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TITLE: BETHE-SALPETER EQUATION:

NUMERICAL EXPERIENCE WITH A HYDROGENLIKE ATOM

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Bethe-Salpeter Equation: Numerical Experience with a Hydrogenlike Atom*

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Abstract

The eigenvalue λ of the Bethe-Salpeter equation,

$$\lambda\phi = \frac{1}{\pi^2} GS\phi \quad ,$$

with the value of E_b calculated from the Sommerfeld expression,

$$E_b = \frac{1}{2} mc^2 \left(1 - \frac{1}{\sqrt{1 + (1/137.03802)^2}}\right) \quad ,$$

substituted into the propagator S, and only the one-photon-exchange term included in G, is, for the $^1S_0$ state,

$$\lambda = 133.9963 \pm 0.0003.$$

The difference between λ and 137.03802 is due to the appearance of a logarithmic term in

$$E_b = \frac{1}{4} mc^2 \alpha^2 \left(1 - \frac{4\alpha}{\pi} \ln \frac{2}{\alpha} + \ldots\right) \quad ,$$

which is valid for the Bethe-Salpeter equation, $^1S_0$ state, one photon exchange. The crossed two photon exchange contribution brings λ up to

$$\lambda = 136.5 \pm (0.1?) \quad ,$$

so that this contribution cancels the logarithmic term, but not exactly.

I. INTRODUCTION

In a paper in Physical Review with this same title,[1] we reported that the binding energy E_b of an equal mass m hydrogen atom resulting from the Bethe-Salpeter equation with one photon exchange differs from the E_b resulting from the Dirac equation with a Coulomb potential in that logarithmic terms occur:

$$E_b = \frac{1}{4} mc^2\alpha^2 \left(1 - \frac{4\alpha}{\pi} \ln \frac{2}{\alpha} + \ldots\right) \quad . \tag{1}$$

The numerical results substantiating the analytic work were expressed in the following way. The Bethe-Salpeter equation is

$$\lambda\phi = \frac{1}{\pi^2} GS\phi \quad , \tag{2}$$

where ϕ is the Bethe-Salpeter amplitude, G the sum of all two particle irreducible graphs (in these calculations only the one photon exchange term is included), S the direct product of the fermion propagators, and $\lambda = 4\pi/e^2 = 1/\alpha$ is a factor removed from the one photon exchange graph (for more details see reference 1). S contains E_b, and we put in the value calculated from the Sommerfeld expression

$$E_b = \frac{1}{2} mc^2 \left(1 - \frac{1}{\sqrt{1 + \alpha^2}}\right) \quad , \tag{3}$$

with the accepted value $\alpha = 1/137.03802$. Were there no difference between the Bethe-Salpeter E_b and the Dirac E_b we should then find the eigenvalue $\lambda = 137.03802$. We do not find agreement because of the logarithmic term in Eq. (1).

In ref. 1 we reported $\lambda = 134.18 \pm 0.01$, and in this work we find $\lambda = 133.9963 \pm 0.0003$ by the use of improved numerical techniques. Our

motivation for making this improvement is the fact that should we want to calculate the Lamb shift by these techniques, such accuracy is required. Our general method, which is a straightforward solution of the Bethe-Salpeter equation, is conceptually simpler than the usual methods,[2] but not yet nearly so accurate or complete.

We would like to explain a little more our motivation. We have calculated λ with the crossed two photon exchange graph included in G. We find[3] $\lambda = 136.5$. Thus our results underscore Salpeter's remark[4] that the calculation done in a straightforward manner converges very slowly as more and more graphs are included in G. In calculating the Lamb shift it is not necessary to include the crossed two photon exchange graph and similar graphs because the ladder approximation yields a spectrum with the same degeneracy as the Dirac equation with a Coulomb potential.[5] Only the level splitting resulting from the self energy graph shown in Fig. 1 is desired, and this splitting may be calculated by including in G the one photon exchange graph and the self energy graph. The matrix elements of the self energy graph may be read from the book by Jauch and Rohrlich[6] since they give the full off-shell matrix elements required (Jauch and Rohrlich give the part arising from the bubble in the fermion line and this has to be multiplied by the one photon exchange part which is easily calculated by the methods of reference 1). Our interest in doing the calculation in this way arises from our belief that the infrared problem does not arise. We believe this because the bound state problem is off-shell and it may be observed that Jauch and Rohrlich's result has no divergence off shell. As the binding energy goes to zero, the problem moves toward the on-shell case, so terms like log $(1/\alpha)$ should occur. We wonder how these logarithmic terms cancel against the one shown in Eq. (1), which according to the result $\lambda = 136.5$ is only partially cancelled by the crossed two photon exchange

part, and it is hard to see how the crossed three photon exchange parts will make up the difference. Perhaps it is precisely the self energy graph which makes up the difference (the photon self energy graph also has a part to play). We hasten to add in conclusion that we have not yet done the calculations, but the calculations are possible (Jochem Fleischer has done similar calculations in the strong interaction case[7,8]). We may hope not to remain forever ignorant of the facts. Many[9] are aware of just how awkward the usual treatment of the infrared problem is, and if it is true that it does not really arise in a full but straightforward approach, it would be a matter of interest.

