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DERIVATION OF THE EQUATIONS OF CONSERVATION OF MASS, MOMENTUM, AND ENERGY OF COMPRESSIBLE FLUID MECHANICS IN BOTH LAGRANGIAN AND EULERIAN FORMS FROM AN INTEGRAL VIEWPOINT

by

Philip L. Browne

ABSTRACT

This report derives, then shows the equivalence of, the Lagrangian and Eulerian equations by use of Reynolds' Transport Theorem. The differential forms of the equations are also deduced from the integral forms. Finally, some common simplifications of the equations are derived.

I. INTRODUCTION

The three fundamental equations of compressible fluid motion are those based on

(a) conservation of mass (the equation of continuity),

(b) conservation of momentum (the equation of motion), and

(c) conservation of energy (total).

These can be derived and written in the so-called <u>Lagrangian</u> or <u>Eulerian</u> forms with the distinction not always made clear, resulting in confusion, especially for those unfamiliar with the subject. Also, these equations may be written as <u>differential</u> equations, <u>holding at a point in time and space</u>, or in a form involving <u>integrals over some element of volume over some period of time</u>. Use of the integrals for derivation of the equations seems to the author to be more <u>intuitively</u> physical and offers a more comprehensible means of obtaining the finite difference approximations for numerical work,[1] especially when more than one dimension is being considered.

In the following pages an attempt will be made to start with the physical or intutitive (integral form) approach. Definitions of the meaning of Lagrangian and Eulerian will be made which are quite intuitive, and from these the meanings of certain operators (derivatives) will be made. For each of the three conservative laws, a Lagrangian integral form and an Eulerian integral form will be written from a physical approach. An attempt will then be made to show the equivalence of these two forms and to derive some corresponding differential forms, which will also be shown to be equivalent.

From a pure mathematics point of view, one should probably use tensors and tensor notation to achieve the most general forms of the equations for all types of coordinate systems. However, since this is written for physical understanding, the work and proofs have been limited to vector notation (to be true in two and three dimensions), and in a Cartesian coordinate system. The only tensor notation used is $x_i x_i = x_1 x_1 + x_2 x_2 + x_3 x_3$. The discussion may seem lengthy and detailed, but it is intended to be understandable and comprehensive. An attempt has been made in each case to begin with a physically visualizable Lagrangian integral form of the law and then derive a Lagrangian differential form. Next, starting with a physically visualizable Eulerian integral form of the same law, one is led to an Eulerian differential form of the law. The equivalence of the two integral forms and the two differential forms is shown. Other things are then pursued, such as simplification of the law of conservation of momentum by the conservation of mass equation and simplification of the energy equation by the mass and momentum equations.

II. EULERIAN AND LAGRANGIAN VIEWPOINTS

By the Eulerian approach we mean that in which at a point (x_1, x_2, x_3) in space at time t, one observes

(a) the changes that occur in time as one remains fixed at (x_1, x_2, x_3) (This change denoted by $\frac{\partial}{\partial t}$) (1)

(b) the variations that exist in space at that particular time (denoted by $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$) .

By the Lagrangian approach we mean that in which one follows or rides along with a particle or set of particles and observes the rate of change that occurs (denoted by $\frac{d}{dt}$, often called the material or total or hydrodynamic or Lagrangian derivative). (2)

Since one of our objectives is to relate the various conservation laws in the two points of view, it is of fundamental importance that we first relate the various derivatives defined above.

III. THE RELATION OF LAGRANGIAN AND EULERIAN DERIVATIVES

As defined above, the material derivative is the time rate of change of a scalar or vector function, $\vec{A}$, as one moves along with a particle. Thus, for a particle which has coordinates (x_1, x_2, x_3) and velocities (v_1, v_2, v_3)* at the time t, and hence moves to the point $(x_1 + v_1\Delta t, x_2 + v_2\Delta t, x_3 + v_3\Delta t)$ at time $t + \Delta t$, the strict mathematical definition of $\frac{d\vec{A}}{dt}$ is given by

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \left[\frac{\vec{A}(x_1 + v_1\Delta t, x_2 + v_2\Delta t, x_3 + v_3\Delta t, t + \Delta t) - \vec{A}(x_1, x_2, x_3, t)}{\Delta t}\right] . \quad (3)$$

Making the Taylor Series expansion of the first term

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \Big[\vec{A}(x_1, x_2, x_3, t) + \frac{\partial \vec{A}}{\partial x_1} v_1\Delta t + \frac{\partial \vec{A}}{\partial x_2} v_2\Delta t + \frac{\partial \vec{A}}{\partial x_3} v_3\Delta t + \frac{\partial \vec{A}}{\partial t} \Delta t + 0_1(\Delta t)^2$$

$$+ 0_2(\Delta t)^3 + \ldots . - \vec{A}(x_1, x_2, x_3, t)\Big]$$

$$= \lim_{\Delta t \to 0} \left[v_1 \frac{\partial \vec{A}}{\partial x_1} + v_2 \frac{\partial \vec{A}}{\partial x_2} + v_3 \frac{\partial \vec{A}}{\partial x_3} + \frac{\partial \vec{A}}{\partial t} + 0_1(\Delta t) + 0_2(\Delta t)^2 \ldots\right] ,$$

where 0_1, 0_2, etc., are functions of order Δt, $(\Delta t)^2$ etc. As we approach the limit $\Delta t \to 0$,

$$\boxed{\frac{d}{dt}[\vec{A}(x_1, x_2, x_3, t)] = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right)\vec{A} = \left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i}\right)\vec{A} \quad .} \quad (4)$$

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* The v_i are actually defined in terms of $\frac{d}{dt}$, that is, $v_i = \frac{dx_i}{dt}$ = time rate of change of x_i, x_i moving along with the particle. (See next page).

This describes the material or Lagrangian derivative, $\frac{d}{dt}$, in terms of the Eulerian derivatives, $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial t}$, in both vector and tensor notation.

A pertinent example. In any coordinate system, if $\vec{A} = \vec{r}$ = the position vector of a given particle in the fluid, then by the physical definition of the velocity of the particle,

$$\vec{v} = \frac{d\vec{r}}{dt} = \text{time rate of change of } \vec{r}\text{, moving with particle.}$$

Proof in Cartesian coordinates. Let $\vec{A} = \vec{r} = \vec{i}_k x_k$. Then by (4),

$$\frac{d\vec{r}}{dt} = \frac{\partial\vec{r}}{\partial t} + v_i \frac{\partial}{\partial x_i} (\vec{i}_k x_k) \quad .$$

Now by definition of $\frac{\partial}{\partial t}$ in (1), $\frac{\partial\vec{r}}{\partial t} = 0$, that is, change in $\vec{r}$ if one stays fixed at position $\vec{r}$ in space. Also, since the unit vectors $\vec{i}_k$ are constant in space and time and since the coordinates x_i are independent of each other,

$$\frac{\partial}{\partial x_i} (\vec{i}_k x_k) = \vec{i}_k \frac{\partial x_k}{\partial x_i} = \vec{i}_i \quad .$$

This gives $\frac{d\vec{r}}{dt} = 0 + v_i\vec{i}_i = \vec{v}$, (4a)

which is none other than the velocity, which makes sense.

