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# LATTICE GAS HYDRODYNAMICS IN TWO AND THREE DIMENSIONS 

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#### Abstract

Hydrudraamical phenomena can be simulated by discrete lattice gam models obeing cellular automata rales (U. Frisch, B. Haslacher, and Y. Pomean, Phys. Rev. Lett. 58, 1606, (1098); D. d'Homièren, P. Lallemand, and U. Frisch, Europhys. Lett. 2, 291, (1986)). It is here showa for a class of D-dimensional lattice gas models how the macrodynamical (large-scale) equations for the densities of microscopieally conserved quantities can be syitematically derived from the undertying exact "miciodynamica" Boolean equations. With suitable restrictions on the crystallographic symmetries of the lattice and after proper limits are taken, various standard duid dynamical equations are obtained, includingap in the ivcompressible Navien-Stokes equations in two and three dimensions. The triolppart coefficients appearing in the macrodynamical equations are obtained using variants of ductuatica-dissipation and Boltumann formalisms adapted to fully discrete situations.


## 1. Introductlon

It is known that wind or water tunnels can be indifferently used for testing low Mach number Lows, provided the Reynolds aumbers are identical. Indeed, two luids with quite different microcopic structures can bave the same macroscopic behaviour. This is because the form of the macroscopic equations is entirely governed by the microscopic conservation laws and symmetries. The values of tie transport coefficjents, such as the viscosity may depend on the details of the microphysics. Still, two fows with similar geometries and identical nondimensionalized values for the relevant transport coeflicients are related by similarity.

Recently, such observations have led to a new simulation strategy for luid dynamics: Actitimus microworld models obeying discrete cellolar automata rules have been found, such that two and three-dimensional Guid dynamica are recovered in the macroscopic limit. ${ }^{(1,2)}$ Cellular antomata, introduced by von Neumann and Ulam, ${ }^{(3)}$ are constituted of a lattice, each site of which can have a linite number of states (usually coded by Boolean variables); the automaton evolves in discret: steps, the sites being simultaneously updated by a deterministic or nondeterministic rale. Typically, only a finite nomber of neighbours are involved in the updating of any site. A very popalar example is Conway's Game of Life. ${ }^{(4)}$ In recent years there has been a renewal of intereat in this subject (see e.g. Ref. 5-7), in particular since cellular automata can be implemented in masstvely parallel hardware. ${ }^{(a, 9)}$

The class of cellular automata used for the simulation of fluid dynamics are here called "lattice gas models". Historically, they emerged from attempts to construct discrete models of Auids with varying motivations. The am of Molecular Dyamics is to simulate the real microworld in order for example to calculate transport coefficients; one concentrates mass and momentum in discrete pasticles with continuons time, positions and velocities and arbitrary interactions. ${ }^{(10-18)}$ Discrete velocity models, introduced by Broadwell ${ }^{(14)}$ (see also Rets. 15-19), have been used mostly to unlersiand rarefied gas dyamics; the velocity set is now Inite; apace and time are atill continoous and the evolution is probar bilistic, being soverned by Boltzmand acattering rales. The tirst lattice gas model (now known a HPP) with diacrete time, positions and velocities and fully determinintic evolotion was introduced by Hardy, de Pazeis and Pomean, ${ }^{(90,91)}$ soe abo related work in Rus. 22. The EPF model, a presentation of which will be postponed to section 2, was introduced to anatyze, in as simple a tramewort ar posible, fundamertal queations in Statistical Mechanica, such a ergodirity and the divergence of transpart coemelents in tw dimenajons. (11) The BPP model leads to sonnd weves, which have been observed in simulations on the MT cellular automaton machine. ${ }^{(8)}$ The diffintiles of the HPP model In coping with tuil liaid dymamics were overcome by Frisch, Basslechas an 1 Pomeau ${ }^{(1)}$ for the two-dimensional Navier-Stokes equationse; models adapted to the three-dimensional case were introdoced by dThumieres, Lallemand and Fisch. ${ }^{(9)}$ This han ied to rapid devel opment of the subject. (93-48) These papers are mostly concerned with lattice gas models leading to the Navier-Stokes equations. A namber of other probleins are known to te amenable to Lattice gas models. Dynamical Ising models with suand waves, ${ }^{(40)}$ Buoy. ancy effects, ${ }^{(49)}$ Seismic P-waves, ${ }^{(49)}$ Magnetohydrodyamics, ${ }^{(49-81)}$ Reaction-Diffusion models, ${ }^{(32-84)}$ Interface and Combustion phenomenc, ${ }^{(39.30)}$ Burgers' modei. ${ }^{(39)}$

The alm of this paper is to present in detail and without onnecessary restrictions
the theory leading trom a simple class of D-dimensional "one-speed" lattice gas models to the continumm macroscopic eqnations of Auid dynamica in two and three dimensions. The extension of our approach to malti-speed models, lnehding for example eero-velocity "rest-particles", to quite staightforward; there will be occasional brief comments on such models. We now outline the paper in some detail while emphasizing some of the key steps. Note that some knowledge of Nonequilibrium Statistical Mechanics is helpful for reading this paper, but we have tried to make the paper self-contained.

Section 2 is devoted to varions lattice gas models and their symmetries. We begin - ith the simple fully deterministic HPP model (square lattice), we then go to the FHP model (trian;alar lattice) which may be formulated with deterministic or nondeterministic collision rules; Anally, we consider a grneral class of (usualiy) nondeterministic one-speed models containing the pseado-4D tace-centered-hypereabic (FCHC) model ased in three dimensions. ${ }^{(8)}$ In this section, we also introduce various abstract symmetry assumptions, which hold for all three models (HPP, FHP, and FCHC), and which will be very naseful in reducing the complexity of the subsequent algebra.

In section 3 we introduce the "microdynamical equations", the Boolean equivalent of Hamilton's equations in ordinary Statistical Mechanics. We then proceed with the probabilistic description of an ensemble of realizations of the lattice gas; at this level, the evolution is governed by a (discrete) Lionville equation for the probability distribation function.

In section 1 we show that there are equilibrium statiatical solations with no equal time correlations between sites. Uader some mildly restrictive assumptions, at each site, a Fermi-Dtrac diatibution is obtained for the mean popalations, which is noiversa, i.e. undependent of colliaion rales. Thin distribation is parametrized by the mean values of the collision invariants (usually, mase and momentam).

Locally, man and momentum are discrete, but their mean valoen, the density and mass current, can be toned contincously, just a in the "real world". Furthermore, apace and time cas be regarded as contmaons by conoldeding local equilibria, slowhy varying in apace and thme (section 8). The matching of these equilibria leads to macroscople PDE's for the conserved quartition.

The resalting "macrodymamieal equations", for the denality and mane current, are not in general invariant under arbitrary rotations. However, to section 0 we show that the relevant terms in the macrowcopic equations beeome botropic $a$ soon as the lattice gas has a sufifiently large cryotallographic symmetry groap (as ib the cace for the FBP and pseado-4 models, bit not for the EPP model).

When the seccary mpmetries bold, Auid dynamical equations are dertived in section 7. We consider varions limite involving large scales and times and amall velocities (compared to particle speed). In one limit we obtain the equations of scalar sound weves; in another llinit we obtaln the incompremible Navier-Stokee equations in two and three djmensions. It is noteworthy that Gallean inverinace, which den aot told at the microncopic level, is restored to these limits.

In section 8 we show bow to determine the viscosities of hattice gases. They can be expressed in terms of equilibrium apace-time correlation functions via an adaptation to lattice gases of Ductuation-dissipation relations. This io befe done with a view-pcint of
"noisy" by drodynamics, which also brings out the crossover pecalerities of two dimensions, namely a residual weak scale-dependence of transport coeflicients at large geales. Alterantively, Iuctuation-disipation relations can be obtained trom the Liouvilte equation with a Green-Kubo formalism. ${ }^{(11)}$ Folly explicit expressions for the viscosities can be derived vis the "Lattice Eoltemann Approximation", not needed for any earlier steps. This is a Anite-diference variant of the discrete-velocity Boltzmann approximation. The latter, which assumes continoous space and time variables, is valid only at low densities, while its lattice variant seems to capture most of the finite-density effects (with the exception of two-dimensional crossover effects). Further studies of the Lattice Boltrmann Approcimation may be fond in Ref 59 . Implications for the question of the Reynolds nomber are discussed at the end of the section.

Section 9 is the conclasion. Varions questions are left for the appendices: detailed technica' proof, inclasion of body forces, catalog of results for various FHP models, proof of an F-theorem for the Latice Boltemann Approximation (due to M. Hénon).

## 2. Determiniatic and nondeterminiatic lattice gan modele

### 2.1 The EPP model

Let us begin with a hearistic constroction of the HPP model ${ }^{(20-99)}$ Consider a two dimensional square lattice with anit lattic: constant as shown in Ig. 1. Partices of unit mass and unit speed are moving along the lattice links and are bocated at the nodes at integer times. Not more than one particie is to be fonnd at a gtven time and node, moring in a given direction (exclasion primeiple). When two and exactly two particlet arrive at a node trom opporite diroctions (head-on collisions), they immetiattly leave the node in the two other, previonsly anoccupied, directions (1). 2). These determinitic collision laws obviously conserve mass (particle namber) and momentum and are the only nontrivial odea with theae properties. Forthermore, they have the same discrete lovariance group as the lattice.

The abore defatioo can be formalized an follows. We take an $L$ by $L$ aquare lattice, periodically wrapped around (a nonemential asomption, made for corvenieace). Eventoaly, we ahall lot $L \rightarrow \infty$. At ench node, labelled by the dincrete rector $r_{*}$, there are four celt labedled by an index $i$, dellaed moduto tonr. The celb are saociated to the anit veetont $e_{i}$ conaecting the aode to tis four nearest nelighboun (i increan countercjoctwise). Each cel ( $\left.r_{*}, i\right)$ has two states coded with a Boolean ratable $n_{1}\left(r_{*}\right)=1$ : "occupied" and $n_{1}\left(r_{*}\right)=0$ : "onoccupied". A cellalar antomaton apdating rale is defned on the Sooleas Ield $n .=\left\{n_{i}\left(r_{*}\right), i=1, \ldots, 4, r_{*} \in\right.$ Lattice $\}$. It has two steps. Step one is collision: at earh aode the four-bit states $(1,0,1,0)$ and $(0,1,0,1)$ are exchanged; all other stacs are left anchanged. Step two is proparation: $n_{i}\left(r_{*}\right) \rightarrow n_{i}\left(r_{*}-c_{i}\right)$. Thin twontep rale io applied at each tonteger time $i_{*}$. As example of implementation of the rete, to wich arrowistand for cell-occnpation, lo showa to Ap. In and lb.

Collisions in the BPP model conserve mass and momentum tocally, whereas propa gation conserves them globally (actually, along each lattice line). If we atribute to each particle a kinetic energy $1 / 2$, the total kinetic enerry is also conserved. Enerfy conservar tion is however indistingoishable trom mess conservation and wil not play any dynamical
role. Models haviag an enesty conservation lav independent of mass conservation will not be considered in this paper (see Rets. 2,27).

The dynamies of the EPP model are invariant under all discrete tranaformations that conserve the square lattice: discrete translations, rotations by $\pi / 2$, mirror symmetries with reapect to a lattice line. Furthermore, the dynamics are invariant under duality, that is exchange of 1's and 0's (particles and holes).

## 4. 8 The FEP modele

The FHP models I, II, and III (see below), introduced by Frisch, Hasslacher and Pomean ${ }^{(1)}$ (see also Refs. 25-30,32,35-42,14) are variants of the BPP model with a larger invariance group residing on a triangular luttice with anit lattice constant (18. 3). Each node is now connected to its six neighboars by nuit vectors $c_{i}$ (with $i$ delned modulo six) and is thas endowed with a six-bit state (or seven, cl. below). Updating invoives again propagation (deined as for HPP) and collisions.

In constructimg collision rules on the triangular lattice, we must pay attention to the Eollowing
Determiniatic vs. mondeterministic rules. For a head-on collision with occupied "input channela" $(i, i+3)$, there are two possible pairs of occapied "output channelo" guch that mass and momentum are conserveil, namely ( $i+1, i+1$ ) and ( $i-1, i-4$ ) (see fig. 4a). We can decide always to make th: same choice; we thea bave a deterministie model, which is chiral, that is oot invariant under mirror-nymmetry. Alternatively, we can make a nondeterministic (random) choice, with equal probabilities to restore mirror-symmetry. Finally, we can make a preado-randon ehoice, dependent, for example, on the parity of a tim: or space fudex.
Spurions conservation laws. Bead-on collisions conserve, haddition to tota particie sumber, the diference of particle aumbers in any pair of opposite direetions $(i, i+8)$. Thus, headon collisions on a sriaggular lattice consarve a total foor scalar quantities. This means thet ba addition to mans and momentmm conservation there is a spurions coasenration law. The largeseale dyramics of such a model will difer drantically trom ordinary hydrodysamich, askes the spasious comaurvation law in removed. One way to achieve this Is to introduce tripis collicions $(i, i+2, i+1) \rightarrow(i+1, i+2, i+8)$ (mee is. ib).

Several model an be construetod on the trinagaler latice. Ite simplest set of collinios ralee with no spmione conservithon law, which will be ealled FEip. I, bavolves
 under duality (particlo-hole exchasge), bat can be made so by betusion of the duals of the head-on colliaioss (eve If. Ac). Fimelly, the set of collision rules can be aaturated (exhanoted) by inctesion of head-oa collisions with a "upectator", (sa) that in, a particle which remains anafected in a collimom; Ag. Id in an wample of a headon colision with a spectator.

The modd FXPP-I is a seven bit variant of Flip-I beluding a eeso-velocity "rentu particle", the additional collinion rules of A. te, and variants of the headoon and triple collisions of ays ta and tb with a spectator rest-parikic. Bhary collinion on rest-particles remove apurious coaservations, and do su more efterently at low denaities thas triple collisions. Finally, model FHP-III is a collision-aaturated version of FHP-II. ${ }^{(88)}$ For simplicity we have chosen not to cover the theory of models with rest-particles in detail.

The dyamics of the FHP models are buvariant sader all diacrete tranoformations that conserve the triangular latice discrete trasslations, rotation by $\mathrm{s} / \mathrm{\$}$, misror-ypmmetries with respect to a lattice line (except for the chiral varianto).

### 8.9 The face-centared-hypercebte 4-D and the pacudo-4-D modete

Three dimensional regular lattices do not heve enough symmetry to ensure macroscopic isotrogy. ${ }^{(1.2 .36)}$ A suitable foor-dimensional model has been introdoced by d'Hamieres, Lallemand and Frisch. ${ }^{(9)}$ Its basic lattice is the lace-centered-hyperen bir (FCHC), defined as the set of signed integers $\left(x_{1}, x_{1}, x_{3}, x_{4}\right)$ such that $x_{1}+x_{3}+x_{3}+x_{4}$ beven. Eech node is consected via links of length $c=\sqrt{2}$ to 24 nearest neistboars, having two coordinates difering by $\pm 1$. Thas the FCHC model has 24 -bit states. The 24 possible velocity vectors are again denoted $e_{i}$; for the index $i$ there is no outstanding ordering and we ahall leave it anspecified. Propagetion on the FCHC lattice gas in as asual. Collinion rules should conserve mass and four-momentum while avoiding sparions conservations. This can be achieved with jast binary collisions, bat betrer strategies are known. Nondeterministic rales involving transition probabilitias are needed to ensure that the collisions and the Latice have the same invaringce gronp (precise deinitions are pootponed to seetion 2.4).

The allowed transformations of the FCBC model are discrete tranalations and those isometries generated by permotations of coordinates, reveral of one or several coordinates and symmetry with reapect to the hyperplane $x_{1}+x_{9}+x_{3}+x_{4}=0$.

The preado-4-D model ${ }^{(9)}$ may be riewed as the three-dimensional projection of an FCHC model with wit periodicity in the $\varepsilon_{4}$-direction (see fs. 5). It residen on an ordinary cubic lattice with unit lattice ccostent. The full forndimensional discrete velocity atructure is preserved a follows. There is one communication chanad to the 12 next nearest neighboars (correspooding to the twelve valocity vectors soch that $v_{1}$, the fourth component of the reloctry, ranhea) and there are two communication channela to the six aearest-neighboars (corresponding reapectively to velocities with $v_{4}= \pm 1$ ). During the propagation phace, particles with $v_{1}= \pm 1$ move to acerest zeighbour nodes, while particlee with $v_{4}=0$ move to sect-Reareat seidiborrs. The collibion strategy is the same as for the FCZC model, $s$ that fousmomentem is conserved. The fourth componeat in aot a opariousty conserved quantity, becanse, ha the incomprasible limit, it doen aot effectively couple beck to the other consarved quantitian. (2)

### 0.44 gesorral sleev of mondeternindetis molele

In mont of thla papar ve shell work with a clase of models (seaerally sondetermbistic) ancompacing all the above oso-speed models. The relevant common appects of all those modela are now Hoted: Thers in a reguin lattice, the nodes of which are connected to aearest aeighbouss throagh liaks of equal lagith; all velocity directions are fa some sense equivalent and the velocity set in tuveriant ongier reveral, at asch node thare in a cell asociated to esch posalble relocity which ean be occapled by coe particle at most; particles are madistinguishable; partiches are marched forvard by succeantrety applyise propagetion anc colloion roles; colinions are parely beal, tave the same funarinees as the velocity set and cooser. only mans and momeatum.

We aow give a more formal definition of these obespeed models a celloler antomata Let us begin with the ceometrical aspecti. We tate a D-dimensional Bravis Lattice $\mathcal{L}$ in $R^{D}$ of inite extension $O(i)$ in ell directions (eventually, $L-\infty$ ); the paition vector
$r_{*}$ of any aode of such a lattice is a linear combination with integer coefilicients of $D$ independent generating vectors. ${ }^{(30)}$ We furthermore assume that there exists a set of $b$ "velocity vectors" $c_{i}$ with components $c_{i a}(\alpha=1, \ldots, D),{ }^{1}$ having equal modulus $c_{1}$ the particle speed, such tibat:
(i) for any $r_{*} \in \mathcal{L}$, the set of the $r_{*}+c_{i}$ 's is the set of gearest neighbours of $r_{*}$;
(ii) any two nodes can be connected via a inite chain of nearest neighbours;
(iii) for any pair ( $c_{i}, c_{j}$ ), there exists an element in the "crystallographic" group $\mathcal{G}$ of isometries globally preserving the set of velocity vectors, which maps $c_{i}$ into $c_{j} ;$
(iv) for any velocity vector $c_{i}$, we denote by $\mathcal{G}_{i}$ the subgroup of $g$ which leaves $c_{i}$ invariant and thus leaves its orthogonal byperplane $\Pi_{;}$globalh invariant; we assume that (a) there is no non-vanishing vector in $\Pi_{i}$ invariant under all the elements of $g_{n}$, (b) the only linear transformations within the space $\Pi_{i}$ commuting with all the elements of $\mathcal{G}_{i}$ are proportional to the identity.

Now, we construct the automaton. To each node $r_{*}$, we attach a b-bit state $n\left(r_{*}\right)=$ $\left\{n_{i}\left(r_{*}\right), i=1, \ldots, b\right\}$, where the $n_{i}$ 's are Boolean variables. The updating of the "Boolean feld" $n($.$) involves two successive steps: collision, followed by propagation. We choose this$ particular order for technical convenience; after a large number of iterations it will beeome irrelevant which step was first. ${ }^{2}$ Propagation is defined as

$$
\begin{equation*}
n_{i}\left(r_{\star}\right) \rightarrow n_{i}\left(r_{\star}-c_{i}\right) . \tag{2.1}
\end{equation*}
$$

The spatial shiftiog by $c_{i}$ is performed on a periodically ${ }^{3}$ wrafped around lattice with $O(L)$ sites in any direction; eventually $L \rightarrow \infty$. Collinion is the simultaneons application at each node of nondeferministic transition rules trom an in-state $s=\left\{a_{i}, i=L_{1}, \ldots, b\right\}$ to an out-state $0^{\prime}=\left\{\rho_{i}^{\prime}, i=1, \ldots, b\right\}$. Each transition is assigned a probability $A\left(0 \rightarrow o^{\prime}\right) \geq 0$, normalized to one $\left(\sum_{\infty} A\left(0 \rightarrow a^{\prime}\right)=1 \forall 0\right)$, and depending only on 1 and $f^{\prime}$ and not on the node. The following assumptions are made.
(v) Conservation laws: the only collections of $b$ real numbers $a_{i}$ such that

$$
\begin{equation*}
\sum_{i}\left(a_{i}^{\prime}-a_{i}\right) A\left(s \rightarrow \theta^{\prime}\right) a_{i}=0, \quad V_{1}, s^{\prime}, \tag{2.2}
\end{equation*}
$$

ase linear combinatioms of 1 (for all $i$ ) and of $c_{i 1}, \ldots, c_{i D}$, i.e. are associated to mass and momeatom.
${ }^{1}$ In this paper Greek and Romas indices refer respectively to components and velocity labels. Summation over repeated Greek indices, but not Roman ones, is implicit.

