LA-UR -80-2605

TITLE: NEW DEVELOPMENTS IN DIFFERENCING THE SPHERICAL GEOMETRY NEUTRON TRANSPORT EQUATION

AUTHOR(S): Warren F. Miller, Jr.

MASTER

SUBMITTED TO: GAMNI International Symposium on Numerical Methods in Engineering, Paris, France, December 1-5, 1980.

University of California

LOS ALAMOS SCIENTIFIC LABORATORY
Post Office Box 1663 Los Alamos, New Mexico 87545
An Affirmative Action/Equal Opportunity Employer

NEW DEVELOPMENTS IN DIFFERENCING THE SPHERICAL GEOMETRY NEUTRON TRANSPORT EQUATION

Warren F. Miller, Jr.

Deputy Associate Director for Nuclear Programs
Los Alamos Scientific Laboratory, Los Alamos, New Mexico USA

ABSTRACT

Early differencing methods due to Carlson, Lathrop, and others have continued to be used to approximate the spherical geometry neutron transport equations. Nonphysical depressions in the scalar flux profiles continue to cause problems when these early techniques are used. Recent developments, however, provide better understanding of the behavior of these methods and have led to a simple approach to improve numerical solutions.

1. INTRODUCTION

It has been observed for several years that nonphysical depressions in the scalar flux spatial profile can appear near a spherical system origin when production, diamond-difference, discrete-ordinates, neutron transport computer codes are used.[1,2] Such depressions have also been observed in solutions from the newer spatial finite element, discrete-ordinates code ONETRAN.[3] Even when such depressions do not appear, the origin flux can be disconcertingly inaccurate, exhibiting a first-order error in the spatial variable.

Reed and Lathrop[4] associated origin scalar flux depressions with the spatial truncation error. They studied this error as well as the angular truncation error with the purposes of improving accuracy and eliminating these depressions. The results of their work were weighted-diamond-difference schemes for use in both the angular and the spatial variables. Using their approach, they reported elimination of the origin scalar flux depression for a sample problem. Their angular weighted-diamond scheme, however, has not been extensively used since it does not allow the transport computer code user the freedom of picking his own quadrature data. Their spatial weighted-diamond scheme is not used due to substantial inaccuracies for coarse grids.

In the present work, the discrete-ordinates equations are first developed and an expression for the angular truncation error is determined. It is shown that one class of scalar flux depressions at the system origin is due to this angular truncation error, and thus, cannot be eliminated by any convergent spatial difference scheme. The spatially differenced equations are then derived using diamond differencing and we demonstrate that the spatial truncation error is spatially nonuniform and becomes first-order at the origin. The second class of scalar flux depressions is due to this spatial truncation error. We develop a new improvement to the diamond-difference equations due to Alcouffe and Miller yielding second-order errors at the system origin. These theoretical developments are provided in Section II. Section III provides numerical results and conclusions.

II. THEORY

A. The Discrete-Ordinates Approximation

The spherical geometry transport equation, for a given energy group, may be written as

$$\mu \frac{\partial \psi}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial \psi}{\partial \mu} + \sigma(r)\psi(r,\mu) = \sigma_s(r)\phi(r) + Q(r) = S(r) \qquad (1)$$

or, in conservation form,[5] as

$$\frac{\mu}{r^2} \frac{\partial(r^2\psi)}{\partial r} + \frac{\partial[(1-\mu^2)\psi]}{r\partial\mu} + \sigma(r)\psi(r,\mu) = S(r) \quad . \qquad (2)$$

In Eqs. (1) and (2), the inhomogeneous term, Q(r), includes external fission, and scattering sources to the energy group. For the purposes of this study, there is no loss in generality in assuming that this source is isotropic. This group source, Q(r), is a function of the flux itself, and the equation is solved iteratively. It is assumed that an interation cycle (called an outer iteration) on the fission and group scattering source has just been completed and Q(r) is known. In Eq. (1), the scalar flux, $\phi(r)$, is defined as

$$\phi(r) = \frac{1}{2}\int_{-1}^{1} d\mu' \psi(r,\mu'), \tag{3}$$

The scattering process within the energy group is also assumed to be isotropic. For a given outer iteration cycle, an inner iteration procedure is used to solve Eq. (1) or Eq. (2) since S(r) is a function of $\psi(r,\mu)$ through $\phi(r)$. Thus, it is further assumed that an inner iteration cycle has just been completed and S(r) is known.