IV. REYNOLDS' TRANSPORT THEOREM

This theorem (Aris[2]), at least to me, concerns the meaning of the material derivative, $\frac{d}{dt}$, when applied to an integral of some scalar or vector function over a given volume. In short, if $\vec{A}(x_1, x_2, x_3, t)$ is either a scalar or vector function, then Reynolds' Transport Theorem is

$$\frac{d}{dt} (\int_V \vec{A} dV) = \int_V [\frac{d\vec{A}}{dt} + \vec{A}(\nabla \cdot \vec{v})] dV \tag{5}$$

$$= \int_V \frac{\partial\vec{A}}{\partial t} dV + \int_S \vec{A}(\vec{v} \cdot d\vec{S}) \tag{5a}$$

[(5a) comes from the divergence thereom]. Thus for a volume, V, which is imbedded in and moves with the fluid, the total time rate of change of $\int_V \vec{A} dV$ is given by (5). The first term represents the time rate of change in $\vec{A}$ inside the volume, V, as it moves. The second term represents the amount of $\vec{A}$ which passes through the surface, S, of V, in unit time, as it moves. $\vec{A}$ is measured per unit volume, and $\vec{v} \cdot d\vec{S}$ (velocity x surface = change in volume per unit time) is the volume swept out, $d\vec{S}$ being in the direction of the outward drawn normal and having the magnitude of the area element.

Proof, Aris[2] has a very neat proof (P. 85) using Jacobians, etc. However, I prefer to give a proof which uses the physical notions of $\frac{d}{dt}$, $\int dV$, etc. Since V moves with the fluid, then the fundamental meaning $\frac{d}{dt}$ in (2), (3) may be utilized, that is,

$$\frac{d}{dt} \int_V \vec{A} dV = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \{[\int_{V(t+\Delta t)} \vec{A} dV]^{t+\Delta t} - [\int_{V(t)} \vec{A} dV]^t\} \quad . \tag{6}$$

But an integral is defined as the limit of sum, that is,

$$\int_V \vec{A} dV = \lim_{\substack{\Delta V_\ell \to 0 \\ n \to \infty}} \sum_{\ell=1}^{n} \vec{A}_\ell \Delta V_\ell \quad .$$

Now we can write (6) as

$$\frac{d}{dt} \int_V \vec{A} dV = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [(\lim_{\Delta V_\ell \to 0} \sum_{\ell=1}^{n} \vec{A}_\ell \Delta V_\ell)^{t+\Delta t} - (\lim_{\Delta V_\ell \to 0} \sum_{\ell=1}^{n} \vec{A}_\ell \Delta V_\ell)^t] \quad ,$$

where we assume that the ΔV_ℓ are also imbedded in and move with the fluid (that is, $\Delta V_\ell^{t+\Delta t}$ contains the same material as ΔV_ℓ^t). The limits may now be interchanged so that

$$\frac{d}{dt} \int_V \vec{A} dV = \lim_{\Delta V_\ell \to 0} \sum_{\ell=1}^{n} \{\lim_{\Delta t \to 0} \frac{1}{\Delta t} [(\vec{A}_\ell \Delta V_\ell)^{t+\Delta t} - (\vec{A}_\ell \Delta V_\ell)^t]\}$$

$$= \lim_{\Delta V_\ell \to 0} \sum_{\ell=1}^{n} [\frac{d}{dt} (\vec{A}_\ell \Delta V_\ell)]$$

$$= \lim_{\Delta V_\ell \to 0} \sum_{\ell=1}^{\infty} [\frac{d\vec{A}_\ell}{dt} \Delta V_\ell + \vec{A}_\ell \frac{d(\Delta V_\ell)}{dt}] \quad . \tag{7}$$

In the limit the first term approaches $\int_V \frac{d\vec{A}}{dt}\, dV$, but the second term is a bit more difficult to interpret. Since we have assumed that the ΔV_ℓ move with the fluid, then for each ΔV_ℓ

$$\frac{d(\Delta V_\ell)}{dt} = \begin{array}{l}\text{rate at which volume is swept out by the surface of } \Delta V_\ell \\ \text{(that is, surface integral of velocity)}\end{array}$$

$$= \int_{S(\Delta V_\ell)} \vec{v} \cdot d\vec{S} \quad .$$

Now by the divergence theorem $\quad \int_S \vec{v} \cdot d\vec{S} = \int_V (\nabla \cdot \vec{v})dV \quad ,$
so

$$\frac{d(\Delta V_\ell)}{dt} = \int_{\Delta V_\ell} (\nabla \cdot \vec{v})dV \quad .$$

In the limit, as $\Delta V_\ell \to 0$, by the mean value theorem,

$$\frac{d(\Delta V_\ell)}{dt} = (\nabla \cdot \vec{v})_\ell \, \Delta V_\ell \quad ,$$

we can now write (7) as

$$\frac{d}{dt} \int_V \vec{A} dV = \int_V \left[\frac{d\vec{A}}{dt} + \vec{A}(\nabla \cdot \vec{v})\right] dV \quad ,$$

which completes the proof.

<u>Other Forms of the Transport Theorem</u>

If we substitute in (5) for the material derivative $\frac{d\vec{A}}{dt}$ as defined in (4),

$$\frac{d}{dt} \int_V \vec{A} dV = \int_V \left[\frac{\partial\vec{A}}{\partial t} + (\vec{v} \cdot \nabla)\vec{A} + \vec{A}(\nabla \cdot \vec{v}) \right] dV \quad . \tag{8}$$

Now, if in Cartesian coordinates we define, when $\vec{A}$ is either of the scalars, ρ, ρE, or vector $\rho\vec{v}$, an operator

$$\nabla \cdot (\vec{v}\vec{A}) = \frac{\partial}{\partial x_i} (v_i \vec{A}) = v_i \frac{\partial \vec{A}}{\partial x_i} + \vec{A} \frac{\partial v_i}{\partial x_i} = (\vec{v} \cdot \nabla)\vec{A} + \vec{A}(\nabla \cdot \vec{v}) \tag{9}$$

(see Appendix A).

