TITLE: AN INSTABILITY LOCALIZED AT THE INNER
SURFACE OF AN IMPLODING SPHERICAL SHELL

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ABSTRACT

It is shown that in an imploding spherical shell the surface instabilities are of two different types. The first, which occurs at the outer surface, is the Rayleigh-Taylor instability. The second instability occurs at the inner surface. This latter instability is not as disruptive as R-T modes, but it has three basic properties which differ considerably from those of the R-T instability: (1) it is oscillatory at early times; (2) it grows faster in the long wavelength modes; (3) it depends on the equation of state. It is further shown that this new instability is driven by amplified sound waves in the shell.

BASIC EQUATIONS

We consider an imploding spherical shell that obeys the following ideal fluid equations:

$$\rho\left(\frac{d\underset{\sim}{v}}{dt}\right) = -\underset{\sim}{\nabla}p \ , \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho\underset{\sim}{v}) = 0 \ , \tag{2}$$

$$\frac{d}{dt}(p/\rho^{\gamma}) = 0 \tag{3}$$

A self-consistent description of the shell motion can be obtained by introducing Sedov's hypothesis[8] of self-similar motion, which in the Lagrangian representation is simply given as

$$R(r_0,t) = r_0 \ f(t) \ . \tag{4}$$

$$\rho(r_0,t) = \rho_0(r_0) \ f^{-3}, \qquad p(r_0,t) = p_0(r_0) \ f^{-3\gamma} \ . \tag{5}$$

THE UNPERTURBED MOTION

Using the time-dependent pressure in Eq. (1), we obtain the unperturbed motion

$$\ddot{f}(t)f^{3\gamma-2}(t) = -\frac{1}{\rho_0 r}\frac{d}{dr}(\rho_0^\gamma(r)) = -\frac{1}{t_c^2} = -1 , \tag{6}$$

The time-dependent part of Eq. (6) yields on integration

$$\dot{f} = \left(\frac{2}{\alpha}(f^{-\alpha} - 1)\right)^{1/2} , \tag{7}$$

where the initial values $f(0) = 1$ and $\dot{f}(0) = 0$ have been used. Here $\alpha = 3(\gamma - 1)$. When $\gamma = 5/3$, Eq. (7) can be integrated once again to give

$$f^2 = 1 - t^2 , \tag{8}$$

where $0 < t < 1$.

$$\rho_0(r) = \hat{\rho}_0((r^2 - r_-^2)/(r_+^2 - r_-^2))^{1/(\gamma - 1)} , \tag{9}$$

where r_+ and r_- are the outer and inner radii of the shell;

$$p_o(r) = \hat{p}_o \left((r^2 - r_-^2)/(r_+^2 - r_-^2) \right)^{\gamma/(\gamma - 1)} , \tag{10}$$

where $\hat{p}_o$ is a constant pressure at $r = r_+$.

STABILITY ANALYSIS

When a perturbation ξ is introduced, the position vector of a fluid element in Lagrangian variables becomes $\underset{\sim}{R} + \underset{\sim}{\xi}$. A straightforward, though somewhat tedious, calculation[7,11] shows that $\underset{\sim}{\xi}$ obeys the equation

$$f^{(3\gamma-1)} \ddot{\underset{\sim}{\xi}} = \frac{(\gamma - 1)}{2}(r^2 - r_-^2)\underset{\sim}{\nabla}\sigma + (\gamma - 1)\sigma \underset{\sim}{r} + \underset{\sim}{r} \times \underset{\sim}{\omega} + (\underset{\sim}{r} \cdot \underset{\sim}{\nabla})\underset{\sim}{\xi} , \tag{11}$$

where $\sigma = \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}$ and $\underset{\sim}{\omega} = \underset{\sim}{\nabla} \times \underset{\sim}{\xi}$.