An angular mesh is imposed on the domain $-1 \le \mu \le 1$ and the mesh edges are denoted by $\mu_{m\pm1/2}$ and $\mu_{m-1/2} < \mu_m < \mu_{m+1/2}$. We initially insist that μ_m be the midpoint of the interval. We denote

$$\psi_m(r) = \psi(r,\mu_m) \tag{4}$$

$$m = 1, 2, \ldots, M,$$

and define the normalized mesh intervals by

$$W_m = \frac{1}{2}\left(\mu_{m+1/2} - \mu_{m-1/2}\right). \tag{5}$$

$$m = 1, 2, \ldots M.$$

The discrete-ordinates approximation to Eq. (2) is then

$$\frac{\mu_m}{r^2}\frac{d(r^2\psi_m)}{dr} + \frac{\left(\alpha_{m+1/2}\psi_{m+1/2} - \alpha_{m-1/2}\psi_{m-1/2}\right)}{rW_m} + \sigma(r)\psi_m(r) = \tilde{S}(r) \tag{6}$$

$$m = 1, 2, \ldots, M.$$

where

$$\tilde{S}(r) = \sigma_N(r)\tilde{\phi}(r) + Q(r) = \sigma_N(r)\sum_{m'=1}^{M} W_{m'}\psi_{m'}(r) + Q(r), \tag{7}$$

and

$$\alpha_{m+1/2} - \alpha_{m-1/2} = 2W_m\mu_m \,, \qquad (8)$$

$$m = 1, 2, \ldots, M.$$

In Eqs. (8) $\alpha_{1/2}$ is taken to be zero due to neutron conservation arguments[4] and, if the mesh points and intervals are symmetric ($\mu_m = -\mu_{M-m+1}$ and $W_m = W_{M-m+1}$), as is usually the case, $\alpha_{M+1/2} = 0$. Note in Eq. (7) that the scalar flux integral [Eq. (3)] is approximated using a quadrature rule

$$\phi(r) \simeq \tilde{\phi}(r) = \sum_{m'=1}^{M} W_{m'}\psi_{m'}(r). \qquad (9)$$

Thus, W_m, μ_m are simultaneously used as quadrature weights and points as well as angular mesh intervals and mesh points.

Each of Eqs. (6) is one equation in two unknowns, $\psi_{m+1/2}(r)$ and $\psi_m(r)$ assuming the cell edge flux $\psi_{m-1/2}(r)$ is known. Thus an additional equation is required. We invoke the angular diamond equation.[1]

$$\psi_m(r) = \frac{1}{2}\left(\psi_{m+1/2}(r) + \psi_{m-1/2}(r)\right), \qquad (10)$$

$$m = 1, 2, \ldots, M,$$

as the other needed relationship. The set of Eqs. (6) and (10) are solved in the order of small values of m, corresponding to the most negative values of μ_m, to large values of m corresponding to the most positive values of μ_m.

For m = 1, a relationship with which to determine $\psi_{1/2}(r)$, associated with the direction $\mu = -1$, is needed. This equation, for the so-called starting direction flux, is obtained by setting $\mu = -1$ in Eq. (1) resulting in the slab geometry transport equation

$$\frac{-d\psi_{1/2}}{dr} + \sigma(r)\,\psi_{1/2}(r) = S(r). \qquad (11)$$

We have assumed that the μ derivative of ψ is bounded at $\mu = -1$. This is the case for a nonsingular source--the situation considered in transport codes.