Then (8) may be written

$$\frac{d}{dt} \int_V \vec{A} dV = \int_V \left[\frac{\partial \vec{A}}{\partial t} + \nabla \cdot (\vec{v}\vec{A})\right] dV \quad . \tag{10}$$

Also in Cartesian coordinates it is possible to suggest a divergence type theorem for $\nabla \cdot (\vec{v}\vec{A})$ (see Appendix B).

$$\int_V \nabla \cdot (\vec{v}\vec{A})\, dV = \int_S \vec{A}(\vec{v} \cdot d\vec{S}) \tag{11}$$

so that (10) becomes

$$\frac{d}{dt} \int_V \vec{A} dV = \frac{\partial}{\partial t} \int_V \vec{A} dV + \int_S \vec{A}(\vec{v} \cdot d\vec{S}) \quad . \tag{12}$$

The $\frac{\partial}{\partial t}$ can be moved in or out of the integral at will because of its definition in (1) which says that $\frac{\partial}{\partial t}$ means observing a change in time while remaining fixed in space. The form (12) in effect states that for a scaler or vector function $\vec{A}$, the time rate of change of $\int_V \vec{A} dV$ as V moves with the fluid is given by the time rate of change of $\int_V \vec{A} dV$ for V fixed in space plus a flux of $\vec{A}$ through the surface S of V.

V. THE EQUATION OF CONSERVATION OF MASS

The mass enclosed by a volume, V, is given by the integral $\int_V \rho dV$. The rate of mass flow out through a surface S is given by $\int_S \rho \vec{v} \cdot d\vec{S}$, where $\vec{v}$ is the velocity of the fluid with respect to the element $d\vec{S}$.

<u>Lagrangian Forms</u>. If V is imbedded in the fluid and moves with it, then the rate of change ($\frac{d}{dt}$) of mass in V as one follows V, is zero, that is,

$$\boxed{\frac{d}{dt} \int_V \rho dV = 0} \tag{13}$$

because no mass is lost or gained through the surface S of V. This is an <u>integral Lagrangian</u> form of the equation of conservation of mass.
Applying the Reynolds' Transport Theorem, (5), with $\vec{A} = \rho$,

$$\int_V \left[\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{v})\right] dV = 0 \quad . \tag{14}$$

Since this holds for any volume, V, used, the integrand must vanish, or

$$\boxed{\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{v}) = 0 \quad .} \tag{15}$$

This is a differential Lagrangian form of the equation of conservation of mass.

Eulerian Forms. If V is fixed in time and space, then the rate of change $(\frac{\partial}{\partial t})$ of mass in V equals the mass lost through the surface, or

$$\boxed{\begin{aligned} \frac{\partial}{\partial t} \int_V \rho\, dV &= -\int_S \rho\vec{v} \cdot d\vec{S} \\ &= -\int_V \nabla \cdot (\rho\vec{v})dV \quad , \end{aligned}} \tag{16}$$

the last step using the divergence theorem. This is an integral, Eulerian form of the equation. Since V is constant in time, $\frac{\partial}{\partial t}$ may be taken inside the integral to give

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{v})\right] dV = 0 \quad , \tag{17}$$

and since this is true for any V,

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{v}) = 0 \quad ,} \tag{18}$$

which is a differential Eulerian form. It could also be written in other ways by expanding $\nabla \cdot (\rho\vec{v})$ by (9).

Equivalence of Forms

Using the Reynolds Transport Theorem as written in (12) with $A = \rho$ we see that the integral forms (13), (16) are equivalent.

The differential forms (15), (18) are equivalent, for if the material derivative, $\frac{d\rho}{dt}$, in (15) is expanded by (4),

$$\frac{\partial\rho}{\partial t} + (\vec{v} \cdot \nabla)\rho + \rho(\nabla \cdot \vec{v}) = 0 \quad . \tag{19}$$

Using $\vec{A}=\rho$ in (9), we have

$$\nabla \cdot (\rho\vec{v}) = (\vec{v} \cdot \nabla)\rho + \rho(\nabla \cdot \vec{v}) \qquad (20)$$

and we see that (20) in (19) gives (18), which proves that the differential forms (15), (18) are equivalent.

VI. THE EQUATION OF CONSERVATION OF MOMENTUM

The momentum enclosed by a volume, V, is given by the integral $\int_V \rho\vec{v}dV$. The rate of actual mass flow across a surface $d\vec{S}$ is given by $\rho\vec{v} \cdot d\vec{S}$. This carries with it across the surface a momentum/time of $\vec{v}(\rho\vec{v} \cdot d\vec{S})$. There is another term which is usually thought of as the force acting on volume, V, due to the pressure acting on its surface, namely $-\int_S Pd\vec{S}$, which may also be interpreted as a rate of momentum flow across S caused by the random motion of the particles. (See Appendix C.)

Lagrangian Forms. If V is imbedded in the fluid and moves with it, then the rate of change ($\frac{d}{dt}$) of momentum in V as one follows V is given by

$$\boxed{\frac{d}{dt} \int_V \rho\vec{v}dV = -\int_S Pd\vec{S}} \qquad (21)$$

because no actual mass is lost or gained through the surface S of V. If one prefers to think of $-\int_S Pd\vec{S}$ as representing the force on V, then (21) is an expression of Newton's Law, force = rate of change of momentum. Equation (21) is an integral Lagrangian form. Using the transport theorem, (5), with $\vec{A} = \rho\vec{v}$, and the well-known

$$\int_S Pd\vec{S} = \int_V \nabla PdV \quad , \qquad (22)$$

one can write (21) as

$$\int_V[\frac{d}{dt} (\rho\vec{v}) + \rho\vec{v} (\nabla \cdot \vec{v})] dV = -\int_V \nabla PdV \quad . \qquad (23)$$

Since this is true for any V,

$$\boxed{\frac{d}{dt} (\rho\vec{v}) + \rho\vec{v}(\nabla \cdot \vec{v}) = -\nabla P \quad ,} \qquad (24)$$

a <u>differential</u> <u>Lagrangian</u> form of conservation of momentum.

<u>Eulerian Form.</u> If V is fixed in time and space, then the rate of change $(\frac{\partial}{\partial t})$ of momentum in V is given by the momentum lost through the surface, namely,

$$-\int_S (\rho\vec{v})(\vec{v}\cdot d\vec{S}) \text{ or } -\int_S \vec{v}[(\rho\vec{v})\cdot d\vec{S}]$$

plus the pressure integral, so that

$$\boxed{\frac{\partial}{\partial t}\int_V \rho\vec{v}dV = -\int_S(\rho\vec{v})(\vec{v}\cdot d\vec{S}) - \int_S P d\vec{S} \quad ,} \tag{25}$$

which is an <u>integral</u> <u>Eulerian</u> form.