II. THE IMPROVED NUMERICAL TECHNIQUES

We restrict the calculation to the positive energy states, so that Eq. (3) of reference 1 becomes

$$\lambda\phi(p,ip_4) = \frac{1}{\pi^2}\int_0^\infty dq \int_0^\infty dq_4\, K(p,ip_4,q,iq_4)\, S(q,iq_4)\, \phi(q,iq_4) \quad . \tag{4}$$

The numerical approach to any such equation is to replace the integrations by finite sums over a mesh. Symbolically,

$$\int dq \int dq_4 = \Sigma\Sigma + \int\int - \Sigma\int + \Sigma\int - \Sigma\Sigma \quad ,$$

$$= \Sigma\Sigma + (\int - \Sigma)\int + \Sigma(\int - \Sigma) \quad . \tag{5}$$

$(\int - \Sigma)\int$ may be referred to as a correction for the q integration and $\Sigma(\int - \Sigma)$ as a correction for the q_4 integration. In these corrections, we do not need to use the exact $\phi(q,iq_4)$, which is unknown anyhow, but we must keep the equations homogeneous. As proposed in reference 1, we make use of the fact that $\phi(q,iq_4)$ is nearly independent of q_4, and in the q_4 correction we use

$$\phi(q,iq_4) = \phi(q,0) \quad . \tag{6}$$

In the q correction, we make use of this fact and the fact that we know approximately how $\phi(q,0)$ depends on q; namely, it must depend on q in the same way that the nonrelativistic $\phi(q)$ does. Thus in the q correction we use

$$\phi(q,iq_4) = \frac{q}{\frac{q^2}{2m} + \frac{1}{2} E_b} \; \frac{\frac{p^2}{2m} + \frac{1}{2} E_b}{p} \phi(p,0) \quad . \tag{7}$$

In detail, our finite sum approximation to Eq. (4) is

$$\begin{aligned} \pi^2 \lambda \phi(p,ip_4) = & \sum_q \sum_{q_4} K(p,ip_4,q,iq_4) \, S(q,iq_4) \, \phi(q,iq_4) \\ & + T(p,p_4) \, \phi(p,0) + \sum_q R(p,p_4,q) \, \phi(q,0) \quad , \end{aligned} \tag{8}$$

where

$$\begin{aligned} R(p,p_4,q) &= \left[\int_0^\infty dq_4 - \sum_{q_4} \right] K(p,ip_4,q,iq_4) \, S(q,iq_4) \quad , \\ &= R_1 - R_2 \quad , \end{aligned} \tag{9}$$

and

$$T(p,p_4) = \frac{\frac{p^2}{2m} + \frac{1}{2}E_b}{p}\left[\int_0^\infty dq - \sum_q\right]\frac{q}{\frac{q^2}{2m} + \frac{1}{2}E_b} R_1(p,p_4,q) \quad , \tag{10}$$

$$= S_1 - S_2 \quad .$$

The integrals can be done analytically (otherwise there would be no point in writing them down) provided we use some nonrelativistic kinematics in K and S, namely

$$K(p,ip_4,q,iq_4) = \frac{1}{4}\ln\left[\frac{(p+q)^2 + (p_4+q_4)^2}{(p-q)^2 + (p_4+q_4)^2}\,\frac{(p+q)^2 + (p_4-q_4)^2}{(p-q)^2 + (p_4-q_4)^2}\right] \quad ,$$

$$S(q,iq_4) = \frac{1}{(\frac{q^2}{2m} + \frac{1}{2}E_b)^2 + q_4^2} \quad . \tag{11}$$

Of course we use these "nonrelativistic" approximations only in corrections which vanish as the mesh size is refined. We find

$$R_1(p,p_4,q) = \frac{\pi}{4}\frac{1}{\frac{q^2}{2m} + \frac{1}{2}E_b}\ln\frac{(\frac{q^2}{2m} + \frac{1}{2}E_b + p + q)^2 + p_4^2}{(\frac{q^2}{2m} + \frac{1}{2}E_b + |p-q|)^2 + p_4^2} \quad ,$$

$$S_1(p,p_4) = f(p,i\sqrt{\frac{E_b}{m}},ip_4) + f(p,i\sqrt{\frac{E_b}{m}},-ip_4) - f(p,-i\sqrt{\frac{E_b}{m}},ip_4)$$

$$- f(p,-i\sqrt{\frac{E_b}{m}},-ip_4) \quad ,$$

$$f(p, i\sqrt{\frac{E_b}{m}}, ip_4) = 137.03802\ \frac{\pi}{2i}\ \frac{\frac{p^2}{2m} + \frac{1}{2}E_b}{p}\left[\frac{F\left(\frac{p}{m},0,i\sqrt{\frac{E_b}{m}}\right) - F\left(\frac{p}{m},0,-1-\sqrt{(1)}\right)}{i\sqrt{\frac{E_b}{m}} + 1 + \sqrt{(1)}}\right.$$

$$- \frac{F\left(\frac{p}{m},0,i\sqrt{\frac{E_b}{m}}\right) - F\left(\frac{p}{m},0,1-\sqrt{(1)}\right)}{i\sqrt{\frac{E_b}{m}} - 1 + \sqrt{(1)}}$$

$$+ \frac{F\left(\frac{p_{max}}{m}, \frac{p}{m}, i\sqrt{\frac{E_b}{m}}\right) - F\left(\frac{p_{max}}{m}, \frac{p}{m}, -1-\sqrt{(1)}\right)}{i\sqrt{\frac{E_b}{m}} + 1 + \sqrt{(1)}}$$

