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The Structure of a Magnetically Driven Plane Shock Wave in a Plasma

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The Structure of a Magnetically Driven Plane Shock Wave in a Plasma

by

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CONTENTS

THE STRUCTURE OF A MAGNETICALLY DRIVEN, PLANE SHOCK WAVE IN A PLASMA

by

T. A. Oliphant and Martha S. Hoyt

ABSTRACT

This report describes numerical calculations of the structure of a plane shock wave driven by a transverse magnetic field. The basic MHD theory is described in detail. Progressively more difficult examples are discussed in sequence starting from the simple Thomas shock wave in air and ending with a charge neutral plasma shock wave driven by a transverse magnetic field.

I. INTRODUCTION

The purpose of the present work is to provide a basic orientation in the methods of computing shock structure. We will apply the methods to simple models for air and plasma. For air, we will use the model used by Thomas,[1] namely a simple gas with rigid sphere interactions. We shall treat the plasma here using a very simple model. First, we assume that there is no charge separation so that the plasma moves along as a single fluid and we will need only one momentum equation. Second, we assume that the electrons and ions are always at equilibrium at the same temperature so that we will need only one energy equation.

We will set up the equation for the basic models in the next four sections. Then we will begin consideration of our special applications in Section VI. The sequence of applications will progress from the simplest to the most complicated case.

Since our primary interest is plasma rather than air, we will set up our basic theory for plasma and in the application simply indicate what changes have to be made to obtain the analogous results for air.

II. THE BASIC MODEL

For our basic model for the plasma we will use the hydromagnetic equations as obtained from lowest-order Chapman-Enskog theory.[2] We will refer to a discussion by Burgers[3] for our basic equations. Hereafter, we will designate this reference by the letter B. Since we rule out charge separation and temperature differences between the electronic and ionic components, we can use the equations for the flow of the gas as a whole. Thus we use the equation under heading (A) on pages 128-129 of B. Throughout this section the units are all Gaussian. Changes will be indicated when they are made in late sections.

The mass conservation law is, from $(5\text{-}27)_B$,

$$\frac{D\rho}{Dt} + \rho\epsilon = 0, \qquad (2\text{-}1)$$

where

$$\epsilon = \frac{\partial u_i}{\partial x_i}, \qquad (2\text{-}2)$$

where ρ is the density and $\vec{u}$ is the velocity of the total fluid.

For the momentum conservation law we will use

$(5\text{-}28)_B$. We drop the gravity term and, since there is no charge separation, we also drop the $\rho_e E_h^*$ term. Thus, we write

$$\rho \frac{Du_i}{Dt} + \frac{\partial p_{ij}}{\partial x_j} - (\vec{J} \times \vec{B})_i = 0. \tag{2-3}$$

We do not divide the last term by c as Burgers does because we write J in emu whereas he writes it in esu. The pressure tensor is given by

$$p_{ij} = \delta_{ij} p + P_{ij}, \tag{2-4}$$

where p is the scalar pressure,

$$p = nkT. \tag{2-5}$$

For the deviator components P_{ij} of the pressure tensor, lowest-order Chapman-Enskog theory gives

$$P_{ij} = -\mu \epsilon_{ij}, \tag{2-6}$$

where μ is the viscosity coefficient of the total fluid and

$$\epsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\,\delta_{ij}\epsilon. \tag{2-7}$$

Thus, (2-3) can be written,

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}(\mu \epsilon_{ij}) + (\vec{J} \times \vec{B})_i. \tag{2-8}$$

For our <u>energy conservation</u> law we use $(5\text{-}29)_B$. Using (2-6) we obtain

$$\frac{D}{Dt}\left(\frac{3}{2}p\right) + \frac{5}{2}\,p\epsilon - \frac{1}{2}\,\mu\epsilon_{ij}\epsilon_{ij} + \frac{\partial q_i}{\partial x_i} - E_i^* J_i = 0, \tag{2-9}$$

where

$$E_i^* = E_i + (\vec{u} \times \vec{B})_i. \tag{2-10}$$

The field E_i^* is the effective electric field felt by the moving plasma. This field is related to the current density by

$$E_i^* = \eta\, J_i, \tag{2-11}$$

where η is the electrical resistivity of the total fluid. Thus, the last term in (2-9) is seen to be the Joule-heating term. The current $\vec{J}$ is related to the magnetic field $\vec{B}$ through Ampere's law,

$$J = \frac{1}{4\pi}\,\nabla \times \vec{B}. \tag{2-12}$$

According to lowest-order Chapman-Enskog theory, the heat flux, q_i, is given by

$$q_i = -\kappa \frac{\partial T}{\partial x_i}, \tag{2-13}$$

where κ is the thermal conductivity of the total fluid and T is its temperature. Using (2-10) through (2-13) we write (2-9) in the form

$$\frac{D}{Dt}\left(\frac{3}{2}p\right) + \frac{5}{2}\,p\epsilon - \frac{1}{2}\,\mu\epsilon_{ij}\epsilon_{ij} - \frac{\partial}{\partial x_i}\left(\kappa \frac{\partial T}{\partial x_i}\right) - \eta\left(\frac{1}{4\pi}\,\nabla \times B\right)^2 = 0. \tag{2-14}$$

To complete our theory we need to obtain an equation for the penetration of the magnetic field into the plasma. Combining (2-10) and (2-11) we get

$$J = \frac{\vec{E}^*}{\eta} = \frac{1}{\eta}\,[E + (\vec{u} \times \vec{B})]. \tag{2-15}$$

Combining (2-12) and (2-15) we get

$$\eta(\nabla \times \vec{B}) = 4\pi\,[\vec{E} + (\vec{u} \times \vec{B})]. \tag{2-16}$$

Taking the curl of both sides, we obtain

$$\nabla \times [\eta(\nabla \times \vec{B})] = 4\pi\,[\nabla \times \vec{E} + \nabla \times (\vec{u} \times \vec{B})]. \tag{2-17}$$

Next, we use the emf equation,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{2-18}$$

Hence (2-17) becomes

$$\frac{\partial \vec{B}}{\partial t} = -\frac{1}{4\pi}\,\nabla \times [\eta(\nabla \times \vec{B})] + \nabla \times (\vec{u} \times \vec{B}). \tag{2-19}$$

Equations (2-1), (2-8), (2-14), and (2-19) constitute our basic set of equations of motion.

III. TRANSPORT COEFFICIENTS

In Section II we developed the basic equations of motion for the plasma as a whole. Included in the equations were the transport coefficients μ, κ, and η for the gas as a whole. For the theory discussed above these transport coefficients are obtained simply by adding together the contribution from the electronic and ionic components.

$$\mu = \mu_e + \mu_i, \tag{3-1}$$

$$\varkappa = \varkappa_e + \varkappa_i, \tag{3-2}$$

$$\eta = \eta_e + \eta_i. \tag{3-3}$$

The dependence of these transport coefficients on the hydrodynamic variables has been given by Spitzer.[4] Hereafter we will refer to this reference as S.

An important quantity which enters into all the transport coefficients is $\ln\Lambda$. This quantity is obtained in the consideration of the encounters of the moving charged particles and is discussed in pages 120-131 of S. The expression for Λ derived in S can be written

$$\Lambda = \Lambda_o \frac{T^{3/2}}{\rho^{\frac{1}{2}}}, \tag{3-4}$$

where the temperature is given in eV, and ρ is given in cgs units. From this point on, all temperatures will be given in eV. Here,

$$\Lambda_o = \frac{3}{2Z_i e^3} \left[\frac{(kT_o)^3 m_A A_i}{\pi}\right]^{\frac{1}{2}}. \tag{3-5}$$

The quantities appearing in Λ_o, as well as in the constants obtained in the remainder of this section, are summarized in Table I.

The number density of particles is obtained from ρ by

$$n_i = \frac{\rho}{m_A(A_i + \nu A_e)}, \tag{3-6}$$

$$n_e = \nu n_i . \tag{3-7}$$

For the most part we ignore νA_e in comparison to A_i in our transport coefficients.