2 For deterministic lattice gases, such as RPP, it is poosible to bring ont the reversibir ity of the apdatiog rule by defining the atate of the antomaton at ball-integer times, With particles located at the middle of links conneeting nearest-neighbour nodes; opdating then comprises kalf a propagation, followed by collinion, followed by another hat propasation. ${ }^{(20)}$

- Other boundary conditions at the lattice edge can also be nsed, for example windtungel" conditions. ${ }^{(25,24,26)}$
(vi) Invariance onder all isometries preserving the velocity set

$$
\begin{equation*}
A\left(g(o) \rightarrow g\left(0^{\prime}\right)\right)=A\left(s \rightarrow g^{\prime}\right), \quad \forall g \in \mathcal{G}, \quad \forall_{0}, s^{\prime} \tag{2.3}
\end{equation*}
$$

(vii) Semi-detailed balance

$$
\begin{equation*}
\sum_{0} A\left(0 \rightarrow 0^{\prime}\right)=1, \quad \forall 0^{\prime} . \tag{2.4}
\end{equation*}
$$

Various comments are now in order. Semi-detailed balance, also used in discrete velocity Boltrmand models, ${ }^{(16)}$ means that if before collision all states have equal probabilities, they stay so after collision. It is trivially satisfed when the collision rule is deterministic and one-to-one. There exists also a stronger assumption, detailed balance (that is $A\left(o \rightarrow s^{\prime}\right)=A\left(0^{\prime} \rightarrow s\right)$ ), which will not be needed here. The HPP, FHP, and FCHP lattice gases satisfy the above essumptions (i) through (iv). The proofs are given in Appendix A. The other assumptions ( $\mathbf{v}$ ) through (vii) hold by construction with the exception of the chiral versions of FHP. The latter do not satisty (vi) becanse the collision rules are not invariant under the mirror-symmetries with respect to velocity vectors.

The invariance assumptions introduced above have important consequences for the tramsformation properties of vectors and tensors. The following definitions will be ased. A tensor is said to be $g$-imvariant if it is invariant under any isometry in $g$. A set of $i$ dependent tensors $\left\{T_{i}=t_{i o_{1}} \alpha_{2} \ldots \alpha_{0}, i=1, \ldots, b\right\}$ is said to be $g$-invariant if any isometry in $\boldsymbol{G}$ changing $c_{i}$ into $c_{j}$, changes $T_{i}$ into $T_{j}$. Note that this is stronger than global invariance under the group 9 . The relocity moment of order $p$ is delned as $\sum_{i} c_{i a_{1}} c_{i a_{2}} \ldots c_{i a_{p}}$.

We now list the transformation properties following from $G$-invariance. The proofs are given in Appendix B.
P1 Parity-invariance. The set of velocity vectors is invariant ander space-reversal.
P2 Any set of i-dependent vectors $v_{i a}$, which is $g$-invariant, is of the form $\lambda e_{i a}$.
P8 Any set of i-dependeat tensors $f_{i a \beta}$, which in $g$-invariant, in of the form deiaci $+\mu \delta_{a \beta}$. P4 betropy of second onder tensory. Any $G$-invariant tensor $t_{a \beta}$ is of the form $\mu \delta_{\alpha \beta}$.
PS Any G-invariant thind order teasor vanishes.
P6 Velocity moments. Odd order velocity moments vanish. The second order velocity moment is given by

$$
\begin{equation*}
\sum_{i} c_{a} c_{i \beta}=\frac{b c^{2}}{D} \delta_{a \beta} \tag{2.5}
\end{equation*}
$$

There is, in genera, ao closed form expression for even order velocity moments beyond. second order, with the assumption made ap to this point (ef. also section 6).

## 3. Mierodynamics and probabllintic description

### 3.1 Mictodymanical equattore

It is possible to give a compact representation of the "microdynamics", describing the application of the updating rules to the Boolean feld. This is the cellular automaton analog of Hamilton's equations of motion in Classical Statististical Mechanics. We begin with the HPP lattice gas (section 2.1). Let $n_{i}\left(t_{*}, r_{\star}\right)$, as defined in section 2.1, denote the HPP Boolean feld at the discrete time $t_{\star}$. With $i$ labelling the four cells of an HPP node, the collision rule can be formulated as follows: If before, $\boldsymbol{i}$ and $\boldsymbol{i}+2$ are empty and $i+1$ and $i+3$ are occupied, then after, the opposite holds; if before, $i+1$ and $i+3$ are empty and $i$ and $i+2$ are occupied, then after, the opposite holds; otherwise, the content of cell $i$ is left anchanged. Thas, the updating of the Boolean field may be written

$$
\begin{equation*}
n_{i}\left(t_{\star}+1, z_{\star}+c_{i}\right)=\left(n_{i} \wedge \neg\left(n_{i} \wedge n_{i+2} \wedge \neg n_{i+1} \wedge \neg n_{i+3}\right)\right) \vee\left(n_{i+1} \wedge n_{i+3} \wedge \neg n_{i} \wedge \neg n_{i+2}\right), \tag{3.1}
\end{equation*}
$$

where the whole r.h.s. is evaluated at $t_{\star}$ and $r_{\star}$. The symbols $\Lambda_{1} V$, and $\neg$ stand for AND, OR, and NOT, respectively. It is known that any Boolean relation can be recoded in arithmetic form ( $\wedge$ becomes multiplication, $\rightarrow$ becomes one minus the variable, etc.). In this way we obtain

$$
\begin{equation*}
n_{i}\left(t_{*}+1, r_{\star}+e_{i}\right)=n_{i}\left(l_{\star}, r_{\star}\right)+\Delta_{i}(n) \tag{3.2}
\end{equation*}
$$

The "collision function" $\Delta_{i}(n)$, which can take the values $\pm 1$ and 0 , desribes the change in $n_{i}\left(t_{*}, r_{*}\right)$ due to collisions. For the HPP model, it depends onty on $i$ and on the set of $n$;'s at $t_{*}$ and $r_{*}$ denoted $n$. It is given by

$$
\begin{equation*}
\Delta_{i}(n)=n_{i+1} n_{i+3}\left(1-n_{i}\right)\left(1-n_{i+2}\right)-n_{i} n_{i+2}\left(1-n_{i+1}\right)\left(1-n_{i+3}\right) . \tag{3.3}
\end{equation*}
$$

Equation (3.2) (with $\Delta_{i}(n)$ given by (3.3)) will be called the microdynamical EPPP equation. It holds for arbitrary $i$ (modulo four), for arhitrary integer $t_{*}$, and for arbitrary $r_{\star} \in \mathcal{L}$ ( $\mathcal{L}$ designates the lattice).

It is easy to extend the microdynamical formalism to other models. For FHP-I (section 2.2), we find that the collision function may be written ( $i$ is now defined modulo six)

$$
\begin{gather*}
\Delta_{i}(n)=\xi_{i+c_{+}} n_{i+1} n_{i+1}\left(1-n_{i}\right)\left(1-n_{i+2}\right)\left(1-n_{i+s}\right)\left(1-n_{i+b}\right) \\
+\left(1-\xi_{i++_{*}}\right) n_{i+2} n_{i+s}\left(1-n_{i}\right)\left(1-n_{i+1}\right)\left(1-n_{i+s}\right)\left(1-n_{i+1}\right) \\
-n_{i} n_{i+s}\left(1-n_{i+1}\right)\left(1-n_{i+2}\right)\left(1-n_{i+1}\right)\left(1-n_{i+s}\right)  \tag{3.4}\\
+n_{i+1} n_{i+s} n_{i+s}\left(1-n_{i}\right)\left(1-n_{i+2}\right)\left(1-n_{i+1}\right) \\
-n_{i} n_{i+3} n_{i+1}\left(1-n_{i+1}\right)\left(1-n_{i+3}\right)\left(1-n_{i+s}\right) .
\end{gather*}
$$

Here, $\xi_{1, \ldots}$ denotes a time- and site-dependent Boolean variable which takes the value one when head-on colliding particles are to be rotated couterclockwise and zero otherwise (remember, that there are two possible outcomes of such collisions). For the theory, the simplest is to assign the two values equal probabilities and to assume all the छ's to be independent. In practical implementations other choices are often more convenient.

We now give the microdynamical equation for the general class of nondeterministic models defned in section 2.4. Propagation is as before. For the collision phase at a given node, it is convenient to sum over all $2^{b}$ in-states $0=\left\{a_{i}=0\right.$ or $\left.1, i=1, \ldots, b\right\}$ and $2^{b}$ out-states $\rho^{\prime}$. The nondeterministic transitions are taken care of by the introduction at each time and node and for any pair of siates ( $f, \mathrm{~s}^{\prime}$ ) of a Boolean variable $\xi_{\mathrm{n}, \mathrm{\prime}}$ (time and apace labels omitted for couciseness). We assume that

$$
\begin{equation*}
\left\langle\xi_{00^{\prime}}\right\rangle=A\left(0 \rightarrow 0^{\prime}\right), \quad \forall 0, o^{\prime}, \tag{3.5}
\end{equation*}
$$

Where $A\left(\otimes \rightarrow \theta^{\prime}\right)$ is the transition probability iutroduced in section 2.4; the angular brackets denote averaging. We also assume that

$$
\begin{equation*}
\sum_{0} \xi_{00}=1, \quad \forall s . \tag{3.5}
\end{equation*}
$$

Since the $\xi$ 's are Boolean, eq. (3.6) means that, for a given in-state $s$ and a given realization of $\xi_{\text {on }}$, one and only one out-state $g^{\prime}$ is obtained. It is now clear that the microdvnamical equation can be written as

$$
\begin{equation*}
n_{i}\left(t_{\star}+1, r_{\star}+c_{i}\right)=\sum_{0, \phi^{\prime}} d_{i}^{\prime} \xi_{00} \prod_{j} n_{j}^{\theta_{j}}\left(1-n_{j}\right)^{\left(1-\sigma_{j}\right)} . \tag{3.7}
\end{equation*}
$$

The factor $f_{i}^{\prime}$ ensures the presence of a particie in the cell $i$ after the collision; the various factors in the product over the index $j$ ensure that before the collision the pattern of $n j$ 's matches that of aj's. Using (3.7) and the identity

$$
\begin{equation*}
\sum_{j} n_{i} \prod_{j} n_{j}^{\theta_{j}}\left(1-n_{j}\right)^{\left(1-\theta_{j}\right)}=n_{i}, \tag{3.8}
\end{equation*}
$$

we can rewrite the microdynamical equation in a form that brings ont the collision function

$$
\begin{align*}
& n_{i}\left(f_{x}+1, r_{*}+c_{i}\right)=n_{i}+\Delta_{i}(n) \\
& \Delta_{i}(n)=\sum_{0, o^{\prime}}\left(o_{i}^{\prime}-\theta_{i}\right) \xi_{00^{\prime}} \prod_{j} n_{j}^{\theta_{j}}\left(1-n_{j}\right)^{\left(1-\theta_{j}\right)} . \tag{3.9}
\end{align*}
$$

In the sequeì it will often be useful to have a compact notation. We deflne the collision operator

$$
\begin{equation*}
C: n_{i}\left(\mathbf{r}_{\star}\right) \mapsto n_{i}\left(\mathbf{r}_{\star}\right)+\Delta_{i}\left(n\left(\mathbf{r}_{*}\right)\right), \tag{3.10}
\end{equation*}
$$

the streaming operator

$$
\begin{equation*}
S: n_{i}\left(p_{k}\right) \mapsto n_{i}\left(p_{*}-c_{i}\right), \tag{3.11}
\end{equation*}
$$

and the evolution operator, the composition of the latter

$$
\begin{equation*}
\mathcal{C}=S \circ C . \tag{3.12}
\end{equation*}
$$

The entire updating can now be written as

$$
\begin{equation*}
n\left(t_{*}+1_{1} .\right)=\varepsilon n\left(t_{*} \cdot\right), \tag{3.13}
\end{equation*}
$$

where the point in the second argument of the $n$ 's stands for all the space variables.
An interesting property of the microdynamical equation, not shared by the Hamilton equations of ordinary Statistical Mechanics, is that it remains meaningful for an infnite lattice, since the updaing of any given rode involves only a finite number of neighbours.

### 9.2 Goneervation relatione

Conservation of mass and momentum at each node in the collision proress can be expressed by the following relations for the collision function:

$$
\begin{array}{ll}
\sum_{i} \Delta_{i}(n)=0, & \forall n \in\{0,1\}^{b} \\
\sum_{i} c_{i} \Delta_{i}(n)=0, & \forall n \in\{0,1\}^{b} \tag{3.15}
\end{array}
$$

This implies important conservation relations for the Boolean field

$$
\begin{align*}
\sum_{i} n_{i}\left(t_{*}+1, r_{*}+c_{i}\right) & =\sum_{i} n_{i}\left(t_{*}, r_{*}\right)  \tag{3.16}\\
\sum_{i} c_{i} n_{i}\left(t_{*}+1, r_{*}+c_{i}\right) & =\sum_{i} c_{i} n_{i}\left(t_{*}, r_{*}\right) \tag{3.17}
\end{align*}
$$

### 3.3 The Liouville equation

We now make the transition, traditional in Statistical Mechanics, trom a deterministic to a probabilistic point of view. This can be obscured by the fact that some of our models are already probabilistic. So, let us assume for a while that the evolution operator is deterministic and invertible (as is the case for hPP).

Asouming that we have a Anite latice, we define the phase space $\Gamma$ so the set of all possibie ascigments $o()=.\left\{n_{i}\left(r_{*}\right), i=1, \ldots, b, r_{*} \in \mathcal{L}\right\}$ of the Booleas feld $n_{i}\left(r_{*}\right)$. $A$ particular assignment of the Boolean feld will be called a confgaration. We now consider at time $f_{1}=0$ an ensemble of initial conditions, each endowed with a probability $P(0,0()) \geq$.0 , such that

$$
\begin{equation*}
\sum_{0(\cdot) \in T} P(0,0(.))=1 \tag{3.18}
\end{equation*}
$$

We let each confguration in the ensemble evolve according to the automaton updating rule, i.e. with the evolution operator $\mathcal{E}$ of eq. (3.13). The latter being, here, invertible, conservation of probability is expressed as

$$
\begin{equation*}
P\left(t_{*}+1,0(.)\right)=P\left(t_{\infty}, \varepsilon^{-1} d(.)\right) \tag{3.19}
\end{equation*}
$$

This equation is clearly the analog of the Liouville equation of Statistical Mechanics, and will be given the same name. Alternatively, the Liouville equation can be writien

$$
\begin{equation*}
P\left(t_{\star}+1, S o(.)\right)=P\left(t_{\star} \mathcal{C}^{-1} s(.)\right) \tag{3.20}
\end{equation*}
$$

To derive this we have used (3.12) and put the streaming operator in the l.h.s., a form that will be more convenient subsequently.

In the nondeterministic case, we must enlarge the probability space to include, not only the phase space of initial conditions, bat the space of all possible choices of the Boolean variables $\xi\left(00^{\prime}\right)$, which at each time and each node select the unique transition from a given in-state a ( $d$. section 3.1). Since the $\xi$ 's are independently chosen at esch time, the entire Boolean feld $n\left(t_{*}\right.$, ) is a Markov procese (with deterministic rulea, this process is degenerate). What we shall continue to call the Liouville equation, is actually the Chapman-Kolmogorov equation for this Markov process, namely

$$
\begin{equation*}
P\left(t_{\star}+1, S s^{\prime}(.)\right)=\sum_{\theta(.) \in \Gamma} \prod_{r-\in L} A\left(\Delta\left(r_{*}\right) \rightarrow \theta^{\prime}\left(r_{*}\right)\right) P\left(t_{*}, \theta(.)\right) . \tag{3.21}
\end{equation*}
$$

This equation just expresses that the probability at $t_{*}+1$ of a given (propagated) configuration $\rho^{\prime}($.$) is the sum of the prohabilities at L_{4}$ of all poosible original confgurations 0 (.) times the transition probability. The latter is a product, because we assumed that the $\xi^{\prime} s$ are chosen independently at each node. In the deterministic case $A\left(0\left(r_{*}\right) \rightarrow \theta^{\prime}\left(r_{*}\right)\right)$ selects the unique conffuration $\mathcal{E}^{-1} e^{\prime}($.$) , su that eq. (3.20) is recovered.$

### 3.4 Nean grantities

Having introduced a probablistic deacription, we now tarn to mean quantities. For an "observable" $q\left(n\left(t_{*},.\right)\right)$, which depends on the Boolean field at a single time, the mean is given by averacing over $P\left(t_{*}, n().\right)$

$$
\begin{equation*}
\left\langle q\left(n\left(t_{n}, .\right)\right)\right\rangle=\sum_{r(.) \in r} q(\cdot(.)) P\left(t_{x}, d(.)\right) . \tag{3.22}
\end{equation*}
$$

An important role will be played in the sequel by the following mean quantitiea: the mean popalation

$$
\begin{equation*}
N_{1}\left(l_{*}, r_{*}\right)=\left\langle n_{1}\left(l_{*}, r_{*}\right)\right\rangle \tag{3.23}
\end{equation*}
$$

the density, and the mass current (mean momentum)

$$
\begin{equation*}
\rho\left(t_{*}, r_{*}\right)=\sum_{i} N_{i}\left(t_{*}, r_{*}\right), \quad J\left(t_{*}, r_{*}\right)=\sum_{i} c_{i} N_{1}\left(t_{*}, r_{*}\right) . \tag{3.24}
\end{equation*}
$$

Note that these are mean quantities per node, not per unit area or volume. The density per cell is defined as $\sigma=\rho / b$. Finally, the mean velocity $u$ in defined by

$$
\begin{equation*}
J\left(t_{*}, r_{-}\right)=\rho\left(t_{*}, r_{*}\right) u\left(t_{*}, r_{*}\right) \tag{3.25}
\end{equation*}
$$

Note that under duality (exchange of particles and holes) $\rho$ changes into $b-\rho, d$ into $1-d, J$ into $-J$, and $u$ in to the "mean hole-velocity" $u_{B}=-u d /(1-d)$.

Averaring of the microdyramical conservation relations (3.16) and (3.17) leads to conservation relations for the mean populations

$$
\begin{align*}
\sum_{i} N_{i}\left(l_{*}+1, r_{*}+c_{i}\right) & =\sum_{i} N_{i}\left(l_{*}, r_{*}\right),  \tag{3.26}\\
\sum_{i} c_{i} N_{i}\left(l_{*}+1, r_{*}+c_{i}\right) & =\sum_{i} c_{i} N_{i}\left(l_{*}, r_{*}\right) . \tag{3.27}
\end{align*}
$$

## 4. Equillbrium solutions

It has been shown by Hardy, Pomeau, and de Pazzis(20) that the BPP model has very simple statistical equilibrium solutions (which they call in rariant states) in whirh the Boolean variables at all the celle are independent. Such equilibrium solutions are the lattice gas equivalent of Maxwell states in Classical Statistical Mechanics and therefore are crucial for deriving hydrodynamics. There are simiiar resalts for the general class of nondeterministic models introduced in section 2.4, wich are now discussed.

