THE CONTINUUM ARGUMENT

Let v(x, t) be a vector-valued field referred to a fixed origin in space, which we identify with the velocity of a "macroscopic" fluid cell. The cell is not small enough to notice a particle structure for the fluid, but it is small enough to be treated as a mathematical point and still agree with physics.

To derive the properties of a flow defined by the vector field, one now invokes the generalized Stokes theorem:

$$\oint_{\partial\Sigma} \mathbf{A} = \oint_{\Sigma} d\mathbf{A},$$

where Σ is a generalized surface or volume, $\partial\Sigma$ the boundary of Σ, $\mathbf{A}$ an n-differential form and $d\mathbf{A}$ an $(n+1)$-differential form. This very general theorem has two familiar forms: one is the classical Stokes theorem from one to two dimensions,

$$\oint_{\partial\Sigma} \mathbf{A}\cdot \mathbf{d}\ell = \oint_{\Sigma} \nabla\times\mathbf{A}\cdot d\mathbf{S},$$

and the other is the Gauss law from two to three dimensions,

$$\oint_{\partial\Sigma} \mathbf{A}\cdot d\mathbf{S} = \int_{\Sigma} (\nabla\cdot\mathbf{A})dV,$$

where ℓ is a curve, $\mathbf{S}$ is a surface, and V is a volume in three-dimensional Euclidean space $\mathbf{R}^3$.

Conservation Laws and Euler's Equation. First, we deal with the idea of continuity, or conservation of flow. If ρ is the density, or mass per unit volume, then the mass of the fluid in volume V (that is, Σ), is equal to $\int_{\Sigma}\rho dV$. A two-dimensional surface in $\mathbf{R}^3$ has an outward normal vector $\mathbf{n}$ which is defined to be positive. The total mass of fluid flowing out of a volume Σ can be written as

$$\oint_{\partial\Sigma} \rho\mathbf{v}\cdot d\mathbf{S} = \int_{\partial\Sigma} \rho\mathbf{v}\cdot \mathbf{n}dS.$$

Continuity of the flow implies a balance between the flow through the surface and the loss of fluid from the volume. That is, the decrease in mass in the volume must equal the outflow of fluid mass through the surface of the volume, which implies by the Gauss law that

$$\oint_{\partial\Sigma} \rho\mathbf{v}\cdot d\mathbf{S} = -\partial_t \int_{\Sigma}\rho dV = \int_{\Sigma} \nabla\cdot(\rho\mathbf{v})dV.$$

This gives the first evolution equation for a fluid, the continuity, or mass-conservation, equation:

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{1}$$

Now we introduce the idea of pressure p as the force exerted by the fluid on a unit surface area of an enclosed volume and use Newton's second law, $\mathbf{F} = ma$. The total force acting on a volume of fluid due to the remainder of the fluid is given by $-\oint_{\partial\Sigma} pd\mathbf{S}$. Using Stokes theorem we can write

$$-\oint_{\partial\Sigma} pd\mathbf{S} = -\oint_{\Sigma} \nabla p dV.$$

The translation of $\mathbf{F} = m\mathbf{a}$ to a continuous medium is

$$-\nabla p = \rho d\mathbf{v}/dt,$$

where $d\mathbf{v}/dt$ is a total derivative. The chain rule on $d\mathbf{v}(\mathbf{x},t)/dt$ gives

$$\rho d\mathbf{v}/dt = \rho\{\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}\}$$

Substituting this result into the equation for $-\nabla p$ yields Euler's equation for an ideal, dissipation-free fluid:

$$\partial_t\mathbf{v} = -(\mathbf{v}\cdot\nabla)\mathbf{v} - \frac{1}{\rho}\nabla p. \tag{2}$$

THE CONTINUUM ARGUMENT (continued)

One can generalize Euler's equation to a form more useful for a dissipative fluid. For this we look at the flux of momentum through a fluid volume. The momentum of fluid passing through an element dV is $\rho\mathbf{v}$, and its time rate of change expressed in components is

$$\partial_t(\rho v_i) = (\partial_t \rho)v_i + \rho(\partial_t v_i).$$

We can rewrite $\partial_t \rho$ and $\partial_t v_i$ as spatial derivatives by using Eqs. 1 and 2. Then

$$\partial_t(\rho v_i) = -\partial_k \Pi_{ik}, \tag{3}$$

where the momentum flux tensor $\Pi_{ik} \equiv p\delta_{ik} + \rho v_i v_k$.

