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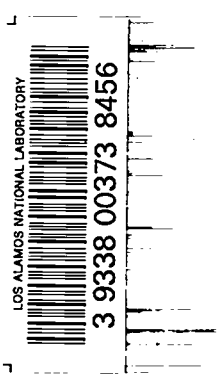
On Frequency Discretization and Conservative Difference Equations for Radiative Transfer



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On Frequency Discretization and Conservative Difference Equations for Radiative Transfer

by

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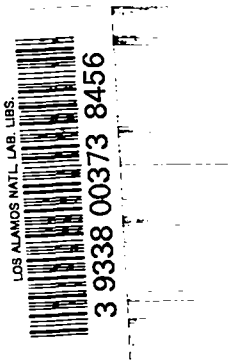

ABSTRACT

In a previous publication, LAMS-2795, On the Computation of the Transport of Radiation (1962), we obtained finite difference approximations to the transport equation by the method of characteristics. In this paper that method is generalized to obtain completely conservative difference equations. We also propose a method of frequency discretization which allows retention of a large part of the information ordinarily lost if the absorption coefficient is simply averaged. It is shown that the difference equations behave correctly in the limit of large or small absorption coefficient.

1. The Method

The transport of photons in a medium which is in local thermodynamic equilibrium and in which scattering can be neglected is described by the transport equation:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\omega}\cdot\nabla\right) N = \sigma(B - N) \tag{1.1}$$

where $N = N(r,t,\vec{\omega},\nu)$ is the specific photon intensity at the space time point (r,t) in the direction $\vec{\omega}$, at the frequency ν. B is the Planck function

$$B(T,\nu) = \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT} - 1\right)^{-1}$$

$T = T(r,t)$ is the temperature of the matter. The absorption coefficient σ depends on temperature and frequency. T and N are further related by the energy balance equation:

$$\begin{aligned}\frac{\partial E_M}{\partial t} &= -\int_0^\infty \sigma\left(4\pi B - \int N d\omega\right) d\nu \\ &= -\left(\frac{\partial E_R}{\partial t} + \operatorname{div}\vec{F}\right)\end{aligned} \tag{1.2}$$

where

$$E_R = \int_0^\infty d\nu \frac{1}{c}\int N d\omega$$

$$\vec{F} = \int_0^\infty d\nu \int N \vec{\omega}\, d\omega$$

E_M = material energy density

Let the transport operator be

$$L = \frac{1}{c}\frac{\partial}{\partial t} + \vec{\omega}\cdot\nabla$$

We will discuss only two cases: the slab,

$$L = \frac{1}{c}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial r} \tag{1.3}$$

and the sphere,

$$L = \frac{1}{c}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu} \tag{1.4}$$

The parameter μ varies from -1 to 1 and

$$\int N d\omega = 2\pi \int_{-1}^{1} N d\mu$$

$$\int N\vec{\omega}\, d\omega = 2\pi \int_{-1}^{1} N\mu d\mu$$

To obtain a difference approximation to the transport equation we shall treat the left and right sides of (1.1) separately. The requirement that the equations be conservative will determine the left side approximation, and the need for correct limiting behavior will determine the right side. This approach was suggested to us by [2].

To obtain an approximate operator $\bar{L}$ for L, we appeal to the method of discrete ordinates as formulated by Carlson and Lathrop in [1].

We urge the reader to consult that report for motivation and details. Here, we shall only write down the appropriate $\bar{L}$ in the notation of [1]. For a given lattice of points (r_i, t_i, μ_m), and for any function $f(r,t,\mu)$, let

$$f_i = f(r_i, t_{s+\frac{1}{2}}, \mu_m)$$

$$f_s = f(r_{i+\frac{1}{2}}, t_s, \mu_m)$$

$$f_{m+\frac{1}{2}} = f(r_{i+\frac{1}{2}}, t_{s+\frac{1}{2}}, \mu_{m+\frac{1}{2}})$$

Then for the slab,

$$\bar{L}N = \frac{N_{s+1} - N_s}{c\Delta t} + \mu_m \frac{N_{i+1} - N_i}{r_{i+1} - r_i} \tag{1.5}$$

with auxiliary condition

$$N_{s+1} + N_s = N_{i+1} + N_i \tag{1.6}$$

See [4] for a discussion of the stability of this scheme.