We now limit the discussion to the case of incompressible, irrotational perturbations for which the surface instabilities occur. That is, we choose the perturbations such that $\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} = 0$ and $\underset{\sim}{\nabla} \times \underset{\sim}{\xi} = 0$. This implies that $\underset{\sim}{\xi} = \underset{\sim}{\nabla}\chi$ with $\nabla^2 \chi = 0$. In spherical coordinates

$$\chi(r,\theta,\phi) = \sum_{\ell m} \left[Q_+^\ell(t)r^\ell + Q_-^\ell(t)r^{-(\ell + 1)} \right] Y_{\ell m}(\theta,\phi) . \tag{12}$$

To obtain $Q_\pm^\ell(t)$, we expand the perturbations in spherical harmonics[11]

$$\underset{\sim}{\xi} = \sum_{\ell m} \left[\xi_1^{\ell m}(r,t)\hat{a}_1 + \xi_2^{\ell m}(r,t)\hat{a}_2 + \xi_3^{\ell m}(r,t)\hat{a}_3 \right] , \tag{13}$$

$$\hat{a}_1 = \hat{e}_r Y_{\ell m} , \qquad \hat{a}_2 = r\underset{\sim}{\nabla}Y_{\ell m} , \qquad \hat{a}_3 = \underset{\sim}{r} \times \underset{\sim}{\nabla}Y_{\ell m} . \tag{14}$$

TIME-DEPENDENT EQUATION

The Classical Rayleigh-Taylor Instability:

$$g(t) = \ddot{R}(t) = r \, \ddot{f}(t) = - r \, f^{-3\gamma+2} \, , \tag{16}$$

t. :n Eq. (15) can be written as

$$\ddot{Q}_{\pm}^{\ell}(t) + \left[\frac{3}{2} \mp (\ell + \frac{1}{2})\right]\frac{|g(t)|}{R(t)} \, Q_{\pm}^{\ell}(t) = 0 \, , \tag{17}$$

Notice that $Q_{-}^{\ell}(t)$ modes localized at the inner surface are oscillatory in the static limit, whereas $Q_{+}^{\ell}(t)$ modes are unstable.

The growth rate of the unstable mode is given by

$$\gamma_t^2 = |g_o| \, (\frac{m}{R_o}) = |g_o|k_\theta \, , \tag{18}$$

This is the growth rate of the classical Rayleigh-Taylor instability for static media.

SOLUTIONS FOR AN ARBITRARY TIME

$$Q_\pm^\ell(t) = c_1 \mathcal{F}_\pm(\ell,\gamma,t) \div c_2 \mathcal{G}_\pm(\ell,\gamma,t) . \tag{19}$$

Here

$$\mathcal{F}_\pm(\ell,\gamma,t) = {}_2F_1\left[\frac{1}{4} + \frac{2+id_\pm}{4\alpha}, \frac{1}{4} + \frac{2-id_\pm}{4\alpha}, \frac{1}{2}; 1 - f^{-\alpha} \right] \tag{20}$$

and

$$\mathcal{G}_\pm(\ell,\gamma,t) = \left(1 - f^{-\alpha}\right)^{1/2} {}_2F_1\left[\frac{3}{4} + \frac{2+id_\pm}{4\alpha}, \frac{3}{4} + \frac{2-id_\pm}{4\alpha}, \frac{3}{2}; 1 - f^{-\alpha} \right], \tag{21}$$

where $d_\pm = \left[8\alpha\{\frac{3}{2} \mp (\ell + \frac{1}{2})\} - (\alpha + 2)^2\right]^{1/2}$.

As in Ref. 10, we obtain the stability criteria by evaluating the asymptotic limits:

$$\lim_{t \to 1} \frac{\mathcal{F}_\pm(\ell,\gamma,t)}{f(t)} \simeq a_\pm^\ell \; f^{(\alpha-2+id_\pm)/2} + \text{c. c.} \tag{22}$$

and

$$\lim_{t \to 1} \frac{\mathcal{G}_\pm(\ell,\gamma,t)}{f(t)} \simeq b_\pm^\ell \; f^{(\alpha-2+id_\pm)/2} + \text{c. c.} \; , \tag{23}$$