Since the viscosity is contributed mainly by the ions, we drop μ_e in (3-1). The result for $\mu = \mu_i$ is given on page 146 of S. The weak field approximation μ_w of μ is isotropic, and from $(5\text{-}54)_S$ we have

$$\mu_w = C_{\mu 1} \frac{T^{5/2}}{\ln\Lambda}, \tag{3-8}$$

where

$$C_{\mu 1} = \frac{0.406\, m_i^{\frac{1}{2}} (kT_o)^{5/2}}{Z_i^4 \, e^4}. \tag{3-9}$$

The viscosity $\mu_\perp$ transverse to a strong magnetic field is given by

$$\mu_\perp = C_{\mu 2} \frac{\rho^2 \ln\Lambda}{T^{\frac{1}{2}} B^2}, \tag{3-10}$$

where B is in gauss. Here

$$C_{\mu 2} = \frac{2}{5}\left(\frac{\mu}{m_i kT_o}\right)^{\frac{1}{2}} (Z_i ec)^2 . \tag{3-11}$$

Let us obtain a formula for the viscosity transverse to an intermediate field by the following sort of interpolation

$$\mu = \frac{\mu_w}{1 + \zeta}. \tag{3-12}$$

In the strong field limit, $f >> 1$, and

$$\mu_\perp = \frac{\mu_w}{f_\mu} .$$

Hence

$$\zeta = \frac{\mu_w}{\mu_\perp} = \frac{C_{\mu 1}}{C_{\mu 2}} \left(\frac{T^{3/2} B}{\rho \ln\Lambda}\right)^2 . \tag{3-13}$$

TABLE I

Physical Constants

Symbol	Quantity	Value	Units
Z_i	ionic charge number	1 for deuterium	
e	electronic charge	4.8029×10^{-10}	esu
T_o	1 eV in °K	11593.	°K/eV
k	Boltzmann's constant	1.3804×10^{-16}	erg/°K-particle
m_A	atomic mass unit	1.6604×10^{-24}	gm
m_e	electron mass	0.9107×10^{-27}	gm
ν	valence of ions	1 for deuterium	
δ_T	Theoretical constant from Spitzer, p. 145	0.225	
R_o	gas constant	0.96385×10^{12}	$\frac{\text{erg}}{\text{eV mode}}$
$R = \frac{R_o}{A}$	gas constant		$\frac{\text{erg}}{\text{eV gm}}$
A_i	atomic number of deuterium	2.01473	
$A_e = \frac{m_e}{m_A}$	atomic number of the electron		
c	velocity of light	2.99793×10^{10}	cm/sec

Hence, (3-12) can be written

$$\mu = \frac{C_{\mu 1} \frac{T^{5/2}}{\ln\Lambda}}{1 + \frac{C_{\mu 1}}{C_{\mu 2}} \left(\frac{T^{3/2} B}{\rho \ln\Lambda}\right)^2} . \tag{3-14}$$

Next, let us consider the thermal conductivity. The relevant results are given on pages 144-145 of S. Here the electronic contribution, κ_e, is predominant so we drop κ_i. The weak magnetic field approximation κ_w of κ is

$$\kappa_w = C_{\kappa 1} \frac{T^{5/2}}{\ln\Lambda} , \tag{3-15}$$

where

$$C_{\kappa 1} = 20 \left(\frac{2}{\pi}\right)^{3/2} \frac{\delta_T (kT_o)^{7/2}}{e^4 m_e^{\frac{1}{2}}} . \tag{3-16}$$

The thermal conductivity $\kappa_\perp$, transverse to a strong magnetic field is given by

$$\kappa_\perp = C_{\kappa 2} \frac{\rho^2 \ln\Lambda}{B^2 T^{\frac{1}{2}}} , \tag{3-17}$$

where

$$C_{\kappa 2} = \frac{8}{3} \left(\frac{\pi k}{T_o}\right)^{\frac{1}{2}} \frac{(ec)^2}{m_e^{3/2}} . \tag{3-18}$$

We make the same sort of interpolation here as we did with μ. Thus,

$$\kappa = \frac{C_{\kappa 1} \frac{T^{5/2}}{\ln\Lambda}}{1 + \frac{C_{\kappa 1}}{C_{\kappa 2}} \left(\frac{T^{3/2} B}{\rho \ln\Lambda}\right)^2} . \tag{3-19}$$

Next, we consider the electrical resistivity. Since the resistive effects involve mainly the electrons, we drop the ionic contribution. The relevant result is given on pages 138-139 of S. We obtain

$$\eta = C_\eta \frac{\ln\Lambda}{T^{3/2}} , \tag{3-20}$$

where

$$C_\eta = \left(\frac{\pi}{2kT_o}\right)^{3/2} \frac{m_e^{\frac{1}{2}} (ec)^2}{2(0.582)} . \tag{3-21}$$

Finally, we consider the mean free path. It is given by

$$\ell = \frac{1}{n\sigma_d} ; \tag{3-22}$$

but σ_d given by Glasstone and Lovberg[5] is

$$\sigma_d = \frac{2\pi e^4}{\left(\frac{3}{2}kT_o\right)^2} \frac{\ln\Lambda}{T^2} . \tag{3-23}$$

Thus,

$$\ell = \frac{\left(\frac{3}{2}kT_o\right)^2 T^2}{2\pi n e^4 \ln\Lambda} . \tag{3-24}$$

Thus, we write

$$\ell = C_\ell \frac{T^2}{\rho \ln\Lambda} , \tag{3-25}$$

where

$$C_\ell = \frac{\left(\frac{3}{2}kT_o\right)^2 A_i m_A}{2\pi e^4} . \tag{3-26}$$

IV. EQUATIONS FOR STEADY-STATE, PLANE SHOCK

Having obtained all of our basic equations and transport coefficients, we now specialize to the case of a steady-state, plane shock wave traveling in the x-direction. For this special case

$$\frac{D}{Dt} = u \frac{d}{dx} , \tag{4-1}$$

$$\epsilon = \frac{du}{dx} , \tag{4-2}$$

and

$$\epsilon_{ij} = \begin{pmatrix} \frac{4}{3}\frac{du}{dx} & 0 & 0 \\ 0 & -\frac{2}{3}\frac{du}{dx} & 0 \\ 0 & 0 & -\frac{2}{3}\frac{du}{dx} \end{pmatrix} . \tag{4-3}$$

Equation (2-1) reduces to

$$\frac{d}{dx}(\rho u) = 0 . \tag{4-4}$$

We will now begin to compare our equations with the equations of Thomas,[1] referred to hereafter as T. (4-4) can easily be put into the form of $(1)_T$ by differentiation.

$$u \frac{d\rho}{dx} = -\rho \frac{du}{dx} . \tag{4-5}$$

We assume that we will have a driving field, $\vec{B}$, directed in the z-direction, but varying only in the x-direction. Then, (2-12) can be written

$$\vec{J} = \hat{y}\left(-\frac{1}{4\pi}\frac{\partial B}{\partial x}\right) = \hat{y}\, J, \tag{4-6}$$

where

$$\vec{B} = \hat{z}\, B \,. \tag{4-7}$$

With the help of (4-6), we reduce (2-3) to

$$\rho u \frac{du}{dx} = -\frac{dP}{dx}, \tag{4-8}$$

where

$$P = p + \frac{B^2}{8\pi} - \tilde{\mu}\frac{du}{dx}, \tag{4-9}$$

and

$$\tilde{\mu} = \frac{4}{3}\mu \,. \tag{4-10}$$

Using (4-4), we easily put (4-10) into the form of $(2)_T$.

$$u\frac{d}{dx}(\rho u) = -\rho u\frac{du}{dx} - \frac{dP}{dx} \,. \tag{4-11}$$

Note that, although this equation is formally identical to $(2)_T$, we include the additional effect of the magnetic field by the added term in (4-9).

Similarly, we write the energy equation (2-14) in the form,

$$\frac{d}{dx}\left(\frac{3}{2}pu\right) + P\frac{du}{dx} + \frac{dq}{dx} - \frac{B^2}{8\pi}\frac{du}{dx} - \eta\left(\frac{1}{4\pi}\frac{dB}{dx}\right)^2 = 0. \tag{4-12}$$

We define the internal energy density to be

$$E = \frac{3}{2}p + \frac{B^2}{8\pi}, \tag{4-13}$$

and also note that

$$p = nkT_oT \,. \tag{4-14}$$

Hence, (4-12) can be written

$$\frac{d}{dx}(uE) + P\frac{du}{dx} + \frac{dq}{dx} - \frac{d}{dx}\left(\frac{uB^2}{8\pi}\right) - \frac{B^2}{8\pi}\frac{du}{dx} - \eta\left(\frac{1}{4\pi}\frac{dB}{dx}\right)^2 = 0. \tag{4-15}$$

This is in the form of $(3)_T$. However, not only do we have the altered definitions of P and E, but additional magnetic terms as well.