Associated with Eq. (1) is the boundary condition at the outer radius, $r = R$,

$$\psi(R,\mu) = \Psi(\mu) \qquad \mu < 0 \qquad (12a)$$

where $\Psi(\mu)$ is known. The analogous conditions for Eqs. (6) are

$$\psi_m(R) = \Psi_m \qquad (12b)$$

$$m = 1, 2, \ldots, M/2.$$

Although no boundary condition is required at $r = 0$, for the development to follow, we must consider the form of Eq. (1) at the origin. We multiply by r and let $r \to 0$ to yield

$$\left.\frac{\partial \psi}{\partial \mu}\right|_{r=0} = 0, \qquad (13a)$$

so that the angular flux is isotropic at the origin. Analogously, multiplying Eqs. (6) by r, letting $r \to 0$ and using Eqs. (8) and (9) yields

$$\psi_m(r)\Big|_{r=0} = \psi_{m+1/2}(r)\Big|_{r=0} = \psi_{m-1/2}(r)\Big|_{r=0} \qquad (13b)$$

$$m = 1, 2, \ldots, M.$$

To determine angular truncation errors in Eqs. (6) we follow the argument presented in Reference 4. Namely, for a particular m, we seek to determine the error in the equations satisfied by $\psi_m(r)$, $\psi_{m+1/2}(r)$ and $\psi_{m-1/2}(r)$. That is, to what order truncation error does Eq. (6) approximate Eq. (1)? Although this is not a truncation error in the sense of specifically determing the error in the *solution* of the equations,[4] it is one approach to viewing trucation errors and, as shown in numerical results,[4] apparently provides an accurate measure of the order of the truncation error of the solution.

We expand the angular cell edge flux in a Taylor series about the point, μ_m. Then

$$\psi_{m\pm1/2}(r) = \psi_m(r) \pm W_m \left.\frac{\partial \psi}{\partial \mu}\right|_{\mu=\mu_m} + \frac{W_m^2}{2} \left.\frac{\partial^2 \psi}{\partial \mu^2}\right|_{\mu=\mu_m} + \ldots \qquad (14)$$

Inserting Eq. (14) into Eq. (6) and using Eq (8) yields

$$\mu_m \frac{\psi_m}{r} + \frac{1}{r}(\psi_{m+1/2} + \psi_{m-1/2}) \left.\frac{\partial \psi}{\partial \mu}\right|_{\mu=\mu_m} + \sigma_t \psi_m = \tilde{S}(r) = 0 \frac{W_m^2}{r}. \qquad (15)$$

But from Eq. (8)

$$\psi_{m+1/2} = \sum_{m'=1}^{m} (\psi_{m'} W_{m'}) = \sum_{m'=1}^{m} \left\{ \int_{\mu_{m'-1/2}}^{\mu_{m'+1/2}} d\mu(\psi) + \psi W_{m'}^3 \right\}$$

$$= \int_{-1}^{\mu_{m+1/2}} d\mu\ (\psi) + \psi W_m^2 = \frac{1}{2}(1 - \mu_{m+1/2}^2) + \psi W_m^2. \qquad (16)$$

Also from Eq. (7)

$$\tilde{S}(r) = \sigma_s(r)\sum_{M'=1}^{M} W_{m'}\psi_{m'}(r) + Q(r)$$

$$= \frac{\sigma_s}{2}(r)\int_{-1}^{1} d\mu\,\psi(r,\mu') + Q(r) + OW_m^2 = S(r) + OW_m^2 \quad . \tag{17}$$

Combining these equations yields

$$\mu_m \frac{\partial \psi_m}{\partial r} + \frac{(1-\mu_m^2)}{r}\left.\frac{\partial \psi}{\partial \mu}\right|_{\mu=\mu_m} + \sigma_t(r)\psi_m(r) = S(r) + OW_m^2 + O\frac{W_m^2}{r} \tag{18}$$

The $OW_m^2 + O\,W_m^2/r$ error from this analysis assumes that μ_m is the midpoint of the angular interval. It is straightforward to show, however, that if the μ_m are selected so that

$$\mu_m = \frac{\mu_{m+1/2} + \mu_{m-1/2}}{2} + OW_m^2, \tag{19}$$

the truncation error is still $OW_m^2 + O\,W_m^2/r$. The popular Gauss Quadrature set, commonly used with discrete ordinate codes satisfies Eq. (19).