For the sphere

$$\bar{L}N = \frac{N_{s+1} - N_s}{c\Delta t} + \mu_m \frac{(A_{i+1}N_{i+1} - A_iN_i)}{V} + \frac{\alpha_{m+\frac{1}{2}}N_{m+\frac{1}{2}} - \alpha_{m-\frac{1}{2}}N_{m-\frac{1}{2}}}{Vw_m} \tag{1.7}$$

with auxiliary conditions

$$N_{s+1} + N_s = N_{i+1} + N_i = N_{m+\frac{1}{2}} + N_{m-\frac{1}{2}} \tag{1.8}$$

where

$$V = \frac{4\pi}{3}\left(r_{i+1}^3 - r_i^3\right)$$

$$A_i = 4\pi r_i^2$$

and the α's are non-negative and satisfy

$$\frac{\alpha_{m+\frac{1}{2}} - \alpha_{m-\frac{1}{2}}}{Vw_m} = -\frac{\mu_m(A_{i+1} - A_i)}{V}$$

$$\alpha_{\frac{1}{2}} = \alpha_{M+\frac{1}{2}} = 0 \tag{1.9}$$

The cosines μ_m are chosen symmetric about zero, and the weights w_m satisfy

$$\begin{cases} \Sigma\, w_m = 2 \\ \Sigma\, w_m \mu_m = 0 \\ \Sigma\, w_m \mu_m^2 = \frac{2}{3} \end{cases} \tag{1.10}$$

It follows from (1.5), (1.7), and (1.9) that

$$\overline{L}N = \gamma(N_P - N_Q) \tag{1.11}$$

where N_P and N_Q are linear combinations of N's at mesh points, each with non-negative coefficients adding up to one. For example, in the sphere, if $\mu_n > 0$

$$N_P = \frac{\frac{1}{c\Delta t} N_{s+1} + \frac{\mu A_{i+1}}{V} N_{i+1} + \frac{\alpha_{m+\frac{1}{2}}}{Vw_m} N_{m+\frac{1}{2}}}{\frac{1}{c\Delta t} + \frac{\mu A_{i+1}}{V} + \frac{\alpha_{m+\frac{1}{2}}}{Vw_m}}$$

$$N_Q = \frac{\frac{1}{c\Delta t} N_s + \frac{\mu A_i N_i}{V} + \frac{\alpha_{m-\frac{1}{2}}}{Vw_m} N_{m-\frac{1}{2}}}{\frac{1}{c\Delta t} + \frac{\mu A_i}{V} + \frac{\alpha_{m-\frac{1}{2}}}{Vw_m}}$$

$$\gamma = \frac{1}{c\Delta t} + \frac{\mu A_{i+1}}{V} + \frac{\alpha_{m+\frac{1}{2}}}{Vw_m}$$

$$= \frac{1}{c\Delta t} + \frac{\mu A_i}{V} + \frac{\alpha_{m-\frac{1}{2}}}{Vw_m}$$

Let us now turn to the right side of (1.1). Introduce the characteristic parameter s, where

$$\frac{d}{ds} = \frac{1}{c}\frac{\partial}{\partial t} + \omega \cdot \nabla$$

Then on any characteristic curve

$$N(s) = N_0 e^{-\int_0^s \sigma\, ds'} + \int_0^s \sigma B\, e^{-\int_{s'}^s \sigma\, ds''} ds'$$

Let Q be some point in the (r,μ,t) space. Let $s(Q) = 0$, and measure off along the characteristic through Q a positive length Δs. Then

$$\int_0^{\Delta s} \sigma(B - N)ds = N_0\left(e^{-\int_0^{\Delta s} \sigma\, ds} - 1\right) + \int_0^{\Delta s} \sigma B\, e^{-\int_s^{\Delta s} \sigma\, ds'} ds \tag{1.12}$$

where N_0 is N at Q.

