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ABSTRACT

Classical field equations are derived from quantum fields to obtain a different and possibly simpler description of a collisionless plasma. The method is to take the simultaneous limit, $\hbar$, e, m $\to$ 0, of charged scalar fields and the electromagnetic field. Laplace transforms for perturbations in a uniform relativistic plasma are compared with corresponding results from the Maxwell-Vlasov equations. For the nonlinear case, a distribution function defined on the classical fields is shown to satisfy the Vlasov equation.

I. DERIVATION

We are interested in finding a different description of a collisionless plasma, in particular, one with about the same physical content as the standard description but which may, in some cases, have a simpler mathematical structure. It is useful to examine a derivation of the standard description the Vlasov equation. An ensemble average over classical ionized particles gives the BBGKY hierarchy of equations.[1] When second order correlations are neglected, the first equation of the hierarchy reduces to the Vlasov equation. It is generally assumed that in the limit e, m $\to$ 0, both e/m and $n_e m$ remaining constant, where n_e, e, and m are the particle density, charge, and mass, respectively, second order correlations disappear. We may then write

$$S_{cp} = \lim_{e,m \to 0} \sum S_c \qquad (1)$$

where S_c represents the equations for classical particles, S_{cp} the equations for a collisionless plasma, and Σ an ensemble average. Because the classical system itself is a limiting case ($\hbar \to 0$) of a quantum system S_Q,

$$S_{cp} = \lim_{e,m \to 0} \sum \lim_{\hbar \to 0} S_Q \quad . \qquad (2)$$

Here we investigate the case where some of the limits are exchanged. The initial limit is

$$S'_{cp} = \lim_{e,m,\,\hbar \to 0} S_Q \; , \qquad (3)$$

where the ratios of e, m, and $\hbar$ remain constant.

Other derivations of plasma phenomena from quantum systems[2-4] have corresponded to the limit $\hbar \to 0$. Because these limits are singular, their exchange may not give the same results. The results must be investigated in each case. We neglect spin effects; S_Q is represented by charged scalar fields (ϕ_q) for each charge species (q) and the electromagnetic field (A). The equations are

$$\partial^\mu \partial_\mu A^\nu = 4\pi \sum_q i \tag{4}$$

$$\cdot \alpha_q \left[\phi_q^* (\partial^\nu + i\alpha_q A^\nu)\phi_q - \phi_q (\partial^\nu - i\alpha_q A^\nu)\phi_q^*\right] ,$$

$$\left(\partial^\mu \partial_\mu + \mu_q^2 + 2\, i\alpha_q\, A_\mu \partial^\mu - \alpha_q^2 A_\mu A^\mu\right)\phi_q = 0 ,$$

where $\alpha_q = e_q/\hbar c$ and $\mu_q = m_q\, c/\hbar$. The Lorentz gauge $\partial_\mu A^\mu = 0$ is used. We use Gaussian units. The fields are quantized by commutation rules such as

$$\left[\partial_0 \phi_q(\vec{x},t),\ \phi_q(\vec{y},t)\right] = -\, i\, c\, \hbar\ \delta(\vec{x} - \vec{y})$$

$$\left[\partial_0 A_\mu(\vec{x},t), A_\nu(\vec{y},t)\right] = -\, i\ 4\pi\ c\ \hbar\ \delta(\vec{x} - \vec{y})\ g_{\mu\nu} \tag{5}$$

where $g_{00} = 1 = -g_{11} = -\, g_{22} = -\, g_{33}$.

The fields differ from their standard representation (ϕ_s) in that $\phi = \hbar\phi_s$. The fields, ϕ, have the same dimensionality as the electromagnetic field; the square of their derivatives gives an energy density.

The field equations are invariant under the limit; the commutators disappear. We may then interpret the fields "classically" as complex number functions. The new system is deterministic although it has finite de Broglie wavelengths. We are interested in the case where the de Broglie wavelength is small compared to any other scale length. It may be considered an infinitesimal; its precise value is not important if it is small enough. It appears reasonable that these equations correspond to a collisionless plasma. The collisionless approximation smooths over particle effects. Conversely, the quantization adds particle-like or discrete effects to the fields.