At this point, one might ask if the error in the flux is unbounded as r approaches zero. The exact flux satisfied Eq. (13a) at the origin. Expanding $\psi_{m\pm1/2}$ of Eq. (13b) in a Taylor series about μ_m, however, yields Eq. (13a) with no truncation error. The origin flux is, then, bounded and isotropic. The exact flux is not obtained, however, since when Eq. (11) is solved for all r (including r = 0), a second order angular truncation error is made in approximating the source using a quadrature rule. For a pure absorber, ($\sigma_s = 0$) we would expect the exact origin flux. This is verified for a sample problem in Section III.

The nonuniformity of the angular error as a function of the spatial variable can, however, cause unexpected behavior in the spatial scalar flux profile. Namely, a nonphysical scalar flux depression can occur in the vicinity of the system origin. This anamoly is due to the discrete-ordinates approximation itself and no spatial difference scheme that converges to Eq. (6) can eliminate it. Note that since for $\sigma_s = 0$, the discrete-ordinates equations give the exact solution at the system origin, this behavior should actually be viewed as a nonphysical *rise* in the scalar flux as one proceeds away from the origin. A demonstration of this anamolous behavior, for a sample vacuum problem, is provided in the Appendix. Numerical results in Section III indicate that this class of depressions can also occur when $\sigma_s > 0$.

B. Spatial Differencing

1. Diamond Difference Equations

We next impose a spatial mesh on the domain $0 \le r \le R$ and let the mesh edges be denoted by $r_{I+1/2}$ and $r_{I-1/2} < r_I < r_{I+1/2}$. For simplicity,

we let r_i be the midpoint of the interval and all mesh intervals, Δr, be equal. We denote

$$\psi_{mi} = \psi(r_i, \mu_m) \qquad (20)$$

$$m = 1, 2, \ldots, M$$
$$i = 1, 2, \ldots, I.$$

The particle balance equation is derived by operating on Eqs. (6) with

$$\frac{4\pi}{V_i}\int_{r_{i-1/2}}^{r_{i+1/2}} dr r^2,$$

with V_i the spherical shell volume, and making suitable approximations. Making the usual assumption that spatially dependent cross sections are approximated by suitable averages, this equation is

$$\frac{\mu_m}{V_i}\left(A_{i+1/2}\psi_{mi+1/2} - A_{i-1/2}\psi_{mi-1/2}\right)$$

$$+ \frac{\left(A_{i+1/2} - A_{i-1/2}\right)}{2V_i W_m}\left(\alpha_{m+1/2}\psi_{m+1/2i} - \alpha_{m-1/2}\psi_{m-1/2i}\right) + \sigma_i\psi_{mi} = S_{mi} \qquad (21)$$

$$m = 1, 2, \ldots, M$$
$$i = 1, 2, \ldots, I$$

with $A_{i\pm1/2}$ the spherical shell areas at $r_{i\pm1/2}$. For $\mu_m > 0$, for example, $\psi_{mi-1/2}$ is known from the solution in the previous spatial cell while $\psi_{m-1/2i}$ is known for the previous angular cell. Thus, coupled with Eqs (21) and (10), we require another set of relationships, the spatial-diamond equations

$$\psi_{mi} = \frac{1}{2}\left(\psi_{mi+1/2} + \psi_{mi-1/2}\right) \qquad (22)$$

$$m = 1, 2, \ldots, M$$
$$i = 1, 2, \ldots, I.$$

The spatial difference approximation to Eq. (11) is obtained by setting $\psi_{m-1/2} = 0$, and $\psi_{m+1/2i} = \psi_{mi}$ in Eq. (21) and using Eqs. (8) (22) yielding

$$\frac{-\left(A_{i+1/2} + A_{i-1/2}\right)}{2V_i}\left(\psi_{1/2,i+1/2} - \psi_{1/2,i-1/2}\right) + \sigma_i\psi_{1/2,i} = Q_i \qquad (23)$$

$$i = 1, 2, \ldots, I.$$

In Eq. (23), we have set $\mu_m = -1$. Equations (23) and (22), the latter for $m = 1/2$, are the starting direction diamond-difference equations.