$$- \frac{F\left(\frac{p_{max}}{m}, \frac{p}{m}, i\sqrt{\frac{E_b}{m}}\right) - F\left(\frac{p_{max}}{m}, \frac{p}{m}, -1-\sqrt{(2)}\right)}{i\sqrt{\frac{E_b}{m}} + 1 + \sqrt{(2)}}$$

$$+ \frac{F\left(\frac{p}{m}, 0, i\sqrt{\frac{E_b}{m}}\right) - F\left(\frac{p}{m}, 0, -1+\sqrt{(1)}\right)}{i\sqrt{\frac{E_b}{m}} + 1 - \sqrt{(1)}}$$

$$- \frac{F\left(\frac{p}{m}, 0, i\sqrt{\frac{E_b}{m}}\right) - F\left(\frac{p}{m}, 0, 1-\sqrt{(1)}\right)}{i\sqrt{\frac{E_b}{m}} - 1 - \sqrt{(1)}}$$

$$+ \frac{F\left(\frac{p_{max}}{m}, \frac{p}{m}, i\sqrt{\frac{E_b}{m}}\right) - F\left(\frac{p_{max}}{m}, \frac{p}{m}, -1+\sqrt{(1)}\right)}{i\sqrt{\frac{E_b}{m}} + 1 - \sqrt{(1)}}$$

$$\left. - \frac{F\left(\frac{p_{max}}{m}, \frac{p}{m}, i\sqrt{\frac{E_b}{m}}\right) - F\left(\frac{p_{max}}{m}, \frac{p}{m}, -1+\sqrt{(2)}\right)}{i\sqrt{\frac{E_b}{m}} + 1 - \sqrt{(2)}}\right],$$

$$F(x_2,x_1,z) = \frac{1}{2}\log\left(w^2 + \left(\frac{z-z^*}{2i}\right)^2\right)\Bigg|_{x_1-\frac{z+z^*}{2}}^{x_2-\frac{z+z^*}{2}}$$

$$+ i \arctan\frac{w}{\frac{z-z^*}{2i}}\Bigg|_{x_1-\frac{z+z^*}{2}}^{x_2-\frac{z+z^*}{2}} ,$$

$$\frac{\pi}{2} > \arctan > -\frac{\pi}{2} ,$$

$$\sqrt{(1)} = \sqrt{1 - \frac{2}{m}\left(\frac{1}{2}E_b + p + ip_4\right)} ,$$

$$\sqrt{(2)} = \sqrt{1 - \frac{2}{m}\left(\frac{1}{2}E_b - p + ip_4\right)} ,$$

p_{max} = largest value of p used in the p_{mesh} .

Methods by which these formulae were derived are outlined in Appendix I.

We transform everything to the u,v meshes from the p,p_4 meshes as in reference 1. The various Σ need to have their correct weight factors and the Jacobians of the transformations as described in reference 1.

Because the corrections are not exact, the results will still depend on mesh size and the parameters a and b introduced as corrections to integration schemes in reference 1 (see Eq. (20) of reference 1).

We have, of course, wondered why our error in the calculation reported in reference 1 is about 20 times larger than we thought. To find out, we have repeated the calculation of reference 1 with still finer mesh sizes. We see that $d\lambda/d$ (mesh size) = 0 for the a,b and the finest mesh sizes used in reference 1, and this led to the illusion that λ was

independent of mesh size to the stated accuracy for these meshes. But as the calculations with still finer meshes show, it is only an illusion. This danger always exists and can never be completely eliminated by numerical means; it still exists for this improved calculation.

III. RESULTS

All results are gathered in Fig. 2 and Table I. The best extrapolated value is

$$\lambda = 133.9963 \pm 0.0003 \quad .$$

The danger referred to just above is reduced by the fact that we have results approaching this limit from below and above, as shown in Fig. 2.

IV. CONCLUSION

We believe that we have shown that it is possible to solve the Bethe-Salpeter equation by the use of straightforward numerical techniques to an accuracy required by the precision measurement of atomic physics.

APPENDIX I.

To evaluate P_1, expand the argument of the logarithm in Eq. (11) thusly:

$$[\ ] = \frac{((p+q)^2 + p_4^2 + q_4^2)^2 - 4p_4^2 q_4^2}{((p-q)^2 + p_4^2 + q_4^2)^2 - 4p_4^2 q_4^2} ,$$

$$= \frac{(q_4^2 - r_1)(q_4^2 - r_2)}{(q_4^2 - r_3)(q_4^2 - r_4)} ,$$

where r_1, r_2, r_3, and r_4 depend on p, q, and p_4. Using $\ell n\ (AB/CD) = \ell n\ A + \ell n\ B - \ell n\ C - \ell n\ D$ leads to integrals of the form

$$\int_0^\infty \frac{dq_4}{a^2 + q_4^2}\ \ell n(q_4^2 + c^2) = \frac{\pi}{a}\ \ell n\ (a + c) .$$