where

$$a_\pm^\ell = \frac{\Gamma(\tfrac{1}{2})\Gamma(\frac{-id_\pm}{2\alpha})}{\Gamma(\tfrac{1}{4} + \frac{2-id_\pm}{4\alpha})\Gamma(\tfrac{1}{4} - \frac{2+id_\pm}{4\alpha})}, \quad b_\pm^\ell = 1 \; \frac{\Gamma(\tfrac{3}{2})\Gamma(\frac{-id_\pm}{2\alpha})}{\Gamma(\tfrac{3}{4} + \frac{2-id_\pm}{4\alpha})\Gamma(\tfrac{3}{4} - \frac{2+id_\pm}{4\alpha})} \; . \tag{24}$$

SUMMARY OF THE ANALYSIS

1) $\ell = 0$ mode.

For $\gamma < 5/3$, the limits of $\mathcal{F}_+/f$ and $\mathcal{G}_+/f$ are finite.

If $\gamma < 5/3$, $\mathcal{F}_-/f$ and $\mathcal{G}_-/f$ diverge asymptotically, which signals instability.

2) The Q_+^ℓ modes with $\ell \geq 1$.

In the asymptotic limit, both $\mathcal{F}_+/f$ and $\mathcal{G}_+/f$ diverge as f^κ as $f \to 0$, where κ is real and negative.

3) The asymptotic limits of both $\mathcal{F}_-/f$ and $\mathcal{G}_-/f$ of Q_-^ℓ diverge for $\ell \geq 1$ and $\gamma < 5/3$.