Finally, for the present geometry (2-19) is written

$$\frac{1}{4\pi}\frac{d}{dx}\left(\eta\frac{dB}{dx}\right) - \frac{d}{dx}(uB) = 0. \tag{4-16}$$

Our basic equations are now (including the equation of state) (4-4), (4-8), (4-13), (4-14), (4-15), and (4-16).

V. THE RANKINE-HUGONIOT CONDITIONS

The first integrals of the equations of Section IV lead to the Rankine-Hugoniot conditions. (4-4) and (4-8) are integrated just as in the case of Thomas to give

$$\rho u = a, \tag{5-1}$$

$$au + P = b. \tag{5-2}$$

However, (4-15) is not quite so easy to integrate. Skipping to (4-16), we obtain

$$\frac{\eta}{4\pi}\frac{dB}{dx} - uB = d. \tag{5-3}$$

In integrating (4-15) we have to make use of (5-2) and (5-3). We obtain

$$uE + ub - \frac{1}{2}au^2 + q - \frac{uB^2}{4\pi} - \frac{Bd}{4\pi} = c \,. \tag{5-4}$$

If we set B = 0, these equations reduce exactly to the corresponding results of Thomas. We now write the relation

$$nkT_o = \rho\mathcal{R}, \tag{5-5}$$

where n is the density in atoms/cc, k is Boltzmann's constant in erg/^{o}K, ρ is the density in grams/cc, and $\mathcal{R}$ is the gas constant in erg/(gram eV). Making use of (5-5), we write (4-14) as

$$p = \rho \mathcal{R} T \,. \tag{5-6}$$

Making use of (2-13) and (4-9), we write (5-2) through (5-4) in the form,

$$\tilde{\mu}\,\frac{du}{dx} = au + a\mathcal{R}\frac{T}{u} + \frac{B^2}{8\pi} - b, \tag{5-7}$$

$$\kappa\,\frac{dT}{dx} = \frac{3}{2}\,a\mathcal{R}T + ub - \frac{1}{2}\,au^2 - \frac{uB^2}{8\pi} - \frac{Bd}{4\pi} - c, \tag{5-8}$$

$$\frac{\eta}{4\pi}\,\frac{dB}{dx} = uB + d \,. \tag{5-9}$$

Let us now consider a schematic plot of the density across the shock wave in Fig. 1.

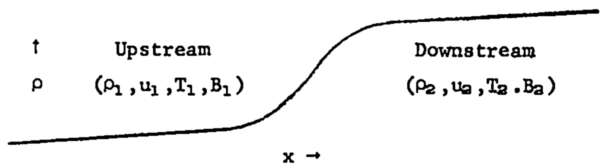

Fig. 1. The density profile of a shock wave.

Since we are talking about the time-independent form of the shock wave, we are automatically considering ourselves to be located in the frame of reference traveling with the shock wave. The (upstream) fluid in the ambient state seems to be flowing to the right into the shock wave with velocity $u_1 > 0$. If the ambient fluid is actually at rest, then u_1 is equal to the magnitude of the velocity of the shock wave. Of course, the actual shock wave would be traveling to the left. From (5-1), we see that ρu is the same for any value of x. Thus,

$$\rho u = \rho_1 u_1 = \rho_2 u_2 = a. \tag{5-10}$$

For convenience we will often leave the constant a in the other equations. In the region away from the shock wave, u and T are constant so that the left-hand sides of (5-7) through (5-9) vanish. This allows us to evaluate b, c, and d.

$$b = au_i + a\mathcal{R}\frac{T_i}{u_i} + \frac{B_i^2}{8\pi}, \tag{5-11}$$

$$c = \frac{3}{2}\,a\mathcal{R}T_i + bu_i - \frac{1}{2}\,au_i^2 - \frac{u_iB_i^2}{8\pi} - \frac{B_id}{4\pi}, \tag{5-12}$$

$$d = -u_iB_i, \tag{5-13}$$

where i is equal to either 1 or 2. We can therefore write (5-7) through (5-9) in the form

$$\tilde{\mu}\,\frac{du}{dx} = a(u - u_i) + a\mathcal{R}\left(\frac{T}{u} - \frac{T_i}{u_i}\right) + \frac{B^2 - B_i^2}{8\pi}, \tag{5-14}$$

$$\kappa\,\frac{dT}{dx} = \frac{3}{2}\,a\mathcal{R}(T-T_i) + b(u-u_i) - \frac{1}{2}\,a(u^2-u_i^2) - \frac{\left(uB^2-u_iB_i^2\right)}{8\pi} - \frac{d}{4\pi}\,(B-B_i), \tag{5-15}$$

$$\frac{\eta}{4\pi}\,\frac{dB}{dx} = uB - u_iB_i. \tag{5-16}$$

Equations (5-10) through (5-13) are equivalent to the Rankine-Hugoniot relations connecting the variables (u_1, ρ_1, T_1, B_1) ahead of the shock to those (u_2, ρ_2, T_2, B_2) behind the shock wave. Connections with the more usual forms of these relations are given below. Equations (5-10) and (5-14) through (5-16) describe the behavior of the variables (u, p, T, B) as we cross the shock front.

To obtain the more usual forms of the Rankine-Hugoniot relations, we rewrite (5-11) and (5-12) in terms of pressure and energy variables

$$b = au_i + P_i, \tag{5-17}$$

$$c = u_i(E_i + P_i) + \frac{1}{2}\,au_i^2 \,. \tag{5-18}$$

With a little algebra we can show that these relations are equivalent to

$$P_2 - P_1 = \rho_1 u_1 (u_1 - u_2), \tag{5-19}$$

$$P_2(u_1-u_2) = \rho_1 u_1\left[e_2 - e_1 + \frac{1}{2}\,(u_1-u_2)^2\right], \tag{5-20}$$

$$e_i = \frac{E_i}{\rho_i} \tag{5-21}$$

is the specific energy. With suitable changes of definition we see that (5-10), (5-19), and (5-20) are formally equivalent to the Rankine-Hugoniot relations as given by Cole.[6] By some further manipulations, we can write the energy equation in the following form.

$$\frac{B_1^2}{16\pi}\left(\frac{\rho_2}{\rho_1}\right)^2 + \left[\frac{P_1}{(2/3)} + \frac{1}{2}\,(P_1+P_2) - \frac{B_1^2}{16\pi}\right]\left(\frac{\rho_2}{\rho_1}\right) = \frac{P_2}{(2/3)} + \frac{1}{2}\,(P_1+P_2) \,. \tag{5-22}$$

Now, let us assume that we know the ambient conditions. Then, let us assume a shock strength, $\tilde{P}$, defined by

$$\tilde{P} \equiv \frac{P_2}{P_1} . \tag{5-23}$$

It is then clear that (5-22) gives us the means of calculating the quantity

$$\tilde{\rho} \equiv \frac{\rho_2}{\rho_1} . \tag{5-24}$$

It is, of course, easier to assume $\tilde{\rho}$ and calculate $\tilde{P}$. The velocity u_1 which is positive and of the magnitude of the shock speed is given by

$$u_1 = \sqrt{\frac{\rho_2}{\rho_1} \frac{(P_2 - P_1)}{(\rho_2 - \rho_1)}} . \tag{5-25}$$

The downstream velocity, u_2, is given by (5-10) which we write as

$$u_2 = \frac{\rho_1}{\rho_2} u_1 . \tag{5-26}$$

Finally, we have

$$B_2 = \frac{\rho_2}{\rho_1} B_1 . \tag{5-27}$$

Equations (5-22), (5-25), and (5-26) with B_1 set to zero agree with the γ-law relations with $\gamma = 1\text{-}\frac{2}{3}$. The details of the comparison are given in Section VI.