In the absence of an electromagnetic field, the charge density due to positive energy plane wave solutions is

$$\rho\ (\vec{x},\ t) = \alpha \sum_{\ell} \exp\ [i\,\vec{\ell}\cdot\vec{x}]$$

$$\cdot \sum_{k} a_k a_{k-\ell}^*\ \omega_{k-\ell}\ \exp[i(\omega_{k-\ell} - \omega_k)t] + \text{c.c.}\ ,$$

where a_k and ω_k are the amplitude and frequency of the plane wave with wave number k. For a uniform plasma the coefficients for $\ell \neq 0$ should disappear. This will be true over an ensemble average if the different Fourier coefficients (a_k) are uncorrelated. In an individual plasma, the number of terms in the sum over $k = O\ (1/\hbar^3)$. Then, for uncorrelated a_k, the density coefficient for $\ell \neq 0$ is $O(\hbar^{3/2})$. In a second limit ($\hbar \to 0$, i.e., α and μ tend to infinity), the density (averaged over any finite volume) becomes uniform in an individual plasma for uncorrelated a_k. In general, if the plasma changes over a scale length ℓ_1, there should be correlations between wave numbers for

$$|\Delta k| = O(1/\ell_1).$$

II. COMPARISON WITH THE MAXWELL-VLASOV EQUATIONS

A. Linear Perturbations

We will compare solutions for the initial value problem in a uniform relativistic plasma with no zero order electromagnetic field. Longitudinal and transverse waves are done separately with the coupling between them ignored. The charged particle fields are separated into zero order and first order parts. Calculations for each charged particle species are done separately.

$$\phi = \phi_0 + \phi_1,$$

$$\phi_0 = \sum_k a_k \exp\ [i(\vec{k}\cdot\vec{x} - \omega_k t)]\ , \tag{6}$$

$$\phi_1 = \sum_k b_k(t)\ \exp\ [i\vec{k}\cdot\vec{x}]\ ,$$

where $\omega_k > 0$. For longitudinal waves, A has a time-like part

$$A^0 = \sum_k A_k^0(t)\ \exp[i\vec{k}\cdot\vec{x}]\ , \tag{7}$$

and a space-like part

$$A^a = \sum_k e_k^a A_k(t)\ \exp[i\vec{k}\cdot\vec{x}],\ a = 1,\ 2,\ 3 \tag{8}$$

where e_k is a unit vector parallel to k. The Lorentz gauge gives

$$-\ i\ kA_k(t) = \frac{\partial}{\partial t} A_k^0\ (t).$$

For transverse waves there is only a space-like part with each polarization perpendicular to k. The linearized equations are

$$\frac{1}{4\pi}\partial_\mu \partial^\mu A^\nu = -\ i\ \alpha\ \phi_0\ \partial^\nu \phi_1^* - i\ \alpha\ \phi_1\ \partial^\nu \phi_0^*$$

$$-\ \alpha^2 A^\nu \phi_0\ \phi_0^* + \text{c.c.}\ , \tag{9}$$

$$\left(\partial^\mu\partial_\mu + \mu^2\right)\phi_1 + 2\ i\ \alpha\ A^\mu\ \partial_\mu\ \phi_0 = 0\ . \tag{10}$$

For longitudinal case the Laplace transform of Eq. (10) gives

$$\left(\frac{s^2}{c^2} + k^2 + \mu^2\right)\ b_k(s)\ = \frac{1}{c^2}\ \left(s\ b_{k0} + b'_{k0}\right)$$

$$-\ 2\ i\ \frac{\alpha}{c} \sum_\ell A_\ell^0(s + i\ \omega_{k-\ell})$$

$$\cdot\, a_{k-\ell}\left[s - \frac{\vec{k}\cdot\vec{\ell}}{\vec{\ell}\cdot\vec{\ell}}\ (s + i\omega_{k-\ell})\right]$$

$$-\ 2\ i\ \frac{\alpha}{c} \sum_\ell A_{\ell 0}\ a_{k-\ell}\left(\frac{\vec{k}\cdot\vec{\ell}}{\vec{\ell}\cdot\vec{\ell}} - 1\right)\ , \tag{11}$$

where

$$b_{k0} \equiv\ b_k(t=0),\ b'_{k0} \equiv \partial\ \frac{b_k}{\partial t}\ (t = 0),$$

$$A_{\ell 0} \equiv A_\ell^0\ (t = 0)\ ,\ A'_{\ell 0} \equiv \frac{\partial A_\ell^0}{\partial t}\ \ (t = 0)\ .$$