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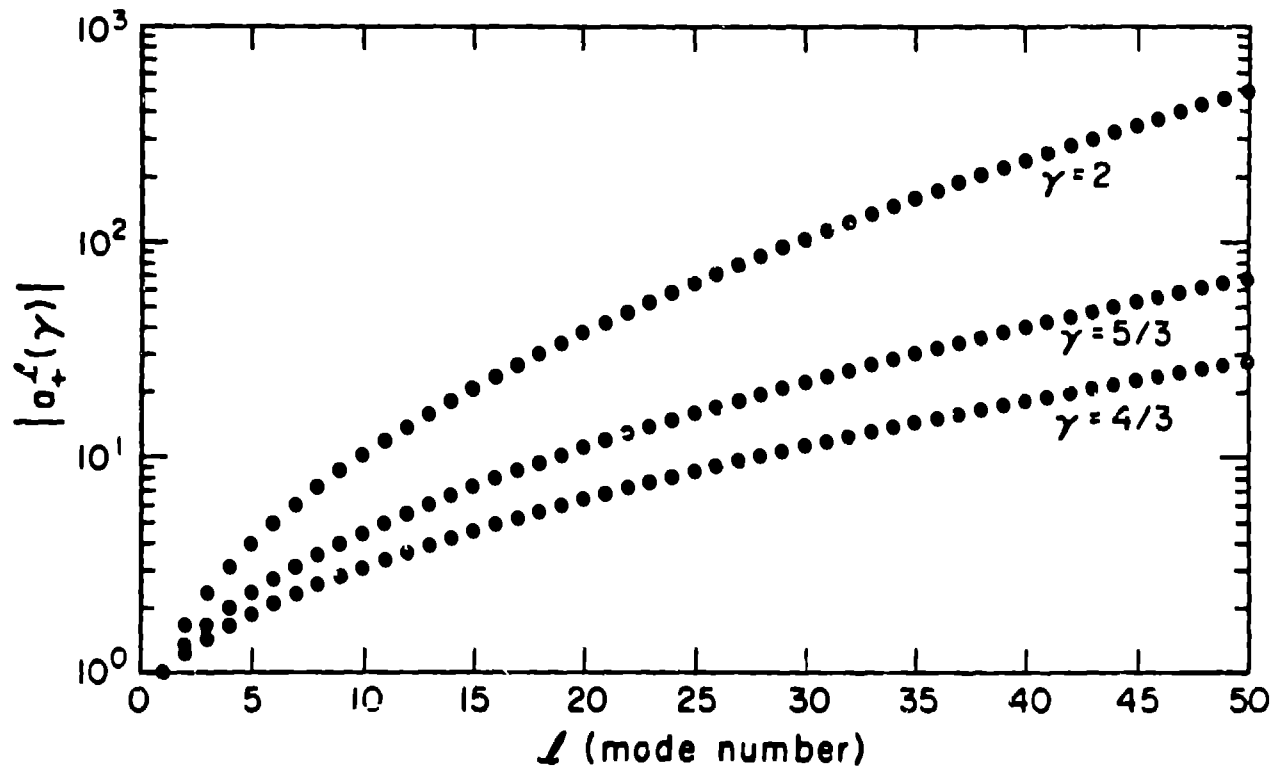12. S. J. Han, Phys. Fluids <u>25</u>,1723(1982).
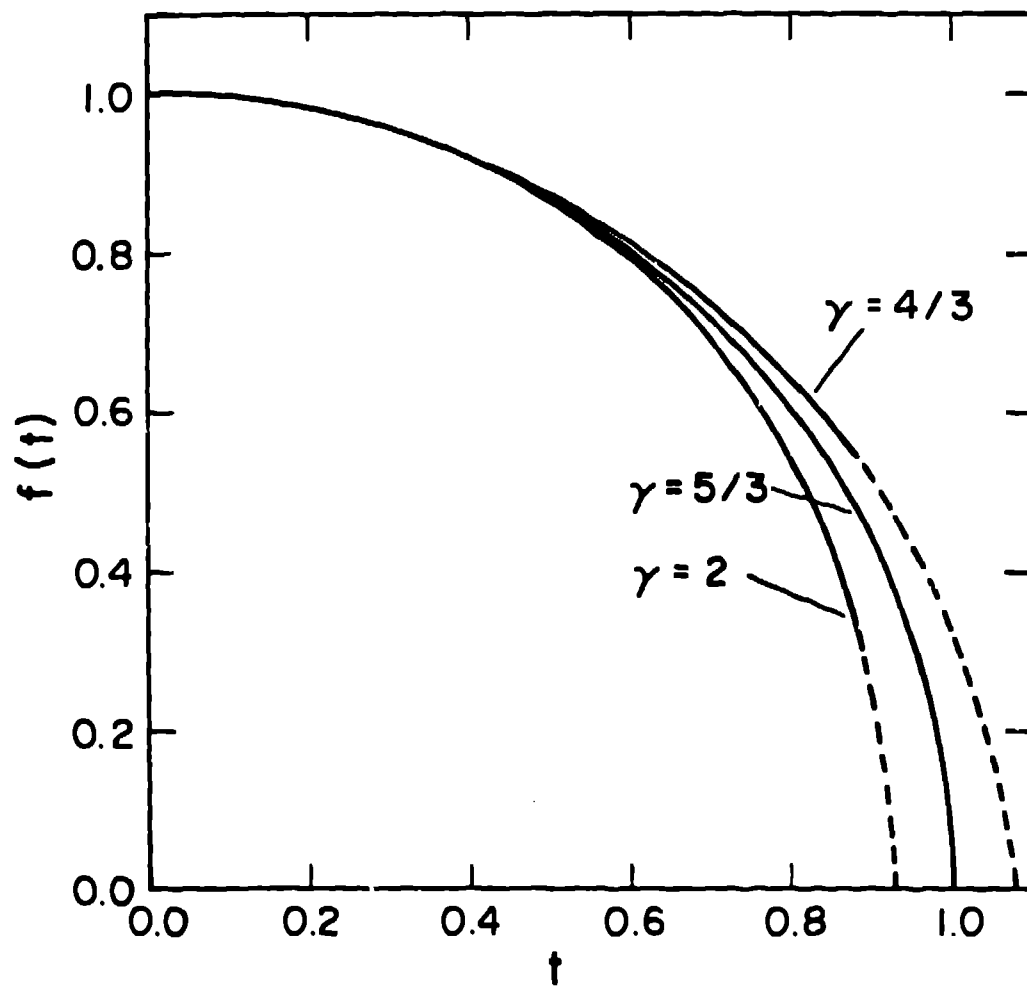