The evaluation of S_1 requires the evaluation of

$$\frac{1}{4\pi} \int_0^\infty \frac{q\ dq}{\left(\frac{q^2}{2m} + \frac{1}{2} E_b\right)^2} \ell n \frac{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + p + q\right)^2 + p_4^2}{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + |p - q|\right)^2 + p_4^2}$$

$$= \frac{m}{4\pi} \int_0^p \frac{\frac{q}{m}\, dq}{\left(\frac{q^2}{2m} + \frac{1}{2} E_b\right)^2} \ell n \frac{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + p + q\right)^2 + p_4^2}{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + p - q\right)^2 + p_4^2}$$

$$+ \frac{m}{4\pi} \int_p^\infty \frac{\frac{q}{m}\, dq}{\left(\frac{q^2}{2m} + \frac{1}{2} E_b\right)^2} \ell n \frac{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + p + q\right)^2 + p_4^2}{\frac{q^2}{2m} + \frac{1}{2} E_b + q - p^2 + p_4^2} ,$$

$$= \frac{m^2}{\pi} \int_0^p \frac{dq}{q^2 + m E_b} \left[\frac{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + p + q\right)\left(\frac{q}{m} + 1\right)}{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + p + q\right)^2 + p_4^2} \right.$$

$$\left. - \frac{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + p - q\right)\left(\frac{q}{m} - 1\right)}{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + p - q\right)^2 + p_4^2} \right]$$

$$+ \frac{m^2}{\pi} \int_p^\infty \frac{dq}{q^2 + m E_b} \left[\frac{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + p + q\right)\left(\frac{q}{m} + 1\right)}{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + p + q\right)^2 + p_4^2} \right.$$

$$\left. - \frac{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + q - p\right)\left(\frac{q}{m} + 1\right)}{\left(\frac{q^2}{2m} + \frac{1}{2} E_b + q - p\right)^2 + p_4^2} \right] ,$$

where the last equality results from an integration by parts. We then proceed by breaking various expressions into partial fractions; for example, in the first term on the right of the preceding equation,

$$\frac{x}{x^2 + A^2} = \frac{1}{2} \left[\frac{1}{x + iA} + \frac{1}{x - iA} \right] ,$$

$$x = \frac{q^2}{2m} + \frac{1}{2} E_b + p + q ,$$

$$A = p_4 ,$$

$$\frac{1}{x^2 + B^2} = \frac{1}{2iB} \left[\frac{1}{x - iB} - \frac{1}{x + iB} \right] ,$$

$$x = q ,$$

$$B = \sqrt{m E_b} ,$$

and in this way we arrive at the expression for S_1 given in the text in terms of four f functions, where

$$f = \frac{1}{\sqrt{mE_b}} \frac{m^2}{4\pi i} \left\{ \int_0^p \frac{dq}{q - i\sqrt{mE_b}} \left[\frac{\frac{q}{m} + 1}{\frac{q^2}{2m} + \frac{1}{2} E_b + p + q + ip_4} \right.\right.$$

$$\left. - \frac{\frac{q}{m} - 1}{\frac{q^2}{2m} + \frac{1}{2} E_b + p - q + ip_4} \right]$$

$$+ \int_0^{p_{max}} \frac{dq}{q - i\sqrt{mE_b}} \left[\frac{\frac{q}{m} + 1}{\frac{q^2}{2m} + \frac{1}{2} E_b + p + q + ip_4} \right.$$

$$\left.\left. - \frac{\frac{q}{m} + 1}{\frac{q^2}{2m} + \frac{1}{2} E_b + q - p + ip_4} \right] \right\} .$$

The four denominators quadratic in q have roots which depend on p, p_4,

$$r_1 = r_3 = [- 1 \pm \sqrt{(1)}] m \quad ,$$

$$r_2 = [1 \pm \sqrt{(1)}] m \quad ,$$

$$r_4 = [- 1 \pm \sqrt{(2)}] m \quad ,$$

where $\sqrt{(1)}$ and $\sqrt{(2)}$ are given in the text. Further reduction by partial fractions leads to

$$\sqrt{\frac{m}{E_b}}\ \frac{1}{4\pi i}\left\{\int_0^{\frac{p}{m}} d\left(\frac{q}{m}\right)\left[\left[\frac{1}{\frac{q}{m}-i\sqrt{\frac{E_b}{m}}}-\frac{1}{\frac{q}{m}+1+\sqrt{(1)}}\right]\frac{1}{i\sqrt{\frac{E_b}{m}}+1+\sqrt{(1)}}\right.\right.$$

$$\left.-\left[\frac{1}{\frac{q}{m}-i\sqrt{\frac{E_b}{m}}}-\frac{1}{\frac{q}{m}-1+\sqrt{(1)}}\right]\frac{1}{i\sqrt{\frac{E_b}{m}}-1+\sqrt{(1)}}\right]$$

$$+\int_{\frac{p}{m}}^{\frac{p_{max}}{m}} d\left(\frac{q}{m}\right)\left[\left[\frac{1}{\frac{q}{m}-i\sqrt{\frac{E_b}{m}}}-\frac{1}{\frac{q}{m}+1+\sqrt{(1)}}\right]\frac{1}{i\sqrt{\frac{E_b}{m}}-1+\sqrt{(1)}}\right.$$

$$\left.\left.-\left[\frac{1}{\frac{q}{m}-i\sqrt{\frac{E_b}{m}}}-\frac{1}{\frac{q}{m}+1+\sqrt{(2)}}\right]\frac{1}{i\sqrt{\frac{E_b}{m}}+1+\sqrt{(2)}}\right]\right\} ,$$

+ a similar expression with $-\sqrt{(1)}$ and $-\sqrt{(2)}$, which explains the source of the sixteen F functions in the text. The F are all of the form

$$F(x_2,x_1,z) = \int_{x_1}^{x_2} \frac{dx}{x-z} ,$$

and the result broken into obviously real and obviously imaginary parts is given in the text.