The following approximate form is assumed for σ and B:

$$\sigma(s) = \begin{cases} \sigma_- , & 0 \le s \le \frac{\Delta s}{2} \\ \sigma_+ , & \frac{\Delta s}{2} \le s \le \Delta s \end{cases}$$

$$B(s) = \begin{cases} B_- + D_- s, & 0 \le s \le \frac{\Delta s}{2} \\ B_+ + D_+(s - \Delta s), & \frac{\Delta s}{2} \le s \le \Delta s \end{cases}$$

Split the last term of (1.12) into two parts

$$\int_0^{\Delta s} = \int_0^{\frac{\Delta s}{2}} + \int_{\frac{\Delta s}{2}}^{\Delta s}$$

In the last part replace σ by σ_+ and in the first part replace σ by σ_-. (The latter is not quite correct, but any more accurate recipe will lead to complications.) With the assumed form of B we find that

$$\begin{aligned}\int_0^{\Delta s} \sigma B e^{-\int^{\Delta s} \sigma\, ds'} ds &= B_+ \left[1 - e^{-\sigma_+ \frac{\Delta s}{2}}\right] \\ &+ D_+ \left[\frac{\Delta s}{2} e^{-\sigma_+ \frac{\Delta s}{2}} - \frac{1}{\sigma_+}\left(1 - e^{-\sigma_+ \frac{\Delta s}{2}}\right)\right] \\ &+ B_- \left[e^{-\sigma_- \frac{\Delta s}{2}} - e^{-\sigma_- \Delta s}\right] \\ &+ D_- \left[\frac{\Delta s}{2} e^{-\sigma_- \frac{\Delta s}{2}} - \frac{1}{\sigma_-}\left(e^{-\sigma_- \frac{\Delta s}{2}} - e^{-\sigma_- \Delta s}\right)\right]\end{aligned} \tag{1.13}$$

Now, in (1.1) we will replace $\sigma(B - N)$ by

$$\frac{1}{\Delta s} \int_0^{\Delta s} \sigma(B - N) ds$$

as computed above. Then

$$\Delta s\, \gamma(N_P - N_Q) = N_0 \left(e^{-\int_0^{\Delta s} \sigma\, ds} - 1\right) + \int_0^{\Delta s} \sigma B e^{-\int_s^{\Delta s} \sigma\, ds'} ds$$

It turns out for the slab that the linear combinations in N_P and N_Q define two points on the same characteristic and that the distance

between these points is γ^{-1}, i.e.

$$N_Q = N_O$$
$$\Delta s = \gamma^{-1} \tag{1.14}$$

This is not the case for the sphere, but we will nevertheless make this assumption as an approximation. Thus

$$N_P = N_Q e^{-\int_0^{\Delta s} \sigma \, ds} + \int_0^{\Delta s} \sigma B e^{-\int_0^{\Delta s} \sigma \, ds'} ds \tag{1.15}$$

The above is the generalized characteristic difference equation. It differs from the equation considered in [3] in that N_P and N_Q are now defined so as to conserve photons.

Turning to the problem of frequency discretization, let

$$N^g = \int_{\nu_g}^{\nu_{g+1}} N d\nu$$

Then

$$N_P^g = \int_{\nu_g}^{\nu_{g+1}} N_Q e^{-\int_0^{\Delta s} \sigma \, ds} + \int_{\nu_g}^{\nu_{g+1}} \int_0^{\Delta s} \sigma B e^{-\int_s^{\Delta s} \sigma \, ds'} ds$$