VI. THE ZERO MAGNETIC FIELD CASE

We will see that certain scaling properties occur as soon as we discard the magnetic field. For a perfect γ-law gas with no magnetic field, Bethe[7] writes (changing notation appropriately)

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)p_2 + (\gamma - 1)p_1}{(\gamma - 1)p_2 + (\gamma + 1)p_1} , \tag{6-1}$$

$$u_1 = \sqrt{\frac{(\gamma + 1)p_2 + (\gamma - 1)p_1}{2\rho_1}} , \tag{6-2}$$

$$u_1 - u_2 = \frac{2(p_2 - p_1)}{\sqrt{2\rho_1 \left[(\gamma+1)p_2 + (\gamma-1)p_1\right]}} . \tag{6-3}$$

Setting B_1 equal to zero in (5-22), (5-25), and (5-26) and performing some simple manipulations, we obtain

$$\frac{\rho_2}{\rho_1} = \frac{4p_2 + p_1}{p_2 + 4p_1} , \tag{6-4}$$

$$u_1 = \sqrt{\frac{4p_2 + p_1}{3\rho_1}} . \tag{6-5}$$

$$u_1 - u_2 = \frac{p_2 - p_1}{\sqrt{\frac{\rho_1}{3} (4p_2 + p_1)}} \tag{6-6}$$

Setting $\gamma = 1\frac{2}{3}$ in (6-1) through (6-3), we obtain (6-4) through (6-6) so that the results of Section V reduce appropriately to the perfect gas case. The shock strength reduces to

$$\tilde{p} = \frac{p_2}{p_1} . \tag{6-7}$$

In the remainder of this section we will use (6-1) through (6-3) instead of (6-4) through (6-6) in order to facilitate comparison with previous work for gases of general γ. However, we must bear in mind that for the plasma case γ is understood to be equal to $1\frac{2}{3}$. From (6-1) through (6-3), we obtain

$$\rho \equiv \frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)\, \tilde{p} + (\gamma - 1)}{(\gamma - 1)\, \tilde{p} + (\gamma + 1)} , \tag{6-8}$$

$$u_1 = \sqrt{\frac{p_1}{2\rho_1} \left[(\gamma + 1)\, \tilde{p}\, (\gamma - 1)\right]} , \tag{6-9}$$

$$\tilde{u} \equiv \frac{u_2}{u_1} = 1 - \frac{2(\tilde{p} - 1)}{(\gamma + 1)\, \tilde{p} + (\gamma - 1)} . \tag{6-10}$$

From the equation of state (5-6) we obtain

$$T \equiv \frac{T_2}{T_1} = \frac{\tilde{p}}{\tilde{\rho}} . \tag{6-11}$$

If the problem can be scaled appropriately, then we need not specify ambient conditions such as (p_1, ρ_1, T_1), but may represent a whole class of problems in terms of scaled variables by choosing only the shock strength, $\tilde{p}$. We will then be able to express a multitude of results involving the three ambient parameters by the same scaled curves. We will show that air and plasma (in our simple model) can be scaled in this way.

For zero magnetic fields, (5-7) and (5-8) reduce to

$$\tilde{\mu} \frac{du}{dx} = au + a\mathcal{R} \frac{T}{u} - b, \tag{6-12}$$

$$\varkappa \frac{dT}{dx} = \frac{a\mathcal{R}\, T}{\gamma - 1} + ub - \frac{1}{2} au^2 - c. \tag{6-13}$$

We have inserted a general γ here which gives agreement with the equation of Thomas. The constants can be written

$$a = \rho_i u_i \ , \tag{6-14}$$

$$b = a\left(u_i + \mathcal{R}\,\frac{T_i}{u_i}\right) \ , \tag{6-15}$$

$$c = \frac{a\mathcal{R}T_i}{\gamma - 1} + bu_i - \frac{1}{2}\,au_i^2 . \tag{6-16}$$

Let us introduce the scaled velocity, ω, and temperature, θ, defined by

$$u = \frac{b}{a}\,\omega \ , \tag{6-17}$$

$$T = \frac{1}{\mathcal{R}}\left(\frac{b}{a}\right)^2 \theta . \tag{6-18}$$

We can now write (6-12) and (6-13) in the form

$$\frac{\tilde{\mu}}{a}\frac{d\omega}{dx} = \omega + \frac{\theta}{\omega} - 1 \ , \tag{6-19}$$

$$\frac{1}{a}\left(\frac{2\kappa}{N\mathcal{R}}\right)\frac{d\theta}{dx} = \theta - \frac{1}{N}\left[(1-\omega)^2 - 1 + \frac{2ac}{b^2}\right] . \tag{6-20}$$

The factor f defined by Thomas is given by

$$\kappa = \frac{3}{4}\,f\,\tilde{\mu}\,\frac{N}{2}\,\mathcal{R} \ . \tag{6-21}$$

In (6-20) and (6-21) we have used the number of degrees of freedom, N, of the gas atoms which is related to γ by

$$\gamma = 1 + \frac{2}{N} \ . \tag{6-22}$$

Here f can be anything since we have as yet specified no relation between κ and $\tilde{\mu}$.

Now, let us scale our x-variable in terms of the mean free path, ℓ.

$$x = \ell s . \tag{6-23}$$

(6-19) and (6-20) then become

$$\left(\frac{\tilde{\mu}}{a\ell}\right)\frac{d\omega}{ds} = \omega + \frac{\theta}{\omega} - 1, \tag{6-24}$$

$$\frac{3}{4}\,f\left(\frac{\tilde{\mu}}{a\ell}\right)\frac{d\theta}{ds} = \theta - \frac{1}{N}\left[(1-\omega)^2 + \alpha\right] , \tag{6-25}$$

where

$$\alpha = \frac{2ac}{b^2} - 1 \ . \tag{6-26}$$

To be able to scale (6-24) and (6-25), we must write the factor $\tilde{\mu}/a\ell$ in terms of the scaled variables ω and θ. Thomas has used the hard-sphere model for air. The model is discussed by Chapman and Cowling (Reference 2, 791 and p. 101). The expressions obtained for $\tilde{\mu}$ and ℓ in this model are

$$\tilde{\mu} = \frac{4}{3}\,\frac{\nu_o}{\pi^{3/2}}\,\frac{(mkT)^{1/2}}{\sigma^2} \tag{6-27}$$

and

$$\ell = \frac{1}{\sqrt{2}\,\pi\,n\,\sigma^2} \ , \tag{6-28}$$

where σ is the collision cross section and ν_o is a dimensionless number 0.998. From (6-27) and (6-28) we obtain

$$\frac{\tilde{\mu}}{a\ell} = \frac{4}{3}\,\nu_o\sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\theta}}{\omega} \ . \tag{6-29}$$

Thus, the factor appearing in (6-24) and (6-25) is dimensionless and allows a convenient scaling. From Section III we have, for the case of zero magnetic field,

$$\tilde{\mu} = \frac{4}{3}\,C_{\mu 1}\,\frac{T^{5/2}}{\ell n\Lambda} \tag{6-30}$$

and

$$\ell = C_\ell\,\frac{T^2}{\rho\,\ell n\Lambda} \ . \tag{6-31}$$

Thus,

$$\frac{\tilde{\mu}}{a\ell} = \frac{4}{3}\,\frac{C_{\mu 1}}{C_\ell}\,\frac{T^{1/2}}{u} \ . \tag{6-32}$$

Using (6-17) and (6-18), we get

$$\frac{\tilde{\mu}}{a\ell} + C\,\frac{\theta^{1/2}}{\omega} \ , \tag{6-33}$$

where C is the dimensionless constant,

$$C = \frac{4}{3}\,\frac{C_{\mu 1}}{C_\ell}\,\frac{1}{\sqrt{\mathcal{R}}} = 1.5115 \ . \tag{6-34}$$

For air,

$$C = \frac{4}{3}\nu\sqrt{\frac{2}{\pi}} = 1.0617 . \tag{6-35}$$

Hence, we write the differential equations for both together in the form

$$C\,\frac{\sqrt{\theta}}{\omega}\,\frac{d\omega}{ds} = \omega + \frac{\theta}{\omega} - 1, \tag{6-36}$$

$$\frac{3}{4}\,f\,C\,\frac{\sqrt{\theta}}{\omega}\,\frac{d\theta}{ds} = \theta - \frac{1}{N}\left[(1-\omega)^2 + \alpha\right]. \tag{6-37}$$

The differential equation for the integral curve in (ω,θ) space is obtained by eliminating ds between (6-36) and (6-37).

$$\frac{d\theta}{d\omega} = \frac{4}{3f}\,\omega\,\frac{\theta - \frac{1}{N}\left[(1-\omega)^2+\alpha\right]}{\omega^2 - \omega + \theta} . \tag{6-38}$$