APPENDIX II

In this appendix we document the very complicated calculations including the crossed two photon exchange graph.

The crossed two pion exchange graph for nucleon-nucleon scattering has been evaluated independently by at least two groups,[10] and the algebra has been checked by use of the algebraic manipulating code REDUCE.[8] The result is that the positive energy-positive energy part of G is

$$G(pp_0qq_0) = \alpha \frac{pq}{E(p)E(q)} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \; I(x_1,x_2,x_3,p,p_0,q,q_0) \; .$$

One factor of α has been removed to include in the eigenvalue λ. We have

$$I = I_{0,2}\, T_1 + I_{1,2}\, T_2 + I_{0,1}\, T_3 \quad ,$$

$$T_1 = E^+ (m^2 + P_1P_2) + E^- z_{12} - E(p)(m^2P^+ + q^2b^-) - E(q)(m^2P^+ + p^2a^-) + m^2(p^2 + q^2) + pqz_3 \quad ,$$

$$T_2 = pq(-m^2 + P_1P_2 + z_{12} + 2m^2 - E(q)b^- - E(p)a^- + z_3 E^- \quad ,$$

$$T_3 = 2E(p)E(q) - m^2 \quad ,$$

$$P_1 = E(1 - x_1 - x_2) + (x_2 + x_3)\, q_0 + (1 - x_1 - x_3)p_0 \quad ,$$

$$P_2 = E(1 - x_1 - x_2) - (1 - x_2 - x_3)q_0 - (x_1 + x_3)p_0 \quad ,$$

$$E^- = E(p)E(q) - m^2 \quad ,$$

$$E^+ = E(p)E(q) + m^2 \quad ,$$

$$P^+ = P_1 + P_2 \quad ,$$

$$a^- = E(1 - x_1 - x_2) - (1 - x_1 - x_2 - 2x_3)q_0 \quad ,$$

$$b^- = E(1 - x_1 - x_2) + (1 - x_1 - x_2 - 2x_3)p_0 \quad ,$$

$$z_{12} = (x_1 + x_3)(1 - x_1 - x_3)p^2 + (x_2 + x_3)(1 - x_2 - x_3)q^2 \quad ,$$

$$z_3 = pq\,[(x_1 + x_3)(x_2 + x_3) + (1 - x_2 - x_3)(1 - x_1 - x_3)] \quad ,$$

$$I_{j,1} = -\frac{1}{A}\,Q_j(s) \text{ for } A \neq 0\,, \quad j = 0,1 \quad ,$$

$$I_{0,1} = \frac{1}{B},\ I_{1,1} = 0 \text{ for } A = 0 \quad ,$$

$$I_{j,2} = \frac{1}{A}\left[\frac{s^j}{s^2 - 1} - j\,Q_0(s)\right] \text{ for } A \neq 0,\ j = 0,1 \quad ,$$

$$I_{0,2} = \frac{1}{B^2}\,,\ I_{1,2} = 0 \text{ for } A = 0 \quad ,$$

$$s = -\frac{B}{A} \quad ,$$

$$A = 2pq\,[x_3(1 - x_1 - x_2 - x_3) - x_1x_2] \quad ,$$

$$B = x_1(1 - x_1)\,\{(E + p_0)^2 - p^2\} + x_2(1 - x_2)\,\{(E - q_0)^2 - q^2\}$$

$$+ x_3(1 - x_3)\{(p_0 - q_0)^2 - p^2 - q^2\} - 2[x_1x_2(E + p_0)(E - q_0)$$

$$+ x_1x_3\,\{(E + p_0)(p_0 - q_0) - p^2\} + x_2x_3(E - q_0)(p_0 - q_0) - q^2]$$

$$- \mu^2\,(1 + x_1 + x_2) \quad ,$$

$$Q_0(s) = \frac{1}{2}\log\frac{s + 1}{s - 1} \quad ,$$

$$Q_1(s) = s\,Q_0(s) - 1 \quad ,$$

m = fermion mass,

μ = boson mass .

When the γ_5's appropriate to the pseudoscalar case are replaced by γ_μ's appropriate to the vector case, one would think that the above

expression is without relevance. However, the main part of the γ_μ vertex is γ_0 (this is the nonrelativistic limit and constitutes a very accurate approximation when we restrict ourselves to the positive-energy states). Now consider a fermion line in the crossed two photon exchange graph (see Fig. 3). One of the γ_0's may be brought through the propagator (as may one of the γ_5's in the pseudoscalar case), and use made of $\gamma_0^2 = 1$ or $\gamma_5^2 = 1$. Since

$$\not{p} = \gamma_0 p_0 - \underline{\gamma} \cdot \underline{p} \quad ,$$

bringing through the γ_0 or γ_5 is the same except that the sign of p_0 is changed in the γ_5 case and not in the γ_0 case: that is, the two cases differ only in the sign of p_0. In the calculations in Ref. 7 above, the external energy is carried by the nucleon line,