It is important to note how the origin point is handled in the process of solving Eqs. (10), (21), (22), and (23). One first solves the starting

direction diamond-difference equations proceeding from the outer boundary to the origin. One then solves Eq. (21), for m = 1 corresponding to the most negative value for μ_m. Again the solution process proceeds from the outer boundary to the origin. At the origin a, so called, reflection condition is normally used. That is, the direction corresponding to spectral reflection (in this case the direction of most positive μ_m) is solved next and the origin value of the angular flux for this direction is set equal to that just calculated for the m = 1 direction. The calculation then proceeds outward for this direction, $\mu = \mu_M$. One then solves in the next most negative direction and proceeds in an analogous way. Note that using this procedure, the origin angular flux at μ_{M-m+1}, m = 1, 2,..., M/2 is always set equal to the corresponding origin flux at μ_m, m = 1, 2, ...,M/2. All origin angular flux values, however, are not equal to one another, in violation of Eq. (13b).

To analyze the truncation errors in Eq. (21), we use the expansion

$$\psi_{mi\pm1/2} = \psi_{mi} \pm \frac{\Delta r}{2}\frac{\partial\psi}{\partial r}\bigg|_{im} + \frac{\Delta r^2}{8}\frac{\partial^2\psi}{\partial r^2}\bigg|_{im} + \ldots \tag{24}$$

We insert Eq. (24) into Eq. (21) yielding

$$\frac{\mu_m}{2V_i}(A_{i+1/2} + A_{i-1/2})\frac{\partial\psi_m}{\partial r}\bigg|_{r=r_i} + \frac{(A_{i+1/2} - A_{i-1/2})}{2V_i}(1-\mu_m^2)\frac{\partial\psi}{\partial\mu}\bigg|_{\mu=\mu_m} + \sigma_i\psi_{mi} = S_{mi} + \ldots \tag{25}$$

Since

$$\frac{(A_{i+1/2} + A_{i-1/2})\Delta r}{2V_i} = 1 + O\frac{\Delta r^2}{r^2} \tag{26}$$

We have a spatial truncation error of $O\,\Delta r^2/r^2$ added to a $O\,\Delta r^2$ error resulting from Eq. (22).

Equation (21), then has a nonuniform spatial truncation error. We again seek the truncation error at the origin. Equation (21) for the origin cell, ($r_{i+1/2} = \Delta r$), is

$$3\mu_m\psi_{m\,1/2} + \frac{3}{2w_m}(\alpha_{m+1/2}\psi_{m+1/2,1} - \psi_{m-1/2}\psi_{m-1/2,1}) + \sigma_1\psi_{m1}\Delta r = S_{m1}\Delta r \tag{27}$$

$$m = 1, 2, \ldots, M .$$

But, using Eqs. (8), (22), and

$$\psi_{m\,3/2} = \psi_{m1} + O\Delta r, \tag{28}$$

we obtain

$$\psi_{m+1/2,1} = \psi_{m-1/2,1} + O\Delta r. \tag{29}$$

Thus, we expect a first order error in the origin flux. A similar analysis yields a first order error in the starting direction origin flux as well. This first order error is demonstrated in the numerical results.

This first order error can cause a flux depression at the origin quite independent from that due to the discrete-ordinates approximation. This latter class of depressions is much more common but eventually disappears as the spatial mesh is refined.

2. Alcouffe-Miller Correction

R. E. Alcouffe and W. F. Miller, Jr.[7] have recently developed a correction that yields second-order spatial truncation errors for all values of r, including r = 0. The correction entails first changing the starting direction difference equations. We note from Eq. (11) that the starting direction equation is precisely the slab transport equation. In lieu of the traditional differencing, given by Eq. (23), we use slab geometry differencing.[1] Operating on Eq. (11) with

$$\frac{1}{\Delta r}\int_{r_{i-1/2}}^{r_{i+1/2}} dr$$

and approximating the cell average flux with the cell center flux yields

$$\frac{-(\psi_{1/2i+1/2} - \psi_{1/2i-1/2})}{\Delta r} + \sigma_i\psi_{1/2i} = S_i . \tag{30}$$

In addition to Eq. (30) we use the diamond equation

$$\psi_{1/2i} = 1/2(\psi_{1/2i+1/2} + \psi_{1/2i-1/2}). \tag{31}$$

By Taylor expansion, it is straightforward to show that the truncation error is $O\Delta r^2$. This is a decided improvement over the truncation error of Eq. (23), $O\Delta r^2/r^2 + O\Delta r^2$.