The transform of Eq. (9) for $\nu = 0$ is

$$\frac{1}{4\pi}\ \left(\frac{s^2}{c^2} + k^2\right) A_k^0\ (s) = \frac{1}{4\pi c^2}\left(s\ A_{k0} + A'_{k0}\right)$$

$$+\ \frac{i\alpha}{c}\sum_\ell\ a_{k+\ell}\ b_{\ell 0}^*$$

$$-\ i\ \frac{\alpha}{c}\ \sum_\ell a_{k+\ell}\ \ (s\ +\ i\omega_{k+\ell})\ \ b_\ell^*(s + i\omega_{k-\ell})$$

$$+\ \frac{\alpha}{c}\ \sum_\ell\ \ a_{\ell-k}^*\ \omega_{\ell-k}\ b_\ell(s\ -\ i\omega_{\ell-k})$$

$$-\ \alpha^2\ \sum_{\ell,j}\ \ A_\ell^0(s\ +\ i\omega_j\ -\ i\omega_{\ell+j-k})\ a_j a_{\ell+j-k}^*\ +\ \text{c.c.}\ \ . \tag{12}$$

The substitution of Eq. (11) into Eq. (12) couples the Fourier modes of A^0. The coupling between A_k^0 and A_ℓ^0 is of the type $a_j\ a_{j+\ell-k}^*$. For a uniform plasma the coupling $= O(\hbar^{3/2})$ since the different a_j are incoherent. The coupling disappears in the limit $\hbar \to 0$, and we have

$$A_k^0(s) = S/D(k,s)\ \ , \tag{13}$$

where S, the source term, depends on initial conditions, and D(k,s) is the Laplace transform "dispersion relation."

$$D(k,s) = \left(\frac{s^2}{c^2} + k^2\right) \left\{ 1 + \frac{8\pi\alpha^2}{c^2} \sum_j |a_j|^2\, 4\omega_j^2 \right.$$

$$\times \frac{\left.\left[1 - \frac{(\vec{j}\cdot\vec{k})^2 c^2}{k^2\omega_j^2} - \left(s^2 + c^2k^2\right)/4\omega_j^2\right]\right\}}{\left[4\,\omega_j^2\,(s+i\vec{k}\cdot\vec{u})^2/c^4 - \left(\frac{s^2}{c^2} + k^2\right)\right]}, \qquad (14)$$

where u is the velocity corresponding to j. With the substitution

$$|a_j|^2\,\omega_j \rightarrow n_0 f_0(\vec{p})\,\hbar c^2/2, \quad \vec{p} = \hbar\vec{j} \qquad (15)$$

where n_0 is the particle density, p is momentum, and f_0 is the momentum distribution function,

$$\lim_{\hbar\to 0} D(k,s) = \left(\frac{s^2}{c^2} + k^2\right)$$

$$\cdot \left\{ 1 + \omega_p^2 \int \frac{d^3p f_0(p)\left[1 - \left(\frac{\vec{u}\cdot\vec{k}}{c}\right)^2/k^2\right]}{\gamma(s+i\vec{k}\cdot\vec{u})^2} \right\}, \qquad (16)$$

where ω_p is the plasma frequency.

The square brackets are identical to the equivalent Vlasov equation dispersion relation[1]

$$D_v(k,s) = 1 - i\omega_p^2\, m\, \frac{\vec{k}\cdot}{k^2} \int d^3p \left(\frac{\partial f_0(p)}{\partial\vec{p}(s+i\vec{k}\cdot\vec{u})}\right). \qquad (17)$$

This is shown by integration by parts. The same procedure may be carried out for transverse waves. We obtain

$$A_k(s) = S/D(k,s),$$

where

$$D(k,s) = \frac{s^2}{c^2} + k^2 + 8\pi\alpha^2 \sum_j |a_j|^2$$

$$- 8\pi\alpha^2 \sum_j |a_j|^2 \frac{(\vec{j}\cdot\vec{e}_k)^2(s^2c^2 + k^2c^4)}{\omega_j^2(s + i\vec{k}\cdot\vec{u})^2}, \qquad (18)$$

where e_k is the polarization vector.

$$\lim_{\hbar\to 0} D(k,s) = \frac{s^2}{c^2} + k^2 + \frac{\omega_p^2}{c^2} \int \frac{d^3p f_0(p)}{\gamma}$$

$$- \left(\frac{s^2}{c^2} + k^2\right) \omega_p^2 \int \frac{d^3p f_0(p)\left[(\vec{u}/c)\cdot\vec{e}_k\right]^2}{\gamma(s+i\vec{k}\cdot\vec{u})^2}. \qquad (19)$$

Transverse waves in a relativistic plasma have been investigated by Lerche[5] and Felderhof,[6] among others. Apparently, however, there is no explicit expression for the Laplace transform. Comparison can be made only for special cases. If the denominator in the last integral is not too small, the last term is a relativistic correction, and in the nonrelativistic limit we obtain the standard expression

$$\omega^2 = k^2 + \omega_p^2 . \qquad (20)$$

The last term of Eq. (19), for small velocities parallel to $\vec{k}$, produces the Weibel instability.[7]