Fig. 1. The absolute values of $a_+^\ell(\gamma)$ given in Eq. (20) are plotted against the mode number $\ell = 1$ to 50.
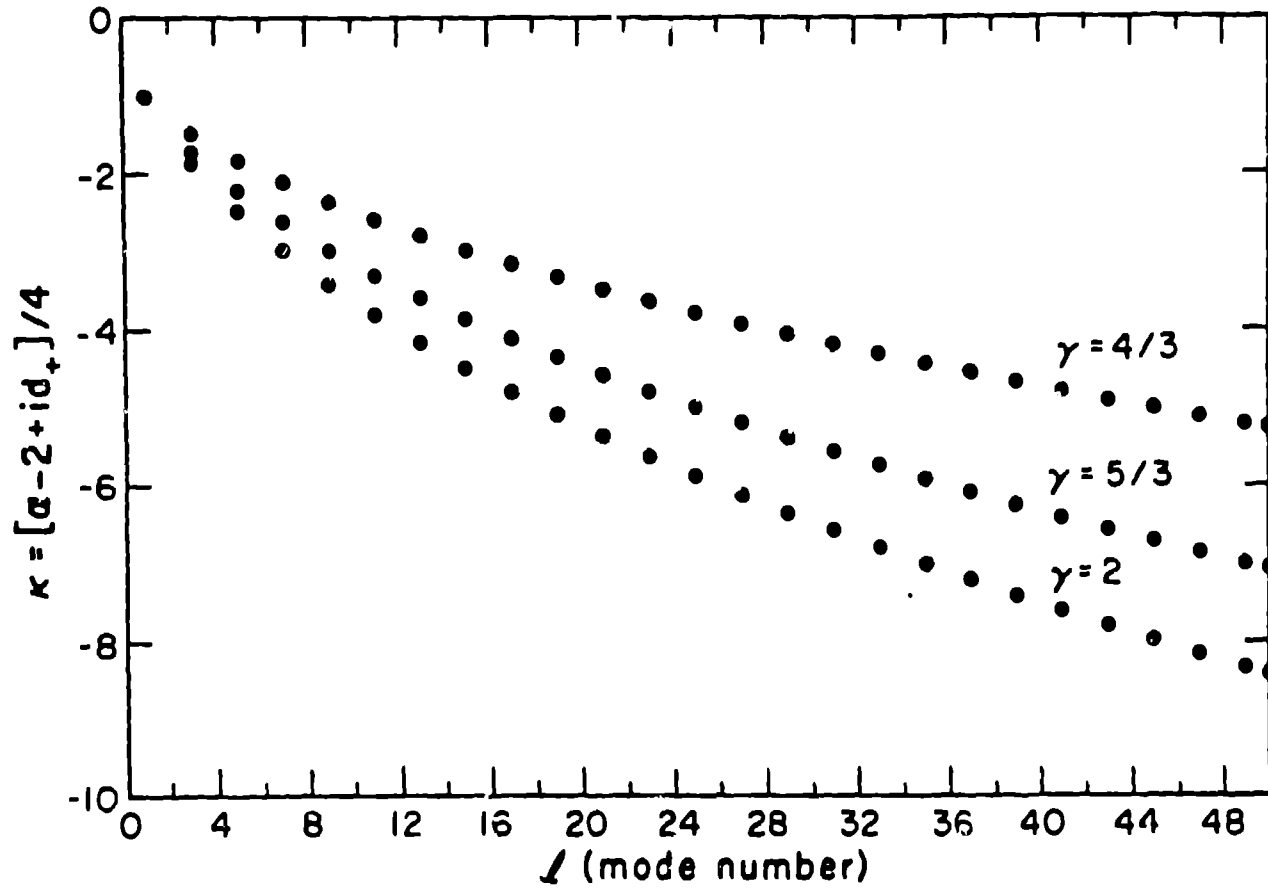
Fig. 2. The exponent of the divergence factor κ is plotted against the mode number.