In the Alcouffe-Miller correction, Eqs. (30) and (31) are used to sweep the mesh from r = R to r = 0. Then at the origin, the flux is set equal to $\psi_{1/2,1/2}$ for all m.

$$\psi_{m,1/2} = \psi_{1/2,1/2} \qquad m = 1, 2, \ldots, M.$$

This assures an isotropic origin flux and a second-order spatial truncation error everywhere. Then for i = 1 and m = 1, 2, ..., M/2, the spatial diamond equation , Eq. (22), is not needed and is not invoked.

III. NUMERICAL RESULTS AND CONCLUSIONS

To demonstrate our results, we solve a simple homogeneous medium problem with $\sigma = 1.0$, $\sigma_s = 0.0$, $Q = 1.0$ everywhere, $N = 4$, vacuum boundary conditions, $R = 4.0$, and for various values of Δr. In Table 1 we see the second order origin flux with the Alcouffe-Miller correction as opposed to the first-order error given by the traditional approach. The errors are calculated using the exact origin flux .98168. We demonstrate elimination of the flux dip with this latter method by also tabulating the edge flux adjacent to the origin ($r = \Delta r$). For the fine mesh case, changing N to 8 or 16 does not alter the origin flux indicating that for a pure absorber, the angular truncation error at the origin goes to zero.

When σ is changed to .01, the depression in the flux persists, as the mesh is refined even with the Alcouffe-Miller correction indicating that the depression is due to the discrete-ordinates approximation itself.

In conclusion, we have shown that a class of flux depressions may appear due to the discrete ordinates approximation and cannot be eliminated with a convergent spatial difference scheme. The more common class of depressions, due to spatial differencing, may be eliminated in most cases by invoking the simple Alcouffe-Miller correction. This correction also yields second-order fluxes for any fixed r. The first class of depressions appears to be important only when σ is quite small.

TABLE 1

SAMPLE PROBLEM DEMONSTRATING DIAMOND DIFFERENCING

No. of Equal Mesh Int.	Origin Scalar Flux		Scalar Flux at $r = \Delta r$		% error in origin flux	
	DD	AM	DD	AM	DD	AM
2	.86165	1.00000	1.1861	1.1450	12.23	1.87
4	.95899	.98765	1.0112	1.0056	2.31	.61
8	.98001	.98320	.98106	.98113	.17	.15
16	.98053	.98207	.98139	.98151	.12	.04
32	.98103	.98178	.98154	.98163	.07	.01
64	.98134	.98171	.98162	.98167	.04	--

DD - Diamond Difference
AM - Diamond Difference with Alcouffe-Miller correction

APPENDIX

We consider the problem of a spherical vacuum with a constant, isotropic source extending to r = R and two angular directions (M = 2). The solution to Eq. (11), for the starting direction flux, is

$$\psi_{1/2}(r) = Q(R - r). \tag{A-1}$$

Using the Gaussian Quadrature, we find the flux for the direction $\mu_1 = -1/\sqrt{3}$ by solving the equation (obtained from Eq. (6)),

$$\frac{-1}{\sqrt{3}r^2}\frac{d(r^2\psi_1)}{dr} + \frac{2a_{3/2}}{rw_1}\psi_1(r) = Q + \frac{a_{3/2}}{rw_1}\psi_{1/2}(r) . \tag{A-2}$$

In deriving Eq. (A-2) we have used Eq. (10). Simplifying and using Eq. (8), we obtain

$$\frac{d\psi_1}{dr} - \frac{2\psi_1}{r} = -\sqrt{3}\,Q - \frac{2}{r}\psi_{1/2} . \tag{A-3}$$

Using an integrating factor of r^{-2} we can write Eq. (A-3) as an integral equation:

$$\psi_1(r) = \int_r^R dr'\left(\sqrt{3}\,Q\left(\frac{r}{r'}\right)^2 + 2\,\frac{r^2}{r'^3}\,\psi_{1/2}(r')\right) . \tag{A-4}$$

