# Reynolds Number

To design and test proposed large-scale equipment, such as airfoils or entire aircraft, it is often much more practical to experiment with scaled-down versions. If such tests are to be successful, however, dynamic similitude must exist between model and field equipment, which, in turn, implies that geometric, inertial, and kinematic similitude must exist.

The Navier-Stokes equations (Eqs. 9 and 10 in the main text) are a good starting point for deriving the relationships needed to establish dynamic similitude. First, we look at the case of laminar flow. Ignoring body force and pressure effects, we examine the momentum conservation relationship for steady, laminar, incompressible, two-dimensional flow, equating just the advection and diffusion terms in the x-direction:

$$\underbrace{\frac{\partial uu}{\partial x} + \frac{\partial vu}{\partial y}}_{\text{Advection}} = \underbrace{\nu_{\mathrm{m}} \left[\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right]}_{\text{Diffusion}} . \quad (1)$$

Here $u$ and $v$ are the $x$ and $y$ components of the velocity and $\nu_{\mathrm{m}}$ is the molecular kinematic viscosity (the ratio of fluid viscosity to fluid density $\mu_{\mathrm{m}}/\rho$). Advection has to do with kinematic effects, that is, the transport of fluid properties by the motion of the fluid, and thus accounts for momentum transport *along* streamlines; the diffusion terms represent viscous effects that cause momentum to diffuse *between* streamlines, thereby tending to diminish any sharp velocity gradients.

We can write Eq. 1 in dimensionless form by introducing a length scale $L$ and a fluid velocity in the free stream $u_0$. The result is

$$u_0 \left[\frac{\partial \mathbf{uu}}{\partial \mathbf{x}} + \frac{\partial \mathbf{vu}}{\partial \mathbf{y}}\right] = \frac{\nu_{\mathrm{m}}}{L} \left[\frac{d^2 \mathbf{u}}{d\mathbf{x}^2} + \frac{d^2 \mathbf{u}}{d\mathbf{y}^2}\right], \quad (2)$$

where the highlighted variables are dimensionless. This portion of the momentum equation can thus be uniquely characterized by the ratio of the coefficients multiplying the dimensionless advection and diffusion terms. The ratio, called the Reynolds number

$$R = \frac{u_0 L}{\nu_{\mathrm{m}}}, \quad (3)$$

can be thought of as a comparative measure of inertial and viscous (diffusive) effects within the flow field. To achieve dynamic similitude in two different laminar-flow situations, the Reynolds numbers for both must be identical.

What happens if we increase the flow speed to the point that viscous dissipation can no longer stabilize the flow, and the macroscopic balance between mean-flow inertia and viscous effects breaks down? At this point there is a transition from purely laminar flow to turbulence. In similar flows, the transition occurs at a specific Reynolds number characteristic of the flow geometry. For instance, any fluid traveling inside a circular pipe—regardless of the specific fluid or conduit being used—experiences the onset of turbulence at $R \cong 2000$.

At or near this "critical" Reynolds number, inertial contributions to mean-flow momentum that cannot be dissipated by viscous stresses must be absorbed by newly formed turbulent eddies. The presence of turbulence energy is often described in terms of an effective turbulence viscosity $\nu_{\mathrm{t}}$, defined as the ratio of the turbulence-shear, or Reynolds, stress to the mean-flow strain rate. With this in mind, an effective *turbulence* Reynolds number—one that includes molecular viscous effects—is

$$R_{\mathrm{eff}} = \frac{u_0 L}{\nu_{\mathrm{t}} + \nu_{\mathrm{m}}} \quad (4)$$

Molecular viscous effects are overwhelmed if $\nu_{\mathrm{t}} \gg \nu_{\mathrm{m}}$. In those instances the exact value of the kinematic viscosity $\nu_{\mathrm{m}}$ is immaterial, and flow behavior is dominated by turbulence effects.

Although a turbulence Reynolds number may be entirely adequate for research on macroscopic flows, the analysis of turbulence *substructure* requires a third Reynolds number, a local turbulence Reynolds number based not on $L$ and $u_0$ but on representative eddy size $s$ and eddy velocity $u'$:

$$R_{\mathrm{s}} = \frac{u's}{\nu_{\mathrm{m}}}. \quad (5)$$

Note that the molecular kinematic viscosity $\nu_{\mathrm{m}}$ is retained in this definition. The choice of molecular viscosity to characterize the dissipative mechanisms responsible for tearing eddies apart is based on the ultimate transformation of turbulence into heat energy. Molecular processes are, in the end, dominant at the smallest scales, and $R_{\mathrm{s}}$ is a relative measure of the loss of kinetic energy from an eddy of a given size to heat. For the smallest eddies in a flow system, $R_{\mathrm{s}} \cong 1$; that is, all the energy of the eddy is dissipated into heat. ■