B. Nonlinear Case

Here we attempt to find a functional of each particle field which corresponds to a distribution function, and which satisfies the Maxwell-Vlasov equations in a self-consistent way. One difficulty is that a continuous charge distribution does not, in general, satisfy the Vlasov equation, e.g.,

$$f(\vec{p},\vec{x},t) = g(\vec{x},t)\,\delta[\vec{p}-\vec{p}(\vec{x},t)],$$

where

$$\vec{p}(\vec{x},t) = \vec{p}(\vec{x},0) + [-(\vec{v}\cdot\vec{\nabla})\vec{p} + (\vec{E} + \vec{v} \times \vec{B})]\ t + O(t^2) \ , \qquad (21)$$

and v is the particle velocity. The delta function factor satisfies the Vlasov equation, but g, the charge density, does not.

$$\frac{\partial g}{\partial t} + \vec{\nabla} \cdot [\vec{v}(\vec{x},t)\ g] = 0 \ . \qquad (22)$$

A point particle distribution,

$$f(\vec{p},\vec{x},t) = \sum_i \delta[\vec{x} - \vec{x}_i(t)]\ \delta[\vec{p} - \vec{p}_i(t)] \ , \qquad (23)$$

does formally satisfy the Vlasov equation.

We look for solutions of the field equations which correspond more closely to "particles" than to a continuous fluid. Consider uncorrelated wave packets with radius = $O(r_1)$, where $r_1 \to 0$ in the limit $\hbar \to 0$. The wave number spread $\Delta k_1 = O\ (1/r_1)$, which gives a velocity dispersion = $\Delta k_1 \hbar = O(\hbar/r_1)$. To maintain a radius = $O(r_1)$ requires that $\hbar = O(r_1{}^2)$. The charge of the wave packet, e_1, is $O(r_1{}^3)$, therefore self forces do not expand the wave packet to a radius greater than $O(r_1)$.

We define

$$f(\vec{p},\vec{x},t) = \int d^3k_1\ W_2(\vec{k}_1 - \vec{k}_0)[g_1 g_2 + c.c.] \ , \qquad (24)$$

where

$$g_1 = \int d^3x_1\ \exp\ [-i\vec{k}_1\cdot\vec{x}_1]\ W_1(\vec{x}-\vec{x}_1)\ \phi(\vec{x}_1,t) \ ,$$

and

$$g_2 = \int d^3x_1\ \exp[i\vec{k}_1\cdot\vec{x}_1]\ W_1(\vec{x}-\vec{x}_1)$$

$$[-i\partial_t - \alpha A_0(\vec{x}_1,t)]\phi^*(\vec{x}_1,t),$$

and

$$\vec{k}_0 = \vec{p}/\hbar + \alpha\vec{A} \ .$$

$$W_1(\vec{x}-\vec{x}_1) = \text{const},\ |\vec{x}-\vec{x}_1| < r_3 \ ,$$

$$= 0,\quad |\vec{x}-\vec{x}_1| > r_3 + r_2 \ .$$

$$W_2(\vec{k}-\vec{k}_1) = \text{const},\ |\vec{k}-\vec{k}_1| < k_3 \ ,$$

$$= 0,\quad |\vec{k}-\vec{k}_1| > k_3 + k_2 \ .$$

$$\lim_{\hbar\to 0}\quad \frac{r_1}{r_2},\ \frac{r_2}{r_3},\ \frac{\Delta k_1}{k_2},\ \frac{k_2}{k_3},$$

$$r_3,\ \hbar k_3,\ \hbar k_3^2 r_3,\ r_3^2 k_3 \to 0 \ .$$

W_1 and W_2 have continuous first order derivatives. We assume ϕ has only positive energy solutions. A more complicated definition is necessary if there are also negative energy solutions. The distribution function is ensemble averaged because the number of wave packets in the support of $W_1 \times W_2$ goes to zero in a single system. In the region where $W_1 \times W_2$ is constant, the function acts as a counter of wave packets because it reduces (by Parseval's formula) to an integral over charge density. Only the asymptotic overlap of these wave packets in the boundary region of $W_1 \times W_2$ contribute to the derivatives of f. Derivatives of ϕ of $O(1)$ of these wave packets may then be neglected. Since the volume of the boundary region over the total volume goes to zero, we may neglect derivatives of ϕ of $O(1)$ in general. This is important since derivatives of $O(1/\hbar)$ reflect the dynamics of charged particles, but derivatives of $O(1)$, which are necessary for charge conservation, are not directly connected with the particle dynamics.