Inserting Eq. (A-1) into Eq. (A-4) yields

$$\psi_1(r) = Q\left\{R - (2 - \sqrt{3})\,r - (\sqrt{3} - 1)\,\frac{r^2}{R}\right\} . \tag{A-5}$$

Using the angular diamond-difference equation, Eq. (10),

$$\psi_{3/2}(r) = 2\psi_1(r) - \psi_{1/2}(r) = Q\left\{R - (3 - 2\sqrt{3})r - \frac{2}{R}(\sqrt{3} - 1)r^2\right\} . \tag{A-6}$$

We next consider the direction $\mu_2 = +1/\sqrt{3}$ and repeat the above procedure. Analogous to Eq. (A-4) is

$$\psi_2(r) = \int_0^r dr'\left(\sqrt{3}\,Q\,\frac{r'^2}{r^2} + 2\psi_{1/2}(r'\ \frac{r'^2}{r^2}\right)$$

$$= Q\left\{R + \frac{1}{3}(5\sqrt{3} - 6)r - (\sqrt{3} - 1)\,\frac{r^2}{R}\right\} . \tag{A-7}$$

Now from Eq. (9)

$$\phi(r) = \frac{1}{2}\left(\psi_1(r) + \psi_2(r)\right) = Q\left\{R + \frac{2}{\sqrt{3}}(2 - \sqrt{3})\,r - (\sqrt{3} - 1)\,\frac{r^2}{R}\right\} . \tag{A-8}$$

It is clear from Eqs. $\{(A-5) - (A-8)\}$ that the flux is isotropic at the origin. However, note that

$$\frac{d\phi(r)}{dr} = 0 \text{ at } r = .211325\ R\ .$$

Also since

$$\frac{d^2\phi(r)}{dr^2} < 0$$

this is a maximum. Hence a scalar flux depression exists in the vicinity of the origin for this problem.

It is not difficult to show that the exact solution to this problem (the solution to Eq. (1)) is

$$\psi(r,\mu) = S\left(r\mu + \sqrt{R^2 - r^2(1-\mu^2)}\right)\ .$$

Then

$$\phi(r) = \frac{1}{2}\int_{-1}^{1} d\mu\, \psi(r,\mu) = \frac{S}{2}\left\{R + \frac{R^2-r^2}{2r}\ \ell n\left(\frac{R+r}{R-r}\right)\right\}\ .$$

Note that now

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=0} = 0$$

and

$$\left.\frac{\partial^2\phi}{\partial r^2}\right|_{r=0} < 0\ ,$$

so that there is no depression in the exact scalar flux at the origin. Note also that the discrete-ordinates solution is exact at the origin.

REFERENCES

[1]K. D. Lathrop, "DTF-IV, a FORTRAN-IV Program for Solving the Multigroup Transport Equation with Anisotropic Scattering," Los Alamos Scientific Report, LA-3373 (1965).

[2]W. W. Engle, Jr., "A Users Manual for ANISN," K-1693, Union Carbide Corporation (1967).

[3]T. R. Hill, "ONETRAN: A Discrete-Ordinates Finite Element Code for the Solution of the One-Dimensional Multigroup Transport Equation," Los Alamos Scientific Report, LA-5990-MS (1975).

[4]W. H. Reed and K. D. Lathrop, "Truncation Error Analysis of Finite Difference Approximations to the Transport Equation," Nucl. Sci. Eng., 41, 237-248 (1970).

[5]B. G. Carlson and K. D. Lathrop, "Transport Theory--The Method of Discrete Ordinates," in Computing Methods in Reactor Physics, Greenspan, Kelber, and Okrent, Eds. (Gordon and Breach, New York, 1968), Chap. III.

[6]E. Isaacson and H. B. Keller, Analysis of Numerical Methods (John Wiley and Sons, New York, 1966).

[7]R. E. Alcouffe and W. F. Miller, Jr., "Improvements in Spherical Geometry Discrete Ordinates Solutions at the System Origin," Nucl. Sci. Eng. (to be submitted).