### 4.1 Steady solutrone of the Liouville equation-

We are interested in equilibrium solutions, that is steady-state solutions of the Liouville equation (3.21) for a finite, periodically wrapped around lattice. Collisions on the lattice are purety local (their impact parameter is zero). This suggests the existence of equilibrium solutions with no single-time spatial correlations. The lattice properties being translationinvariant, the distribation should be the same at each node. Thus we are looking for equilibriom solutions of the form

$$
\begin{equation*}
P(o(\cdot))=\prod_{r_{-}} p\left(o\left(r_{*}\right)\right) \tag{4.1}
\end{equation*}
$$

Where $p(0)$, the probability of a given state, is node-independent. Maximization of the entropy (ef. Appendix F) suggests that $p(o)$ should be competety factorized over all cells, that is, of the form

$$
\begin{equation*}
p(0)=\prod_{i} N_{i}^{0}\left(1-N_{i}\right)^{(1-0,1)} \tag{4.2}
\end{equation*}
$$

Note that $N_{i}^{* \prime}\left(1-N_{i}\right)^{\left(1-\theta_{i}\right)}$ is the probability of a Boolean va iable with mean $N_{1}$.
Now, we must check that there are indeed solutions of $:=$ form that we have been spessing. Substitation of $P(o()$.$) given by (4.1) with p(0)$ given by (4.2) into the Liourille equation (3.21) leads to

$$
\begin{equation*}
\prod_{j} N_{j}^{\bullet_{j}^{\prime}}\left(1-N_{j}\right)^{\left(1-0_{j}^{\prime}\right)}=\sum_{0} A\left(0 \rightarrow 0^{\prime}\right) \prod_{j} N_{j}^{0_{j}}\left(1-N_{j}\right)^{\left(1-0_{j}\right)}, \quad \forall 0^{\prime}, \tag{4.3}
\end{equation*}
$$

where $N$, is the mean population of cell $i$, independent of the node and of the time.
Eq. (4.3) is a set $2^{\text {b }}$ (the number of diferent states) equations for $b$ anknowns. The fect that it actually possesses solutions is nontrivial Furthermore, these solutions can be completely described. Such results follow from the

Lerama. The following statements are equivalent:
(a) The $N_{i}$ 's are a solution of (4.3).
(b) The $N_{i}$ 's are a solution of the set of $b$ equations

$$
\begin{equation*}
\sum_{o o^{\prime}}\left(0_{i}^{\prime}-0_{i}\right) A\left(0 \rightarrow o^{\prime}\right) \prod_{j} N_{j}^{0 j}\left(1-N_{j}\right)^{\left(1-o_{j}\right)}=0, \quad \forall i \tag{4.4}
\end{equation*}
$$

(c) The $N_{i}$ 's are given by the Fermi-Dirac distribution

$$
\begin{equation*}
N_{i}=\frac{1}{1+\exp \left(h+\mathbf{q} \cdot \mathrm{e}_{i}\right)}, \tag{4.5}
\end{equation*}
$$

Where $h$ is an arbitrary real number and $q$ is an arbitrary $D$-dimensional vector.
The proot of the equivalence is siven in Appendix C; it makes ase of semi-detailed balance and the absence of spurious invariants. The most important consequence of the lemmes is the
Univerallity theorem. Nondeterministic lattice gas models satistying scmi-detailed bat ance and bsving no spurious invariants admit universal equilibrium solutions, completely factorired over all nodes and all cells, with mean popalations given by the Fermb-Dirac distribution (4.5), depending only on the density $\rho$ and the mass current $J=\rho u$, and independent of the transition probablilities $A\left(0 \rightarrow 0^{\prime}\right)$.

The proof follows trom the observation that the Lagrange mu!tipliers $h$ and $q$ of the Fermi-Dirac distribution can be calculated in terms of the density and the mass current through the relations

$$
\begin{align*}
& p=\sum_{i} N_{i}=\sum_{i} \frac{1}{1+\exp \left(h+q \cdot c_{i}\right)},  \tag{4.6}\\
& \rho u=\sum_{i} N_{i} e_{i}=\sum_{i} e_{i} \frac{1}{1+\exp \left(h+q \cdot q_{i}\right)} . \tag{4.7}
\end{align*}
$$

For the HPP model, this set of equationa is reducible to a cubic polynomial equation, no that explicit molutions are known. ${ }^{(20)}$ For the FHP model, explicit solutions are known onty for apecial cases. ${ }^{(0)}$

It is not partleularly surprising, for modela that have a bulltin exclusion principle (not more than one particle per cell), to obtaln a Fermi-Direc diatribution at equilibrium. Note that the factorized equilibrium sontions remaln meaningtul on an infaite latice. There is no pronf at the moment that the only equillibrium solutions which are relevant to the limit of intaite laticices are of the above form, namely cumpletely factorized (which ther implies she Ferml-Dirac distribution). There is atrong aumerical ovidence, for those models that have been slmulated, that the Fermi-Dirac is the onty relevant one. ${ }^{(0,98,25)}$

## 4. Low-apaes equibstris

In the "real world" equilibrium distributions with different mean velocities are simply related by a Galilean transformation. Galileau inmpiance does not bold at the microsenpic level on a iantice gas; therefore there is no simple relation between the equillbria with vanishing and nonvanishing mean velocity. For subsequent derivations of luid dyamical
equations, we shall only need equilibria with iow speeds, that is with $u=|u|<c$, the particle speed. Such equilibria can be calculated perturbatively in powers of $u$.

We write the equilibrium distribution as

$$
\begin{equation*}
N_{i}=f_{\mathbf{r}}\left(h(p, u)+q(\rho, u) \cdot c_{i}\right), \tag{4.8}
\end{equation*}
$$

where we have used the Fermi-Dirac function

$$
\begin{equation*}
f_{r 0}(x)=\frac{1}{1+e^{x}} \tag{4.9}
\end{equation*}
$$

We observe that

$$
\begin{equation*}
u=0 \Rightarrow N_{i}=\frac{\rho}{b}=d . \tag{4.10}
\end{equation*}
$$

Indeed, by assumption (iii) of section 2.4, there exist an isometry of the lattice exchanging any two velocity vectors $c_{i}$ and $c_{j}$; the vector $u=0$ being also trivially invariant, the mean population $N_{i}$ is independent of $i$. Thus $f_{r D}(h(\rho, 0))=d$ and $g(\rho, 0)=0$.

Furthermore, it follows trom parity-invariance $\left(u \rightarrow-u, c_{i} \rightarrow-c_{i}\right)$ that

$$
\begin{equation*}
h(\rho,-u)=h(\rho, u), \quad q(\rho,-u)=-q(\rho, u) . \tag{4.11}
\end{equation*}
$$

We now expand $h$ and $q$ in powers of $u$

$$
\begin{align*}
& h(\rho, u)=h_{0}+h_{g} u^{2}+O\left(u^{4}\right)  \tag{4.12}\\
& f_{c}(\rho, u)=\xi_{1} u_{c}+O\left(u^{s}\right),
\end{align*}
$$

where $h_{0}, h_{2}$, and $g_{1}$ depend on $\rho$. The fact that $h_{g}$ and $g_{1}$ are scalars rather than second order tensors is a consequence of the isotropy of second order tensors (property P 4 of section 2.4). We substitate (4.12) into (4.8) and expand the mean popalations in powers of $u$

$$
\begin{equation*}
N_{i}=f_{T D}+q_{1} f_{r D}^{\prime} u \cdot c_{i}+h_{2} f_{T D}^{\prime} u^{2}+\frac{1}{2} q_{i}^{2} f_{r D}^{\prime \prime}\left(u \cdot c_{i}\right)^{2}+O\left(u^{8}\right) . \tag{4.13}
\end{equation*}
$$

Here, $f_{r b}, f_{r D}^{\prime}$, and $f_{r D}^{\prime \prime}$ are the values at $h_{0}$ of the Fermi-Dirac function and its Arst and second dertvatives. From (4.13) wa calculate the density $\rho=\sum_{i} N_{i}$ and the mass current on $=\sum_{i} c_{i} N_{i}$, using the velocity moment relations (P6 of section 2.4). Identification sives $h_{0}, h_{3}$, and $q_{1}$ in terms of $\rho$. This ia than used to calculate the equilibrium mean population up to second order in u; we obtain

$$
\begin{equation*}
N_{d}^{\infty}(\rho, u)=\frac{\rho}{b}+\frac{\rho D}{\alpha^{2} b} c_{a} u_{a}+\rho G(\rho) Q_{i \alpha \beta} u_{a} u_{\beta}+O\left(u^{b}\right), \tag{4.14}
\end{equation*}
$$

where

$$
\begin{equation*}
G(\rho)=\frac{D^{2}}{2 c^{1} b} \frac{b-2 \rho}{b-\rho} \text { and } Q_{i a \rho}=c_{i \alpha} c_{i \beta}-\frac{e^{2}}{D} \delta_{a \beta} . \tag{4.18}
\end{equation*}
$$

In (4.14) the superscript "eq" atreasen that the mean popalation are evaluated at equilibriam.

Note that the coemeient $G(\rho)$ of the quadratic term manishes for $p=b / 2$, that is, when the deasity of particlen and holes are the same. This resalt, which holds more generally for the coeflucients of any even power of $u$, foliows by duallty: $N_{i}^{\text {ef }}$ soes into $1-N_{i}^{\text {ue }}$ and $u$ into $-u$ at $\rho=6 / 2$. It does not matier whether or not the collision rules are duality-invariant. as long as they satisty semf-detalled balance, since the equilibrium is then univirsal.

## 6. Macrodynamical equations

In the "reaj world", Auid dynamics may be viewed as the glueing of local thermodynamic equilibris with siowly varying parameters. ${ }^{(61.09)}$ Lattice gases also admit equilibrium solutions. ${ }^{4}$ These have continuously adjustable parameters, the mean values of the conserved quantities, namely mass and momentum. On a very large lattice, we can set up local equilibria with density and mass curtant slowly changing in space and time. From the conservation reiations we shall derive by a multi-scale technique macrodynamical equations, that is PDE's lor the large scale and long time behaviour of density and mass current

We consider a lattice gas satisfying all the assumptions of section 2.4. We denote by $\rho\left(r_{*}\right)$ and $u\left(r_{*}\right)$ the density and (mean) velocity at lattice node $r_{*}$. We assume that these quantities are changing on a spatial scale $\epsilon^{-1}$ (in units of lattice constant). This requires that the lattice size $L$ be itself at least $O\left(\epsilon^{-1}\right)$. Eventually, we let $\epsilon \rightarrow 0$. The spatial change is assumed to be suffciently regular to allow interpolations for the purpose of calculating derivatives. ${ }^{5}$ When time and space are treated as continuons, they are denoted $t$ and $\mathbf{r}$. We further assume that the density is $O(1)$ and that the velocity is small compared to the particle apeed c. ${ }^{7}$ We expect the following phenomena;

1) relaxation to local equilibrium on time scale $e^{0}$,
2) density perturbations propafating as sound waves on time scale $\epsilon^{-1}$,
3) diffustive (and posibiby advective) effects on time scale $e^{-1}$.

We thus use a three time formalism: $t_{m}$ (discrete), $t_{1}=c t_{*}$, and $t_{2}=\epsilon^{2} l_{\text {, }}$, the latter two being treated as continuous variables. We use two space variables: $\mathrm{r}_{*}$ (discrete) and $r_{1}=\epsilon r_{*}$ (contincous).

Let us denote by $N_{i}^{(0)}\left(r_{*}\right)$ the mean equilibrium populations based on the local value of $\rho$ and $u$. They are given by (4.14). The actual mean populations $N_{i}(t, r) ~ w i l l ~ b e ~ c l o s e ~$ to the equilibrium ralues and may be expanded io powers of $\epsilon$ :

$$
\begin{equation*}
N_{i}=N_{i}^{(0)}(t, r)+e N_{i}^{(1)}(t, r)+O\left(c^{q}\right) . \tag{5.1}
\end{equation*}
$$

The corrections should not contribute to the local values of denaity and mean momentum; thus

$$
\begin{equation*}
\sum_{i} N_{i}^{(1)}(t, r)=0 \quad \text { and } \quad \sum_{i} c_{i} N_{i}^{(1)}(t, r)=0 \tag{5.2}
\end{equation*}
$$

We now start from the cxact conservation relations (3.26) and (3.27) and expand both the $N_{i}^{\prime}$ 's and the Boite diferences in powers of $c$. Note that all Anite diferences must be expanded to second order, otherwise, the viscous terme are not correctly captured. Time and space derivatives will be denoted $\partial_{1}$ and $\partial_{p}=\left\{\partial_{a}, a=1, \ldots, D\right\}$. For the multi-scale formalism, we mate the substitutions

$$
\begin{equation*}
\partial_{1} \rightarrow c \partial_{1_{1}}+c^{2} \partial_{1_{2}} \text { and } \partial_{\mathrm{r}} \rightarrow \mathrm{c} \partial_{r_{1}} . \tag{5.3}
\end{equation*}
$$

[^0]The components of $\partial_{\mathrm{r}_{1}}$ will be denoted $\partial_{1 a}$.
To leading order, $O(\epsilon)$, we obtain

$$
\begin{equation*}
\partial_{i_{1}} \sum_{i} N_{i}^{(0)}+\partial_{1 \beta} \sum_{i} e_{i \beta} N_{i}^{(0)}=0, \tag{5.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\partial_{i_{1}} \sum_{i} \mathrm{c}_{i \alpha} N_{i}^{(0)}+\partial_{1 \beta} \sum_{i} \mathrm{c}_{i \alpha} \mathrm{c}_{i \beta} N_{i}^{(0)}=0 \tag{5.5}
\end{equation*}
$$

We now substitate the equilibrium values (4.14) for the $N_{i}^{(0)}$ 's and use the velocity moment relations P6 of section 2.4. We obtain the "macrodynamical Euler equations"

$$
\begin{equation*}
\partial_{t_{1} \rho} \rho+\partial_{1 \rho}\left(\rho u_{\beta}\right)=0, \tag{8.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\partial_{1_{1}}\left(\rho u_{\alpha}\right)+\partial_{1 \beta} P_{\alpha \beta}=0 \tag{5.7}
\end{equation*}
$$

$P_{a \beta}$ is the momentum-inux tensor, ${ }^{\text {a }}$

$$
\begin{align*}
P_{a \beta} & \equiv \sum_{j} c_{i \alpha} c_{i \beta} N_{i}^{o \phi} \\
& =\frac{\rho^{2}}{D} \rho \delta_{a \beta}+\rho G(\rho) T_{a \beta \gamma \delta} u_{\gamma} u_{6}+O\left(u^{4}\right), \tag{5.8}
\end{align*}
$$

with

$$
\begin{equation*}
T_{a \beta \gamma \delta}=\sum_{i} c_{i \alpha} c_{i \beta} Q_{i \gamma 6} \tag{5.9}
\end{equation*}
$$

and $G(\rho)$ and $Q_{\text {iyd }}$ given by (4.15) of section 4. Note that the correction term in the s.b.s. of (5.8) is $O\left(u^{4}\right)$ rather than $O\left(u^{3}\right)$; indeed, it follows from the parity-invariance of the lattice ges that Arst order spatial derivative terms do not contain odd powers of u.

We now proceed to the next order, $O\left(e^{2}\right)$. We expand (3.26) and (3.27) to second order, collecting all $O\left(e^{2}\right)$ terms, we obtals

$$
\begin{align*}
& \partial_{i_{2}} \sum_{i} N_{i}^{(0)}+\frac{1}{2} \partial_{i_{1}} \partial_{i_{1}} \sum_{i} N_{i}^{(0)}+\partial_{i_{1}} \partial_{1 \beta} \sum_{i} c_{i \beta} N_{i}^{(0)}+\frac{1}{2} \partial_{1, \rho_{1,} c_{\beta} c_{i \gamma} N_{i}^{(0)}+}^{\partial_{i_{1}} \sum_{i} N_{i}^{(1)}+\partial_{1 \beta} \sum_{i} c_{i \beta} N_{i}^{(1)}=0}
\end{align*}
$$

and

$$
\begin{align*}
& \partial_{i_{1}} \sum_{j} c_{i \alpha} N_{i}^{(0)}+\frac{1}{2} \partial_{i_{1}} \partial_{i_{1}} \sum_{i} c_{i \alpha} N_{i}^{(0)}+\theta_{i_{1}} \partial_{1 \beta} \sum_{i} c_{i \alpha} c_{i \beta} N_{i}^{(0)}+\frac{1}{2} \theta_{1 \beta} \partial_{1 \gamma} c_{i \alpha} c_{i \beta} c_{i \gamma} N_{i}^{(0)}+ \\
& \partial_{i_{1}} \sum_{i} c_{i \alpha} N_{i}^{(1)}+\theta_{1 \beta} \sum_{i} c_{i \alpha} c_{i \beta} N_{i}^{(1)}=0 \tag{8.11}
\end{align*}
$$

[^1]By (5.2) $\sum_{i} N_{i}^{(1)}=0$ and $\sum_{i} c_{i a} N_{i}^{(1)}=0$. For the $N_{i}^{(0)} s_{s}$, we substitate their low-speed equilibriam form (4.14), leaving out $O\left(u^{2}\right)$ terms. Reexpressing derivatives of $\rho$ and pu with respect to $t_{1}$ in terms of space derivatives, using (5.6)-(5.7), we obtain

$$
\begin{equation*}
\partial_{t, \rho} \rho=0, \tag{5.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\partial_{i_{j}}\left(\rho u_{a}\right)+\partial_{1 \beta}\left(\sum_{i} c_{i a} c_{i \beta} N_{i}^{(1)}+\frac{D}{2 c^{2} b} T_{a \beta \gamma \delta} \partial_{1 \gamma}\left(\rho u_{\beta}\right)\right)=O\left(u^{2}\right) . \tag{5.13}
\end{equation*}
$$

Eq. (5.12) tells us that there is no mass diffusion (there is a single species of particles). Eq. (5.13) describes the momentum diffusion over long ( $O\left(\epsilon^{-3}\right)$ ) time-scales. It has two contributions. The term involving $T_{a \beta \gamma}$ comes from particle propagation and we shall comment on it later.

The other term in (5.13) involves the deviations $N_{i}^{(1)}$ from the equilibriom mean populations. $N_{i}^{(1)}$ vanishes when the equilibrium is naiform. It must therefore be a linear combination of gradients (with respect to $r_{1}$ ) of $\rho$ and $\rho \mathbf{p}$. Linear response theory is needed to calculate the coefficients. At this point, we shall coly make use of symmetry arguments to reduce the number of coefficients. We assume that $u$ is small, so that, to leading order equilibria are invariant under the isometry group $\mathcal{G}$ of the lattice (see section 2.4). Since the gradient oi $\rho$ is a vector and the gradient of $\rho u$ is a second order tensor, properties P2 and P3 of section 2.4 allow us to write

$$
\begin{equation*}
N_{i}^{(1)}=\sigma e_{i \alpha} \partial_{1 \alpha} \rho+\left(\psi e_{i \alpha} e_{i \beta}+\chi \delta_{\alpha \beta}\right) \partial_{l \alpha}\left(\rho u_{\beta}\right) . \tag{5.14}
\end{equation*}
$$

By eq. (5.2), we have $\sigma=0$ and $c^{2} \psi+D \chi=0$. Note that $\psi$ should depend on $\nu$, but not on $u$, since it in evaluated at $u=0$. Substitating the expresaion for $N_{i}^{(1)}$ into ( 5.13 ), we obtain

$$
\begin{equation*}
\partial_{i_{1}}\left(\rho u_{a}\right)+\partial_{1 \beta}\left[\left(\psi(\rho)+\frac{D}{2 \alpha^{2 b}}\right) T_{a \beta \gamma_{6}} \partial_{17}\left(\rho u_{6}\right)\right]=O\left(u^{2}\right) . \tag{5.15}
\end{equation*}
$$

In the sequel, it will be more convenjent to collapse the set of four equations, governing the evolation of $\rho$ and $\rho u$ on $O\left(\epsilon^{-1}\right)$ and $O\left(c^{-2}\right)$ time-acalee, into a patr of equations, written tin terms of the original variables $!$ and $r$ (in their continuons version). We thus obtaln the macrodynamical equations

$$
\begin{align*}
& \partial_{i} \rho+\partial_{\beta}\left(\rho u_{\beta}\right)=0  \tag{5.16}\\
& \partial_{1}\left(\rho u_{a}\right)+\partial_{\beta}\left(\rho G(\rho) T_{\alpha \beta \gamma 6} u_{7} u_{6}+\frac{\alpha^{2}}{D} \rho_{\alpha A}\right)+\partial_{\beta}\left[\left(\psi(\rho)+\frac{D}{2 c^{2} b}\right) T_{\alpha \beta \gamma 6} \partial_{7}\left(\rho u_{b}\right)\right] \\
& =O\left(\epsilon u^{d}\right)+O\left(\varepsilon^{2} u^{2}\right)+O\left(e^{3} u\right) . \tag{5.17}
\end{align*}
$$

The equivalence of (5.16) and (5.17) to (5.6), (5.7), (5.12), and (5.15) follows by (5.3). Note that (5.16) is the standard density equation of Auid mechanices and that ( 5.17 ) already bas a strong resemblance to the Navier-Stokes equations.