The meaning of the momentum flux tensor can be seen immediately by integrating Eq. 3 and applying Stokes theorem.

$$\partial_t \int_\Sigma \rho v_i d\Sigma = -\int_\Sigma \partial_k \Pi_{ik} d\Sigma = -\oint_{\partial\Sigma} \Pi_{ik} n_k dS.$$

So

$$\partial_t \int_\Sigma \rho v_i dV = -\oint_{\partial\Sigma} \Pi_{ik} n_k dS,.$$

where the left-hand side is the rate of change of the ith component of momentum ρv_i in the volume and $\Pi_{ik} n_k d\Sigma$ is the ith component of momentum flowing through dS. Therefore, Π_{ik} is the ith component of momentum flowing in the kth direction. This is more easily seen by writing

$$\Pi_{ik} n_k = p\delta_{ik} n_k + \rho v_i v_k n_k = p\mathbf{n} + \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n}).$$

Equations 1, 2, and 3 are the basic formalism for classical Newtonian ideal fluids (fluids with no dissipation) and are also true for flows in general.

Classical Dissipative Fluids—The Navier-Stokes Equations. The general Euler's equation is $\partial_t(\rho v_i) = -\partial_k \Pi_{ik}$, where Π_{ik} is now the momentum stress tensor. The form of this tensor changes if the fluid is dissipative, for example, if viscous forces convert the energy in the flow into heat. Traditionally, Π_{ik} is modified in the following way. Take $\Pi_{ik} = p\delta_{ik} + \rho v_i v_k$ and introduce an unknown tensor σ'_{ik} that describes the effects of viscious stress. Then rewrite the momentum stress tensor as

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \rho\sigma'_{ik} \equiv \sigma_{ik} + \rho v_i v_k,$$

where $\sigma_{ik} = p\delta_{ik} - \rho\sigma'_{ik}$ is called the stress tensor and σ'_{ik} the viscosity stress tensor.

The form of σ'_{ik} can be deduced on general grounds. First we assume that the gradient of the velocity changes slowly so σ_{ik} is linear in $\partial_k v_i$. Moreover, σ'_{ik} is zero for $\mathbf{v} = 0$, and under rotation it must vanish since uniform rotation produces no overall transport of momentum. The unique form that has these properties is

$$\sigma'_{ik} = a(\partial_k v_i + \partial_i v_k) + b\delta_{ik}\partial_j v_j,$$

where a and b are unknown coefficients. It is usually written in the form

$$\sigma'_{ik} = \nu(\partial_k v_i - \partial_i v_k - 2/3\delta_{ik}\partial_j v_j) + \zeta\delta_{ik}\partial_j v_j,$$

where ν is the kinematic shear viscosity and ζ is the kinematic bulk viscosity.

For an incompressible fluid (the density is constant so $\rho = \rho_0$) this tensor simplifies, and Euler's equation goes over to the incompressible Navier-Stokes equations:

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{v}\nabla^2\mathbf{v} \quad \text{and} \quad \nabla\cdot\mathbf{v} = 0.$$

In tensor notation we have

$$\partial_t v_i + (v_j\partial_j)v_i = -\frac{1}{\rho}\partial_i p + \nu\frac{\partial^2}{\partial_k\partial_k}v_i \quad \text{and} \quad \partial_k v_k = 0.$$

In Part II we end the theoretical discussion of the lattice gas by giving the incompressible limit of the lattice gas Navier-Stokes equations. ■