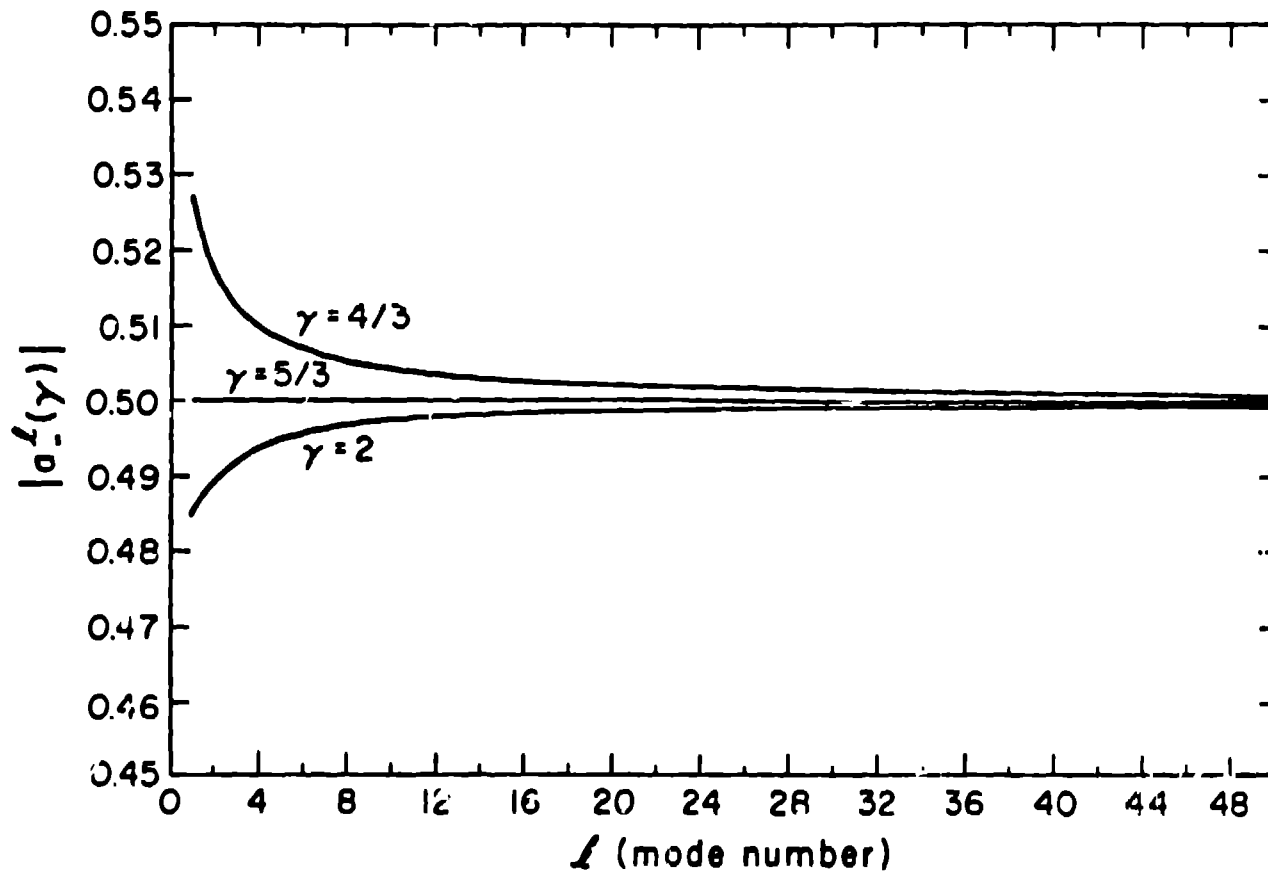
Fig. 3. The absolute values of $a_-^{\ell}(\gamma)$ for $\gamma = 4/3$, 5/3, and 2 are plotted against the mode number $\ell = 1$ to 50.
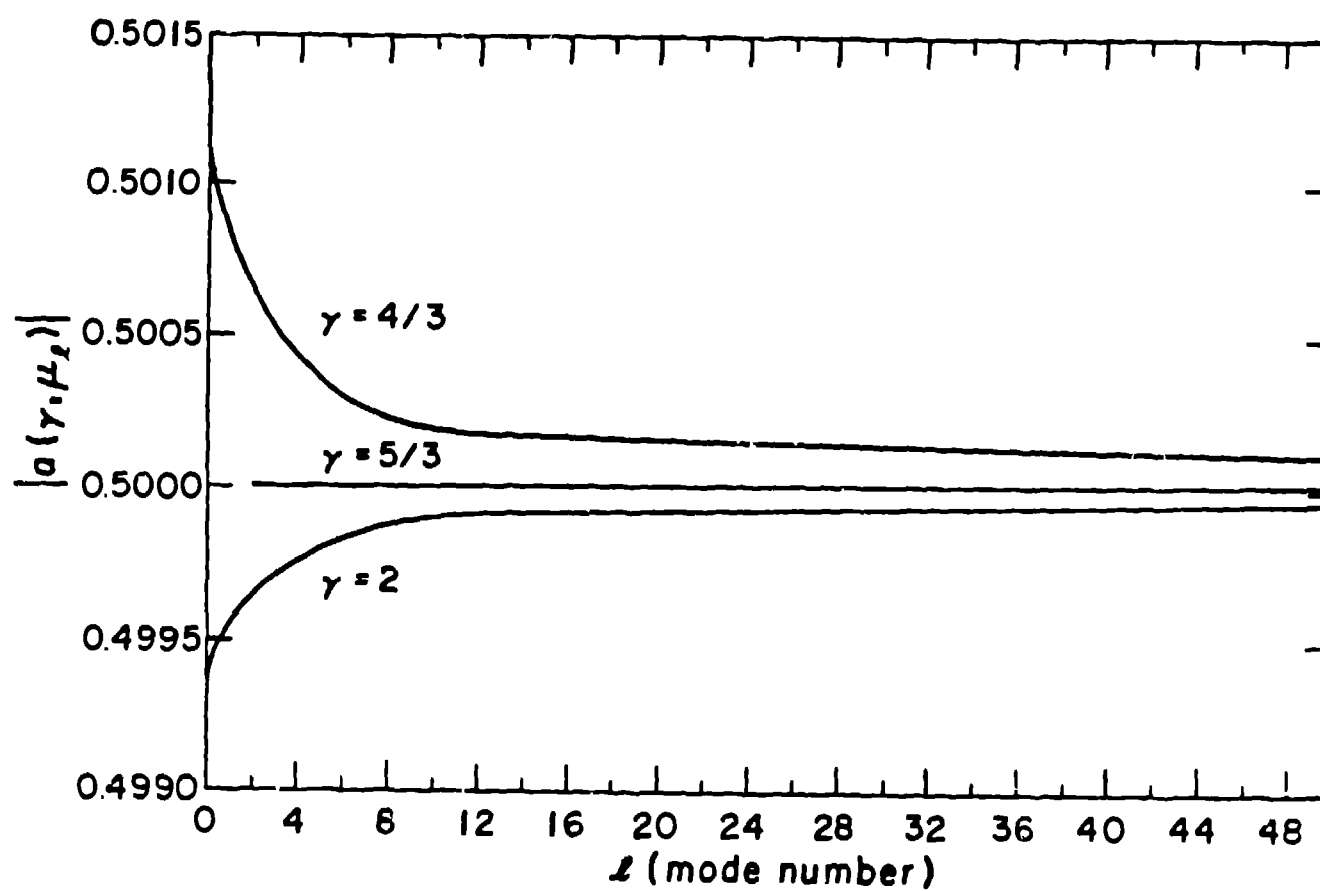
Fig. 4. The absolute values of $a(\gamma, \mu_\ell)$ for $\gamma \sim 4/3$, 5/3, and 2 are plotted against the mode number ℓ. Here μ_ℓ is the smallest eigenvalue for a given ℓ with $r_+ = 7.0$ cm and $r_- = 5.0$ cm.