## 6. Recovering leotropy

The macrodynamical equations (5.16)-(5.17) are not fulis isotropic. The presence of a latice with discrete rotational symmetries is still felt through the tensor

$$
\begin{equation*}
\eta_{\alpha \beta \gamma \delta}=\sum_{i} e_{i \alpha} c_{i \beta} Q_{i \gamma \delta}=\sum_{i} e_{i \alpha} c_{i \beta}\left(e_{i \gamma} e_{i \delta}-\frac{c^{2}}{D} \delta_{7 \delta}\right) . \tag{6.1}
\end{equation*}
$$

appearing in both the nonlinear and diffustive terms of (5.17). Furthermore, the higher order terms in the r.h.s. of (5.17) have no reason to be isotropic. This should not worry as since they will eventually turn out to be irrelevant. Contrary to translational discreteness, rotational discreteness cannot go away under the macroscopic limit; the latter involves large scales bnt not in any way "iarge angles" since the group of rotations is compact.

We have seen in section 2.4 that tensors up to third order, having the same invariance group $g$ as th discrete velocity set are isotropic. Not so for tensors of fourth order such as Tasyo. Indeeo for the HPP model (section 2.1) explicit calculation of the momentum-fux tensor, given by ( 5.8 ), is quite straightforward. The result is

$$
\begin{gather*}
P_{11}=\rho G(\rho)\left(u_{1}^{2}-u_{2}^{2}\right)+\frac{\rho}{2}+O\left(u^{4}\right), \quad P_{99}=\rho G(\rho)\left(u_{2}^{2}-u_{1}^{3}\right)+\frac{\rho}{2}+O\left(u^{4}\right)  \tag{0.2}\\
P_{19}=P_{21}=0 \tag{6.3}
\end{gather*}
$$

with

$$
\begin{equation*}
G(\rho)=\frac{2-\rho}{1-\rho} \tag{6.4}
\end{equation*}
$$

The only second order tensors quadratic in the velocity being $u_{a} u_{p}$ and $u \cdot u \delta_{a \beta}$, the tensor $P_{a \in}$ is not isotropic.

In order to eventually obtain the Navier-Stokes equations, the tensor $T_{a \beta \text { g }}$ given by (6.1) must be isotropic that is, invariant ander the full orthogonal sroup. This tensor is pairwise symmetrical in $(a, \beta)$ and $(\gamma, \delta)$; from (6.1), it follors that it satistiea

$$
\begin{equation*}
\sum_{\gamma} T_{a \beta \gamma \gamma}=0, \quad \sum_{a \beta} T_{a \beta a \beta}=b c^{4}\left(1-\frac{1}{D}\right) \tag{6.5}
\end{equation*}
$$

When the tensor $T_{a A y}$ is isotropic, these properties uniquety constrain it to be of the following form:

$$
\begin{equation*}
T_{a \beta y \theta}=\frac{b c^{4}}{D(D+2)}\left(\delta_{a \gamma} \delta_{\beta \delta}+\delta_{a \delta} \delta_{D 7}=\frac{2}{D} \delta_{a \beta} \delta_{7 \delta}\right) \tag{6.6}
\end{equation*}
$$

For general group-theoretical material concernlag the isotropy of tenson with discrete symmetries in the consext of lattice gases, we refer the reader to Ref. 36. Crucial observations for obeaining the two and three-dimensional Navier-Stokes equations ase the Leotropy of pairwise symmetrical teusors for the triangular FHP lattice in two dimensions
and the face-centered-hypercabic ( FCHC ) lattice in four dimensions, and thas also for the psendo-4D three-dimensional modei We give now eiementary proofs of these reaults.

In two dimensions, it is convenient to consider $T_{a, y d}$ as a linear map from the space $E$ of two-by-two real symmetrical matrices into itselt:

$$
\begin{equation*}
T: \quad A_{\alpha \beta} \mapsto T_{\alpha \beta \gamma \theta} A_{\gamma \delta} . \tag{6.7}
\end{equation*}
$$

A basis of the space $E$ is formed by ihe matrice; $P_{1}, P_{2}$, and $P_{3}$, associated to the orthogonal projections onto the $x_{1}$-axds and onen two ubher directions at $2 \pi / 3$ and $4 \pi / 3$. In this representation, an arbitrory $E$-motrix may be written as

$$
\begin{equation*}
A=x_{1} P_{1}+x_{3} P_{2}+x_{s}{ }_{3} s_{1} \tag{6.8}
\end{equation*}
$$

and $T$ becomes a three-by-three matrix $T_{a},(a, b=1,2,3)$. The key observation is that the texagoual group (rotations by multiples of $x / 3$ ) beromes the permatation group of $P_{1}$, $P_{3}$, and $P_{3}$. Thus $T_{a b}$ is invariant under arbitrary permutations of the coordinates, i.e. is of the form

$$
\begin{equation*}
x_{a b}=\phi \operatorname{din} q(1,1,1)+x 1_{a b} \tag{6.9}
\end{equation*}
$$

Where diag $(1,1,1)$ is the diagonal matrix with entries one and $l_{06}$ is the matrix with all entries equal to one, and $\phi$ and $\chi$ are artitiary cealarr. From (6.8) we have

$$
\begin{equation*}
\operatorname{tr}(A)=x_{1}+x_{2}+x_{2} \tag{6.10}
\end{equation*}
$$

where tr denotes the trace. We alo note that

$$
\begin{equation*}
P_{1}+P_{9}+P_{3}=(z / 2) I, \tag{6.11}
\end{equation*}
$$

Where $I$ is the identity (eheck it for the anit veetort of the $x_{1}$ and $x_{2}$ axis). Using (6.10) and (6.11), we can rewrite (6.9)

$$
\begin{equation*}
T: \quad A \mapsto \phi A+\frac{3}{2} \times \operatorname{tr}(A) I . \tag{0.12}
\end{equation*}
$$

Reverting to tensor notations, this becomes

$$
\begin{equation*}
T_{a \beta 7 d}=\frac{\phi}{2}\left(\delta_{a \gamma} \delta_{g \phi}+\delta_{a \delta} \delta_{B \gamma}\right)+\frac{3 x}{2} \delta_{a \beta} \delta_{7}, \tag{0.13}
\end{equation*}
$$

Which is obviously isotropic.
We turn to the four-dimensional case, asing the FCHC model of section 2.3. Invariance under permutations of coordinates and reversal of any coordinate implies that the most general possible form for $T_{a A 70}$ is

$$
\begin{equation*}
T_{a \beta, \delta}=\phi \delta_{a \delta} \delta_{A T} \delta_{7 \delta}+\chi\left(\delta_{a,} \delta_{A 6}+\delta_{a \delta} \delta_{A y}\right)+\psi \delta_{a g} \delta_{7 \delta} . \tag{0.14}
\end{equation*}
$$

The $\chi$ and $\psi$ terms are already isotropic. The vanishing of $\phi$ is a consequence of the invariance of the velocity set onder the symmetry $\Sigma$ with respect to the hyperplane $x_{1}+$ $x_{3}+x_{3}+x_{4}=0$, that is

$$
\begin{equation*}
x_{a}-x_{\alpha}-\sigma, \quad \sigma=\frac{1}{2} \sum_{a} x_{a} . \tag{6.15}
\end{equation*}
$$

Indeed, consider the vector $v_{\alpha}=(2,0,0,0)$. Contracting the $\phi$ term four times with $v_{\alpha}$, we obtain $16 \phi$; the image of $v_{\alpha}$ under $\Sigma \omega_{o}=(1,-1,-1,-1)$, which contracted four tiraes with the $\phi$ term given $4 \phi$. Thas invariance requires $\phi=0$, which proves isotropy.

We return to the general $D$-dimensional cere, assuming isotropy. Substituting (6.6) into the macrodynamical momentum equation (5.17), we obtain

$$
\begin{align*}
& \left.\partial_{\ell}\left(\rho u_{a}\right)+\partial_{\beta}(\rho \Omega l \rho) u_{\alpha} u_{\theta}\right)+\partial_{\alpha}\left(c_{\beta}^{2} \rho\left(1-\rho(\rho) \frac{u^{3}}{c^{3}}\right)\right) \\
& =\partial_{\beta}\left[\left(\nu_{c}(\rho) \quad\left(\partial_{\alpha}\left(\rho u_{\beta}\right)+\partial_{\beta}\left(\rho u_{\alpha}\right)-\frac{2}{D} \delta_{\alpha \beta} \partial_{\gamma}\left(\rho u_{\delta}\right)\right)\right]+O\left(\epsilon u^{3}\right)+O\left(\epsilon^{2} u^{2}\right)+O\left(\epsilon^{3} u\right),\right. \tag{0.16}
\end{align*}
$$

with

$$
\begin{equation*}
\rho(\rho)=\frac{D}{D+2} \frac{b-2 \rho}{b-\rho}, \quad c_{t}^{2}=\frac{c^{2}}{D}, \quad \nu_{c}(\rho)=-\frac{b c^{4}}{D(D+2)} \psi(\rho), \quad \nu_{p}=-\frac{c^{2}}{2(D+2)} . \tag{0.17}
\end{equation*}
$$

Note that $g(\rho)$ appearing in (6.17) is not the same as $G(\rho)$ introduced in (4.15). Note also that $\psi(\rho)$, which was introduced in section 5, is still to be determined (cf. section 8).

We have now recovered macrocopic botropy; equation (6.16) is very closely related to the fluid dyamical momentum (Navier-Stokes) equations. We postpone all further remarks to the next section.

## 7. Fluid dynamleal rtglmee

Let us rewrite the macrodynamical equations for mass and momentum, dertved in the previous sections in a compact form which bring ont their similarities with the equations of fluid dynamics:

$$
\begin{gather*}
\partial_{1 p}+\delta_{\beta}\left(\rho u_{\beta}\right)=0  \tag{7.1}\\
\partial_{1}\left(\rho u_{a}\right)+\partial_{\beta} P_{a \beta}=\partial_{\beta} S_{a \beta}+O\left(c u^{3}\right)+O\left(\varepsilon^{2} u^{2}\right)+O\left(\epsilon^{3} u\right) . \tag{7.2}
\end{gather*}
$$

The momentum-Aux tensor $P_{a s}$ and the viscous atress tensor $S_{\alpha \rho}$ are given by

$$
\begin{equation*}
P_{a \rho}=c_{0}^{2} \rho\left(1-\rho(\rho) \frac{u^{2}}{c^{2}}\right) \delta_{a \rho}+\rho \rho(\rho) u_{a} u_{\rho} \tag{7.3}
\end{equation*}
$$

and

$$
\begin{align*}
& S_{a \Delta}=\nu(\rho)\left(\partial_{a}\left(\rho u_{\theta}\right)+\partial_{p}\left(\rho u_{a}\right)-\frac{2}{D} \delta_{a \theta} \theta_{7}\left(\rho u_{7}\right)\right)  \tag{7.4}\\
& \nu(\rho)=\nu_{c}(\rho)+\nu_{p}
\end{align*}
$$

Where $g(\rho), c_{\rho}^{3}, \nu_{c}$, and $\nu_{p}$ are deined in (6.17). Their values for the FHP and FCHC models are given below
$\rho(\rho)=\frac{3-\rho}{6-\rho}, \quad c_{\rho}^{2}=\frac{1}{2}, \quad \nu_{c}(\rho)=-\frac{3}{4} \psi(\rho), \quad \nu_{p}=-\frac{1}{8}, \quad$ for FHP
$g(\rho)=\frac{4}{3} \frac{12-\rho}{24-\rho}, \quad c_{0}^{2}=1, \quad \nu_{c}(\rho)=-4 \psi(\rho), \quad \nu_{p}=-\frac{1}{6}, \quad$ for FCHC.
Varions remarks are now in order. When the velocity $u$ is very small, the momentumfux tensor reduces to a diagonal pressure term $p \delta_{\alpha \beta}$ with the pressure given by the "isothermal" relation

$$
\begin{equation*}
p=c_{0}^{2} \rho \tag{7.6}
\end{equation*}
$$

From this, we infer that the speed of sound should be $c_{\theta}$, namely $1 / \sqrt{2}$ for FHP and 1 for FCBC.

The momentum-inax tensor in the "real world" is $P_{\alpha \beta}=p \delta_{\alpha \beta}+\rho u_{\alpha} u_{\beta}$. This form is a consequence of Galiean invariance, which allows one to relate thermodynamic equilibria with vanishing and nonvanishing mean velocities. The lattice gas momentum-fux tensor (7.3) with nonvanishing velocity differs by an additive term in the pressure and a multiplicative density-dependent factor $g(p)$ in the advection term. We shall see later in this section how Galilean invariance can nevertheless be recovered.

Eq. (7.4) is the stress-strain relation for a Newtonian fluid having kinematic viscosity $\nu_{c}+\nu_{p}$ and vanishing bull viscosity. ${ }^{(63)}$ The traceless character of $S_{C A}$ (which implies this vanishing of the bult viscosity) comes from the traceless character of $Q_{i o \rho}$, defined by (4.15); this result would be apset by the presence of rest-particles such as exist in the models FHP-II and III (cf. Appendix E). The kinematic viscosity has two contributions. One is the "collision viscosity" $\nu_{c}$, not yet determined, which depends on the details of the collisions and is positive (cf. section 8). The other one is the "propagation viscosity" $\nu_{p}$, which is negative and does not involve the collisions. The presence of such a negative propagation viscosity is an effect of the lattice discreteness (cf. Ref. 39)

The general strategy for obtaining from (7.1)-(7.2) various standard fuid dynamical equations is to rescale the space, time and velocity variables in such a way as to make undeairable terms irrelevant as $\epsilon \rightarrow 0$. Three diferent regimes will be considered in the following subsections. They correspond respectively to sonad propagation, to sound propagation with slow damping, and to incompressible (Navier-Stokes) fluid dynamies.

## T. 1 Sound propegetton

Consider a weak perturbation of the equilibrium solntion with density $\rho_{0}$ and velocity zero. We write

$$
\begin{equation*}
\rho=\rho_{0}+\rho^{\prime} . \tag{7.7}
\end{equation*}
$$

In a suitable limit we expect that the only relevant terms in (7.1)-(7.2) will be ${ }^{9}$

$$
\begin{align*}
& \partial_{1} \rho^{\prime}+\rho_{0} \nabla \cdot \mathbf{u}=0 \\
& \rho_{0} \partial_{i} \mathbf{u}+c_{1}^{\prime} \nabla \rho^{\prime}=0 . \tag{7.8}
\end{align*}
$$

- From kere on we use vector notation whenever possible.

Formally, this régime is obtained by setting

$$
\begin{equation*}
r=\epsilon^{-1} r_{1}, \quad t=\epsilon^{-1} t_{1}, \quad \rho^{\prime}=\epsilon^{4} \rho_{1}^{\prime}, \quad u=\epsilon^{4} U, \quad a>0 . \tag{7.9}
\end{equation*}
$$

It is then straightforward to check that the leading order terms take the form of eqs. (7.8) (in the rescaled variables). Eliminating $u$ in (7.8), we oblain the scalar wave equation

$$
\begin{equation*}
\frac{\partial^{2}}{\partial t^{2}} \rho^{\prime}-c_{\theta}^{2} \nabla^{2} \rho^{\prime}=0 \tag{7.10}
\end{equation*}
$$

In other words, density and velocity perturbations with amplitudes o(1) on temporal and spatial scales $O(e)$ propagate as sound waves with speed $c_{g} .{ }^{10}$ Since the present régime of undamped sound waves involves only tensors of second order, it also applies to the BPP model

### 7.2 Domped sound

Another régime includes the viscous damping term, so that instead of (7.8) we should have

$$
\begin{align*}
& \partial_{t} \rho^{\prime}+\rho_{t} \nabla \cdot \mathbf{u}=0 \\
& \rho_{0} \partial_{t} \mathbf{u}+c^{2} \nabla \rho^{\prime}=\rho_{0} \nu\left(\rho_{0}\right)\left(\nabla^{2} \mathbf{u}+\frac{D-2}{D} \nabla \nabla \cdot \mathbf{u}\right) . \tag{7.11}
\end{align*}
$$

To obtain this régime we proceed as in section 7.1 and include an additional time $\boldsymbol{f}_{\mathbf{2}}=\epsilon^{\mathbf{2}} \boldsymbol{t}$. Furthermore, in the scaling relation (7.9) we now require $a>1$, that is, $n$ and $\rho^{\prime}$ should be $o(\epsilon)$; otherwise the nonlinear term becomes also relevant. Note that the Jamping is now un a time scale $O\left(\epsilon^{-3}\right)$. Since propagation and damping are on time-scales involving different powers of $\epsilon$, it is not possible to describe them in a single equation withont mixing orders.

### 7.3 Incompresible fluid dymentes:the Havier-Stokes equatione

It is known that many features of low Mach numbern lows in an ordinary gas can be described by the incompressible Navier-Stokes equation

$$
\begin{align*}
& \boldsymbol{\theta}_{\mathfrak{\ell}} \mathbf{u}+\mathbf{u} \cdot \nabla \mathbf{u}=-\nabla \boldsymbol{p}+\nu \nabla^{\boldsymbol{1}} \mathbf{u}  \tag{7.12}\\
& \nabla \cdot \mathbf{u}=\mathbf{0} .
\end{align*}
$$

In the "real world", the incompressible Navicr-Stokes equation can be derived from the full compressible equations, using a Mach number expansion. There are some fine points in this expansion for which we refer the interested reader to Ref. 64. Ignoring these, the easential observation is that, to leading order, density variations become irrelevant everywhere, except in the pressure term; the latter becomes slayed to the nonlinear term by the incompressibility constraint.

