Fields and Spins in

Fields in Higher Dimensions. We describe here how to construct a field in higher dimensions and how such a field is related to fields in the 4-dimensional world in which we live. Higher dimensional fields unify an infinite number of 4-dimensional fields. A typical and simple example of this can be seen from a scalar field (a spin-0 field) in five dimensions. A scalar field has only one component, so it can be written as $\varphi(x,y)$, where x is the 4-dimensional space-time coordinate and y is the coordinate for the fifth dimension. We will assume that the fifth dimension is a little circle with radius R, where R is independent of x. (After this example, we examine the generalizations to more than five dimensions and to fields carrying nonzero spin in the higher dimensions.)

Functions on a circle can be expanded in a Fourier series; thus, the 5-dimensional scalar field can be written in the form

$$\varphi(x,y) = \sum_{n=-\infty}^{\infty} \varphi_n(x)\exp(iny/R)\,, \tag{1}$$

where n is an integer, and $\varphi_n(x)$ are 4-dimensional fields. The Fourier series satisfies the requirement that the field is single-valued in the extra dimension, since Eq. 1 has the same value at the identical points y and $y + 2\pi R$. Usually the wave equation of $\varphi(x,y)$ is a straightforward generalization of the 4-dimensional scalar wave equation (that is, the Klein-Gordon equation) in the limit that interactions can be ignored. The 5-dimensional Klein-Gordon equation for a massless 5-dimensional particle is

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 - \frac{\partial^2}{\partial y^2}\right)\varphi(x,y) = 0\,. \tag{2}$$

The presence of additional terms depends on the details of the Lagrangian, and we ignore them for the present description. It is a simple matter to substitute the Fourier expansion of Eq. 1 into Eq. 2 and use the orthogonality of the expansion functions $\exp(iny/R)$ to rewrite Eq. 2 as an infinite number of equations in four dimensions, one for each $\varphi_n(x)$:

$$\left[\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{n}{R}\right)^2\right]\varphi_n(x) = 0\,. \tag{3}$$

Note the following very important point: for $n = 0$, Eq. 3 is the equation for a massless 4-dimensional scalar field, whereas for $n \neq 0$, Eq. 3 is the wave equation for a particle with mass $|n|/R$. The massless particle, or "zero mode," should correspond to a field observable in our world. The fields with nonzero mass are called "pyrgons," since they are on a "tower" of particles, one for each n. If R is near the Planck length (10^{-33} centimeter), then the pyrgons have masses on the order of the Planck mass. However, it is also possible that R can be much larger, say as large as 10^{-16} centimeter, without conflicting with experience.

The 4-dimensional form of the Lagrangian depends on an infinite number of fields and is very complicated to analyze. For many purposes it is helpful to truncate the theory, keeping a specially chosen set of fields. For example, 5-dimensional Einstein gravity is simplified by omitting all the pyrgons. This can be achieved by requiring that the fields do not depend on y, a procedure called "dimensional reduction." The dimensionally reduced theory should

quences of the symmetries of space-time is so attractive that efforts to generalize the Kaluza-Klein idea have been vigorously pursued. These theories require a more complete discussion of the possible candidate manifolds of the extra dimensions.

The geometry of the extra dimensions in the absence of matter is typically a space with a high degree of symmetry. Symmetry requires the existence of transformations in which the starting point looks like the point reached after the transformation. (For example, the environments surrounding each point on a sphere are identical.) Two of the most important examples are "group manifolds" and "coset spaces," which we briefly describe.

The tranformations of a continuous group are identified by N parameters, where N is the number of independent transformations in the group. For example, $N = 3$ for SU(2) and 8 for SU(3). These parameters are the coordinates of an N-dimensional manifold. If the vacuum values of fields are constant on the group manifold, then the vacuum solution is said to be symmetric.

Coset spaces have the symmetry of a group too, but the coordinates are labeled by a subset of the parameters of a group. For example, consider the space SO(3)/SO(2). In this example, SO(3) has three parameters, and SO(2) is the phase symmetry with one parameter, so the coset space SO(3)/SO(2) has three minus one, or two, dimensions. This space is called the 2-sphere, and it has the geometry of the surface of an ordinary sphere. Spheres can be generalized to any number of dimensions: the N-dimensional sphere is the coset space $[SO(N+1)]/SO(N)$. Many other cosets, or "ratios" of groups, make spaces with large symmetries. It is possible to find spaces with the symmetries of the electroweak and strong interactions. One such space is the group manifold SU(2) × U(1) × SU(3), which has twelve dimensions. More interesting is the lowest dimensional space with those symmetries, namely, the coset space [SU(3) × SU(2) × U(1)]/[SU(2) × U(1) × U(1)], which has dimension 8 + 3 + 1 − 3 − 1 − 1 = 7. (The SU(2) and the U(1)'s in the denominator differ from those in the numerator, so they cannot be "canceled.") Thus, one might hope that (4 + 7 = 11)-dimensional gravity would

Higher Dimensions

describe the low-energy limit of the theory.