$$p_0 = k_0 + E \quad ,$$

where k_0 is a variable integrated over in the $\int d^4k$ required for the crossed two photon exchange graph. But the sign of k_0 does not matter since it is integrated over. One concludes that the pseudoscalar and vector cases (restricted to the γ_0 part) are the same except for the sign of E. Therefore,

$$E = -0.511000698682 \quad ,$$

$$m = 0.51100410 \quad ,$$

$$\mu = 0.$$

For the rest, we make a number of transformations of the variables x_1, x_2, x_3. First,

$$x_1 = zy \quad ,$$

$$x_2 = z(1 - y) \quad ,$$

$$x_3 = (1 - z)x \quad ,$$

transforms

$$\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 I = \int_0^1 dx \int_0^1 dy \int_0^1 dz\, z(1 - z)\, I \quad .$$

Then we transform

$$z = \frac{z_{max}\, w}{1 - z_{max} - (1 - 2z_{max})w} \quad ,$$

which transforms the integral to

$$\int_0^1 dw \int_0^1 dx \int_0^1 dy\, I \frac{z(1 - z)\, z_{max}(1 - z_{max})}{[1 - z_{max} - (1 - 2z_{max})w]^2} \quad ,$$

in which z is a function of w of course.

For each p, p_0, q, q_0 we adjusted z_{max} so that the maximum value of the integrand occurred for $\frac{3}{8} < w < \frac{5}{8}$. z_{max} varied from 2×10^{-7} to 0.5 depending on the values of p, p_0, q, q_0. $z_{max} = 2 \times 10^{-7}$ reflects how close to the boundary the peak of the integrand may occur. We use about 20 mesh points in the w integration and 5 each in the x and y integration.

This calculation has to be joined to the calculation described in Ref. 1. Everything is quite straightforward, and we believe that the above details sufficiently document the entire calculation leading to $\lambda = 136.5$. It was only possible to use the crudest q, q_0 mesh (with $a = 1.377$, $b = 1.168$), which means already that 10,000 integrals of the above form have to be calculated, and the accuracy has to be judged from Fig. 2, plus worries about

the w, x, and y meshes. We believe $\lambda = 136.5$ is good to ± 0.1, but perhaps in the future with still faster computing machines it will be possible to verify this assertion.

References

* Work performed under the auspices of the United States Atomic Energy Commission.

1. J. L. Gammel and M. T. Menzel, Phys. Rev. A 7, 858 (1973).
2. H. Grotch and D. R. Yennie, Rev. Mod. Phys. 41, 350 (1969).
3. The calculations are documented in Appendix II.
4. E. E. Salpeter, Phys. Rev. 87, 328 (1952): see remarks at top of the right hand column of p. 329.
5. We assume this statement is true. We should be delighted to have a reference or a refutation!
6. J. M. Jauch and F. Rohrlich, Theory of Photons and Electrons, Addison-Wesley, Cambridge (Mass.), 1955.
7. J. Fleischer, J. L. Gammel, and M. T. Menzel, Phys. Rev. D 8, 1545 (1973).
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9. See the statement in J. D. Bjorken and S. Drell, Relativistic Quantum Mechanics, McGraw Hill Book Company (New York), 1964, in the last paragraph at the bottom of p. 179.
10. J. Fleischer, J. L. Gammel, and M. T. Menzel, Phys. Rev. D 8, 1545 (1973), and D. Bessis, G. Turchetti, and W. R. Wortman, Phys. Lett. 39B, 601 (1972).

Table I.

Data plotted in Fig. 2.

x	(a)	(b)	(c)	(d)
0.5		134.11478	133.99680	133.99452
1.0	134.15343	134.15835	133.99799	133.99214
1.5			133.99961	133.98955
2.0	134.17475	134.17980	134.00162	133.98691
3.0	134.17630	134.18721	134.00671	133.98180
4.0	134.17725	134.19403		133.97708
6.0	134.18707	134.20536	134.02835	133.96871
8.0		134.19989	134.04619	
12.0	134.06512	134.06277	134.08744	133.94970

Figure Captions

1. The self energy graph.

2. λ vs. q-mesh size. $192/x$ = number of points in q-integration.
   (a) Results reported in Ref. 1. a = 1.377, b = 1.168, 16 point Gaussian q_4 integration.
   (b) q_4-correction applied to (a). Eight points used in Gaussian q_4 integration.
   (c) Both corrections applied. a = 1.5, b = 0, 8 point Gaussian q_4 integration.
   (d) Both corrections applied. a = 1.377, b = 1.168, 8 point Gaussian q_4 integration.