10 We have used bere the Landau $O()$ and $O()$ notation.
${ }^{11}$ The Mach number is the ratio of a characteristic liow velocity to the speed of sound

Just the same kind of expansion (with the same difflculties) can be applied to lattice gas dynamics. We start from (7.1)-(7.2) and freeze the density by setting it equal to the constant and uniform value $\rho_{0}$ everywhere except in the pressure term ohere we keep the density fiuctuaticas. We also ignore all higher order terms $O\left(\epsilon^{3} u\right)$, etc. This produces the following set of equations

$$
\begin{align*}
& \rho_{0} \partial_{t} \mathbf{u}+\rho_{0} \rho\left(\rho_{0}\right) \mathbf{u} \cdot \nabla \mathbf{u}=-c_{0}^{2} \nabla \rho^{\prime}+\rho_{0} \nu\left(\rho_{0}\right) \nabla^{2} \mathbf{u} \\
& \nabla \cdot \mathbf{u}=0 . \tag{7.13}
\end{align*}
$$

The resulting equations (7.13) differ from (7.12) only by the presence of the factor $g\left(\rho_{0}\right)$ in front of the advection term $\mathbf{u} \cdot \nabla \mathbf{u}$. As it stands (7.13) is not Galiean invariant. This of conrse reflects the lack of Galiean invariance at the lattice level Similarly, the vanishing of $g\left(\rho_{0}\right)$ when the density per cell $d=\rho_{0} / b$ is equal to $1 / 2$, i.e. for equal mean numbers of particles and holep, reflects a duality-invariance of the lattice gas without counterpart in the "real world" (cf. end of section 4.2). However, as soon as $d \neq 1 / 2$, it is straightforward to reduce (7.13) to the true Navier-Stokes equations (7.12); it suffices to rescale time and viscosity:

$$
\begin{equation*}
t \rightarrow \frac{t}{g\left(\rho_{0}\right)}, \quad \nu \rightarrow g\left(\rho_{0}\right) \nu . \tag{7.14}
\end{equation*}
$$

Now we show that there is actually a rescaling of variables which reduces the macrodynamical equations to the incompressible Navier-Stokes equations. We set

$$
\begin{equation*}
r=\epsilon^{-1} r_{l}, \quad t=\frac{1}{\rho\left(\rho_{0}\right)} \epsilon^{-3} T, \quad u=\epsilon U, \quad \rho^{\prime}=\frac{\rho_{0} g\left(\rho_{0}\right)}{\epsilon_{0}} \epsilon^{2} P^{\prime}, \quad \nu=\rho\left(\rho_{0}\right) \nu^{\prime} . \tag{5.15}
\end{equation*}
$$

Thus, all the relevant terms are $O\left(\epsilon^{2}\right)$ in (7.1) and $O\left(\epsilon^{3}\right)$ in (7.2). The higher order terms in the r.h.s. of (7.2) are $O\left(\epsilon^{4}\right)$ or smaller. In this way we obtain, to leading order ( $\nabla_{1}$ denotes the gradient with respect to $\boldsymbol{r}_{1}$ )

$$
\begin{align*}
& \partial_{T} \mathrm{U}+\mathrm{U} \cdot \nabla_{1} \mathrm{U}=-\nabla_{1} P^{\prime}+\nu \nabla_{1}^{\prime} \mathrm{U} \\
& \nabla_{\mathrm{L}} \cdot \mathrm{U}=0, \tag{7.16}
\end{align*}
$$

which are exactly the incompressible Navier-Stokes equations.
Varions comments are now made. The expansion leading to (7.16) is a large-seale and low Marb number expansion (the former is here inversely proportional to the latter). It also follows trom the scaling relations (7.15) that the Reynolds number is kept Axed. It is not possible within our tramework to have an asympiotic refime leading to nonlinear compressible equations at Anite Mach number. Indeed, the speed of sound is here a fnite fraction of the particle speed and it is essential that the macrosiopic velocity be small compared to particle speed, so as not to be contaminated by bigher order nonlinearities. It is noteworthy that models can be constructed having mayy rest-particies (uero-velocity) with arbitrarily low speed of sound.

In a pure Navier-Stokes context, the non-Galilean Lvariance at the microscopic level is not a serious difficulty; as we have seen, Galilean invariance is recovered macroscopically,
just by rescaling the time variable. However: when the modela discussed here are generalized to include for example multi-phase low or buoyancy effects, a more serions problem may arise because the advection term of scalar quantities suib as chenical concentrations or temperature involves usually a factor $g(\rho)$ different from that of the nonlinear advection term in the Navier-Stokes equations. Various solutions to this problem have been proposed. (47.05)

There is a variant of our formalism, leading also to the incompressible Navier-Stokes equations, bat in terms of the mass current $\mathrm{J}=\rho \mathrm{u}$ rather than the velocity $u$. The analog of (7.13) (without rescaling) is then

$$
\begin{align*}
& \partial_{J} \mathrm{~J}+\frac{\rho\left(\rho_{0}\right)}{\rho_{0}} \mathrm{~J} \cdot \nabla \mathrm{~J}=-c_{0}^{2} \nabla \rho^{\prime}+\nu\left(\rho_{0}\right) \nabla^{2} \mathrm{~J}  \tag{7.17}\\
& \nabla \cdot \mathrm{~J}=0 .
\end{align*}
$$

Since $J$ and $\rho\left(\rho_{0}\right) / \rho_{0}$ change sign onder duality, (7.17) brings out duality-inneriance. ${ }^{17}$ A more decisive advantage of the f -representation is that it gives a better approcimation to the steady Navier-Stokes equations when the Mach number is only moderately small. This is because in the steady state the continuity equation implies exactly $\nabla \cdot \mathrm{J}=0$.

In three dimensions, when we use the pseudo-4D FCHC model, there are three independent space variables $\mathrm{r}=\left(\boldsymbol{x}_{1}, x_{3}, x_{3}\right)$ bat four velocity components

$$
\begin{equation*}
\mathrm{U}_{f}=\left(\mathrm{U}, U_{4}\right)=\left(U_{1}, \sigma_{2}, U_{3}, U_{4}\right) \tag{7.18}
\end{equation*}
$$

The four-velocity $U_{f}$ satistes the forr-dimensional Navier-Stokes equations with no $x_{4}$ dependence. Thus, the three-velocity U satistes the three-dimensional Navier-Stokes equations (7.16), while $U_{4}$ satisfies (note that the pressure term drops out)

$$
\begin{equation*}
\theta_{T} O_{4}+U \cdot \nabla_{1} \nabla_{4}=\nu^{\prime} \nabla_{!}^{?} \nabla_{4} . \tag{7.19}
\end{equation*}
$$

This is the equation for a passive scalar with unit Schmidt number (ratio of vacosity to defosivity).

Finalty, we refer the reader to Appendix D for the inclusion of body forces in the Navier-Stokes equations.

[^2]
## 8．The viecoslty

All the macroscopic equations derived in section 7 have a aniversal form，which does not depend on the details of collisions．The kinematic shear viscosity $\nu$ ，which we shall benceforth call the viscosity，does not possess this universality．Transport coef⿴囗十丌ients such as the viscosity characterize the linear response of equilibrium solutions to small externally imposed perturbations．It is known in Statistical Mechavics that the relaxation （or dissipation）of external perturbations is connected to the Iuctuations at equilibrium via fuctuation－dissipation relations．Such relations have a coanterpart for lattice gases． Two quite different approaches are known．In section 8．1，following a suggestion already made in Ref．21，we present the＂noisy＂hydrodynamics view－point，in the spirit of Landan and Lisechitz．${ }^{(00,67)}$ Another approach，in the spirit of $K a b o^{(68)}$ and Green，${ }^{(09)}$ using a Liouville equation formalism，may be found in Ref．41．In section 8.2 we introduce the lattice analog of the Boltzmann approcimation，which allows an explicit calculation of the viscosity．In section 8.3 we discuss some implications for the Reynolds numbers of incompressible flows simulated on lattice gases．

## 8．1 Flzetuation－diseipation relation and＂nodey＂hydrodymanice

We frat explain the basic ideas in words．Spontaneons factuations at equilibrium involve modes of all possible scales．The fuctuations of very large scales should have their dy－ namics governed by the macroseopic equations derived in sections 5－7．Such Iuctuations are also expected to be very weak，so that linear hydrodynamics should apply．Large－ acale spontaneons fuctaations are constantly regenerated，and in a random manner，this regeneration is provided by a random force（noise）term which can be identifled and ex－ preased in terms of the luctuating microscopic variables．In this random force has a short correlation－time（i．e．small compared to the life－time of the large－seale fluctoations un－ der investigation），then each harge－scale mode $u$ has its dynamics governed by a Langevin equation ${ }^{13}$ It follows that the variance $\left(v^{2}\right)$ can be expressed in terms of the damping coeflicient $\gamma$（related to the viscosity）and of the time correlation function of the random force．Alternatively，the variance $\left\langle v^{2}\right\rangle$ can be calculated from the known one－time equilib－ rinm properties．Identification gives the viscosity in terms of equilibriam time－correlstion functions．This is the seneral programme that we now carry out for the special case of Latice gues．We restrict ourselves to equilibrium solution with zero mean velocity．

We shall ase in this section the following notation．The density $\rho$ and the mass carrent $J$ are no longer given by their expressions（3．24）in terms of the mean popalations；instead， they are delned to terms of the Auctuating Boolean Aeld

$$
\begin{equation*}
\rho\left(t_{*}, i_{n}\right)=\sum_{1} n_{1}\left(t_{*}, r_{*}\right), \quad J\left(t_{*}, r_{*}\right)=\sum_{1} c_{1} n_{1}\left(t_{*}, r_{*}\right) . \tag{8.1}
\end{equation*}
$$

We denote by $\tilde{n}$ ，the luctuating part of the Boolean feld，defined by

$$
\begin{equation*}
n_{1}\left(t_{*}, r_{*}\right)=d+\dot{n}_{1}\left(t_{*}, r_{*}\right), \tag{8.2}
\end{equation*}
$$

where $d$ is the density per cell．

[^3]We introduce meso-averaged aelds by taking spatial averages over a a distance $\epsilon^{-1}$. ${ }^{14}$ These $\boldsymbol{w i l l}$ be denoted by angular brackets with the subscript ma. The meso-averages of $n_{i}, p$, and $J$ are denoted $n_{i}, p$, and $J$ respectively. Locally, the equilibrium relation (4.14) thould hold approximately for the meso-averaged populations. We thus write

$$
\begin{equation*}
n_{1}=\frac{D}{b}+\frac{D}{e^{2} b} J \cdot c_{i}+\delta_{i}+n_{1}^{(2)}\left(t_{*}, r_{*}\right) . \tag{8.3}
\end{equation*}
$$

$\delta_{;}$represents the (still unknown) inpat 'rom non-hydrodynamic fluctuations; $n_{1}^{(1)}$ is the contribation analogous to $\epsilon N_{i}^{(1)}$ in (5.1), arising from the gralients of meso-averages. Note that in (8.3) we dropped contributions nonlinear in the mass current; indeed, we should be able to determine the viscosity trom just linear hydrodynamics. ${ }^{18}$

We now derive the equations for noisy hydrodynamics. As usual, we start from the microscopic conservation relations (3.16) and (3.17) and we take their meso-averages:

$$
\begin{align*}
& \sum_{i}\left[n_{i}\left(t_{*}+1, r_{*}+c_{i}\right)-n_{i}\left(t_{*}, r_{\star}\right)\right]=0  \tag{8.4}\\
& \sum_{i} c_{i}\left[n_{i}\left(t_{*}+1, r_{*}+c_{i}\right)-n_{i}\left(t_{*}, r_{*}\right)\right]=0 \tag{8.5}
\end{align*}
$$

Substituting (8.3) into (8.5), we obtain

$$
\begin{align*}
& \left.\frac{1}{b} \sum_{i} c_{i}\left|p\left(t_{*}+1, r_{*}+c_{i}\right)-D\left(t_{*}, r_{*}\right)\right|+\frac{D}{c^{2} b} \sum_{i} c_{i} c_{i} \cdot J\left(t_{*}+1, r_{*}+c_{i}\right)-J\left(t_{*}, r_{*}\right) \right\rvert\, \\
& +\sum_{i} c_{i}\left|n_{i}^{(1)}\left(t_{*}+1, r_{*}+c_{i}\right)-n_{i}^{(1)}\left(t_{*}, r_{*}\right)\right|=\mathbf{f}\left(t_{*}, r_{*}\right) \tag{8.6}
\end{align*}
$$

where

$$
\begin{equation*}
f\left(t_{*}, r_{*}\right)=-\sum_{i}^{n} c_{1}\left[\delta_{1}\left(t_{*}+1, r_{*}+c_{i}\right)-\delta_{1}\left(t_{*}, r_{*}\right)\right] \tag{8.7}
\end{equation*}
$$

is the random force. Ueing (8.1), (8.2), (8.3), (8.4), and (8.5), we can also write (to leading orrier in cradiente)

$$
\begin{equation*}
f\left(t_{*}, r_{*}\right)=\left\langle\left.\frac{1}{c^{2} b} \sum_{j}\left(c^{2} e_{i}+D c_{1} \cdot c, c_{i}\right)\left[\tilde{n}_{j}\left(t_{*}+1, r_{*}+c_{i}\right)-\tilde{n}_{:} ; t_{*}+1, r_{*}+c,\right) \right\rvert\,\right\rangle_{m a} . \tag{8.8}
\end{equation*}
$$

The 1.h.s. of (8.6) is expmoded in powers of gredienta (i.e. of c), as we have done in section S. However, we keep paite diferences rather than derivatives in time because of the presence of the rapidly varying random force. Since we only want to identify the shear viscosity (the bult viscosity is zeru), it sumpes to extract the solenoidal part of the

[^4]hydrodynamical equation For this and other reasons it is better to work in Fourier space. We delne the (spatial) Fourier transform of the fluctuating Boolean Ield by
\[

$$
\begin{equation*}
\left.\bar{n}_{1}\left(t_{\infty}, r_{-}\right)=\sum_{k} e^{i k r_{\cdot}} \tilde{n}^{\left(t_{\infty}, k\right.}\right), \tag{89}
\end{equation*}
$$

\]

Where the components of $k$ are multiples of $2 \pi$ over the lattice periodic.ules in the various directions. We smilarly define $\hat{j}$ and $\hat{f}$, the Founer transforms of the mass current and the random force. Their solenoidal parts, projection on the byperplane perpendicular to $\mathbf{k}$, are denoted $\boldsymbol{j}_{\perp}$ and $\hat{\boldsymbol{f}}_{\perp}$.

To leading order in $k$, we obtain trom (8.8), using (2.5)

$$
\begin{equation*}
\hat{f}_{\perp}\left(t_{-}, \mathbf{k}\right)=-\sum_{j} i \mathbf{k} \cdot c_{j}\left(c_{j}-\frac{c_{j} \cdot \mathbf{k} \mathbf{k}}{k^{2}}\right) \hat{n}_{j}\left(t_{-}+1, \mathbf{k}\right) . \tag{8.10}
\end{equation*}
$$

The meso-averaging is just the restriction tha: $k<e$. Fourier transforming (8.6) and taking the solenoidal part, we obtain for small $k$

$$
\begin{equation*}
j_{\perp}\left(t_{+}+1, k\right)-j_{\perp}\left(t_{-}, k\right)+\nu k^{3} j_{\perp}\left(t_{*}, k\right)=\hat{f}_{\perp}\left(t_{-}, k\right) \tag{8.11}
\end{equation*}
$$

This is our discrete Langevin equaticn. Note that $\nu$ is the (total) viscosity $\nu=\nu_{\mathrm{c}}+\nu_{p}$. In principle we must expand to second order in $k$ to obtain the viscous terms, but we could as well have written the l.h.s of (8.11) a priori, since we want to ase (8.11) to determine the viscosity. It is atraightorward to solve the linear Anite-diference equation (8.11). From the solution, we calcalate the variance of $J_{\perp}$ and obtain, when the viscous damping time $1 /\left(\nu k^{2}\right)$ is large compared to the correlation time of the random force

$$
\begin{equation*}
\left.\left.\langle | j_{\perp}\left(l_{2}, k\right)\right|^{2}\right\rangle=\frac{1}{2 \nu k^{2}} \sum_{l_{-}=-\infty}^{1, x+\infty}\left(f_{\perp}\left(l_{-}, k\right) \cdot f_{\perp}^{\prime}\left(f_{-}, k\right)\right\rangle, \tag{8.12}
\end{equation*}
$$

Where the asterisk denotes complex conjugation. The variance of $j_{\perp}$ can aloo be calculated directly using (8.1) and

$$
\begin{align*}
& \left\langle\tilde{n}_{1}\left(t_{*}, \rho_{-}\right) \tilde{n}_{1}\left(t_{-}, 0\right)\right\rangle=\left\langle\tilde{n}_{1}^{3}\right\rangle \delta_{0}, \delta_{\rho_{0}}  \tag{8.13}\\
& \left\langle\hat{n}_{1}^{2}\right\rangle=\left(n_{1}^{2}\right)-\left\langle n_{1}\right\rangle^{s}=d-d^{2} .
\end{align*}
$$

Where $\delta_{\rho}$. denotes a Kronecker delta in the spatial separaticn $\rho_{n}$. We obtain

$$
\begin{equation*}
\left.\left.\langle | j_{\perp}\left(l_{-}, k\right)\right|^{2}\right\rangle=\frac{1}{V} b c^{2} d(1-d) \frac{D-1}{D}, \tag{8.14}
\end{equation*}
$$

where 1 denotes the total number of lateice points in the periodicity volume. Thus. the L.h of $(812)$ is $k$-independent We evaluate the $t h s$ of $(812)$ in the limit $k \rightarrow 0$, using (8 10) We ghp some intermediace steps in which we (i) age the stationarity of the Auctuations at equilibrium. (i) use the isotropy of gecond and lourth order gymmetrical
tensors, (iii) interchange the $\mathbf{k} \rightarrow 0$ limit and the infinite summation over $l_{\mathrm{n}} .{ }^{10}$ Identifying the two expressions (8.12) and (8.14), we obtain for the viscosity

$$
\begin{align*}
\nu & =\frac{D}{2(D-1)(D+2)} \frac{1}{b c^{2}} \frac{1}{d(1-d)} \frac{1}{V} \sum_{t_{0}=-\infty}^{t_{+}=+\infty} \sum_{i j a \beta} Q_{i a \beta} Q_{j a \beta}\left\langle\hat{n}_{i}\left(t_{*}, 0\right) \hat{n}_{j}^{*}(0,0)\right\rangle \\
& =\frac{D}{2(D-1)(D+2)} \frac{1}{b c^{2}} \frac{1}{d(1-d)} \sum_{t_{*}=-\infty}^{t_{n}=+\infty} \sum_{\rho_{*} \in C} \sum_{i j \alpha \beta} Q_{i \alpha \beta} Q_{j \alpha \theta}\left\langle\tilde{n}_{i}\left(t_{*}, \rho_{*}\right) \tilde{n}_{j}(0,0)\right\rangle, \tag{8.15}
\end{align*}
$$

with

$$
\begin{equation*}
Q_{i a \beta}=c_{i \alpha} c_{i \beta}-\frac{c^{2}}{D} \delta_{a \beta} . \tag{8.16}
\end{equation*}
$$

This completes the Inctuation-dissipation calculation of the viscosity. A consequence of the Fouricr-space representation (the upper halt of (8.15)) is the positivity of the viscosity; indeed, the viscosity is, within a positive factor, the time-summation of the antocorrelation of $\sum_{i} Q_{\text {iap }} A_{i}\left(\ell_{n}, 0\right)$.

Seversl comments are now in order. It is easily checked that the $\mathrm{I}_{\mathrm{*}}=0$ contribution to the viscosity (lower part of (8.15)) is $c^{2} /(2(D+2))$, that is, just the opposite of the "propagation viscosity" $\nu_{p}$ introduced in section 7. The viscosity is the sum of the collision vacosity $\nu_{0}$ and of $\nu_{p}$. Using the identity

$$
\begin{equation*}
\sum_{t=-\infty}^{1+\infty} z\left(t_{*}\right)=2 \sum_{t=0}^{1+\infty+\infty} z\left(t_{*}\right)-2(0), \tag{8.17}
\end{equation*}
$$

(for an even function $Z\left(L_{w}\right)$ ), we And that $\nu_{c}$ has a representation similar to (8.15) (lower part), with an additional factor of 2 and the summation over $f_{*}$ extending only from 0 to $\infty$. We thereby recover an expression dertved in Ref. 41, using a diacrete variant of the Green-Kubo formalism. It is reassuring to have two completehy different derivations of the viscosity, since we consider our fuctuation-dissipation dertration somewhat delicate.

It in of interest that the Inctuation-dinoipation dertvation gives directly the (total) viscosity. This angseat that the splitting into collision and propagation riscosities is an artefact of our molti-scale formalim.

There is no cloned form representation of the correlation fanction $\left\langle n_{1}\left(h_{n}, p_{+}\right) n_{j}(0,0)\right\rangle$, except for short times. However, (8.15) is a good starting point for a Monte-Carlo calculation of the visconity (d. Rel 41).