Little can be done about the first integral above except to introduce some mean σ^g, giving

$$N_P^g = N_Q^g e^{-\int_0^{\Delta s} \sigma^g ds} + \int_{\nu_g}^{\nu_{g+1}} \int_0^{\Delta s} \sigma B e^{-\int_{s'}^{s} \sigma \, ds''} ds' \tag{1.16}$$

We propose that the last term in (1.12) be computed "exactly", as follows:

The quantities D represent $\frac{dB}{ds}$, so let

$$D = \frac{dB}{ds} = \frac{dT}{ds}\frac{\partial B}{\partial T} = \bar{D}\frac{\partial B}{\partial T}$$

We will in fact write

$$\bar{D} = \frac{1}{4T^3}\frac{dT^4}{ds}$$

Introduce the following functions

$$g_0 = \int B(T)\, d\nu$$

$$g_1 = \int B(T)\, e^{-\sigma(T)\frac{\Delta s}{2}}\, d\nu$$

$$g_2 = \int \frac{\partial B}{\partial T}\left[\frac{\Delta s}{2} e^{-\sigma(T)\frac{\Delta s}{2}} - \frac{1}{\sigma(T)}\left(1 - e^{-\sigma(T)\frac{\Delta s}{2}}\right)\right] d\nu$$

$$g_3 = \int B(T)\left[e^{-\sigma(T)\frac{\Delta s}{2}} - e^{-\sigma(T)\Delta s}\right] d\nu$$

$$g_4 = \int \frac{\partial B}{\partial T}\left[\frac{\Delta s}{2} e^{-\sigma(T)\frac{\Delta s}{2}} - \frac{1}{\sigma(T)}\left(e^{-\sigma(T)\frac{\Delta s}{2}} - e^{-\sigma(T)\Delta s}\right)\right] d\nu$$

$$g_5 = \frac{\int e^{-\sigma\Delta s}\, B d\nu}{\int B d\nu}$$

Note that these functions depend on one more parameter than do the functions now tabulated in transport codes.

If in (1.13) we make the correspondence

$P \longleftrightarrow +$

$Q \longleftrightarrow -$

then (1.16) becomes

$$N_P^g = g_5 N_Q^g + g_0(P) + g_1(P) + \bar{D}_P g_2(P) + g_3(Q) + \bar{D}_Q g_4(Q) \tag{1.17}$$

There still remains the actual specification of the $g_k(P)$, $g_k(Q)$, and $\bar{D}_P$, $\bar{D}_Q$. This will be determined by the requirement that we obtain a diffusion approximation for large σ, so we leave this point to the next section.

The only remaining point here is the energy balance equation, but we only need to note that

$$2\pi \sum_{\text{angle}} \sum_{g} \bar{L}N^g w_m$$

is our approximation to

$$2\pi \int_0^\infty d\nu \int_{-1}^{1} d\mu \left(\frac{1}{c} \frac{\partial N}{\partial t} + \vec{\omega} \cdot \nabla N \right)$$

and therefore the energy balance has to be

$$2\pi V\Delta t \sum_{\text{angle}} \sum_{g} \bar{L}N^g w_m = - \int_{\substack{\text{space-time} \\ \text{cell}}} \frac{\partial E_m}{\partial t} dV dt \tag{1.18}$$

We now have a completely conservative difference scheme in the following sense: $\bar{L}$ is a locally conservative difference operator, and the exchange of energy between the radiation field and the matter as expressed in the approximate transport equation is exactly matched by the exchange of energy in the energy balance equation. A price has been paid for

this, for we can no longer guarantee that the intensity at each mesh point is positive, since the auxiliary conditions (1.6) involve an extrapolation. We feel this is not serious, because as we shall see in the next section we obtain the correct behavior for large σ. Any negative intensities encountered should be kept during one time step and then set to zero before proceeding to the next step. Another problem that arises is that our equations may have a very poor truncation error, particularly at the center of a sphere.