Now, let us show that, once we specify $\tilde{p}$, our scaled solution is completely determined. From (6-15), we have

$$\frac{a}{b} = \frac{u_1}{u_1^2 + \mathcal{R}T_1} . \tag{6-39}$$

Therefore, using (6-17), we write

$$\omega_1 = \frac{a}{b}\,u_1 = \frac{u_1^2}{u_1^2 + \mathcal{R}\,T_1} . \tag{6-40}$$

Substituting (5-6) and (6-22) into (6-9), we obtain

$$u_1 = \sqrt{\frac{\mathcal{R}T_1}{N}\left[(N+1)\tilde{p} + 1\right]} . \tag{6-41}$$

Substituting (6-41) into (6-40), we obtain

$$\omega_1 = \frac{(N+1)\tilde{p} + 1}{(N+1)(\tilde{p}+1)} . \tag{6-42}$$

From (6-10), we get

$$\omega_2 = \left[1 - \frac{N(\tilde{p} - 1)}{(N+1)\tilde{p} + 1}\right]\omega_1 . \tag{6-43}$$

Directly from (6-19), we obtain

$$\theta_i = \omega_i(1 - \omega_i) . \tag{6-44}$$

Therefore, from $\tilde{p}$ we can calculate ω and θ both before and behind the shock wave.

Using (6-8), (6-10), and (6-11), we can calculate $\tilde{p}$, $\tilde{u}$, and $\tilde{T}$ in the scaled problem. To get back to the unscaled solution, we need to specify two out of the three ambient quantities (p_1, ρ_1, T_1). The third ambient variable is then given by (5-6). From the definition of $(\tilde{p},\tilde{\rho},\tilde{T})$ we then immediately obtain (p_2, ρ_2, T_2). The constants a, b, and c are then obtained from (6-14), (6-15), and (6-16). We can now plot the unscaled solution using the definitions or the scaled variables and other simple relations which are summarized as follows:

$$a = \rho_1 u_1 \tag{6-45}$$

$$b = a u_1 + p_1 , \tag{6-46}$$

$$x = \ell s , \tag{6-47}$$

$$u = \frac{b}{a}\,\omega , \tag{6-48}$$

$$T = \frac{1}{\mathcal{R}}\left(\frac{b}{a}\right)^2 \theta , \tag{6-49}$$

$$\rho = \frac{a}{u} , \tag{6-50}$$

$$p = \rho\,\mathcal{R}T . \tag{6-51}$$

VII. BECKER'S SIMPLIFIED SOLUTION

As quoted by Thomas,[1] Becker[8] shows that when

$$f = \frac{4}{3}\,\frac{N+2}{N} , \tag{7-1}$$

a great simplification occurs. Namely, the solution to (6-41) takes on the simplified form

$$\theta = \frac{1}{\omega_1 + \omega_2}\left[\omega_1\omega_2 - (\omega_1 + \omega_2 - 1)\omega^2\right]. \tag{7-2}$$

Using this relation, we eliminate θ from (6-39), obtaining

$$\frac{d\omega}{ds} = \frac{-(\omega_1 - \omega)(\omega - \omega_2)}{C\sqrt{(\omega_1+\omega_2)\left[\omega_1\omega_2 - (\omega_1+\omega_2-1)\omega^2\right]}} \equiv \varphi(\omega). \tag{7-3}$$

Thus, we have a single differential equation instead of the two simultaneous equations, (6-39) and (6-40). We can integrate (7-3) numerically. There is a slight complication with regard to the initial value of ω. If we take $\omega = \omega_1$ or $\omega = \omega_2$, we must start our integration at some indeterminate large distance from the shock wave, and this is clearly unsatisfactory. Instead, we must find an approximate solution to (7-3), valid when ω deviates from ω_i $(i = 1, 2)$ by a small, but finite, amount. Thus, we expand $\varphi(\omega)$ about ω_i

$$\varphi(\omega) = \frac{\partial\varphi(\omega)}{\partial\omega}\bigg|_{\omega=\omega_i} \Delta\omega \quad , \tag{7-4}$$

where we have used the fact that $\varphi(\omega_i) = 0$. Using the abbreviation

$$\varphi_\omega = \frac{\partial\varphi(\omega)}{\partial\omega}\bigg|_{\omega=\omega_i} \quad , \tag{7-5}$$

we have

$$\varphi_\omega = \frac{\epsilon_i(\omega_1 - \omega_2)}{C\sqrt{(\omega_1 + \omega_2)[\omega_1\omega_2 - (\omega_1 + \omega_2 - 1)\omega_i^2]}} , \tag{7-6}$$

where

$$\epsilon_i = \left\{ \begin{matrix} +1, & i = 1 \\ -1, & i = 2 \end{matrix} \right\} . \tag{7-7}$$

Equation (7-3) becomes

$$\frac{d\omega}{ds} = (\omega - \omega_i)\,\varphi_\omega \; . \tag{7-8}$$

The solution to (7-8) is

$$\omega = \omega_i\left(1 - \epsilon_i C^{\varphi_\omega s}\right) . \tag{7-9}$$

Now, suppose we wish to start our integration at the point, s_o, at which ω deviates from ω_i by some fraction, Υ, such as, for example, $\Upsilon = 0.001$. We simply set

$$s_o = \frac{\log \Upsilon}{\varphi_\omega} . \tag{7-10}$$

Thus, we carry out our integration numerically for increasing (decreasing) s using the exact form (7-3). We integrate until ω has approached ω_j $(j \neq i)$. To do this we must set up a reasonable mesh. A rough estimate of the shock thickness is obtained as follows. We define the center of the shock front to be at the point on the ω vs. s curve at which $\omega = \bar{\omega}$. Then, using (7-3) with ω set equal to $\bar{\omega}$, the shock thickness is given by

$$\Delta \equiv (\omega^1 - \omega^2)\frac{ds}{d\omega}\bigg|_{\omega=\bar{\omega}}$$

$$= C\left\{\frac{(\omega_1 - \omega_2)[2\bar{\omega}(\omega_1\omega_2 - (2\bar{\omega}-1)\bar{\omega}^2]^{\frac{1}{2}}}{(\omega_1 - \bar{\omega})(\bar{\omega} - \omega_2)}\right\} , \tag{7-11}$$

where

$$\bar{\omega} = \frac{\omega_1 + \omega_2}{2} \; . \tag{7-12}$$

For our beginning s value, let us take

$$s_N = s_o \pm K(\omega_1 - \omega_2)\frac{ds}{d\omega}\bigg|_{\omega-\bar{\omega}} \quad , \tag{7-13}$$

where K is an arbitrary number of the order of unity, say 4. For N mech intervals we then have the increment

$$\Delta s = \pm\frac{(s_N - s_o)}{N} \quad . \tag{7-14}$$

We are thus free to carry out our integration using Runge-Kutta, or some very simple scheme such as

$$\omega_{n+1} = \omega_n + \Delta s\; f(\omega_n) \quad . \tag{7-15}$$

Computations using (7-15) have been carried out on the Maniac-II. The cases of air and plasma and the results will be discussed at the end of this section.

We will now discuss the validity of (7-1) for air and for plasma. The factor f can be obtained experimentally for air by measuring $\tilde{\mu}$ and $\varkappa$. Quoting Thomas, we have

$$f_{air} = 1.95 \quad . \tag{7-16}$$

For plasma we use (3-8) for μ and (3-15) for $\varkappa$.

$$\tilde{\mu} = \frac{4}{3} C_{\mu 1}\,\frac{T^{5/2}}{\ell n\Lambda} \quad , \tag{7-17}$$

$$\varkappa = C_{\varkappa 1}\,\frac{T^{5/2}}{\ell n\Lambda} \quad . \tag{7-18}$$

Thus, we obtain

$$f_{plasma} = \frac{2}{N\mathcal{R}}\,\frac{C_{\eta 1}}{C_{\mu 1}} = 2.26 \times 10^2 \tag{7-19}$$

which is a dimensionless constant.

For air, N = 5, so that

$$\frac{4}{3}\ \frac{N+2}{N} = \frac{4}{3}\cdot\frac{7}{5} = 1.867\ , \qquad (7\text{-}20)$$

which is close to (7-16), so the simple approximation is seen to be good for air. On the other hand, for plasma

$$\frac{4}{3}\ \frac{N+2}{N} = \frac{4}{3}\cdot\frac{5}{3} = 2.222\ , \qquad (7\text{-}21)$$

which does not agree well with (7-19), so the simple approximation is quite bad for plasma. Furthermore, generalizations to more complicated shock wave calculations will depart from the simple approximation. Therefore, in the next section we will solve the pair of simultaneous differential equations without the simplification of (7-1).