The gravitational field can be generalized to higher (>5) dimensional manifolds, where the extra dimensions at each 4-dimensional space-time point form a little ball of finite volume. The mathematics requires a generalization of Fourier series to "harmonic" expansions on these spaces. Each field (or field component if it has spin) unifies an infinite set of pyrgons, and the series may also contain some zero modes. The terms in the series correspond to fields of increasing 4-dimensional mass, just as in the 5-dimensional example. The kinetic energy in the extra dimensions of each term in the series then corresponds to a mass in our space-time. The higher dimensional field quite generally describes mathematically an infinite number of 4-dimensional fields.

Spin in Higher Dimensions. The definition of spin in D dimensions depends on the D-dimensional Lorentz symmetry; 4-dimensional Lorentz symmetry is naturally embedded in the D-dimensional symmetry. Consequently a D-dimensional field of a specific spin unifies 4-dimensional fields with different spins.

Conceptually the description of D-dimensional spin is similar to that of spin in four dimensions. A massless particle of spin J in four dimensions has helicities $+J$ and $-J$ corresponding to the projections of spin along the direction of motion. These two helicities are singlet multiplets of the 1-dimensional rotations that leave unchanged the direction of a particle traveling at the speed of light. The group of 1-dimensional rotations is the phase symmetry SO(2), and this method for identifying the physical degrees of freedom is called the "light-cone classification." However, the situation is a little more complicated in five dimensions, where there are three directions orthogonal to the direction of the particle. Then the helicity symmetry becomes SO(3) (instead of SO(2)), and the spin multiplets in five dimensions group together sets of 4-dimensional helicity. For example, the graviton in five dimensions has five components. The SO(2) of four dimensions is contained in this SO(3) symmetry, and the 4-dimensional helicities of the 5-dimensional graviton are 2, 1, 0, -1, and -2.

Quite generally, the light-cone symmetry that leaves the direction of motion of a massless particle unchanged in D dimensions is SO($D-2$), and the D-dimensional helicity corresponds to the multiplets (or representations) of SO($D-2$). For example, the graviton has $D(D-3)/2$ independent degrees of freedom in D dimensions; thus the graviton in eleven dimensions belongs to a 44-component representation of SO(9). The SO(2) of the 4-dimensional helicity is inside the SO(9), so the forty-four components of the graviton in eleven dimensions carry labels of 4-dimensional helicity as follows: one component of helicity 2, seven of helicity 1, twenty-eight of helicity 0, seven of helicity -1 and one of helicity -2. (The components of the graviton in eleven dimensions then correspond to the graviton, seven massless vector bosons, and twenty-eight scalars in four dimensions.)

The analysis for massive particles in D dimensions proceeds in exactly the same way, except the helicity symmetry is the one that leaves a resting particle at rest. Thus, the massive helicity symmetry is SO($D-1$). (For example, SO(3) describes the spin of a massive particle in ordinary 4-dimensional space-time.) These results are summarized in Fig. 5 of the main text.

unify all known interactions.

It turns out that the 4-dimensional fields implied by the 11-dimensional gravitational field resemble the solution to the 5-dimensional Kaluza-Klein case, except that the gravitational field now corresponds to many more 4-dimensional fields. There are methods of dimensional reduction for group manifolds and coset spaces, and the zero modes include a vector boson for each symmetry of the extra dimensions. Thus, in the (4 + 7)-dimensional example mentioned above, there is a complete set of vector bosons for the standard model. At first sight this model appears to provide an attractive unification of all the interactions of the standard model; it explains the origins of the local symmetries of the standard model as space-time symmetries of gravity in eleven dimensions.

Unfortunately, this 11-dimensional Kaluza-Klein theory has some shortcomings. Even with the complete freedom consistent with quantum field theory to add fermions, it cannot account for the parity violation seen in the weak neutral-current interactions of the electron. Witten[1] has presented very general arguments that no 11-dimensional Kaluza-Klein theory will ever give the correct electroweak theory.

Supersymmetry and Gravity in Four Dimensions

We return from our excursion into higher dimensions and discuss extending gravity not by enlarging the space but rather by enlarging the symmetry. The local Poincaré symmetry of Einstein's gravity implies the massless spin-2 graviton; our present goal is to extend the Poincaré symmetry (without increasing the number of dimensions) so that additional fields are grouped together with the graviton. However, this cannot be achieved by an ordinary (Lie group) symmetry: the graviton is the only known elementary spin-2 field, and the local symmetries of the standard model are internal symmetries that group together particles of the same spin. Moreover, gravity has an exceptionally weak interaction, so if the graviton carries quantum numbers of symmetries similar to those of the standard model, it will interact too strongly. We can