2. Limiting Forms

(a) Thick sphere. This is taken to mean that

$$e^{-\sigma \Delta s} B = 0$$

at all frequencies. Then from (1.17) using one frequency interval

$$N_P = g_0(P) + \bar{D}_P g_2(P) \tag{2.1}$$

Let f stand for any one of g_0, g_1, $\bar{D}g_2$, g_3, and $\bar{D}g_4$. Let P_s, P_i, and $P_{m+\frac{1}{2}}$ be the (r,t,μ) points corresponding to N_s, N_i, $N_{m+\frac{1}{2}}$, and let $P^s_{i,m} = (r_i, t_s, \mu_m)$.

Now, N_P is a linear combination of N's at certain diamond points. For example, we might have

$$N_P = a\, N_{s+1} + b\, N_{i+1} + c\, N_{m+\frac{1}{2}}$$

Then we define f(P) in (1.16) as the same linear combination, in this case

$$f(P) = a\, f(P_{s+1}) + b\, f(P_{i+1}) + c\, f(P_{m+\frac{1}{2}}) \tag{2.2}$$

In addition, N satisfies the auxiliary conditions (1.8). If

$$\left\{\begin{aligned} f(P_{s+1}) &= \frac{1}{4}\left[f\left(P^{s+1}_{i+1,m+\frac{1}{2}}\right) + f\left(P^{s+1}_{i,m+\frac{1}{2}}\right) + f\left(P^{s+1}_{i+1,m-\frac{1}{2}}\right) + f\left(P^{s+1}_{i,m-\frac{1}{2}}\right)\right] \\ f(P_{i+1}) &= \frac{1}{4}\left[f\left(P^{s+1}_{i+1,m+\frac{1}{2}}\right) + f\left(P^{s}_{i+1,m+\frac{1}{2}}\right) + f\left(P^{s+1}_{i+1,m-\frac{1}{2}}\right) + f\left(P^{s}_{i+1,m-\frac{1}{2}}\right)\right] \\ f(P_{m+\frac{1}{2}}) &= \frac{1}{4}\left[f\left(P^{s+1}_{i+1,m+\frac{1}{2}}\right) + f\left(P^{s}_{i+1,m+\frac{1}{2}}\right) + f\left(P^{s+1}_{i,m+\frac{1}{2}}\right) + f\left(P^{s}_{i,m+\frac{1}{2}}\right)\right] \end{aligned}\right\} \quad (2.3)$$

then f will also satisfy the auxiliary conditions. It now follows by induction that

$$N_{s+1} = g_0(P_{s+1}) + \bar{D}_{P_{s+1}}\, g_2(P_{s+1}) \quad (2.4)$$

and so on for N_{i+1}, $M_{m+\frac{1}{2}}$.

It is only necessary to arrange that (2.3) hold when $e^{-\sigma\Delta s}\, B$ is zero. Therefore Δs can be held fixed at the value given by (1.14). The $f(P^s_{i,m+\frac{1}{2}})$ will be multiple-valued, the value depending on which cell adjacent to $P^s_{i,m+\frac{1}{2}}$ is being considered, except when $e^{-\sigma\Delta s}\, B = 0$. The remaining variables are defined as follows:

Let T be given at cell centers and integral times, i.e.

$$T^s_{i+\frac{1}{2}}$$

Let

$$T^s_i = \frac{1}{2}\left(T^s_{i+\frac{1}{2}} + T^s_{i-\frac{1}{2}}\right)$$

Whenever B(T) appears in $f(P^s_{i,m+\frac{1}{2}})$ substitute T^s_i for T. Whenever σ appears substitute

$$\frac{1}{2}\left[\sigma_{i-\frac{1}{2}}\left(T_i^s\right) + \sigma_{i+\frac{1}{2}}\left(T_i^s\right)\right.$$