Using the methods discussed in this section, we have calculated the shock wave structure for air and have plotted the results in Figs. 2 and 3. In Fig. 2 we give a plot which compares with Fig. 1 of Thomas. The straight line is drawn through the point $\omega = \bar{\omega}$ on the curve in Fig. 2 and is adjusted to have a slope which satisfies (7-11). A glance at Fig. 2 indicates that the straight line gives a good rough estimate of the shock thickness. Since the Thomas approximation is not good for plasma, we will not discuss the plasma shock structure in this section.

VIII. SOLUTION OF THE SIMULTANEOUS EQUATIONS

As mentioned in the previous section, we will be concerned, in general, with values of f for

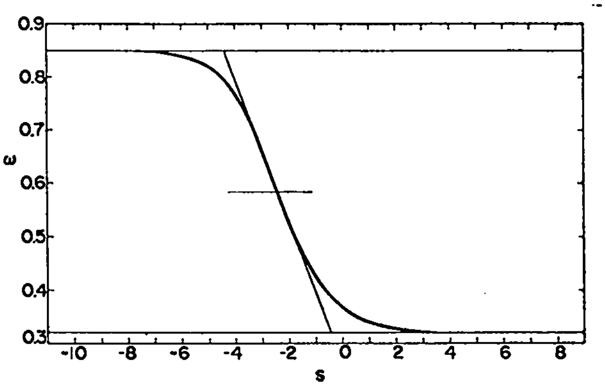

Fig. 2. The scaled velocity ω vs. the scaled distance s for a shock wave in air. The pressure ratio is 4.4981 as in the second case of Thomas.[1]

which (7-1) doesn't hold. In general, there is no simple analytic solution to (6-39), and we have to go back and solve (6-37) and (6-38) simultaneously. We write them in the form

$$\frac{d\omega}{ds} = \varphi(\omega,\theta)\ , \qquad (8\text{-}1)$$

$$\frac{d\theta}{ds} = \Psi(\omega,\theta)\ , \qquad (8\text{-}2)$$

where

$$\varphi(\omega,\epsilon) = \frac{\omega}{C\sqrt{\theta}}\left(\omega + \frac{\theta}{\omega} - 1\right)\ , \qquad (8\text{-}3)$$

$$\Psi(\omega,\epsilon) = \frac{4\omega}{3\ f\ C\sqrt{\theta}}\left\{\theta - \frac{1}{N}\left[(1-\omega)^2 + \alpha\right]\right\}. \qquad (8\text{-}4)$$

The theory of simultaneous, nonlinear differential equations contains complications not found in the theory of single nonlinear differential equations. These complications are inevitable in more refined shock structure calculations. The basic theory is discussed by Minorsky[9] in his treatise on nonlinear mechanics. Of primary importance in this theory is the idea of singular points. If we combine (8-1) and (8-2) into the form

$$\frac{d\omega}{\varphi} = \frac{d\theta}{\Psi}\ , \qquad (8\text{-}5)$$

we get a picture of what we mean by singular points. A singular point (ω_i,θ_i) is a point at which φ and Ψ both vanish. Thus, we see that the asymptotic limits of the shock wave occur at the singular points (ω_i,θ_i), $(i = 1,2)$, corresponding to the far upstream and far downstream, respectively.

We attack (8-1) and (8-2) by a linearization in the neighborhood of a singularity as was done in

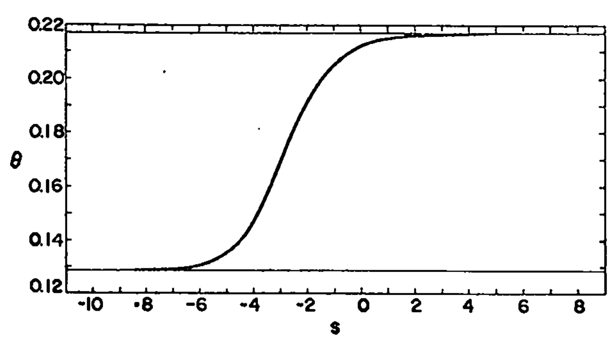

Fig. 3. The scaled temperature θ vs. the scaled distance for the case corresponding to Fig. 2.

the previous section. We will leave the index i undetermined so that either the upstream or downstream singular point can be selected as a starting point for integrating the simultaneous differential equation. More will be said about this later.

We write

$$\varphi = (\omega - \omega_i)\, \varphi_\omega + (\theta - \theta_i)\, \varphi_\theta \,, \tag{8-6}$$

$$\Psi = (\omega - \omega_i)\, \Psi_\omega + (\theta - \theta_i)\, \Psi_\theta . \tag{8-7}$$

For simplicity, let

$$y = \omega - \omega_i, \tag{8-8}$$

$$z = \theta - \theta_i \,. \tag{8-9}$$

Then (8-1) and (8-2) become

$$\frac{dy}{ds} = y\varphi_\omega + z\varphi_\theta \,, \tag{8-10}$$

$$\frac{dz}{ds} = y\Psi_\omega + z\Psi_\theta \,. \tag{8-11}$$

We look for solutions of the form

$$y = A\, e^{\lambda s} \,, \tag{8-12}$$

$$z = B\, e^{\lambda s} \,. \tag{8-13}$$

Substituting (8-12) and (8-13) into (8-10) and (8-11), we obtain

$$\begin{pmatrix} \lambda - \varphi_\omega & -\varphi_\theta \\ -\Psi_\omega & \lambda - \Psi_\theta \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \,, \tag{8-14}$$

where the eigenvalues, λ, are obtained by setting the determinant of the matrix in (8-14) equal to zero. Thus, we obtain

$$\lambda_\pm = \frac{\varphi_\omega + \Psi_\theta \pm \sqrt{(\varphi_\omega + \Psi_\theta)^2 + 4(\varphi_\theta \Psi_\omega - \varphi_\omega \Psi_\theta)}}{2} . \tag{8-15}$$

For each eigenvalue $\lambda_\pm$, we obtain an eigenvector

$$\begin{pmatrix} A_\pm \\ B_\pm \end{pmatrix} .$$

From (8-14) we see that the constants $A_\pm$ and $B_\pm$ are related by

$$B_\pm = \frac{(\lambda_\pm - \varphi_\omega)}{\varphi_\theta}\, A_\pm \,. \tag{8-16}$$

We choose our constant A in the following way. From (8-12) we write our solution in the form

$$\omega = \omega_i + A_\pm\, C^{\lambda_\pm s} \,. \tag{8-17}$$

If $\lambda \gtrless 0$, we have a growing (decaying) exponential, and, hence, we are in the upstream (downstream) region. We have

$$\omega = \omega_i \left(1 - \epsilon_i C^{\lambda_i s} \right), \tag{8-18}$$

where

$$\epsilon_i = \begin{Bmatrix} +1 \text{ for } i = 1 \\ -1 \text{ for } i = 2 \end{Bmatrix} . \tag{8-19}$$

Our beginning value of s is determined by setting

$$C^{\lambda_i s_o} = Y \tag{8-20}$$

or

$$s_o \equiv \frac{\log Y}{\lambda_i} \,, \tag{8-21}$$

which is analogous to (7-10).
We then obtain $\omega(s_o)$ by using (8-18). Comparing (8-17) and (8-18), we see that

$$A_i = -\epsilon_i \omega_i . \tag{8-22}$$

Thus, from (8-16),

$$B_i = -\frac{\lambda_i - \varphi_\omega}{\varphi_\theta}\, \epsilon_i \omega_i \,, \tag{8-23}$$

and (8-13) is written

$$\theta = \theta_i - \frac{\lambda_i - \varphi_\omega}{\varphi_\theta}\, \epsilon_i \omega_i\, C^{\lambda_i s} \,, \tag{8-24}$$

which we use to obtain $\theta(s_o)$. This gives us the means to start our integration at either singular point.

The question arises whether there is any difference in starting from one singular point or the other. Indeed there is, and this decision involves certain general basic properties of such singularities. These properties are discussed fully by Minorsky.[9] Consider the (ω,θ) plane as illustrated in Fig. 4.