We need local solutions for ϕ. Let

$$\phi(\vec{x}_1,t_1) = \sum_k a_k \exp\left\{i[\vec{k}\cdot\vec{x}_1 - \omega(\vec{x}_1,t_1)t_1]\right\} \ . \qquad (25)$$

We assume that A changes slowly over distances of $O(r_1)$. Wavelengths of this size come from collisions between wave packets, and the collision strength vanishes in the limit. We expand ω in an asymptotic series in orders of $\hbar$,

$$\omega(\vec{x},t) = \sum_{j=0}^{\infty} \omega_j(\vec{x},t) \ ,$$

$$\text{where } \omega_j = O(\hbar^{j-1}) \ , \tag{26}$$

and

$$\omega_j(\vec{x},t) = \sum_{n=0}^{\infty} \omega_{jn}(\vec{x}) \ t^n \quad .$$

Then ω may be solved explicitly. When only those terms necessary for the first order derivatives of $f(\vec{p},\vec{x},t)$ are kept, and ω is expanded about $(\vec{x},t)$,

$$\phi(\vec{x}_1,t_1) = \sum_{k} a_k \ \exp\Big\{i[\vec{k}\cdot\vec{x}_1 - (t_1-t)$$

$$(\omega_0 + \alpha v^{\mu}A_{\mu a}(x_1-x)^a)]\Big\} \ , \tag{27}$$

where ω_0, v, and A are defined at $(\vec{x},t)$. (Double Greek indexes indicate a summation over all indexes; double Latin indexes indicate a summation over space-like indexes.)

$$p^a = \hbar(k^a-\alpha A^a) \ .$$

$$p^0 = \left(m^2c^4 + c^2p^ap^a\right)^{1/2} \ . \tag{28}$$

$$v^{\mu} = p^{\mu}/p_0 \ .$$

$$\omega_0 = p_0/\hbar + \alpha A_0 \ .$$

Application of the Vlasov operator to f gives

$$[\partial_t + v^a\partial_a + e(\vec{E} + \vec{v}\times\vec{B})^a \ \partial_{p^a}]f$$

$$= (\partial_t f)_{k_0} + (v^a\partial_a f)_{k_0} - \alpha v^{\mu}A_{\mu a} \ \partial_{k_0^a} f \ . \tag{29}$$

For g_1, the three derivatives in Eq. (29) give (inside the Fourier integral of ϕ), respectively,

$$\text{(a)} \quad - i[\omega_0(k) + \alpha v^{\mu}(k) \ A_{\mu a}(x_1-x)^a]$$

$$\rightarrow - i[\underline{\omega_0(k_0) + \left(\frac{\partial\omega}{\partial k}\right)^a_{k_0}(k_1-k_0)^a} + \left(\frac{\partial\omega}{\partial k}\right)^a_{k_0}(k-k_1)^a$$

$$+ \alpha v^{\mu}(k) \ A_{\mu a}(x_1-x)^a] \ ,$$

$$\text{(b)} \quad - iv^a(k_1-k)^a \ , \tag{30}$$

and

$$\text{(c)} \quad i\alpha v^{\mu} \ A_{\mu a}x_1^{\ a}$$

$$= i\alpha v^{\mu}A_{\mu a}(x_1-x)^a + \underline{i\alpha v^{\mu}A_{\mu a}x^a} \ ,$$

where

$$\left(\frac{\partial\omega}{\partial k}\right)^a_{k_0} = v(k_0)^a \equiv v^a \ .$$

The underscored terms are cancelled by the same terms for g_2. After cancellation we have

$$- i\alpha A_{\mu a}(x-x_1)^a[v(k_0)-v(k)]^{\mu} \ . \tag{31}$$

This cancels to the first approximation, but the remainder is greater than O(1). From g_2 we obtain the negative of term (31). The action of $(-i\partial_t-\alpha A_0)$ on ϕ^* produces a coefficient which multiplies each Fourier component of ϕ^*.

$$\left\{\omega(k_0) - \alpha A_0(\vec{x},t)\right\} + \left\{\omega(k) - \omega(k_0)\right.$$

$$\left. - \alpha[A_0(\vec{x}_l,t) - A_0(\vec{x},t)] + \alpha v^{\mu} A_{\mu a}(x_l - x)^a \right\} \quad . \qquad (32)$$

Term (31) and its negative from g_2 cancel the first braces { } of term (32). This leaves terms of $O(r_3^2 k_3)$, $O(\hbar k_3^2 r_3) = o(1)$. The (logarithmic) derivatives of term (32) itself are of O(1) and may be neglected. Then Eq. (29) is o(1) for $\hbar \to 0$.

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