Since V is fixed in time and space, $\frac{\partial}{\partial t}$ may be moved inside the integral. Converting surface integrals to volume integrals using (11) with $\vec{A} = \rho\vec{v}$, and (22), we have

$$\int_V \frac{\partial(\rho\vec{v})}{\partial t} dV = -\int_V\{\nabla\cdot[\vec{v}(\rho\vec{v})] + \nabla P\}dV \quad . \tag{26}$$

Since this holds for any V, the integrand must be zero or

$$\boxed{\frac{\partial(\rho\vec{v})}{\partial t} = -\{\nabla\cdot[\vec{v}(\rho\vec{v})] + \nabla P\} \quad ,} \tag{27}$$

which is a <u>differential</u> <u>Eulerian</u> form of conservation of momentum.

<u>Equivalence of Forms</u>

The transport theorem, (12), with $\vec{A} = \rho\vec{v}$, demonstrates the equivalence of the integral forms (21) and (25). The differential form (24), with the material derivative $\frac{d}{dt}(\rho\vec{v})$ expanded by (4) gives

$$\frac{\partial(\rho\vec{v})}{\partial t} + (\vec{v}\cdot\nabla)(\rho\vec{v}) + (\rho\vec{v})(\nabla\cdot\vec{v}) = -\nabla P \quad . \tag{27a}$$

By (9) this gives (27) the corresponding Eulerian differential form.

VII. THE EQUATION OF CONSERVATION OF MOMENTUM - SIMPLIFIED BY USE OF THE EQUATION OF CONSERVATION OF MASS

This simplification amounts to expanding the $\frac{d(\rho\vec{v})}{dt}$ or $\frac{\partial(\rho\vec{v})}{\partial t}$ terms and eliminating the $\vec{v}\,\frac{d\rho}{dt}$ or $\vec{v}\,\frac{\partial\rho}{\partial t}$ by means of the conservation of mass equation.

Lagrangian. Expanding $\frac{d(\rho\vec{v})}{dt}$ in (23), a Lagrangian integral form of conservation of momentum, we get

$$\int_V \left[\rho\frac{d\vec{v}}{dt} + \vec{v}\,\frac{d\rho}{dt} + \rho\vec{v}(\nabla \cdot \vec{v})\right] dV = -\int_S P d\vec{S} \quad . \tag{28}$$

Using the Lagrangian differential equation of conservation of mass, (15), two terms drop out, leaving

$$\boxed{\int_V \rho\,\frac{d\vec{v}}{dt}\, dV = -\int_S P d\vec{S} \quad ,} \tag{29}$$

a Lagrangian integral momentum equation.

Converting the surface integral by (22), one can then get the corresponding Lagrangian differential equation

$$\boxed{\rho\frac{d\vec{v}}{dt} = -\nabla P \quad .} \tag{30}$$

This could also have been obtained from the Lagrangian differential momentum equation, (24), simplified by the mass equation, (15).

Eulerian. Expanding $\frac{\partial(\rho\vec{v})}{\partial t}$ and using (Appendix A) in (26), an Eulerian integral form of momentum equation, we get

$$\int_V \left(\rho\,\frac{\partial\vec{v}}{\partial t} + \vec{v}\,\frac{\partial\rho}{\partial t}\right) dV = -\int_V \{\vec{v}[\nabla \cdot (\rho\vec{v})] + \rho[(\vec{v} \cdot \nabla)\vec{v}] + \nabla P\} dV \quad . \tag{31}$$

Using the Eulerian differential equation of conservation of mass, (18), this simplifies to

$$\boxed{\int_V \{\rho[\frac{\partial\vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}]\} dV = -\int_V \nabla P dV \quad ,} \tag{32}$$

an <u>Eulerian</u> <u>integral</u> momentum equation, and the corresponding <u>differential</u> <u>Eulerian</u> form

$$\boxed{\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla P \quad .} \tag{33}$$

This could also be obtained from the Eulerian differential momentum equation (27), simplified by the mass equation, (18), and Appendix A.

VIII. THE EQUATION OF CONSERVATION OF ENERGY

The energy per unit mass in any material is given by

$$E = e + \frac{v^2}{2} \quad , \tag{34}$$

where e is the internal energy per unit mass and $\frac{v^2}{2}$ is the kinetic energy per unit mass. The energy enclosed by a volume, V, is then given by the integral, $\int_V \rho E \, dV$. Also, since $\rho\vec{v}\cdot d\vec{S}$ represents the rate of actual mass flow across a surface element of $d\vec{S}$, this mass then carries with it across the surface an energy/time of $E(\rho\vec{v}\cdot d\vec{S})$. The rate at which work is being done by the material inside V on that outside V is given by $\int_S P\vec{v}\cdot d\vec{S}$, that is, a pressure acting through a distance.

<u>Lagrangian Forms</u>. If V is imbedded in the fluid and moves with it, then the rate of change $[\frac{d}{dt}]$ of energy in V as one follows V is equal to the rate at which the material outside V does work on the material inside V, that is,

$$\boxed{\frac{d}{dt}\int_V \rho E dV = -\int_S P\vec{v}\cdot d\vec{S} \quad .} \tag{35}$$

There is no gain or loss of energy from mass flowing through the surface, S, because V moves with the fluid. This is an <u>integral</u> <u>Lagrangian</u> form of the conservation of energy.

Using the transport theorem, (5), with $A = \rho E$, on the left hand side and the divergence theorem on the right hand side of (35),

$$\int_V \left[\frac{d(\rho E)}{dt} + \rho E(\nabla\cdot\vec{v})\right] dV = -\int_V \nabla\cdot(P\vec{v}) dV \quad . \tag{36}$$

Since this is true for any V,

$$\boxed{\frac{d(\rho E)}{dt} + \rho E(\nabla \cdot \vec{v}) = -\nabla \cdot (P\vec{v}) \quad ,} \tag{37}$$

which is a <u>differential</u> <u>Lagrangian</u> form of the energy equation.

<u>Eulerian Forms</u>. If V is fixed in time and space, then the rate of change $(\frac{\partial}{\partial t})$ of energy is affected by the energy lost through the surface, S, namely $\int_S E(\rho \vec{v} \cdot d\vec{S})$, so that

$$\boxed{\frac{\partial}{\partial t} \int_V \rho E dV = -\int_S E(\rho \vec{v} \cdot d\vec{S}) - \int_S P\vec{v} \cdot d\vec{S} \quad ,} \tag{38}$$

which is an <u>integral</u> <u>Eulerian</u> form of conservation of energy.