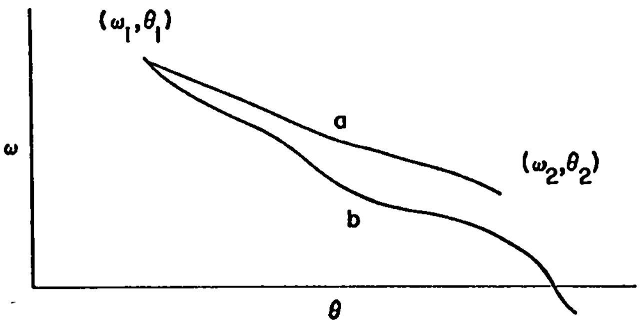

Fig. 4. The integral curves in the (ω,θ) plane.

As the variable s moves from $-\infty$ to $+\infty$, it acts as a parameter of a parameterized arc (Curve a) which moves from (ω_1,θ_1) to (ω_2,θ_2). Such an arc is a solution to (6-39) and is called an <u>integral curve</u> in the (ω,θ) plane. The parameterization in terms of s is the complete solution to the pair (6-37) and (6-38) for the boundary condition of our problem. There are other solutions to this pair of equations which result from other boundary conditions. For example, it may be possible for solutions to start at or near (ω_1,θ_1) and go quite far from (ω_2,θ_2) as illustrated by Curve b of Fig. 2. Indeed this will happen in our problems. Thus, it is expedient to start our numerical integration at point (ω_2,θ_2) instead of (ω_1,θ_1). If we start at (ω_1,θ_1), the slightest numerical error will cause us to miss (ω_2,θ_2) by a considerable margin. By the same token, the error diminishes as we go from (ω_2,θ_2) to (ω_1,θ_1). It is often possible (as in our case) to discover an appropriate direction of integration by trial and error. This is not generally true, however.

Carrying out the differentiations indicated in (8-6) and (8-7), we obtain

$$\varphi_\omega = \frac{1}{C\sqrt{\theta_i}}(2\omega_i - 1) \tag{8-25}$$

$$\varphi_\theta = \frac{1}{C\sqrt{\theta_i}}\ , \tag{8-26}$$

$$\Psi_\omega = \frac{1}{\frac{3}{4}\, f\, C\sqrt{\theta_i}}\left\{\theta_i - \frac{1}{N}\Big[(3\omega_i - 1)(\omega_i - 1) + \alpha\Big]\right\}, \tag{8-27}$$

$$\Psi_\theta = \frac{\omega_i}{\frac{3}{2}\, f\, C\sqrt{\theta_i}}\left[\frac{(1-\omega_i)^2 + \alpha}{N\theta_i} + -1\right]. \tag{8-28}$$

The simultaneous equations were first solved for air. As stated in Section VII, the Thomas approximation should be good for air. Indeed the curves obtained using the simultaneous equations fall close enough to the curves plotted in Figs. 2 and 3 so that the difference is barely detectable on the graphs. Therefore, we give no new plots for air. However, the situation is quite different for plasma for which the curves obtained are plotted in Figs. 5 and 6. The structure is quite different from that obtainable with the Thomas approximation. There is a dual structure to the ω(velocity) profile as shown in Fig. 5. The sharp falloff near the right end of the curve arises from the viscosity dissipative effect. The much more diffuse effect noted over more of the curve arises from the thermal conductivity which allows a thermal wave to propagate far ahead of the viscosity-dominated shock front. This effect was also noticed in running the numerical program for the θ-pinch.[10] The dual structure just described is ruled out in the Thomas approximation.

IX. THE NONZERO MAGNETIC FIELD CASE

If the magnetic field does not vanish, then all the scaling properties introduced starting in Section VI are lost. We must return to Eqs. (5-14) through (5-16) which can be written in the form

$$\frac{du}{dx} = \varphi(u,\ T,\ B), \tag{9-1}$$

$$\frac{dT}{dx} = \Psi(u,\ T,\ B), \tag{9-2}$$

$$\frac{dB}{dx} = \chi(u,\ T,\ B), \tag{9-3}$$

where

$$\varphi(u,T,B) = \frac{1}{\tilde{\mu}}\left[a(u-u_i) + a\mathcal{R}\left(\frac{T}{u} - \frac{T_i}{u_i}\right) + \frac{B^2 - B_i^2}{8\pi}\right], \tag{9-4}$$

$$\Psi(u,T,B) = \frac{1}{\kappa}\left[\frac{3}{2}\,a\mathcal{R}\,(T-T_i) + b(u-u_i) - \frac{1}{2}\,a(u^2-u_i^2) - \frac{(uB^2 - u_iB_i^2)}{8\pi} - \frac{d}{4\pi}\,(B - B_i)\right], \tag{9-5}$$

$$\chi(u,T,B) = \frac{4\pi}{\eta}\,(uB - u_iB_i). \tag{9-6}$$

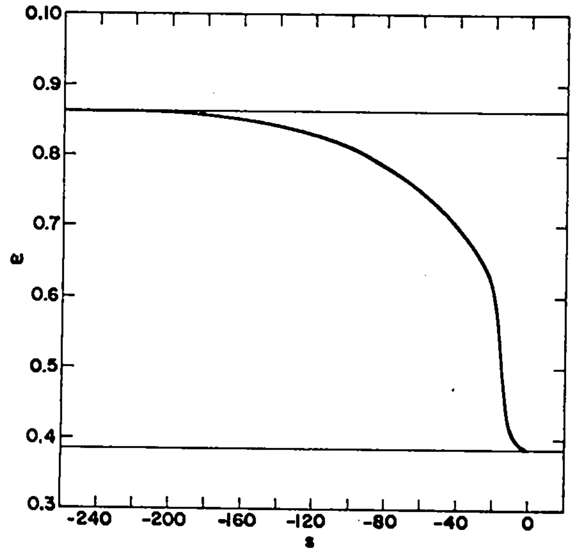

Fig. 5. The scaled velocity ω vs. the scaled distance s for a shock wave in plasma. The pressure ratio is 4.4981 as in the case of air.

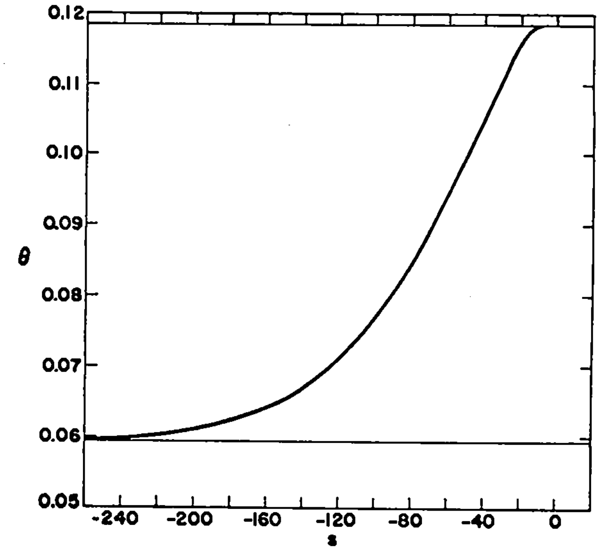

Fig. 6. The scaled temperature θ vs. the scaled distance for the case corresponding to Fig. 5.

Equations (9-1) through (9-3) can be combined in the form

$$\frac{du}{\varphi} = \frac{dT}{\Psi} = \frac{dB}{\chi} . \qquad (9\text{-}7)$$

The singular points (u_i, T_i, B_i) occur where the denominators in (9-7) all vanish.