In ong dertvation we heve dropped all contributions from nonlinear ferms in the mass corrent J . Is this justifed? If we reinstate the nonlinear terms, we obtain, for the solenoidad part of the meso-averaged masa curreat, the Navier-Stokes equations (7.17) of section 8 with the edditional random force, the Fourier representation of which in given by (8.10). On mecroscopic scales this force may be considered as $\delta$-correlated in time. Its spectrum follows, ior small $k$ a $k^{D+1}$ power-law. ${ }^{17}$ The Navier-Stokes equations with this kiod of

[^5]power-law forcing is one of the few problems in nonlinear Statistical Fluid Mechanics which can be syatematically analyzed by renormalization group methods. ${ }^{(70,71)}$ For $D>2$, the nonlinear term is irrelenat for small $k$ so that our calculation of the viscosity is insitimate. At the "crossover" dimension $D=2$, the nonlinear term becomes "marginal"; it produces a renormalieation of the viscosity which is then logarithmically scaie-dependent. Thas, in the limit of infnite scale-separation, the viscosity becomes infaite in two-dimensions. This is an instance of the known divergence of transport coefflients in two-dimensional Statiatical Mechanics. ${ }^{(07.72)}$ Alternatively, the divergence of the yiscoaity in two-dimensions can be viewed as due to the presence of a "long-timetali", proportional to $t_{-}^{-D / 2}$, in the correlation function appearing in (8.15). Aitempts have been made to observe long-time-tails and acale-dependence of the viscosity in Monte-Carlo simulations of Lattice gas modele. (c,91.41.42) This is not easy because (i) the effecis show op only at very long times (or large scales) and may then be hidden ty Monte-Carlo noise (insufficient averaging) (ii) the effects should get weaker as the number $b$ of cells per node increases (ci end of sectlon 8.2).

Finally, the noisy bydrodynamics formalism can be ased to estimate to what extent the microscopic noise contaminates the hydrodynamic macroucopic aignal. Estimates, asauming the signal to be meso-averaged in apace and tlme, have been made in the context of tully developed incompressible two- and three-dimensional turbalence. ${ }^{14}$ It has been fond that in two dimensions noise be relevant only at scales leas than the dissipation scale, while in three dimensions this happens only far ont in the dissipation range. ${ }^{(73)}$

## E. The lattice Doltamane approarmation

Explicit calcalation of transport coeffienta can be done for Laticy gases, using the Boltrman approdimation. In this approximation one asoumes that particles entering a collision process have no prior correlations. The microdyonaleal fonnalism of section 3.1 is particularly well suited for $d$ rtring what we shall call the lattice Boltmano equation. We take the ensemble avarage of eq. (3.9). The Boolean variablea no become the mean popalations $N_{1}$. The average of the collinion function $\Delta_{i}$ can be completely factoricod, thanta to the Boltemann approcimation. We obtaln

$$
\begin{align*}
& N_{i}\left(L_{\infty}+1_{1} \mathbf{r}_{*}+\mathbf{c}_{i}\right)=N_{i}\left(f_{*}, r_{*}\right)+\Delta_{i}^{B 0 / 4} \\
& \Delta_{i}^{\text {LUNA }}=\sum_{0,0}\left(0_{i}^{\prime}-0_{i}\right) A\left(0 \rightarrow 0^{\prime}\right) \prod_{j} N_{j}^{0 j}\left(1-N_{j}\right)^{\left(1-0_{j}\right)} . \tag{8.18}
\end{align*}
$$

Here, all the $N$,'s are ovaluated at $t_{n}$, and $\mathbf{r}_{\mathrm{m}}$. The $A\left(0 \rightarrow \rho^{\prime}\right)$ 's, the transition probabilitios introdaced in section 2.4, are the averages of the Boolens transition variables far. Note that the (Boltzmana) collision function $\Delta$ ant vasiabes at equllibriam.

The Boltumenn approxdmation in ordinary gwee to woclated with low density gitup tions, when the mean-tree path is so large that perticles entering a collbion came moorly from distant uncorrelated regions. The Boltrmano approximation for a lattice gas appeary

[^6]to have a very broad validity, not particularly restricted to low densities. ${ }^{10}$ We shall come back to the matter at the end of this section.

Our lattice Boltrmann equation (8.18) is a Anite difference equation. There is a differential version of it, obtained by Taylor-expanding the finite differences to first order, namely

$$
\begin{equation*}
\partial_{i} N_{i}+c_{i} \cdot \nabla N_{i}=\Delta_{i}^{\mathrm{Boln}}, \tag{8.19}
\end{equation*}
$$

Where $\Delta_{i}^{\text {pell' }}$ is defineds as in (8.18). Boltzmann equations of the form (8.19) have been extensively studled as discrete velocity approximations to the ordinary Boltzmann equation. ${ }^{(14-10,18)}$ The (differential) Boltzmann formalism has been applied to various lattice gas models. ${ }^{(32.36)}$ This formaism correctly captures all hydrodynamic phenomena involving only first order derivatives. Indeed, for these, we have seen that only the equilibrium solutions matter, and the latter are completely factorized. Difusive phenomena involve second order derivatives. Hence, the propagation viscosities (cl. section 7), which are an effect of lattice-discreteness, are not captured by the (diferential) Boltzmann equation. At low densities, where collision viscosities do minate over propagation viscosities, the discrepancy is irrelevant.

We do not intend to engage into extended discassions of the consequences of the latice Boltzmann equation, because most of the derivation of the hydrodynamical equations is independent of this approxdmation. These are however two important results which follow from the lattice Boltrmann equation. The Arst one concerns the irreversible approach to equilibriam. It is derived by adapting an H-theorem formalisin to the folly discrete context (see Appendix F by Hénon).

The second result is an explicit derination of the viscosity. From the Boltemann equation this is unually done by a Chapman-Engkog formalism ${ }^{(74,58)}$ (see also Gatignol's monography Ref. 16). This formaliom is easily adapted to the lattice Boltumann equation. ${ }^{(70)}$ With the general moith-scale formalism of sections $5-7$, we have already covered a substantia traction of the groand. Furthermore an elteraative derivation, which stays completely at the microscopic level is presented in this volome by Bénon who also diecusses consequences of his expicit visconity-formula. ${ }^{(30)}$ We athall therefore be brief.

TLe problem of the viscodity amounts to Anding the coefleient $\phi$ relatios the gradieat of the maes carreat pu to the trat arder pertarbation $N_{i}^{(1)}$ of the mean popalation, through (CL eq. (6.14) of rection 5)

$$
\begin{align*}
& N_{i}^{(1)}=\psi Q_{i a \beta} \partial_{1 a}\left(\rho u_{\beta}\right) \\
& Q_{i a \beta}=c_{i a} c_{i \beta}-\frac{\sigma^{d}}{D} \delta_{a \beta} . \tag{8.20}
\end{align*}
$$

We start trom (5.1) with $N_{i}^{(0)}$ gtren by (4.14). We substitute into the lattice Boltzmand equation (8.18) and identlfy the terms $O(c)$. For thls we Taylor-expand anite differences

[^7]to frst order, use (5.6) and (5.7) to express time-derivatives in terms of space-derivatives, and ignore all terms beyond the linear ones in the velocity. We obtain
\[

$$
\begin{equation*}
\frac{D}{b c^{2}} Q_{i a \beta} \partial_{l a}\left(\rho u_{\beta}\right)=\sum_{j} A_{i j} N_{j}^{(1)} \tag{8.21}
\end{equation*}
$$

\]

Here,

$$
\begin{equation*}
A_{i j}=\left[\frac{\partial \Delta_{i}^{\text {Eolls }}}{\partial N_{j}}\right]_{N_{i}=-/ / b}, \tag{8.22}
\end{equation*}
$$

is the linearized collision matrix, evaluated at the zero-velucity equilibrium, which can be expressed in compact form as ${ }^{(30)}$

$$
\begin{equation*}
A_{i j}=-\frac{1}{2} \sum_{00^{\prime}}\left(o_{i}-a_{i}^{\prime}\right) A\left(0 \rightarrow s^{\prime}\right) d^{p-1}(1-d)^{n-p-1}\left(o_{j}-s_{j}^{\prime}\right), \quad p=\sum_{i} 0_{i} . \tag{8.23}
\end{equation*}
$$

We eliminate $N_{i}^{(1)}$ between (8.20) and (8.21), to obtain

$$
\begin{equation*}
\left[\frac{D}{b c^{2}} Q_{i a A}-\psi \sum_{j} \Lambda_{i j} Q_{j a \beta}\right] \delta_{1 a}\left(\rho u_{\beta}\right)=0 \tag{8.24}
\end{equation*}
$$

This should hold for arbitrary gradients of the mass current. Thus, the quantity between square brackets ranishes. This means that, for any $(\alpha, \beta), Q_{i a p,}$ considered as a vector with components labelled by $i$, is an eigenvector of the linearired collision matrix with eigenvalue $D /\left(b c^{2} \psi\right)$; direct proof of this may be jerived trom the $g$-invariance. From (8.24) we can easily caculate $\psi$; the simplent is to multiply the vanishing square bracket by $Q_{\text {iap }}$ and sum uver $i, \alpha$, and $\beta$. $L$, in addition, we assume the isotropy of fourth order tensors, we can use (0.17) to obtain a closed-form expression for the collision viscosity

$$
\begin{equation*}
\nu_{0}=-\frac{\alpha^{s}}{D+2} \frac{\sum_{i a \beta} Q_{i a \beta}^{2}}{\sum_{i j a \beta} Q_{i a \beta} A_{i j} Q_{j a \beta}} . \tag{8.25}
\end{equation*}
$$

In Appendix E we give explicit formulae calculated from (8.25) for the viscosities of the FHP models (including those with rest-particles which require minor amendements of cur formaliom).

We tinally address the question of the validity of the lattice Boltzmann equation. Comparisons of the viscosities obtained trom simulations ${ }^{(23,27,20)}$ or Monte-Carlo calculations ${ }^{(70)}$ with the predictions of the lattice Boltrmann approximation suggest that the validity of the latter is not limited to low deasities. We know that equilibrium solutions are factorized and that iransport coefficients can be calculated with arbitrarily weak macroscopic gradients. However, thls cannot be the basis for the valldity of the Boltzmana approximation: a weak mascroscopic gradient implies that the probability of changing the state of a given node trom its equilibrium value is small; but when such a change takes place, it produces a atrong microscopic perturbation in its environment. Otherwise there would be no (weak)
divergence of the viscosity in two dimensions; indeed, the Boltumann approcimation does not capture nobe-induced remormalization effects (ci. end of section 8.1). A more bikely explanation of the success of the lattice Boltumann approdimation may be that it is the leading order in some kind of $1 / b$ expansion, where $b$ is the number of velocity cells at each node. At the moment, we cas only support this by the following hearistic argument. Deviations trom Boltzmann require correlations between paricles entering a collision. The latter arise trom previons collisions, ${ }^{30}$ when $b$ is large the weight pertaining to such events ought to be small.

## 0. 3 The Revaolde namer

Knowing the Hinematic shear viscosity in terms of the density and the collision rules, we can calculate the Reynolds number asoociated to a largeracale flow.

A nataral onit of length is the lattice constant (distance of adjacent nodes), which has been taken equal to one for the two-dimensional HPP and FHP models. The fourdimensional FCHC model has a lattice constant of $\sqrt{2}$, but its three-dimensional projected version, the psendo-4-D FCHC model, resides on a culic battice wich has also unit lattice constant. The sime necessary for microscopic information to propagate trom one node to its connecting neighbours defines a natural onit of time. Whe then have a natural onit of velocity: the speed necessary to travel the lattice coastant (or the projected lattice corstant for the pseado-4D model) in a anit time. In these anits, the characteriatic acale and velocity of the fow will be denoted by $l_{0}$ and $v_{0}$.

The stendard delnition of the Reynolds namber is

$$
\begin{equation*}
R=\frac{\text { characteristic seale } \times \text { characteritic velocity }}{\text { binematic abear viscoity }} . \tag{8.28}
\end{equation*}
$$

Ir deriving the Navier-Stokea equations in sertion 7.3. ve resealed apace, time, velocity, pressure and viscosity (cl. eq. (7.16)). The rescaling of space (by e) and of velocity (by $e^{-1}$ ) cencel in the namerator of (8.26). The resealed raconty is $\nu\left(\rho_{0}\right)=\nu\left(\rho_{0}\right) / \rho\left(\rho_{0}\right)$. Heace, the Reynotds number is

$$
\begin{equation*}
R=\ell_{0} m_{0} \frac{\theta\left(\infty_{0}\right)}{v\left(\infty_{0}\right)} . \tag{8.27}
\end{equation*}
$$

In order to operate it as incompreasible rifime, the volocity $x_{0}$ athoald be suall compared to the speod of cound $c_{3}$. The latter is modeldependeat: $c_{1}=1 / \sqrt{3}$ for FHP-I, $c_{0}=\sqrt{3 / 7}$
 therfore reexpres the Reynolds number to terms of the Mach number

$$
\begin{equation*}
M=\frac{\omega_{0}}{e_{0}} \tag{8.28}
\end{equation*}
$$

We obiain

$$
\begin{equation*}
R=M C_{0} R_{+}\left(\infty_{0}\right), \tag{8.29}
\end{equation*}
$$

where

$$
\begin{equation*}
R_{0}\left(\rho_{0}\right)=\frac{c_{0} g\left(\rho_{0}\right)}{\nu\left(\rho_{0}\right)} \tag{8.30}
\end{equation*}
$$

[^8]contains all the focal information.
In fow simalationg osing latice gases, it is of interest to operate at the deasity which maximizes $R_{\text {. }}$. Let as wort this ont for the simplest esse of FHP.I. For the riscosity, we uge the lattice Boltzmann value given in Afpendix E. We have
\[

$$
\begin{equation*}
\rho\left(\rho_{0}\right)=\frac{1}{2} \frac{1-2 d}{1-d}, \quad \nu\left(\infty_{0}\right)=\frac{1}{12 d(1-d)^{s}}-\frac{1}{8}, \quad d=\frac{\rho_{0}}{6} . \tag{8.31}
\end{equation*}
$$

\]

Here, $d$ is the mean deasity per cell. Subatituting in (8.30), we and that

$$
\begin{equation*}
R_{*}^{a x}=\operatorname{mex} R_{m}=0.387, \quad \text { for } d=d_{\text {mas }}=0.187 . \tag{8.32}
\end{equation*}
$$

Results for FHP-D and FHP-II are given in Appendix E. Note that a gain of about a factor 6 is achieved in going from FHP-I to FHP-II, because the latter includes many more collisions. For the preado-4-D FCHC model there is work in progress on the optimiration of collisions. It is already known that $R_{+}^{\text {ean }}$ is at least $8 .{ }^{(77)}$

High Reynolds number incompresible risbleat lows have a whole range of scales. The smallest effectively excited acale is called be dissifation scale and denoted $L_{4}$. It is then of interest to find how many lattice coneicats aro contained in $l_{\text {d }}$, since this will determine how effective latife gaces are in gimalebla; hish Reynolds number Dows. ${ }^{\text {(1.ss) }}$ For this, let $l_{0}$ denote the integral scale of the low. Between $l_{0}, l_{d}$ and the Reynolds number $R$, there is the following relation

$$
\begin{equation*}
\frac{4}{L_{0}}=R^{-m}, \tag{8.33}
\end{equation*}
$$

where in $=1 / 2$ in two dimensions and $m$ as is in three dimensions. Is two dimensions, (8.33) is a consequence of the Batchelorikinctiasn ${ }^{(78,70)}$ phenomenolopeal theory of the enstrophy cacade, which in vill supported by aumarical simulations. ${ }^{(0)}$ In three dimensions, (8.35) tollows from the Kotmoyosor ${ }^{(61)}$ phemonenological theory of the enerty eascade, which is well sapported ${ }^{\text {91 }}$ by axperimontal data ${ }^{(09)}$ Uatas (8.29) and (8.35) and ecouming that $R_{0}$ bae itt madmam ratoe $R_{n}^{n-2}$, ve obtain

$$
\begin{equation*}
L_{4}=\left(M R^{a-N}\right)^{-t} S_{0}^{\dagger}=\left(A^{\prime} R^{a+1}\right)^{-1} R^{t} \text { ln 2D } \tag{8.34}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{4} \pi\left(N R^{n+\infty}\right)^{-4} L_{0}^{f}=\left(N R^{\infty-a}\right)^{-1} R^{f} \text { in 2-D. } \tag{8.35}
\end{equation*}
$$

In all casen, we see tian $l_{\alpha} \rightarrow \infty \times R \rightarrow \infty$, hut minch more glowly in three than in two dimengions. We are thus ansured that at hist Reytolds unombers the separation of scale between the lattice constant and $l_{d}$, neceseary for hydrodynamic bebaviour ia aatiafed. Having it ton well satisted may bowever be \& mixed blensing, as stressed in Ref. 33. Indeed, in hydrodyamic simalations asing lestice gasey it is not desirable to bave too

[^9]much irrelemant microscopic information. Fortunately, in three dimensions there is no serions problem. To Wuatrate this potan, we take $M=0.3$, Mech namber at which compressibility effects can be safely ignored, we take the maximum tnown value $R^{\text {eac }}$ ェ 0 for the FCHC, and we take $l_{0}=10^{\circ}$, a fairly large value which implies a memory requirement of at least 24 gigabits; from (8.35) we find that $\ell_{d}$ is about three lattice constants. In two dimenslons, similar calculations with the FHP models give $l_{d}$ 's of several tens of lattice constants. It is therefore of interest in two dimensions to try to decrease the viscosity, thereby increasing $R_{p}^{\text {ax }}$. One way is to ase the four-dimensional FCBC model projected down to two rather than three dimensione. Note that it in not correct to infer from dimensional analysis that necessarity $R_{m}^{\text {ean }}$ must be $O(1)$. $R^{\text {eax }}$ is very much a function of the complexity of collisions. For exsmple, by going trom FHP-I to FCHC, $R_{n}^{-a}$ increases more than twenty times.

## 9. Conclusion

In Statistical Mechanics there are many instances where two models, microscopicaly quite different, have the same large-scale properties. For example, the laing model and a real Ferromafnet have presumably the same largescale critical behaviour. Similarly, the lattice gases studied in this paper, such a FEIP and FCHC, are mecroscopically mdisticrishable from real Iuida. This provides as with an ettractive alternstive to the traditiona simub tions of Fluid Mechanies. In latice gas simulations, wo just manipulate bits representing occupation of microscople celle. The physical interpretation need not be in terme of particles moving and colliding. The Idea can clearity be extended to include processes such as ctemical reactions or moltt-phace fow. (s3-80) An open quention in wether there are ceilular antomata implementations of proceaces which in the real world do not have a dipcrete microncopic origin, anch as propagation of e.m. wavea. More generally, what are the P.D.E.'s which can be eflecieaty implemeated on ceDolar antomata? We emphaize enfejenth, because there are always brate fore implementations: replace destvatives by Anite diferences on a regratar grid and une latte boating point trencations of the contiovous delds. The result may be vewed an a coltular astomaton, but ooe in which there is so "bt democracy", beoker as there 1 a a ind hiararchical order between the bits.

Our destration of hydrodyamiks from the mierodya amies leaves room for brprovemont. A hey ammption made is section 4.1 may be formalated an follows. Among the inveriant menoures of the milerodyzambeal equations, only the completely factoriced ones (which play the robe, hese, of the microcanomked emsemble) is relevant in the liodit of large Lattices. Om a Anite lattice with determiniatic and invortible opdating ralea, we eqpeet thas there are many other inveriant meanuses. Indoed, phace space is a Anite set and updaling is a permatation of this set; it to thus unillely that there should be a closed orbit going through all pointw. So, we do aot expect the drecrete eqnivalent of an argodic theorem Anyway, ergodle resolts should be irrelevant. On the one hand, on ap $L \times L$ lettice with 6 bits per node its takes $z^{\prime \prime} L^{D}$ opdates to vialt all condsurations (Lf they are aceeasible). On the other hand, we know (trom sirnulationa) that lecal equilibriom is achieved in a few updates and global equilibriom is achieved on a difusive time acale (approcimately $L^{\prime}$ ). We believe that, ou large Lattices, the lactorized equillbrium ditributions constitate some kind of "Axed point" to which there is rapid convergence of the iterated Boolean map
defined by the microdynamical equations of section 3.1. Understanding this process should clarify the mechanism of irreversibility in latice gases and, eventually, in real gases.
Achroolalgementa
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## Appendlx A. Bale symmetries of HPP, PHP, and FCHC modela

We show that the models BPP, FHP, and FCHC, introduced in section 2, satisfy the symmetry assumptions (i) throogt (iv) of section 2.4. Assamptions (i) and (ii) are obvions for all three models. Let as consider (iii) and (iv) successively for the three models.