Since V is now fixed in space, $\frac{\partial}{\partial t}$ may be moved inside the integral. Also, using the divergence theorem on the surface integrals, we have

$$\int_V \frac{\partial(\rho E)}{\partial t} dV = -\int_V [\nabla \cdot (E\rho\vec{v}) + \nabla \cdot (P\vec{v})] dV \quad . \tag{39}$$

Since this is true for any V, the integrand must vanish or

$$\boxed{\frac{\partial(\rho E)}{\partial t} = -[\nabla \cdot (E\rho\vec{v}) + \nabla \cdot (P\vec{v})] \quad ,} \tag{40}$$

which is a <u>differential</u> <u>Eulerian</u> energy conservation equation.

<u>Equivalence of forms</u>. The transport theorem, (12), with A = ρE, immediately gives the equivalence between the two integral forms, (35) and (38).

Starting with the Lagrangian differential form, (37), if one expands the material derivative according to (4)

$$\frac{\partial(\rho E)}{\partial t} + (\vec{v} \cdot \nabla)\rho E + \rho E\,(\nabla \cdot \vec{v}) = -\nabla \cdot (P\vec{v})$$

and then uses (9) with $\vec{A} = \rho E$, the Eulerian form (40) is obtained.

IX. THE EQUATION OF CONSERVATION OF TOTAL ENERGY - SIMPLIFIED BY USE OF THE DIFFERENTIAL EQUATIONS OF CONSERVATION OF MASS AND MOMENTUM TO GIVE THE CONSERVATION OF INTERNAL ENERGY EQUATIONS

The simplification amounts to expanding the various terms and then elimating terms containing the derivatives of ρ and $\vec{v}$ by the equations of conservation of mass and momentum to leave expressions for the derivatives of e, the internal energy.

Lagrangian. Starting with the Lagrangian integral equation, (35), substituting (34), and using the divergence theorem,

$$\frac{d}{dt}\int_V \rho e dV + \frac{d}{dt}\int_V \rho\frac{v^2}{2}\, dV = -\int_V \nabla \cdot (P\vec{v})dV \quad . \tag{41}$$

Now applying the transport theorem (5) to the second term, and (9) to the third term,

$$\frac{d}{dt}\int_V \rho e dV + \int_V\left[\frac{d(\rho\frac{v^2}{2})}{dt} + \rho\frac{v^2}{2}\nabla \cdot \vec{v}\right] dV = -\int_V[(\vec{v} \cdot \nabla)P + P(\nabla \cdot \vec{v})]dV \quad . \tag{42}$$

But

$$\frac{d(\rho\frac{v^2}{2})}{dt} + \rho\frac{v^2}{2}\nabla \cdot \vec{v} = \rho\frac{d(\frac{v^2}{2})}{dt} + \frac{v^2}{2}\left[\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{v})\right] = \rho\frac{d(\frac{v^2}{2})}{dt} \quad , \tag{43}$$

using the differential equation of conservation of mass, (15), which makes the bracket term vanish.

Also,

$$\rho\frac{d}{dt}\left(\frac{v^2}{2}\right) = \rho\frac{d}{dt}\left(\frac{v_i v_i}{2}\right) = \rho v_i \frac{dv_i}{dt} = \rho\vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{v} \cdot \rho\frac{d\vec{v}}{dt}$$

$$= -\vec{v} \cdot \nabla P \tag{44}$$

by the differential equation of conservation of momentum (30). Using (44), (43) in (42),

$$\boxed{\frac{d}{dt}\int_V \rho e dV = -\int_V P(\nabla \cdot \vec{v})dV \quad ,} \tag{45a}$$

or substituting for $\nabla \cdot \vec{v}$ from the differential equation of conservation of mass, (15),

$$\boxed{\frac{d}{dt} \int_V \rho e dV = \int_V P(\frac{1}{\rho} \frac{d\rho}{dt}) dV \quad .} \tag{45b}$$

These are simplified integral Lagrangian equations of conservation of energy (internal energy).

Now applying the transport theorem, (5), to (45b), we get

$$\int_V [\frac{d(\rho e)}{dt} + \rho e \nabla \cdot \vec{v}] dV = \int_V P(\frac{1}{\rho} \frac{d\rho}{dt}) dV \tag{46}$$

or

$$\int_V [\rho \frac{de}{dt} + e \frac{d\rho}{dt} + e\rho(\nabla \cdot \vec{v})] dV = \int_V P(\frac{1}{\rho} \frac{d\rho}{dt}) dV \quad . \tag{47}$$

Once again, the conservation of mass, (15), eliminates the two middle terms to give

$$\int_V \rho \frac{de}{dt} dV = \int_V P \frac{1}{\rho} \frac{d\rho}{dt} dV \quad , \tag{48}$$

which is true for all V, so that

$$\boxed{\frac{de}{dt} = P \left(\frac{1}{\rho^2}\right) \frac{d\rho}{dt} \quad ,} \tag{49}$$

which is a simplified differential Lagrangian form of the energy equation. To put it in a more familiar form, define the specific volume

$$V^* = \frac{1}{\rho} \text{ with } \frac{dV^*}{dt} = - \frac{1}{\rho^2} \frac{d\rho}{dt} \quad , \tag{50}$$

which gives the familiar thermodynamic form of law of conservation of energy with no heat flow ($dQ = 0$),

$$\boxed{\frac{de}{dt} = -P \frac{dV^*}{dt} \quad ,} \tag{51}$$

These simplified differential Lagrangian energy equations can be derived in a similar way from the unsimplified equation, (37), by use of (15), (34), (44), and Appendix A.

Eulerian. It is possible to begin with the unsimplified Eulerian intergral equation, (38), and with steps similar to those used above on the Lagrangian equation to achieve an Eulerian equation in terms of (ρe), but it is much easier to achieve the same result by starting with the simplified Lagrangian integral equation (45a,b). Applying the transport theorem, (5), to (45a) with $A = \rho e$, we get

$$\int_V \left[\frac{d(\rho e)}{dt} + \rho e(\nabla \cdot \vec{v})\right] dV = -\int_V P(\nabla \cdot \vec{v})dV \quad . \tag{52}$$

Now expanding the material derivative by (4),

$$\int_V \left[\frac{\partial(\rho e)}{\partial t} + (\vec{v} \cdot \nabla)\rho e + \rho e(\nabla \cdot \vec{v})\right] dV = -\int_V P(\nabla \cdot \vec{v})dV \quad . \tag{53}$$

Using (9) with $A = \rho e$ and taking $\frac{\partial}{\partial t}$ outside integral,

$$\boxed{\frac{\partial}{\partial t} \int_V (\rho e)dV = -\int_V \nabla \cdot (e\rho\vec{v})dV - \int_V P(\nabla \cdot \vec{v})dV \quad ,} \tag{54}$$

or by the divergence theorem

$$\boxed{\frac{\partial}{\partial t} \int_V (\rho e)dV = -\int_S e(\rho\vec{v} \cdot d\vec{S}) - \int_V P(\nabla \cdot \vec{v})dV \quad .} \tag{54a}$$