Linearizing about the singular points, we have

$$\varphi = (u - u_i)\varphi_u + (T - T_i)\varphi_T + (B - B_i)\varphi_B, \qquad (9\text{-}8)$$

$$\Psi = (u - u_i)\Psi_u + (T - T_i)\Psi_T + (B - B_i)\Psi_B, \qquad (9\text{-}9)$$

$$B = (u - u_i)\chi_u + (T - T_i)\chi_T + (B - B_i)\chi_B. \qquad (9\text{-}10)$$

Let us introduce the vector $\vec{y}$ leaving i understood.

$$\vec{y} = \begin{pmatrix} u - u_i \\ T - T_i \\ B - B_i \end{pmatrix} , \qquad (9\text{-}11)$$

and the matrix,

$$G = \begin{pmatrix} \varphi_u & \varphi_T & \varphi_B \\ \Psi_u & \Psi_T & \Psi_B \\ \chi_u & \chi_T & \chi_B \end{pmatrix}. \qquad (9\text{-}12)$$

Then our set of differential equations (9-1) through (9-3) can be written

$$\frac{d\vec{y}}{dx} = G\vec{y} \quad . \qquad (9\text{-}13)$$

We look for solutions of the form

$$\vec{y} = \vec{v}\, e^{\lambda x} , \qquad (9\text{-}14)$$

where $\vec{v}$ is a vector independent of x. The eigenvalue equation is

$$\det\,(\delta_{ij}\lambda - G_{ij}) = 0. \qquad (9\text{-}15)$$

We find eigenvalues from the roots of the cubic equation (9-15). We label the eigenvalues with a subscript, $\lambda_i (i = 1, 2, 3)$. This allows us to find the corresponding eigenvectors, $\vec{v}_i$. Thus, we have the linearized solution

$$\vec{y} = \vec{v}_i e^{\lambda_i x} . \qquad (9\text{-}16)$$

Carrying out the differentiations indicated in (9-8) through (9-10), we obtain

$$G_{11} = \varphi_u = \frac{a}{\tilde{\mu}} \left(1 - \frac{\mathcal{R}T}{u^2}\right) - \frac{1}{\tilde{\mu}}\frac{\partial\tilde{\mu}}{\partial u} , \qquad (9\text{-}17)$$

$$G_{12} = \varphi_T = \frac{a}{u\tilde{\mu}} - \frac{1}{\tilde{\mu}}\frac{\partial\tilde{\mu}}{\partial T}, \tag{9-18}$$

$$G_{13} = \varphi_B = \frac{B}{4\pi\tilde{\mu}} - \frac{1}{\tilde{\mu}}\frac{\partial\tilde{\mu}}{\partial B}, \tag{9-19}$$

$$G_{21} = \Psi_u = \frac{1}{\kappa}\left(b - au - \frac{B^2}{8\pi}\right) - \frac{1}{\kappa}\frac{\partial\kappa}{\partial u}, \tag{9-20}$$

$$G_{22} = \Psi_T = \frac{3a\mathcal{R}}{2\kappa} - \frac{1}{\kappa}\frac{\partial\kappa}{\partial T}, \tag{9-21}$$

$$G_{23} = \Psi_B = -\frac{(uB + d)}{4\pi\kappa} - \frac{1}{\kappa}\frac{\partial\kappa}{\partial B}, \tag{9-22}$$

$$G_{31} = X_u = \frac{4\pi B}{\eta} - \frac{4\pi}{\eta}\frac{\partial\eta}{\partial u}, \tag{9-23}$$

$$G_{32} = X_T = -\frac{4\pi}{\eta}\frac{\partial\eta}{\partial T}, \tag{9-24}$$

$$G_{33} = X_B = \frac{4\pi u}{\eta} - \frac{4\pi}{\eta}\frac{\partial\eta}{\partial B}. \tag{9-25}$$

To get a simpler calculation let us make the physically unrealistic, but mathematically simplifying, assumption that the transport coefficients have no magnetic field dependence. Furthermore, let us assume that $\ell n\Lambda$ does not vary appreciably. We then have for the matrix elements, G_{ij},

$$G_{11} = \varphi_u = \frac{a}{\tilde{\mu}}\left(1 - \frac{\mathcal{R}T}{u^2}\right), \tag{9-26}$$

$$G_{12} = \varphi_T = \frac{a\mathcal{R}}{u\tilde{\mu}} - \frac{5}{2T}, \tag{9-27}$$

$$G_{13} = \varphi_B = \frac{B}{4\pi\tilde{\mu}}, \tag{9-28}$$

$$G_{21} = \psi_u = \frac{1}{\kappa}\left(b - au - \frac{B^2}{8\pi}\right), \tag{9-29}$$

$$G_{22} = \psi_T = \frac{3a\mathcal{R}}{2\kappa} - \frac{5}{2T}, \tag{9-30}$$

$$G_{23} = \psi_B = -\frac{(uB + d)}{4\pi\kappa}, \tag{9-31}$$

$$G_{31} = X_u = \frac{4\pi B}{\eta}, \tag{9-32}$$

$$G_{32} = X_T = 0 \tag{9-33}$$

$$G_{33} = X_B = \frac{4\pi u}{\eta}. \tag{9-34}$$

We have run numerical examples with a nonvanishing magnetic field using Eqs. (9-1) through (9-3). For small magnetic field (< 10 gauss), the results agree with Figs. 5 and 6. In Figs. 7 through 9 we show the results with an upstream asymptotic magnetic field of 100 gauss. As we see in Fig. 9, the downstream magnetic field has been compressed to > 240 gauss. The effect of the magnetic field is most easily seen in Fig. 7. The sharp portion of the shock front is smoothed out somewhat by the diffused magnetic field.

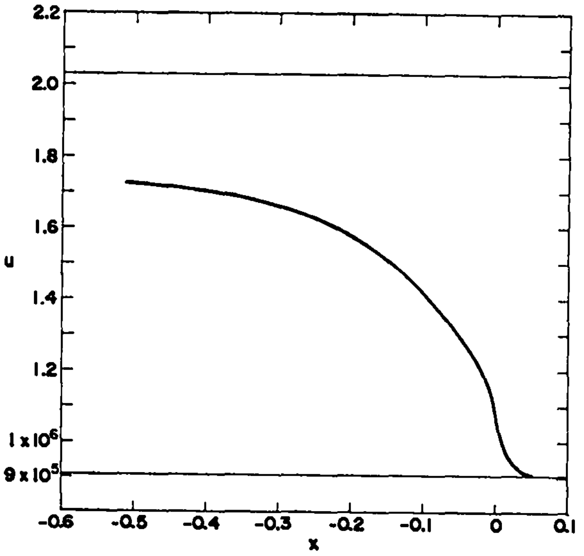

Fig. 7. The actual velocity u vs. the actual distance x for a shock wave in plasma with a magnetic field. The net (plasma plus field) pressure ratio is 4.4981.

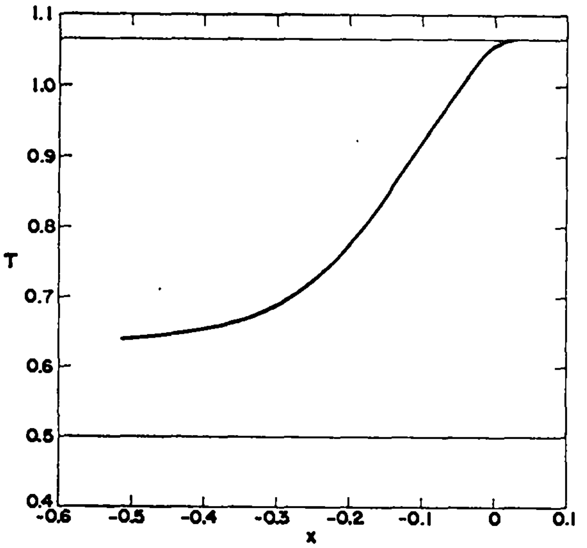

Fig. 8. The actual temperature T vs. the actual distance x for the case corresponding to Fig. 7.

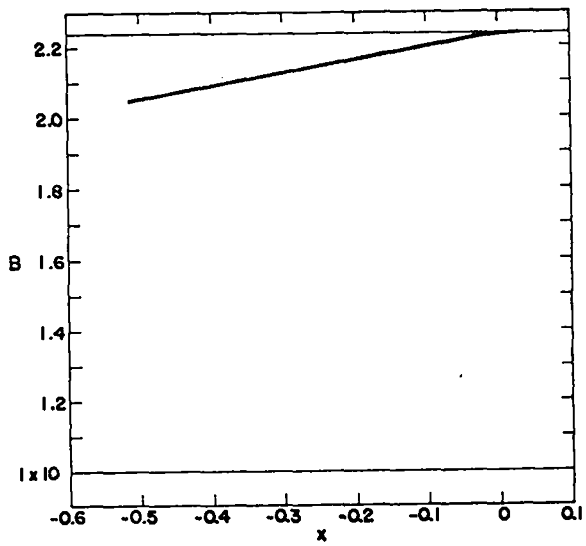

Fig. 9. The transverse magnetic field B vs. the distance x for the case corresponding to Fig. 7.

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