3. A portion of the two photon exchange graph.

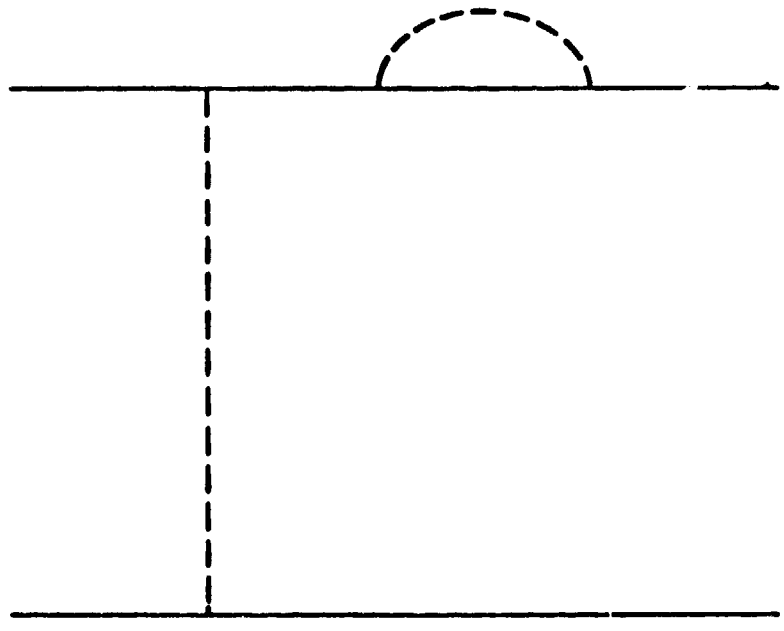
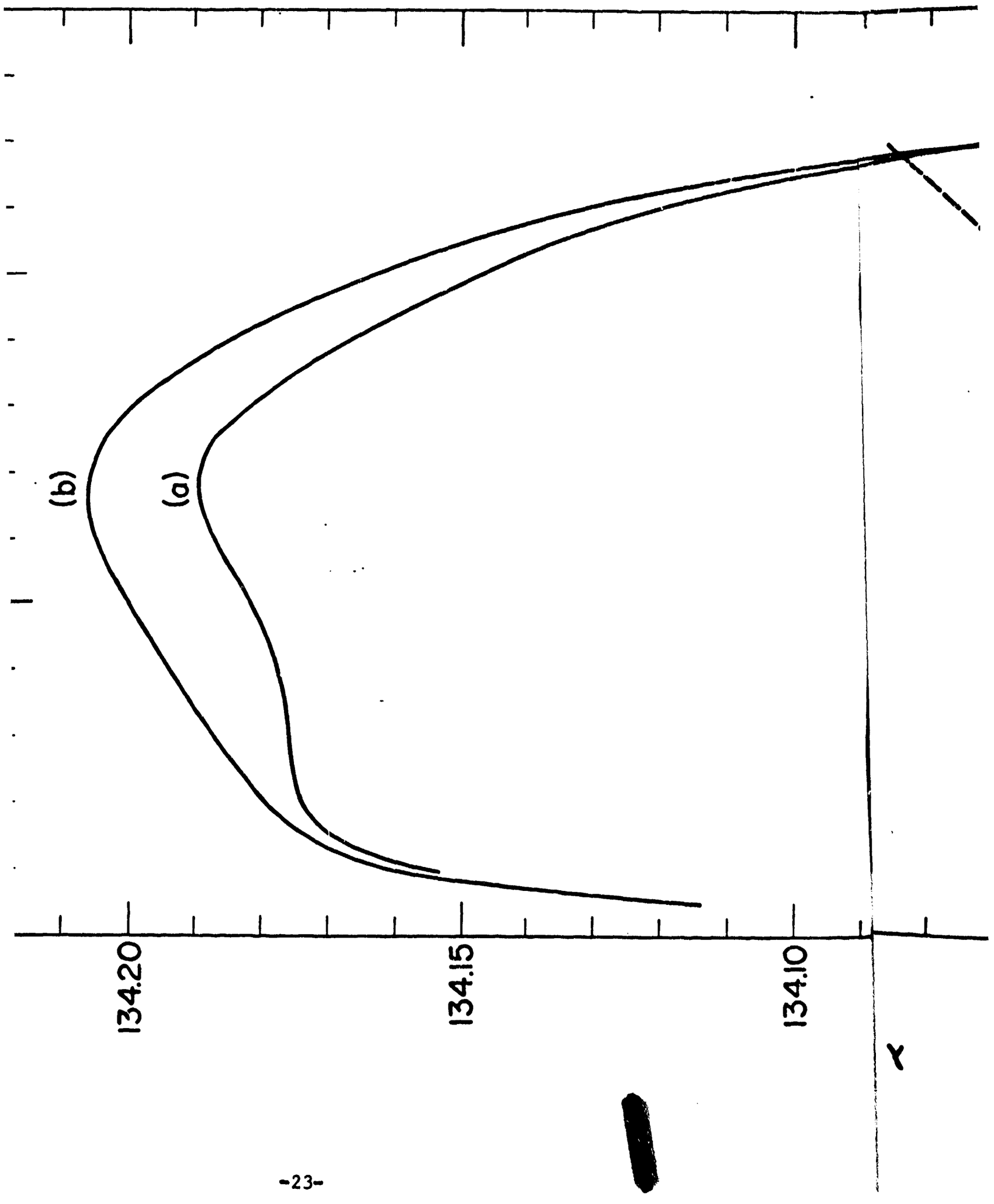