## IPP

Let as take the $x_{1}$ axis in the direction of the vector $c_{1}$. The isometry group $\mathcal{G}$ of the velocity set is generated by permutationg of the $x_{1}$ and $x_{2}$ coordinates and reversals of any of them. Cleariy, any two vectors $c$, and $c$, can be exchanged by some isometry, so that assumptlons (iii) bolds. Consider a particular vector, say, $e_{1}$. The subgronp $\mathcal{G}_{1}$, leaving $c_{1}$ invariant reduces to the identity and reversal of $x_{2}$; this implies parts (a) and (b) of assumption (iv).

## FTP

Let as take the $x_{1}$ axis in the direction of $c_{1}$. The isometry groap $\mathcal{G}$ in now generated by rotations of $\pi / 3$ and reversal of the $x_{3}$ coordinate. Assnmption (iii) is obvious. The subgroup $g_{1}$ reduces again to the identity and the reversal of $x_{2}, s 0$ that (iv) follows.

## rasc

The FCHC lattice was delned in sertion 2.3 with explicit reference to coordinates $z_{1}, z_{2}$, $x_{i}$, and $x_{4}$. In this coordinate system, the velocity set is formed of

$$
\begin{array}{lll}
( \pm 1, \pm 1,0,0), & ( \pm 1,0, \pm 1,0), & ( \pm 1,0,0, \pm 1)  \tag{A.1}\\
(0, \pm 1, \pm 1,0), & (0, \pm 1,0, \pm 1), & (0,0, \pm 1, \pm 1) .
\end{array}
$$

By the orthonornal change of variables

$$
\left(\begin{array}{l}
y_{1}  \tag{A.2}\\
y_{1} \\
y_{3} \\
y_{1}
\end{array}\right)=\frac{1}{\sqrt{2}}\left(\begin{array}{cccc}
1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & -1 & 1
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{1} \\
x_{3} \\
z_{4}
\end{array}\right),
$$

the relocity set becomes

$$
\begin{gather*}
( \pm \sqrt{2}, 0,0,0), \quad(0, \pm \sqrt{2}, 0,0), \quad(0,0, \pm \sqrt{2}, 0), \quad(0,0,0, \pm \sqrt{2}), \\
\left( \pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right) . \tag{A.3}
\end{gather*}
$$

The isometry group $g$ in generated by permutationg and reversals of the $z_{a}$ coordinates and by ithe symmetry with respect to the byperplane $x_{1}+x_{2}+x_{3}+x_{4}=0$, which is conveniently written ba terms of yo coordinates as

$$
\begin{equation*}
\Sigma: \quad\left(y_{1}, y_{2}, y_{3}, y_{4}\right)-\left(-y_{3}, y_{2},-y_{1}, y_{4}\right) . \tag{A.4}
\end{equation*}
$$

Assumption (lii) is obrions in any of the coordinate rystems. As for assumption (tv), let as consider the subsroup $G_{1}$ leaving Lovariant, say, the rector with yo coordiuates $(0,0,0,1 / \sqrt{2})$. The reatriction of $g_{1}$ to the byperplase $y_{1}=0$ is geserated by the identity, pertautations, and reverals of $y_{1}, y_{2}$, and $y_{3}$. Assumptions (a) and (b) follow readily

## Appendix B. Symmetry-related propertles

Using assumptions (i) through (iv) of section 2.4, we prove properties P1-P6
P1 Parity-in rariance. The set of velocity vectors is invariant onder space-reversal.
Indeed, on a Bravaig Lattice, vectory connecting neighbouring nodes come in opposite pairs.
P2 Any set of i-dependent vectors $v_{i \alpha}$, which is $\mathcal{G}$-invariant, is of the form $\lambda \boldsymbol{c}_{\mathrm{i}}$.
We write $v_{i}$ as the sum of its projection on $c_{i}$ and of a vector perpendicular to $c_{i}$. This decomposition being $g$-invariant, the intter vector vanishes by (iv - a).
PS Any set of i-dependent tensors $t_{i a \beta}$, which is $\underline{\varepsilon}$-invariant, is of the form $\lambda c_{i a} c_{i \beta}+\mu \delta_{a \beta}$.
To the tensory $t_{i \alpha \beta}$, we associate the linear operators $T_{i}: x_{\alpha} \mapsto t_{i \alpha \beta} x_{\beta}$. $\mathcal{G}$-iuvariance means that the $T_{i}$ 's commute with any lattice isometry leaving $\mathrm{c}_{\mathrm{i}}$ invariant. We now write the $\mathcal{G}$-invariant decomposition

$$
\begin{equation*}
T_{i}=P_{i} T_{i} P_{i}+\left(I-P_{i}\right) T_{i} P_{i}+P_{i} T_{i}\left(I-P_{i}\right)+\left(I-P_{i}\right) T_{i}\left(I-P_{i}\right), \tag{B.1}
\end{equation*}
$$

-here $I$ is the identity in $R^{D}$ and $P_{i}$ is the orthogonal projection on $c_{i}$. The second opere tor in ( $B .1$ ), applied to an arbitrary vector $w$, sives

$$
\begin{equation*}
\left(I-P_{i}\right) T_{i} P_{i} w=\frac{w \cdot a_{i}}{\sigma^{2}}\left(I-P_{i}\right) T_{i} c_{i} . \tag{B.2}
\end{equation*}
$$

The vectors $\left(I-P_{i}\right) T_{i} e_{i}$ are $G$-invariant and orthogonal to $c_{i}$, and thos vanish by (iva). The third operator in (B.1) vanishes for similar reasons (use the f-invariance of the transposed of the $T i^{\prime}$ ). The fourth operator in ( $B .1$ ) is, by ( fv -b) proportional to $I_{i}$, the identity to the subspace orthogonal to $\mathrm{c}_{i}$. Since $I=I_{i}+P_{i}$, the proof is completed.

We mention that we obtatiod PS by trying to formalize a readt nsea by Henon ${ }^{(9 P)}$ in deriving a cloned-form viecority formala
P 4 Lootropy of secand order tensors. Any $g$-invarinat tensor $t_{a s}$ ts of the form $\mu \delta_{a s}$.
This is a special esse of $P 3$, when there $L$ no $i$-dependence.
PS Any 9 -invariant third order tensor vaciahes.
This follows trom P1 (parity tovariance).
Pe Velocity moments. Odd order velocity moments vaniah. The second order velocity moment is stiven by

$$
\begin{equation*}
\sum_{i} c_{i} e_{i \beta}=\frac{b c^{\prime}}{D} \delta_{a \beta} \tag{B.3}
\end{equation*}
$$

The manishing of odd order moments is a consequence of P1. (B.3) follows trom P4 and the identity

$$
\begin{equation*}
\sum_{i} c_{i a} c_{i 0}=b c^{\pi} \tag{B.4}
\end{equation*}
$$

## Appendix C. Equillbrium coluthone

We prove the
Lemme. The following statements are equivalent:
(a) The $N_{i}$ 's are a solution of

$$
\begin{equation*}
\prod_{j} N_{j}^{\bullet_{j}^{\prime}}\left(1-N_{j}\right)^{\left(1-0_{j}^{\prime}\right)}=\sum_{j} A\left(\bullet \rightarrow \theta^{\prime}\right) \prod_{j} N_{j}^{\bullet_{j}}\left(1-N_{j}\right)^{\left(1-\theta_{j}\right)}, \quad \forall s^{\prime} . \tag{C.1}
\end{equation*}
$$

(b) The $N_{i}$ 's are a solution of the set of $b$ equations

$$
\begin{equation*}
\Delta_{1}(N) \equiv \sum_{00^{\prime}}\left(o_{i}^{\prime}-\theta_{i}\right) A\left(0 \rightarrow 0^{\prime}\right) \prod_{j} N_{j}^{\theta_{j}}\left(1-N_{j}\right)^{\left(1-\theta_{j}\right)}=0, \quad \forall i . \tag{C.2}
\end{equation*}
$$

(c) The $N_{i}$ 's are given by the Fermi-Dirac distribution

$$
\begin{equation*}
N_{i}=\frac{1}{1+\exp \left(h+q \cdot c_{i}\right)} \tag{C.3}
\end{equation*}
$$

where $h$ is an arbitrary real number and $q$ is an arbitrary $D$-dimensional vector.
Proof that (a) implies (b).
We multipty (C.1) by $\mathrm{e}_{;}^{\prime}$ and sum over all states $\mathrm{o}^{\prime}$ to obtain

$$
\begin{equation*}
\sum_{j}{a_{i}^{\prime}} \prod_{j} N_{j}^{d_{j}}\left(1-N_{j}\right)^{\left(1-\bullet_{j}^{\prime}\right)}=\sum_{0 o^{\prime}}{o_{i}^{\prime} A\left(\bullet \rightarrow o^{\prime}\right) \prod_{j} N_{j}^{o_{j}}\left(1-N_{j}\right)^{\left(1-a_{j}\right)} . . . . ~}_{\text {. }} \tag{C.4}
\end{equation*}
$$

In the L.h.s. of (C.4) we change the dummy variable of luto a and decorate it with a factor $A\left(0 \rightarrow \sigma^{\circ}\right)$, summed over $\sigma^{\prime}$, which it one by aormatiention of probability. Tranderring everything into the r.h.s., we obtatn (C.2). Note that the Lh.s of (C.2) resembles the "collision fanction" $\Delta$; of seetion 3.1 (eq. (3.9)), but is evaluated with the mean populations instead of the Boolean popalations $n_{i}$. The relation $\Delta_{1}=0$ expreases that there to no change in the mean popolations onder collindoas.
Proof that (b) lmples (c).
We define

$$
\begin{gather*}
N_{i} \geq \frac{N_{i}}{1-N_{i}}, \\
\Pi \equiv \prod_{j}\left(1-N_{j}\right) . \tag{C.b}
\end{gather*}
$$

Eq. (C.2) may be written

$$
\begin{equation*}
\Delta_{i} / \Pi=\sum_{\infty^{\prime}}\left(0_{i}^{\prime}-\theta_{i}\right) A\left(0 \rightarrow 0^{\prime}\right) \prod_{j} N_{j}^{0,}=0 . \tag{C.7}
\end{equation*}
$$

We now make ase of a trick employed in proving B-Theorems in disesete velocity models (nee Ref. 16, p.29). We multipty (C.7) by los $\mathcal{N}_{i}$, sum over $i$, and ase

$$
\begin{equation*}
\sum_{i}\left(a_{i}^{\prime}-a_{1}\right) \log N_{1}=\log \frac{\Pi_{2} N_{2}^{0_{j}^{\prime}}}{\Pi_{j} N_{j}^{\prime}} \tag{C,8}
\end{equation*}
$$

to obtain

$$
\begin{equation*}
\sum_{00^{\prime}} A\left(\theta \rightarrow \theta^{\prime}\right) \log \left(\frac{\Pi_{j} N_{j}^{\alpha_{j}}}{\Pi_{j} N_{j}^{\sigma_{j}}}\right) \prod_{j} N_{j}^{e_{j}}=0 . \tag{C.9}
\end{equation*}
$$

Semi-detailed balance $\left(\sum_{0} A\left(0 \rightarrow s^{\prime}\right)=\sum_{0^{\prime}} A\left(0 \rightarrow 0^{\prime}\right)=1\right)$ implies that

$$
\begin{equation*}
\sum_{0 o^{\prime}} A\left(0 \rightarrow 0^{\prime}\right)\left(\prod_{j} \dot{N}_{j}^{\theta_{j}}-\prod_{j} \dot{N}_{j}^{\theta_{j}^{\prime}}\right)=0 . \tag{C.10}
\end{equation*}
$$

Combining (C.9) and (C.10), we obtain

$$
\begin{equation*}
\sum_{00^{\prime}} A\left(e \rightarrow s^{\prime}\right)\left[\log \left(\frac{\Pi_{j} \mathcal{N}_{j}^{\varepsilon_{j}^{\prime j}}}{\Pi_{j} \mathcal{N}_{j}^{\theta_{j}^{\prime j}}}\right) \prod_{j} N_{j}^{\theta_{j}}+\prod_{j} \dot{N}_{j}^{\theta_{j}}-\prod_{j} \mathcal{N}_{j}^{\theta_{j}^{\prime}}\right]=0 . \tag{C.11}
\end{equation*}
$$

We make use of the relation $(x>0, y>0)$

$$
\begin{equation*}
y \log \frac{z}{y}+y-z=-\int_{z}^{y} \log \frac{t}{2} d t \leq 0, \tag{C.12}
\end{equation*}
$$

equality being schieved only when $2=y$. The l.h.s. of (C.11) is a linear combination of expressions of the form (C.12) with nonnegative weights $A\left(0 \rightarrow d^{\prime}\right)$. For it tu vanish, we mast have

$$
\begin{equation*}
\prod_{j} \mathcal{N}_{j}^{0_{j}}=\prod_{j} \mathcal{N}_{j}^{d_{j}}, \quad \text { whemever } A\left(0 \rightarrow A^{\prime}\right) \neq 0 . \tag{C.13}
\end{equation*}
$$

This is equivalent to

$$
\begin{equation*}
\sum_{i} \log \left(N_{i}\right)\left(a_{i}^{\prime}-a_{i}\right) A\left(a \rightarrow \theta^{\prime}\right)=0 \quad \forall_{0} \theta^{\prime} . \tag{C.14}
\end{equation*}
$$

(C.13) means that be $N_{i}$ in a collisiot inveriant. We now use saumption $(v)$ of section 2.4, conceratar the abseace of apurious invariate, to conclade that

$$
\begin{equation*}
\log N_{i}=h+q \cdot e_{i} \tag{C.15}
\end{equation*}
$$

Whech is the moat general collinion invariant (a linear combination of the mace invariant and of the $D$ momentum loveriants $)$. Reverting to the mean populations $N_{i}=N_{i} /\left(1+N_{i}\right)$, - obtals (C.s).

Proof that (c) impllet (a).
(C.s) implies

$$
\begin{equation*}
\sum_{\rho} \log \left(N_{1}\right)\left(0,-0^{\prime}\right)=0, \quad \text { whenever } A\left(0-0^{\prime}\right) \neq 0 . \tag{C.10}
\end{equation*}
$$

## This implies

$$
\begin{equation*}
\sum_{0} A\left(0-\theta^{\prime}\right)\left(\prod_{j} N_{j}^{\left(0_{j}-\theta_{j}^{\prime}\right)}-1\right)=0 . \tag{C.17}
\end{equation*}
$$

Using semi-detailed balance, this may be written as

$$
\begin{equation*}
1=\sum_{\rho} A\left(0 \rightarrow O^{\prime}\right) \frac{\prod_{j} \tilde{N}_{j}^{\sigma_{j}}}{\prod_{j} \dot{N}_{j}^{\sigma_{j}^{\prime j}}} . \tag{C.18}
\end{equation*}
$$

Reverting to the $N_{j}$ 's, we obtain (C.1). This completes the proof of the equivalence lemma

Appendix D. Incluston of body-forces
Uniag the game notation as in section 7.5, we wish to obtain a Navier-Stokes equation with a body-foree $f$, that is

$$
\begin{align*}
& \delta_{\mathbf{T}} \mathrm{U}+\mathrm{U} \cdot \nabla_{1} \mathrm{U}=-\nabla_{1} P^{\prime}+\nu^{\prime} \nabla_{1}^{2} \mathrm{U}+\mathbf{f} \\
& \nabla_{\mathbf{L}} \cdot \mathrm{U}=0 . \tag{D.1}
\end{align*}
$$

The force $f$ may depend on space and time and can be velocity-independent (case I; e.g. grarity) or linear in the relocity $U$ (case $\square$; e.g. Coriolis force). The ides is to introdace a bias in the trangition rales so as to sive a net momentom input. Since all the terms in the Navier-Stokes momentum equation are $O\left(\epsilon^{3}\right)$ and the bydrodynamic velocity is $O(\epsilon)$ (Defore rescalios), the bias should be $O\left(c^{2}\right)$ for case I and $O\left(e^{2}\right)$ for case $I$.

We give now the modifed form of the microdynamieal equation (3.9) appropriate for body-forces. We introduce, in addition to the Boolean (transition) variables $\xi_{\text {ef }}$ of rection 3.1, the B, olean variables $\xi_{\text {ow }}$ such that

$$
\begin{equation*}
\left\langle\varepsilon_{0}\right\rangle=B\left(0 \rightarrow o^{\prime}\right) . \tag{D.2}
\end{equation*}
$$

The $B\left(0 \rightarrow g^{\prime}\right)$ 's are a ate of transition probabilities asociated to the body-forre; they eatin'y normalization

$$
\begin{equation*}
\sum_{\sigma} B\left(0 \rightarrow 0^{\prime}\right)=1 \tag{D.3}
\end{equation*}
$$

and mass conservation

$$
\begin{equation*}
\sum_{i}\left(0_{i}^{\prime}-a_{i}\right) B\left(0 \rightarrow \theta^{\prime}\right)=0, \quad \forall_{1} \theta^{\prime} . \tag{D.4}
\end{equation*}
$$

They do not satisfy momentom conservation, semi-detailed balance and $\mathcal{G}$-invariance. The C'ro's are chosea independently at each diacrote time and aode anc the $B\left(0 \rightarrow a^{\prime}\right)$ 's may depead an apace and time; further constratote will be given below. We abo need a Boolean
 apply. The mean of $s$ in getren by

$$
\begin{align*}
& \langle s\rangle=\operatorname{mos}(\infty) c^{\circ} \\
& n=8 \quad \cos L \quad n=2 \text { case } L I . \tag{D.B}
\end{align*}
$$

This will take care of the reatigis ficton arising from the change of variablea (7.15). The modited microdynamical equation how

$$
\begin{align*}
& n_{1}\left(L_{0}+1, r_{*}+a_{i}\right)=n_{1}+\Delta_{i}(n) \\
& \Delta_{i}(n)=\sum_{0, n_{0}}^{\infty}\left(0_{i}^{\prime}-o_{i}\right)\left((1-\xi) \ell_{0 r}+\left\{f_{i, 0}^{\prime}\right) \prod_{j} n_{j}^{0_{j}^{\prime}}\left(1-n_{j}\right)^{\left(1-o_{j}\right)} .\right. \tag{D.6}
\end{align*}
$$

Let as evaluate the body-force resulting from the additional f' ferm. For this we maltiply by $e$, ad average over the equilibrium distribation; dertations from equilibrium
araing trom hydrodynamic gradients are irrelevant. We irnore the s-lactor since it jost provides the araling factor.

We begin with case $I$. The average in then evaluated over the zero-velocity eqpilibrium diatribution with density per cell d; we obtain

$$
\begin{equation*}
f=\sum_{0, d_{1}, 1} c_{i}\left(d_{i}^{\prime}-\theta_{i}\right) B\left(0 \rightarrow s^{\prime}\right)\left(\frac{d}{1-d}\right)^{p}(1-d)^{b}, \quad p=\sum_{j} \theta_{j} \tag{D.1}
\end{equation*}
$$

Where $b$ is the number of cells per node. Equation (D.7) is the additiocal constraint on the $B\left(0 \rightarrow a^{\prime}\right)$ 's for case I. If $f$ is space and/or time-dependent, 20 are the $B\left(0 \rightarrow 0^{\prime}\right)$ 's. It is easy to check that for any given vector P there excat Boolean transition variables $\xi_{0,0}^{\prime}$ of mean $B\left(\circ \rightarrow \theta^{\prime}\right)$ sutisfying (D.7). When $f$ is in the direction of a particular velocity vector, say $c_{i g}$, we can lif particles with velocity $-c_{i}$ into particles with velocity $c_{i_{0}}$ whenever this is possible, while leaving all other particles unchanged. This is done with a probability dependent on the amplitude of the farce. Other directions of the force are handled by superpusition.