These are <u>simplified</u> <u>integral</u>, <u>Eulerian</u> forms of the conservation of energy. The physical interpretation of (54a) might be that for a fixed volume, V, the change in <u>internal</u> energy is given by the loss of internal energy through the surface and the integral of the rate at which P is doing work locally. This latter concept comes from the equation of conservation of mass, (15), which defines

$$(\nabla \cdot \vec{v}) = \left(-\frac{1}{\rho}\frac{d\rho}{dt}\right) = \left(\frac{1}{V^*}\frac{dV^*}{dt}\right) = \begin{matrix}\text{rate of volume change}\\ \text{per unit volume}\end{matrix} \tag{55}$$

so that

$$P(\nabla \cdot \vec{v}) = P(-\frac{1}{\rho}\ \frac{d\rho}{dt}) = P(\frac{1}{V^*}\ \frac{dV^*}{dt}) = \text{rate of work done per unit volume}$$

We can achieve further simplification by starting with the Eulerian integral equation (54), with $\frac{\partial}{\partial t}$ inside the integral

$$\int_V[\rho\ \frac{\partial e}{\partial t} + e\ \frac{\partial \rho}{\partial t}]dV + \int_V \nabla \cdot (e\rho\vec{v})dV = -\int_V P(\nabla \cdot \vec{v})dV \quad . \tag{56}$$

Using (9) with $\vec{A} = e$, $\vec{v} = \rho\vec{v}$,

$$\nabla \cdot [e(\rho\vec{v})] = e\nabla \cdot (\rho\vec{v}) + \rho(\vec{v} \cdot \nabla)e \quad ;$$

we have

$$\int_V\{\rho\ \frac{\partial e}{\partial t} + e[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{v})] + \rho(\vec{v} \cdot \nabla)e\}\ dV = -\int_V P(\nabla \cdot \vec{v})dV \quad .$$

The conservation of mass, (18), eliminates the bracket to give

$$\int_V[\rho\ \frac{\partial e}{\partial t} + \rho(\vec{v} \cdot \nabla)e]dV = \int_V P(\frac{1}{\rho}\ \frac{d\rho}{dt})dV \quad . \tag{57}$$

This holds for any volume, giving

$$\frac{\partial e}{\partial t} + (\vec{v} \cdot \nabla)e = -\frac{P}{\rho^2}\ \frac{d\rho}{dt} = -P\frac{dV^*}{dt} \tag{58}$$

This is a <u>simplified</u> <u>Eulerian</u> <u>differential</u> equation of conservation of energy. It could be obtained more simply from the simplified Lagrangian differential equation, (51), by use of the definition (4) of the material derivative.

COMMENTS AND ACKNOWLEDGMENTS

This is a review of some notes that were written by the author in 1964 to clear up his own confusion about the equations of Lagrangian and Eulerian hydrodynamics. They are not intended to be used by experts in the field, but rather by beginners who want a physical, overall view of the ideas involved in deriving the equations and of the math used in relating the two concepts.

Many people have helped through discussions about the subject, and it is impossible to remember all. However, I would like to mention a few. Conversations with G.N. White helped me get a better grasp of the math and conversations with Patrick J. Blewett helped me get a better grasp of the physics. I am grateful to Eldon J. Linnebur and Dan Carroll for encouraging me to write up some of my notes and allowing me to take time to do so. I have appreciated the help of Tessa Lippiatt in guiding me through the complexities of writing this report. Pearl Lucero typed the final copy very efficiently.

APPENDIX A

EXPANSIONS OF THE OPERATOR $\nabla \cdot (\ )$

$$\vec{A} = \rho: \quad \nabla \cdot (\rho\vec{v}) = \frac{\partial}{\partial x_i} (\rho v_i) = v_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i}$$

$$= (\vec{v} \cdot \nabla)\rho + \rho(\nabla \cdot \vec{v}) = (\vec{v} \cdot \nabla) \vec{A} + \vec{A}(\nabla \cdot \vec{v}) \quad .$$

$$\vec{A} = \rho\vec{v}: \quad \nabla \cdot (\vec{v}\vec{A}) = \nabla \cdot (\vec{v}\rho\vec{v}) = \nabla \cdot \left[\vec{v}(\rho\vec{v})\right] = \frac{\partial}{\partial x_i} \left[\vec{v}(\rho v_i)\right]$$

$$= \vec{v} \frac{\partial}{\partial x_i} (\rho v_i) + \rho v_i \frac{\partial}{\partial x_i} \vec{v} \text{ or } \vec{v}[\nabla \cdot (\rho\vec{v})] + \rho(\vec{v} \cdot \nabla)\vec{v}$$

$$= \vec{v} \left(\rho \frac{\partial v_i}{\partial x_i} + v_i \frac{\partial \rho}{\partial x_i}\right) + \rho v_i \frac{\partial \vec{v}}{\partial x_i}$$

$$= \vec{v} v_i \frac{\partial \rho}{\partial x_i} + \rho v_i \frac{\partial \vec{v}}{\partial x_i} + \rho\vec{v} \frac{\partial v_i}{\partial x_i}$$

$$= v_i \frac{\partial}{\partial x_i} (\rho\vec{v}) + \rho\vec{v} \frac{\partial v_i}{\partial x_i}$$

$$= (\vec{v} \cdot \nabla) \vec{A} + \vec{A} (\nabla \cdot \vec{v}) \quad .$$

$$\vec{A} = \rho E: \quad \nabla \cdot (\vec{v}\vec{A}) = \nabla \cdot [\vec{v}(\rho E)]$$

$$= \frac{\partial}{\partial x_i} (v_i \rho E) = v_i \frac{\partial}{\partial x_i} (\rho E) + \rho E \frac{\partial v_i}{\partial x_i}$$

$$= (\vec{v} \cdot \nabla)\vec{A} + \vec{A}(\nabla \cdot \vec{v}) \quad .$$

APPENDIX B

A DIVERGENCE TYPE THEOREM FOR $\nabla \cdot (\vec{v}\vec{A})$

$$\int_V \nabla \cdot (\vec{v}\vec{A})dV = \int_S \vec{A}(\vec{v} \cdot d\vec{S}) \quad .$$

Proof.

For $\vec{A} = \rho$, ρE, these quantities can be considered scalar coefficients of $\vec{v}$, and the divergence theorem holds directly.