Whenever $\bar{D}$ appears use one of the following:

$$\frac{1}{4\left(T_i^s\right)^3}\,\mu_{m+\frac{1}{2}}\,\frac{\left(T_{i+\frac{1}{2}}^s\right)^4 - \left(T_{i-\frac{1}{2}}^s\right)^4}{\Delta r} \tag{2.5}$$

or

$$\frac{1}{4\left(T_i^s\right)^3}\left[\frac{1}{c}\,\frac{\left(T_i^s\right)^4 - \left(T_i^{s-1}\right)^4}{\Delta t} + \mu_{m+\frac{1}{2}}\,\frac{\left(T_{i+\frac{1}{2}}^s\right)^4 - \left(T_{i-\frac{1}{2}}^s\right)^4}{\Delta r}\right] \tag{2.6}$$

In the above we have used the fact that

$$\bar{D} = \frac{1}{4T^3}\,\frac{dT^4}{ds} = \frac{1}{4T^3}\left(\frac{1}{c}\,\frac{\partial T^4}{\partial t} + \mu\,\frac{\partial T}{\partial r}\right)$$

In (2.5), $\frac{1}{c}\,\frac{\partial T^4}{\partial t}$ is omitted, which will lead to the usual diffusion approximation.

From (2.3) and (2.4) we have (with 2.5)

$$N_{s+1} = \frac{ac}{4\pi}\cdot\frac{1}{2}\left[\left(T_{i+1}^{s+1}\right)^4 + \left(T_i^{s+1}\right)^4\right] \tag{2.7}$$

$$- \frac{ac}{\pi}\,\frac{1}{4}\left(\mu_{m+\frac{1}{2}} + \mu_{m-\frac{1}{2}}\right)\left[\frac{\left(T_{i+\frac{1}{2}}^{s+1}\right)^4 - \left(T_{i+\frac{1}{2}}^{s+1}\right)^4}{\bar{\sigma}_{i+1}^{s+1}\,\Delta r} + \frac{\left(T_{i+\frac{1}{2}}^{s+1}\right)^4 - \left(T_{i-\frac{1}{2}}^{s+1}\right)^4}{\bar{\sigma}_i^{s+1}\,\Delta r}\right]$$

where

$$\bar{\sigma}_i^{s+1} = \left[\int \frac{\partial B}{\partial T}\,\frac{2}{\sigma_{i+\frac{1}{2}}^{s+1} + \sigma_{i-\frac{1}{2}}^{s+1}}\,d\nu\right]^{-1}$$

$$N_{i+1} = \frac{ac}{4\pi} \cdot \frac{1}{2}\left[\left(T_{i+1}^{s+1}\right)^4 + \left(T_{i+1}^{s}\right)^4\right]$$

$$- \frac{ac}{\pi}\frac{1}{4}\left(\mu_{m+\frac{1}{2}} + \mu_{m-\frac{1}{2}}\right)\left[\frac{\left(T_{i+\frac{3}{2}}^{s+1}\right)^4 - \left(T_{i+\frac{1}{2}}^{s+1}\right)^4}{\bar{\sigma}_{i+1}^{s+1}\Delta r} + \frac{\left(T_{i+\frac{3}{2}}^{s}\right)^4 - \left(T_{i+\frac{1}{2}}^{s}\right)^4}{\bar{\sigma}_{i+1}^{s}\Delta r}\right] \tag{2.8}$$

There is no need to write $N_{m+\frac{1}{2}}$, for it drops out of the energy balance. If we arrange that

$$\mu_m = \frac{1}{2}\left(\mu_{m+\frac{1}{2}} + \mu_{m-\frac{1}{2}}\right)$$

then using (2.7) and (2.8) with (1.7), (1.10), and (1.18)the reader can see that we have an implicit diffusion approximation.