Figure 1


(b)
(a)
134.20
134.15
134.10
x

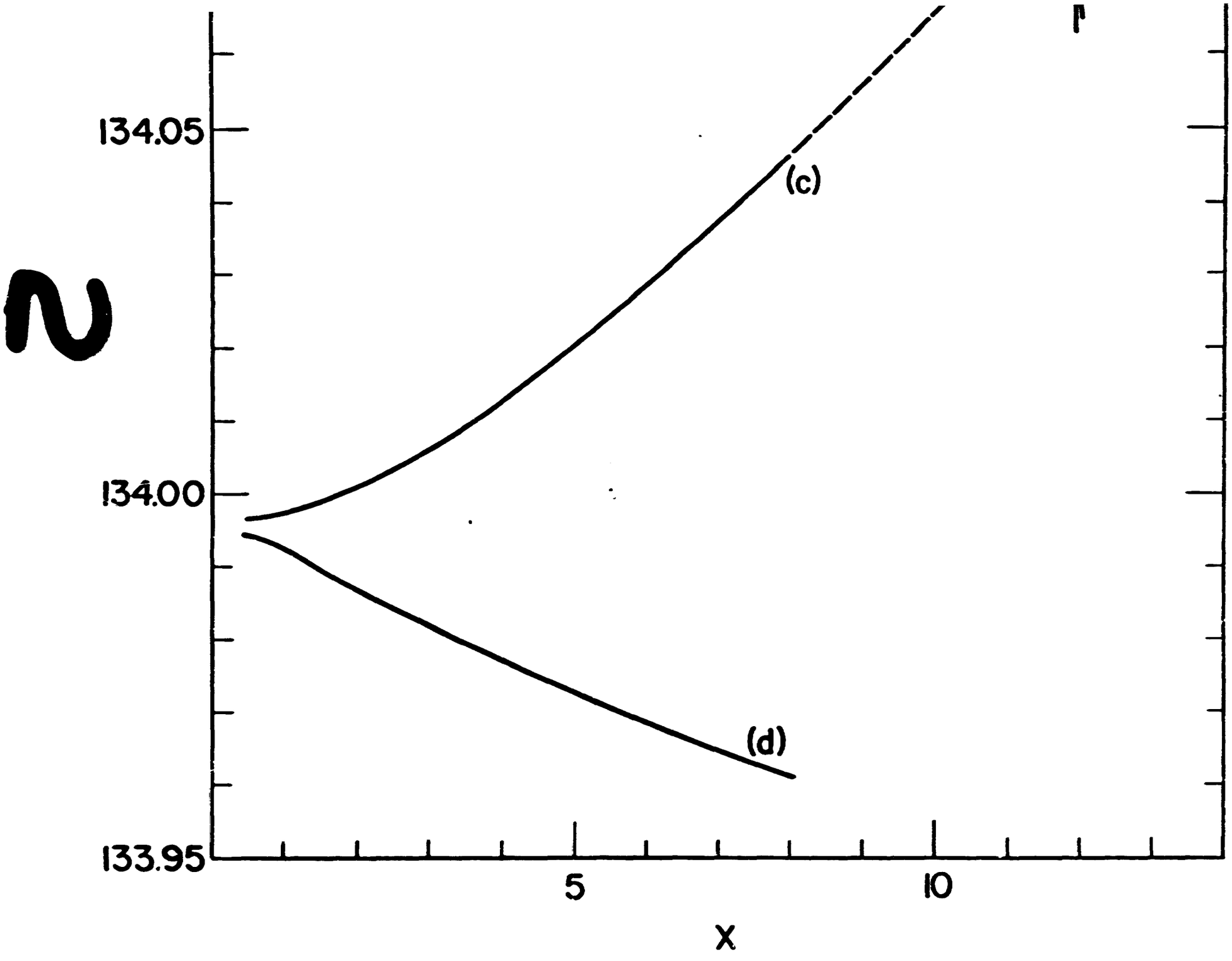


Figure 2

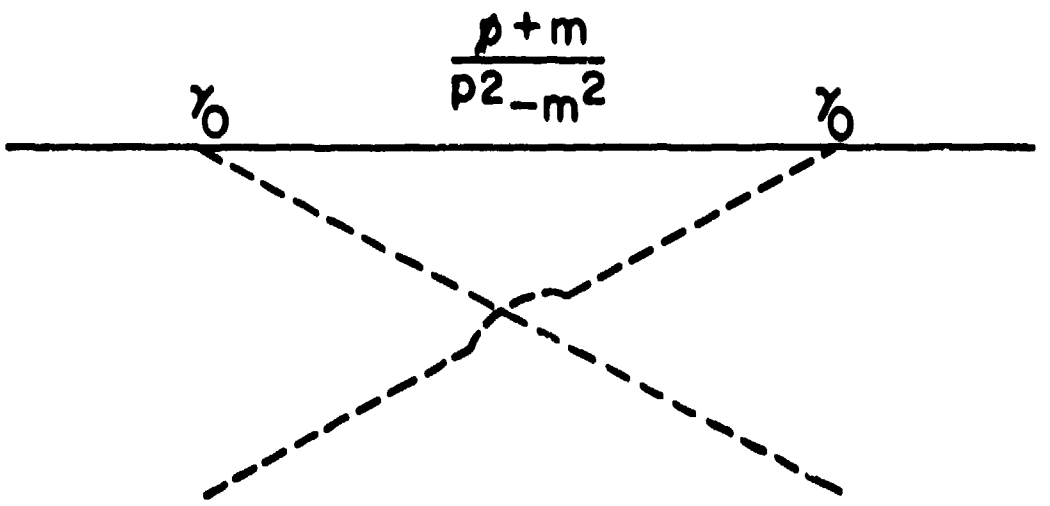


Figure 3