For $\vec{A} = \rho\vec{v}$ from Appendix A,

$$\int_V \nabla \cdot (\vec{v}\vec{A})dV = \int_V \nabla \cdot [\vec{v}(\rho\vec{v})]dV$$

$$= \int_V \frac{\partial}{\partial x_i} [\vec{v}(\rho v_i)]dV$$

$$= \int_{x_3}\int_{x_2}\int_{x_1} \frac{\partial}{\partial x_i} [\vec{v}(\rho v_i)]dx_1 dx_2 dx_3$$

$$= \int_{x_3}\int_{x_2} \{[\vec{v}(\rho v_1)]^+ - [\vec{v}(\rho v_1)]^-\} dS_1$$

$$+ \int_{x_1}\int_{x_3} \{[\vec{v}(\rho v_2)]^+ - [\vec{v}(\rho v_2)]^-\} dS_2$$

$$+ \int_{x_1}\int_{x_2} \{[\vec{v}(\rho v_3)]^+ - [\vec{v}(\rho v_3)]^-\} dS_3 \quad ,$$

where $dS_1 = dx_2 dx_3$, $dS_2 = dx_1 dx_3$, $dS_3 = dx_1 dx_2$

so

$$\int_V \nabla \cdot [\vec{v}(\rho\vec{v})] \, dV = \int_S \rho\vec{v}(\vec{v} \cdot d\vec{S}) \quad .$$

APPENDIX C

THE PRESSURE MOMENTUM TERM

Consider a one-dimensional system (Fig. 1) in which

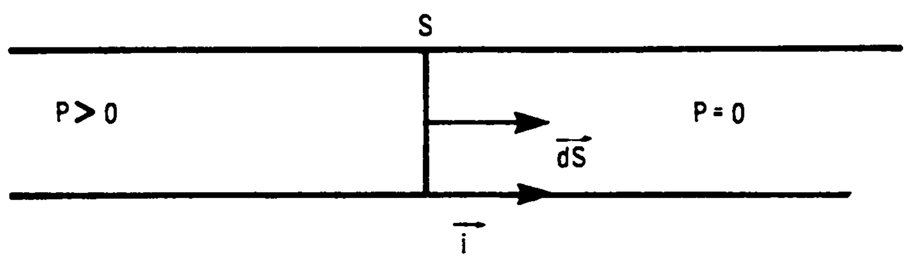

Fig. 1. One-Dimensional Flow

an imaginary surface, S, separates material of pressure P > 0 on the left from material of pressure P = 0 on the right. $P d\vec{S}$ represents the gain of +x momentum on the right from material on the left, so $-P d\vec{S}$ represents the gain (really a loss since it is negative) of momentum of the material on the left. For a more detailed study see Spitzer,[3] pages 94-98.

SUMMARY OF RESULTS

Total or Material Derivative:

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \nabla)\vec{A} \quad .$$

Reynolds' Transport Theorem:

$$\frac{d}{dt} \int_V \vec{A} dV = \int_V \left[\frac{d\vec{A}}{dt} + \vec{A}(\nabla \cdot \vec{v})\right] dV \quad .$$

<u>Lagrangian</u>	<u>Eulerian</u>

Conservation of Mass:

$$\frac{d}{dt} \int_V \rho dV = 0 \quad . \qquad \text{(A-1)}$$

$$\frac{\partial}{\partial t} \int_V \rho dV = -\int_S \rho \vec{v} \cdot d\vec{S} \quad . \qquad \text{(A-I)}$$

Conservation of Momentum:

$$\frac{d}{dt} \int_V \rho \vec{v} dV = -\int_S P d\vec{S} \quad . \qquad \text{(A-2)}$$

$$\frac{\partial}{\partial t} \int_V \rho \vec{v} dV = -\int_S P d\vec{S} - \int_S \rho \vec{v}(\vec{v} \cdot d\vec{S}) \quad . \qquad \text{(A-II)}$$

Conservation of Energy:

$$\frac{d}{dt} \int_V E dV = -\int_S P\vec{v} \cdot d\vec{S} \quad . \qquad \text{(A-3)}$$

$$\frac{\partial}{\partial t} \int_V \rho E dV = -\int_S E\rho\vec{v} \cdot d\vec{S} - \int_S P\vec{v} \cdot d\vec{S} \quad . \qquad \text{(A-III)}$$

<u>Differential Forms</u>

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{v}) = 0 \quad . \qquad \text{(A-1a)}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\vec{v}) \quad . \qquad \text{(A-Ia)}$$

$$\frac{d(\rho\vec{v})}{dt} + \rho\vec{v}(\nabla \cdot \vec{v}) = -\nabla P \quad . \qquad \text{(A-2a)}$$

$$\frac{\partial(\rho\vec{v})}{\partial t} = -\nabla P - \nabla \cdot (\vec{v}\rho\vec{v}) \quad . \qquad \text{(A-IIa)}$$

$$\frac{d}{dt}(\rho E) + \rho E(\nabla \cdot \vec{v}) = -\nabla \cdot (P\vec{v}) \quad . \qquad \text{(A-3a)}$$

$$\frac{\partial(\rho E)}{\partial t} = -\nabla \cdot (\rho E\vec{v}) - \nabla \cdot (P\vec{v}) \quad . \qquad \text{(A-IIIa)}$$

Simplifications

Lagrangian

Using (A-1a) in (A-2a)

$$\rho\frac{d\vec{v}}{dt} + \vec{v}\cancel{\frac{d\rho}{dt}} + \rho\vec{v}\cancel{(\nabla\cdot\vec{v})} = -\nabla P \quad . \qquad \text{(A-2b)}$$

Using (A-1a) in (A-3a)

$$\rho\frac{dE}{dt} + E\cancel{\frac{d\rho}{dt}} + \rho E\cancel{(\nabla\cdot\vec{v})} = -P(\nabla\cdot\vec{v}) - \vec{v}\cdot\nabla P \quad . \qquad \text{(A-3b)}$$

Using (A-2b) in (A-3b) with (4)

$$\rho\frac{de}{dt} + \cancel{\frac{\rho}{2}\frac{dv^2}{dt}} = -P(\nabla\cdot\vec{v}) + \cancel{\vec{v}\cdot(\rho\frac{d\vec{v}}{dt})} \quad . \qquad \text{(A-3c)}$$

Using (A-1a) in (A-3c)

$$\frac{de}{dt} = \frac{P}{\rho^2}\frac{d\rho}{dt} \quad . \qquad \text{(A-3d)}$$

Eulerian

Using (A-Ia) in (A-IIa)

$$\rho\frac{\partial\vec{v}}{dt} + \vec{v}\cancel{\frac{\partial\rho}{dt}} = -\nabla P - \vec{v}[\cancel{\nabla\cdot(\rho\vec{v})}] - \rho(\vec{v}\cdot\nabla)\vec{v} \qquad \text{(A-IIb)}$$