(b) Thin sphere. This is taken to mean that for any function $p(\sigma)$,

$$\int p(\sigma)\, B d\nu = \int [p(0) + p'(0)\sigma]\, B d\nu$$

We will not attempt to give the precise limiting form in this case, but will only indicate what is happening. First, it follows from (1.17) and the definitions of the g_k that if N(incoming) is zero then

$$N_P = O(\sigma)$$

and therefore that N_s, N_i, and $N_{m+\frac{1}{2}}$ are $O(\sigma)$. Then from (1.11)

$$2\pi V\Delta t \sum \bar{L}N\, w_m = 2\pi V\Delta t \sum (\Delta s)^{-1}(N_P - N_Q)\, w_m$$

$$= 2\pi V \Delta t \sum \frac{w_m}{\Delta s}\int d\nu \int_0^{\Delta s} \sigma B ds + O(\sigma^2)$$

Now,

$$\frac{1}{2}\sum \frac{w_m}{\Delta s}\int_0^{\Delta s} \sigma B ds$$

represents an average value of σB. Admittedly it is not an average which one would ordinarily choose, but it is a legitimate average. Thus, the energy balance is

$$4\pi V\Delta t\ (\sigma B)_{\text{average}} = -\int_{\substack{\text{space-time}\\ \text{cell}}} \frac{\partial E_M}{\partial t}\, dtdV$$

which is the correct form.

3. Solving the Equations

For a given temperature distribution (1.17) and the auxiliary conditions form a linear system that is easily solved. For example, in the slab let r_{i_0} be the left boundary, r_{i_1} the right boundary. The boundary conditions will be that N_{i_0} is given for $\mu_m > 0$, N_{i_1} given for $\mu_m < 0$. If t_{s_0} is the initial time then N_{s_0} is also given. If we start with the cell (r_{i_0}, r_{i_0+1}), (t_{s_0}, t_{s_0+1}) for $\mu_m > 0$, in (1.17) N_Q is known and (1.17) and (1.6) determine N_{i_0+1}, N_{s_0+1}. We continue sweeping to the right until all $N_i(\mu_m > 0)$ are computed. A similar sweep from right to left determines all $N_i(\mu_m < 0)$, and all the N_{s_0+1} are also now known, so the next time cycle could start if the temperatures were correct.

Some sort of iteration is necessary to obtain the temperature at the new time t_{s+1}. The following is one possibility. First, in the functions g_k change variables from ν to $u = h\nu/\kappa T$. This brings out a factor T^4 in g_0, g_1, and g_3, and a factor T^3 in g_2 and g_4 which cancels

the T^3 in the denominator of $\bar{D}$. Each $\bar{D}$ contains a difference of fourth powers of T. These should be factored into a difference of T's times a cubic polynomial in T. The other T^4 should be written as TT^3. For the interval (r_i, r_{i+1}) we now have the form

$$N_P = d_{i+\frac{1}{2}} + a_{i-\frac{1}{2}} T^{s+1}_{i-\frac{1}{2}} + b_{i+\frac{1}{2}} T^{s+1}_{i+\frac{1}{2}} + c_{i+\frac{3}{2}} T^{s+1}_{i+\frac{3}{2}} \tag{3.1}$$

where a, b, c, d, and g_5 still depend on T^{s+1}, and d depends on N. Then

$$2\pi V \Delta t \sum_{\text{angle}} \sum_{g} \bar{L}\, N^g w_m = D_{i+\frac{1}{2}} + A_{i-\frac{1}{2}} T^{s+1}_{i-\frac{1}{2}} + B_{i+\frac{1}{2}} T^{s+1}_{i+\frac{1}{2}} + C_{i+\frac{3}{2}} T^{s+1}_{i+\frac{3}{2}} \tag{3.2}$$

By linearizing the contribution from the material energy the entire energy balance equation can be made tri-diagonal in T^{s+1}.

Let $\bar{T}$ be the previous iterate for T^{s+1}. Replace T^{s+1} by $\bar{T}$ in (3.1), and solve for all the N, and also compute the coefficients A, B, C, D by accumulating the appropriate sums. Solve the linearized energy balance equation for T^{s+1}. It is important that (3.1) be computed as it is written; otherwise, loss of significance in computing differences of T^4 will cause trouble. The iteration should be repeated until good energy conservation occurs.

The proposed iteration will work well in thick or near thick problems. It is liable to be slow in thin problems.

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