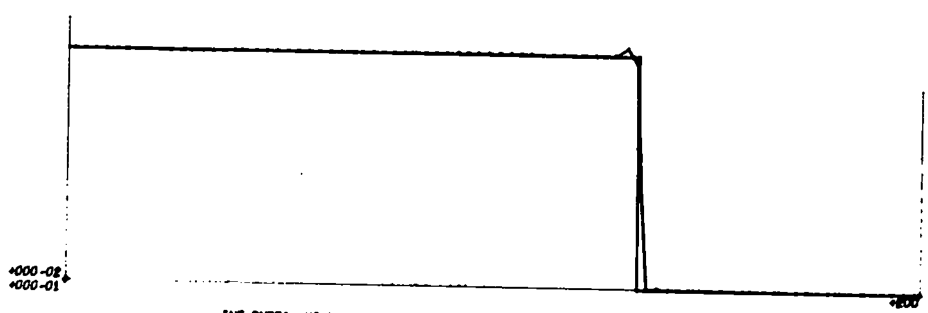
DENSITY VS X

PROB NO = 17 DX= 2.50E-01 IN= 80 TIME= 1.005000E 01



VELOCITY VS X

PROB NO = 17 DX= 2.50E-01 IN= 80 TIME= 1.005000E 01



INT ENERGY VS X

PROB NO = 17 DX= 2.50E-01 IN= 80 TIME= 1.005000E 01

CONCLUSIONS

The problems show that grouping by Group A definitely gives fewer truncation errors than by Group B. The exception, of course, is shown for Type II differencing (one point scheme) where both groupings should be the same.

A comparison of the theoretically expected truncation errors given in Table 2 with the results in Problems 1 to 7 (Group A) is enlightening. That Type II differencing gives about twice the diffusion as that given by Type V is evident. The effect of a smaller magnitude coefficient in the δx^2 term for Types III, IV, and VII as compared to Type I is evident. A similar comparison can be made for the effect of the second order space term between Types I and VI.

In general, the second order time terms in conjunction with accurate space differencing gives a very fast and accurate differencing scheme. A look at the density versus x plot of Problem 17 shows the shock front to be essentially two cells wide. It is suprising that so few terms in the expansions are necessary to produce this near perfect moving step function. It should be noted by observation of Eq. (11) and Table 2 that no viscosity[1] (explicit or otherwise) is used to "smear" the shock and that if any viscous type effects are present, they are produced by the fourth space derivatives.

REFERENCES

1. Longley, H. J., Los Alamos Report LAMS-2379 (1960).

2. Lax, Peter D. and Wendroff, Burton, NYO-9759, 17 Ed. "Difference Schemes with High Order of Accuracy for Solving Hyperbolic Equations" July 31, 1962.