Using (A-Ia) in (A-IIIa)

$$\rho\frac{\partial E}{\partial t} + E\cancel{\frac{\partial\rho}{\partial t}} = -E[\cancel{\nabla\cdot(\rho\vec{v})}] - \rho(\vec{v}\cdot\nabla)E - \nabla\cdot(P\vec{v}) \quad . \qquad \text{(A-IIIb)}$$

Using (4) in (A-IIIb)

$$\rho\frac{\partial e}{\partial t} + \frac{\rho}{2}\frac{\partial v^2}{\partial t} = -\rho(\vec{v}\cdot\nabla)(e + \frac{v^2}{2}) - P(\nabla\cdot\vec{v}) - \vec{v}\cdot\nabla P \quad .$$

Expand and use (A-4)

$$\rho\frac{\partial e}{\partial t} + \rho\vec{v}\cdot\frac{\partial\vec{v}}{\partial t} = -\rho(\vec{v}\cdot\nabla)e - \rho(\vec{v}\cdot\nabla)\frac{v^2}{2} - P(\nabla\cdot\vec{v}) - \vec{v}\cdot\nabla P \quad .$$

Rearrange and use (A-5) and (A-IIb)

$$\rho\frac{\partial e}{\partial t} + \vec{v}\cdot\cancel{\rho\frac{\partial\vec{v}}{\partial t}} = -\rho(\vec{v}\cdot\nabla)e - \cancel{\vec{v}\cdot\rho(\vec{v}\cdot\nabla)\vec{v}} - P(\nabla\cdot\vec{v}) - \cancel{\vec{v}\cdot\nabla P} \quad .$$

or

$$\rho \frac{\partial e}{\partial t} + \rho(\vec{v} \cdot \nabla)e = -P\ (\nabla \cdot \vec{v}) \quad . \qquad \text{(A-IIIc)}$$

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) = \frac{d}{dt}\left(\frac{\vec{v} \cdot \vec{v}}{2}\right) = \frac{d}{dt}\left(\frac{v_i v_i}{2}\right) = v_i \frac{dv_i}{dt} = \vec{v} \cdot \frac{d\vec{v}}{dt} \quad . \qquad \text{(A-4)}$$

$$(\vec{v} \cdot \nabla) \frac{v^2}{2} = v_k \frac{\partial}{\partial x_k}\left(\frac{v_i v_i}{2}\right) = v_k v_i \frac{\partial v_i}{\partial x_k} = v_i v_k \frac{\partial v_i}{\partial x_k} = v_i(\vec{v} \cdot \nabla)v_i$$

$$= \vec{v} \cdot \left[(\vec{v} \cdot \nabla)\vec{v}\right] \quad . \qquad \text{(A-5)}$$

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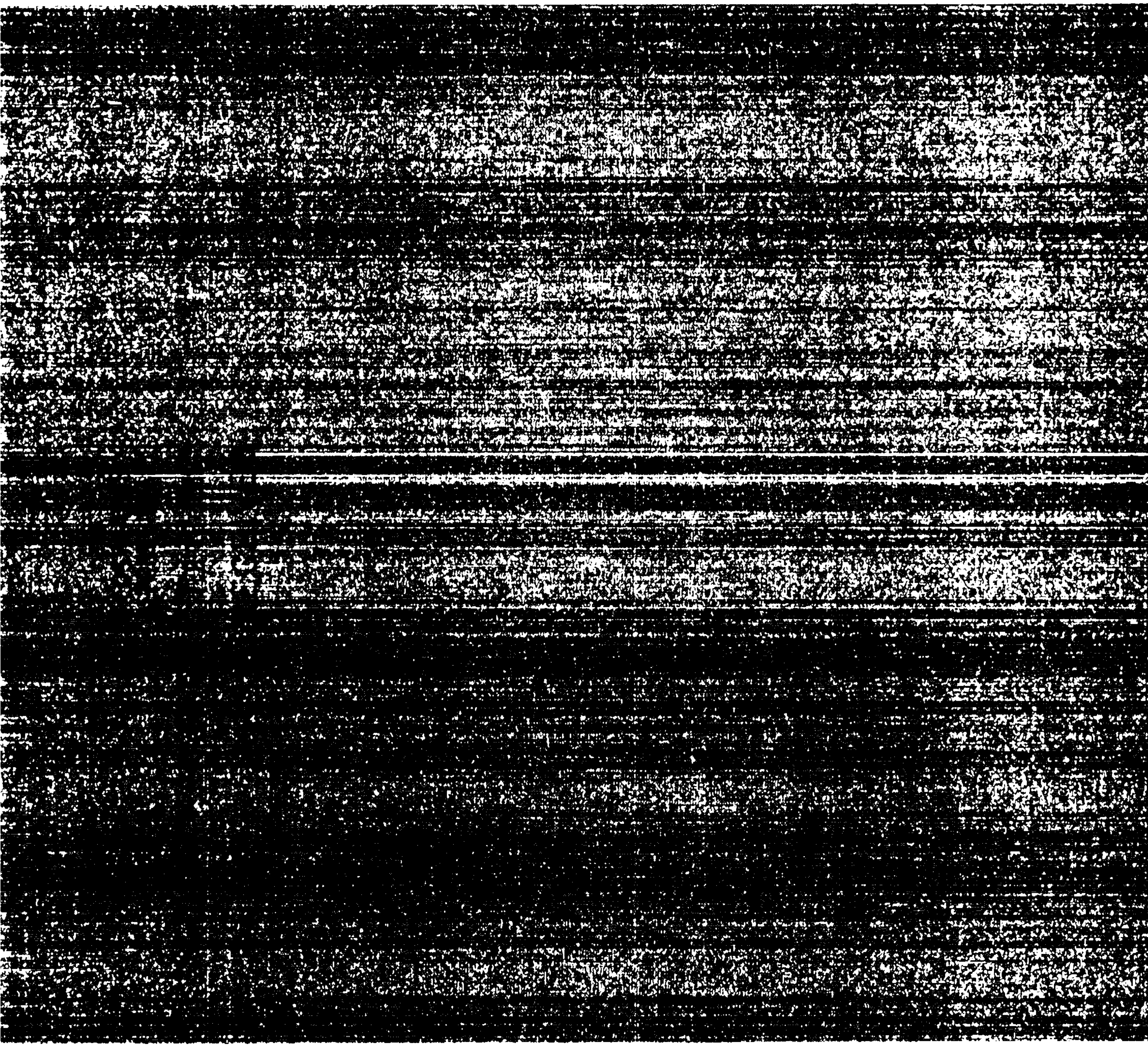

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