We tarn to case II. We wish to obtain a force of the form

$$
\begin{equation*}
f_{a}=C_{a \beta} D_{p} \tag{D.8}
\end{equation*}
$$

Where $C_{O B}$ is a D-dimensional matrix. When the velocity $U$ vanimhes, the body-force should also vanish; this requires

$$
\begin{equation*}
\sum_{i, \infty_{i}} \varepsilon_{i}\left(o_{i}^{\prime}-o_{i}\right) B\left(0 \rightarrow o^{\prime}\right)\left(\frac{d}{1-d}\right)^{\prime}(1-d)^{b}=0, \quad p=\sum_{j} \varepsilon_{j} \tag{D.9}
\end{equation*}
$$

With nonvanishing velocity we most we the corresponding equilibrinm popalations given to relevans arder by (ci. (4.14))

$$
\begin{equation*}
N_{i}=d+\frac{d D}{d^{2}} e_{e} n_{d} . \tag{D.10}
\end{equation*}
$$

Bere we have used the unsealed valocty n . Below, we shall howerve ase U since the sea'log fector is titem care of by the Booloan owitch s. Using (D.10) th (D.B), we find that the average momentam lmparted by for tranaltions in to leadiag order linear im U. Ideatiofing with (D.is), we ind that the $B\left(0 \rightarrow 0^{\prime}\right)$ 's moen satinty the following constralate

$$
\begin{equation*}
C_{o,}=\frac{D}{\sigma^{0}}(1-d)^{b-1} \sum_{0, \delta, i} \epsilon_{i a}\left(0_{i}^{\prime}-a_{i}\right) B\left(0 \rightarrow \theta^{\prime}\right)\left(\frac{d}{1-d}\right)^{\prime} \sum_{j} \theta_{j} c_{j p}, \quad p=\sum_{j} e_{j} \tag{D.11}
\end{equation*}
$$

Equations (D.9)and (D.11) are the additional constrants on the $B\left(0 \rightarrow \theta^{\prime}\right)$ 's for case $\bar{D}$.
As an ulnatration, consider the case of the preado-4-D FCBC model with a Coriolie force $2 \Omega \wedge U$, where $\Omega$ Lo the madreetion. A posolble tmplementation lop the for tranaitions is through rotation by $\pi / 2$ around the $2 a-a d s$ of those particles having their veloedty perpendicular to this axds (with a probability dependent on n).

## Appendix E. Catalog of results for PHP modele

The purpose of this appendbe is to summarize all known anatytic results for the FHP modela, including the modela II and III which bave reat-partieles. Adapting the theory to caser with at most one rest-particle is quite straightforward if one inclodes the reat-particle velocity, namely vector ecro. Onr derivations made extensive use of properties $P 1$ to $P 6$ of section 2.4. With rest-particlea, P1, P2, P4, and P5 are anchanged. In P3, $\lambda$ and $\mu$ have usually diferent values for moving and rest-particles. P6 becomes

$$
\begin{equation*}
\sum_{i} c_{i \alpha} c_{i \beta}=\frac{(b-1) a^{2}}{D} \delta_{\alpha \beta} \tag{E.1}
\end{equation*}
$$

Where $b$ is still the number of bita, so that $b-1$ is the number of particles moving with spoed $c$.

In Table 1 below, we tive resalts in terms of the mean density per cell $d$ for the for lowing quantities: the mean density $p_{0}$, the coefficient $g\left(p_{0}\right)$ rescailing the noolinear term to the Navier-Stokes equation (ct. for example (7.13)), the tinematic shear visconity $\nu$, the kinematic bulk necosity $s$, the maximum value $R_{\text {ax }}$ of the coemedent $R_{n}$ appearing in the Reynolds number (df. (8.20)), and dan, the denaty at which the Reynolds uamber Lo madmum. The viscoaities $\nu$ and $\xi$ are calculated with in the lattice Boltzmann approxImation (ef. section 8.2). pos is the dynamic balk viscosity; when it does not vanish, as is the case with rest-particles, eq. (7.11) becomed

$$
\begin{align*}
& \theta_{1} \rho^{\prime}+\rho_{0} \nabla \cdot u=0 \\
& \rho_{0} \partial_{t} u+c_{0}^{2} \nabla \rho^{\prime}=\rho_{0} \nu\left(\nabla^{\imath} u+\frac{D-2}{D} \nabla \nabla \cdot u\right)+\rho_{0} \delta \nabla \nabla \cdot u . \tag{E.2}
\end{align*}
$$

FHP-I FBP-D FHP-m

| $\infty$ | dd | 7d | $7 d$ |
| :---: | :---: | :---: | :---: |
| - | $\frac{1}{5}$ | $\sqrt{4}$ | $\sqrt{4}$ |
| 0 | 12-4 | $\frac{1}{15} 4-8$ | $\frac{1}{15} \frac{0}{1-4}$ |
| $\nu$ | $\frac{1}{15}$ ग(1-ग) $-\frac{1}{6}$ | $\frac{1}{15}$ ग(1-d) $\frac{1}{1-1 / 79}$ | $\frac{1}{50} \frac{1}{\text { (1-d) }} 1-8 \frac{1}{1-6)} 7^{7}-\frac{1}{8}$ |
| 5 | 0 | $\frac{1}{d} \frac{1}{(1-d)}{ }^{2}-\frac{1}{10}$ | 直 ग(1-J $1-\mathrm{sati-d}-\frac{1}{28}$ |
| $R^{\text {a }}$ | 0.387 | 1.08 | 2.22 |
| duc | 0.187 | 0.170 | 0.285 |

## Appendlx F. An H Thoorem for lettice gesee

 by M. Bénon, Obeervatoire de Nice.
## 1. Dotation and beate equatrone

We number trom 1 to $b$ the cells at a given node ( $b$ in the number of different velocity vectors). It is not necesaary that the velocity moduli are equal. Also it will not be necessary to specity any symmetry for the lattice or for the collision rales. Finally, we will not make use of the conservation of the number of particles or of the romentum, so that the prool is applicable to lattices where these conservation haws are violated.

We write $A_{i}=1$ if particle $i$ is present in the input atete, 0 Hit is absent. An inpat state in thas defloed by $1=\left(a_{1}, \ldots, a_{b}\right)$. The number of distinct input states is $2^{6}$.

We call $P(a)$ the probability of an inpot state $a$. We have

$$
\begin{equation*}
\sum_{0} P(0)=1 \tag{F.1}
\end{equation*}
$$

We call $N_{0}$ the probubility that particie $;$ is present. We have

$$
\begin{equation*}
N_{i}=\sum_{0} 0_{i} P(0), \quad 1-N_{i}=\sum_{0}\left(1-\theta_{i}\right) F(0) . \tag{F.2}
\end{equation*}
$$

We delline in the amme way $o_{i}^{\prime}, o^{\prime}=\left(0_{1}^{\prime}, \ldots, \theta_{b}^{\prime}\right), P^{\prime}\left(\delta^{\prime}\right), N_{i}^{\prime}$ tor the ootpot state.
We call $A\left(0 \rightarrow \theta^{\prime}\right)$ the probability that an mpot state is changed into an output state $d^{\prime}$ by the collision. We have

$$
\begin{equation*}
P^{\prime}\left(\theta^{\prime}\right)=\sum_{0} P(0) A\left(0 \rightarrow \theta^{\prime}\right) \tag{F.3}
\end{equation*}
$$

We here of course

$$
\begin{equation*}
\sum_{\infty} A\left(0 \rightarrow \theta^{\prime}\right)=1 \tag{F.4}
\end{equation*}
$$

where the sum b over all outpat states. We will amome that the collision roles obey semi-detalled belascing, i.e. that we have aleo

$$
\begin{equation*}
\sum_{0} A\left(t-o^{\prime}\right)=1 \tag{F.5}
\end{equation*}
$$

## 2. Loesl theorm



$$
\begin{equation*}
\sum_{\sigma} f\left(P^{\prime}\left(0^{\prime}\right)\left|\leq \sum_{0} f\right| P(0) \mid\right. \tag{F.6}
\end{equation*}
$$

Proot from seneral propertice of conyex functions we have

$$
\begin{equation*}
f\left[\frac{\sum_{f} f(a) P(a)}{\sum_{0} q(0)}\right] \leq \frac{\sum_{f} q(a) f|P(0)|}{\sum_{f} f(0)} \tag{י.7}
\end{equation*}
$$

where the $g(0)$ are arbitrary poritive or zero coeficdents. Tuling $g(0)=A\left(0 \rightarrow \theta^{\prime}\right)$, with $\theta^{\prime}$ stren, and using (F.3) and (F.B), we obtain

$$
\begin{equation*}
f\left|P^{\prime}\left(\theta^{\prime}\right)\right| \leq \sum_{0} A\left(\theta, \theta^{\prime}\right) \delta(P(\theta) \mid . \tag{F.8}
\end{equation*}
$$

Summing over $i^{\prime}$ and uning (F.4), we obtain (F.6).

Lemma 2. The following inequality holds:

$$
\begin{equation*}
\sum_{0} P^{\prime}\left(a^{\prime}\right) \ln P^{\prime}\left(0^{\prime}\right) \leq \sum_{0} P(0) \ln P(0) . \tag{F.9}
\end{equation*}
$$

Proot we apply Lenuma 1 with $f(x)=x \ln x$.
Lemma 3. The following inequality balds:

$$
\begin{equation*}
\sum_{0} P(0) \ln P(0) \geq \sum_{i=1}^{b}\left[N_{i} \ln N_{i}+\left(1-N_{i}\right) \ln \left(1-N_{i}\right)\right] . \tag{F.10}
\end{equation*}
$$

The equality holds if and only if

$$
\begin{equation*}
P\left(a_{1}, \ldots, a_{b}\right)=\prod_{i=1}^{b} N_{i}^{a_{1}}\left(1-N_{i}\right)^{1-b_{1}} . \tag{F.11}
\end{equation*}
$$

Proof (inspired by Ref. 83): The right-hand side of (F.10) can be written, usina (F.2):

$$
\begin{equation*}
\sum_{i=1}^{b} \sum_{i}\left|0_{i} P(0) \ln N_{i}+\left(1-a_{i}\right) P(0) \ln \left(1-N_{i}\right)\right| \tag{F.12}
\end{equation*}
$$

or

$$
\begin{equation*}
\sum_{i} P(c) \ln \left[\prod_{i=1}^{0} N_{i}^{0 \cdot 1}\left(1-N_{i}\right)^{1-o_{i}}\right] . \tag{F.13}
\end{equation*}
$$

Therefore (F.10) can aloo be written

$$
\begin{equation*}
\sum_{0}^{\infty} P(0) \operatorname{le}\left[\frac{\operatorname{Him}_{i}^{0} N_{i}^{e_{1}\left(1-N_{i}\right)^{1-0_{1}}}}{P(0)}\right] \leq 0 . \tag{F.14}
\end{equation*}
$$

We heve, tor any z:

$$
\begin{equation*}
m z \leq z-1 \tag{F.18}
\end{equation*}
$$

where the equality holds oaty $甘 z=1$. Therefore

$$
\begin{equation*}
\text { mo }\left[\frac{\prod_{i n i}^{b} N_{i}^{0}\left(1-N_{i}\right)^{1-0_{1}}}{P(0)}\right] \leq \frac{\prod_{i+1}^{b} N_{i}^{0,}\left(1-N_{i}\right)^{1-0}}{P(0)}-1 \text {. } \tag{F.16}
\end{equation*}
$$

Multiplying this by $P(0)$ and summing oves $s$, we obtain the desired result.
The relation ( $F .11$ ) correspoads to the Boinemany approximation (independence of input particles).

Local H theorarn. I' the collision rules astisfly semi-detailed balascing, and in the Boltemaon approximation, the following inequality bolds:

$$
\begin{equation*}
\sum_{i=1}^{1}\left[N_{i}^{\prime} \ln N_{i}^{\prime}+\left(1-N_{i}^{\prime}\right) \ln \left(1-N_{i}^{\prime}\right)\right] \leq \sum_{i=1}^{1}\left[N_{i} \ln N_{i}+\left(1-N_{i}\right) \ln \left(1-N_{i}\right)\right] . \tag{F.17}
\end{equation*}
$$

Proot trom Lemma 3 we have

$$
\begin{equation*}
\sum_{0} P(0) \ln P(1)=\sum_{i=1}^{b}\left[N_{i} \ln N_{i}+\left(1-N_{i}\right) \ln \left(1-N_{i}\right)\right] . \tag{F.18}
\end{equation*}
$$

Combining with Lemma 2:

$$
\begin{equation*}
\sum_{0} P^{\prime}\left(0^{\prime}\right) \ln P^{\prime}\left(0^{\prime}\right) \leq \sum_{k=1}^{b}\left[N_{i} \ln N_{i}+\left(1-N_{i}\right) \operatorname{lo}\left(1-N_{i}\right)\right] . \tag{F.19}
\end{equation*}
$$

Finally, applying Lemma 3 to the $N_{i}^{\prime \prime B}$ and the $P^{\prime}$ 's, we obtain (F.17).
We remark that both conditions of the theorem are neceasary; one can easily ind counterexamples if one or the other to not antinted. Consider for instance a node of the HPP lattice with probabilities before collision: $P(1,0,1,0)=1 / 2, P(0,1,0,0)=1 / 2$. We have: $N_{1}=1 / 2, N_{1}=1 / 2, N_{3}=1 / 2, N_{4}=0$; The Boltrmens approodmation in not satist ed. We take the aunal BPP colliaion rales. The probabibities after collision are then $P^{\prime}(0,1,0,1)=1 / 2, P^{\prime}(0,1,0,0)=1 / 2$. From this we dednce $N_{1}^{\prime}=0, N_{j}^{\prime}=1, N_{3}^{\prime}=0$, $N_{i}^{\prime}=1 / 2$, and it can be immediately veritied that the lath-hand member of ( $F .17$ ) is larger than the right-hand member.

Similarly, lat us modity the collision roles and krep only one tind of collinion: $(1,0,1,0)$ given $(0,1,0,1)$, bat not converzely. Semb-detailed belanciag is not sationed. Take for inatance $N_{1}=N_{8}=N_{3}=N_{4}=1 / 2$. We aeame that the Boltrmans approdimation holds, therefore $P(0)=1 / 16$ for al o. We dedace: $P^{\prime}(1,0,1,0)=0 ; P^{\prime}(0,1,0,1)=2 / 16$; $P^{\prime}\left(\delta^{\prime}\right)=1 / 16$ tor the other $A^{\prime} ; N_{1}^{\prime}=N_{3}^{\prime}=7 / 16, N_{5}^{\prime}=N_{i}^{\prime}=9 / 16 ;$ and here arain the inequallity (E.17) is violeted.

## 2. Olobel theorefe

Firat we sum (F.17) over all lattice podes. We obtain a sum over all cells at all lattice aodes; their total number will be denoted by r :
$\sum_{j=1}^{p}\left[N^{(n)} \ln N^{(j)}+\left(1-N^{(j)}\right) \operatorname{mg}\left(1-N^{(j)}\right)\right] \leq \sum_{j=1}^{p}\left[N^{(j)} \ln N^{(j)}+\left(1-N^{(\Omega)} \ln \left(1-N^{(j)}\right)\right]\right.$.
Next we remark thet this sum is invariant ander proparation. We can therefore extend the theorem to an abitrary number of time stepa, and we obtaln (with the same bypotheses as for the local theorem):

Global H theoram. The function

$$
\begin{equation*}
\sum_{j=1}^{p}\left[N^{(j)} \ln N^{(\jmath)}+\left(1-N^{(j)}\right) \ln \left(1-N^{(j)}\right)\right] \tag{F.21}
\end{equation*}
$$

is non-increasing as the lattice gas evolves.

## 4. Interpretation in terme of daformetion thiory

Consider a probability distribution over $\nu$ possibie cases: $p_{1}, \ldots, p_{\nu}$. The associated information is

$$
\begin{equation*}
\log \nu+\sum_{i=1}^{\nu} p_{i} \log _{\lambda} p_{i} \tag{F.22}
\end{equation*}
$$

This information has a minimal value 0 if all cases have the same probability: $p_{1}=\cdots=$ $p_{\nu}=1 / \nu$. It has a maximal value $\log _{g} \nu$ if one of the $p_{i}$ in 1 while the others are 0 , i.e. for a deterministic choike between the $\nu$ cases.

We come back to latticen. $P(0)$ represents a probability diatribution on $2^{b}$ cases, and therefore an information

$$
\begin{equation*}
b+\sum_{s} P(0) \log _{3} P(0) \tag{F.23}
\end{equation*}
$$

Thus, Lemma 2 expresses the following property: if semi-detailed balancing is satisfed, then the information contrined ia the $P$ ean onty remain constant or decrease in a collision.

From the P's, we can conpate the $N_{i}$ 's by the formulas (F.2), bat the converne is not generally true; in other worda, the $P^{\prime}$ g contain mose mformation than the $N_{i}$ 's. Lemme 3 expreseen this thet.

In the particalar case of the Boltracius approdimation, the particles are considered as independent, and therefore the $P^{\prime}$ 'coniain no more information than the $N_{i}$ 's. We have then the equality to ( $F .10$ ).

The proot of the local B theorem ean therefore be foterpreted an follows: (i) initially the $N_{i}$ 's are given; the rupresents a stve : toformation; (ii) we compute the correnponding $P^{\prime}$ is the Bolizmans epprocemation; t", toformation does sot chasger (Hil) we compote the collision and obtata the $P^{\prime \prime}$ a; the infornation decreaces of otay conatant; (tv) we compute the $N^{\prime \prime}$ strom the $P^{\prime \prime}$ s: here agois the trformation decrases or ataye conntant.

## FIGURE CAPTIONS

Fis. 1 The HPP model. The black arrows are for c:ll-occupation. In (a) and (b) the lattice is shown at two succesuive times.
Fig. 2 Collision rules for the EPP model.
Fig. 3 The $F H P$ mudel with binary head-on and triple collisions at two successive times.
Fig. 4 Collision rules for the FHP models. (a) Headon collision with two output changels diven equal weights; (b) triple collision; (c) dual of head-on collision ander particietiole exchange; (d) headon collision with spectator; (e) binary collisions involving one rettparticle (represented by a circle).
Fig. 5 The pseudo-4D FCHC model Only the neighbourhood of one node is shown. Along the dotted links, connecting to dext-nearest neigbbours, at most one particle can propagate, with component $v_{4}=0$; along the thick black links, connecting to nearest aeighbours, up to two particles can proparate, with components $v_{4}= \pm 1$.

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[^0]:    4 The qualilication "thermodyname" is not so appropriate since there is no relevant enercy varable

    - Henceforth we shall just write "velocity", since thla mean velocity changes in space.

    6 The interpolations can be done via the Fourier representation if the lattice is periodic.
    ${ }^{9}$ Eventually, we shall asame the velocity to be $O(C)$, but at this point it is more convenient to kecp e and $u$ as independent expansion parameters.

[^1]:    - Actually, this is only the leading order approximation to the momentum-dux.

[^2]:    ${ }^{12}$ In the u-representation duality-invariance is broken because we bave decided to work with the velocity of particles rather than with that of boles.

[^3]:    ${ }^{13}$ For the case of lattice gases，we shall actually ubtain a lnite difference equation．

[^4]:    ${ }^{14}$ Mure precisely, by dropping spatial Fourier components with wavenumber $k>e$.
    is This is not exactly true in two dimensings as we shall see below.

[^5]:    10 This is equivalent to asuming that the viscosity is Inlte, cf. below.
    17 A factor $k^{2}$ comes from the average squared Fourier amplitude and another factor $k^{D-1}$ from the $D$-dimensional volume element.

[^6]:    11 Note that in the incompressible case, only golenoidal noise is relerant.

[^7]:    10 Even at low densities, the Boltrmann approcimation may not be ralid. Indeed, with. out efectively changing the dynamics, we can reduce the density by arbitrary large lactors by having the particles initially located on a sub-lattice with some large periodicity; these Lre however pathologically unstable cond gurations.

[^8]:    30 Collisions penduce correlations whenever the particles are not exsctly at equilibnum

[^9]:    ${ }^{11}$ Small intermitteucy corrections which would siightly increase the exponent m cannot be ruied out.

