Neutrinos that Travel in a Vacuum

In vacuum neutrino oscillations can occur if the neutrino flavor states are “coherent,” linear superpositions of neutrino mass states. Coherent means that the phases of two or more of the mass states are correlated, so that the relative phase between the mass states leads to an interference term in the calculation of quantum probabilities and expectation values. Quantum interference between the mass states becomes an integral part of the neutrino’s description.

In certain respects, this phase phenomenon parallels the relationship between circular and plane polarization for ordinary light. The left and right states of circularly polarized light emerge most naturally from Maxwell equations. All other states can be expressed as linear superpositions of those two states. In particular, plane-polarized light is a superposition of equal parts of two circular states that have a constant relative-phase difference. Changing the relative phase of the states rotates the plane of polarization.

Similarly, we think of the neutrino mass states $|\nu_1\rangle$, $|\nu_2\rangle$, and $|\nu_3\rangle$, with distinct masses $m_1$, $m_2$, and $m_3$, as the analogues of the circularly polarized states. The weak-interaction states, or the flavor neutrinos, $|\nu_e\rangle$, $|\nu_{\mu}\rangle$, and $|\nu_{\tau}\rangle$, are created as coherent, linear superpositions of the mass states and are the analogues of the independent planes of polarization. Because the phase of each mass state $|\nu_k\rangle$ depends on the mass $m_k$ ($k = 1, 2, 3$), each state evolves with a different phase, and therefore the relative phase between the states changes with time. Quite distinct from our analogy with ordinary light, this change can lead to the appearance of different neutrino flavors.

To keep things simple, in this article we shall consider only two neutrino mass states $|\nu_1\rangle$ and $|\nu_2\rangle$. The electron- and muon-neutrino flavor states would then be written as $|\nu_e\rangle = \sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle$ and $|\nu_{\mu}\rangle = \sin \theta |\nu_1\rangle - \cos \theta |\nu_2\rangle$.

The angle $\theta$ characterizes the amount of mixing between the mass states and is known as the intrinsic mixing angle. If $\theta$ is small, the electron neutrino consists primarily of the state $|\nu_1\rangle$ and has only a small admixture of $|\nu_2\rangle$, whereas the muon neutrino would be dominated by $|\nu_2\rangle$ and would have only a small amount of $|\nu_1\rangle$.

We assume that at $t = 0$, the neutrino is created in the distinct superposition that corresponds to the electron neutrino $|\nu_e(0)\rangle = |\nu_1\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle$.

When the neutrino travels through a vacuum, each mass-state component $|\nu_k\rangle$ evolves with its own phase factor $\exp(-ikx/c)$. (We are working in units in which the speed of light $c$ is equal to 1.) We assume each mass state has the same momentum $p$, which is much greater than the masses $m_k$.

Thus, $E_k = \sqrt{p^2 + m_k^2} = p + m_k^2/2p$, where $k = 1, 2$. Because $m_1 \neq m_2$, and hence $E_1 \neq E_2$, the relative phase between $|\nu_1\rangle$ and $|\nu_2\rangle$ will change as $|\nu(t)\rangle$ evolves with time. At an arbitrary time $t$, the neutrino has evolved to the state $|\nu(t)\rangle = \cos \theta e^{-ikx/c} |\nu_1\rangle + \sin \theta e^{-ikx/c} |\nu_2\rangle$.

In general, this linear combination of $|\nu_1\rangle$ and $|\nu_2\rangle$ is neither a pure electron neutrino state $|\nu_e\rangle$ nor a pure muon neutrino state $|\nu_{\mu}\rangle$. Instead, it is a linear superposition of both states.

Quantum mechanics then tells us that, after travelling a distance of $x$ meters, $|\nu(t)\rangle$ could be detected as a muon neutrino. That transformation probability, denoted as $P(v_e \to v_{\mu})$, is given by

$$P(v_e \to v_{\mu}) = |\langle \nu_{\mu}(x) | \nu_1 \rangle|^2 = \sin^2(2\theta) \sin^2(\pi x/x_f).$$

(Even massive neutrinos would be highly relativistic and would travel at nearly the speed of light $c$. We have made the approximation $\pi \approx 6c/x_f$ in our units. See the box “Derivation of Neutrino Oscillation Length” on page 161.) Because of the term $\sin^2(\pi x/x_f)$, the probability oscillates with distance from the source. The parameter $k$ is called the oscillation length and is given in meters by

$$k = \frac{2\pi m_2}{\sqrt{E_1 E_2}} = \frac{2\pi m_2}{1.27\sqrt{m_1 m_2}}. $$

Thus, $E_2 = \sqrt{p^2 + m_2^2} = p + m_2^2/2p$. (We are working in units in which the speed of light $c$ is equal to 1.) We assume each mass state has the same momentum $p$, which is much greater than the masses $m_k$.

For large $\sin^2(2\theta)$, $P(v_e \to v_{\mu})$, the oscillation probability for doing so may be small, independent of neutrino energy. Quantum interference between the mass states becomes an integral part of the neutrino’s description. The MSW effect provides us with one way to explain that deficit. Instead, the data suggest that a resonance phenomenon might be responsible for this deficit.

Neutrinos that Travel through Matter

The work of Wolfenstein, and then Mikheyev and Smirnov, showed that the oscillation probability could increase dramatically because of an additional phase difference due to the neutrino interacting with the matter it travels through. The origin of this phase shift can be understood by analogy with a well-known phenomenon in optics: When light travels through a medium, it sees a refractive index $n$ that is a function of the light’s frequency and the medium’s composition. The value of $n$ is given by

$n = 1 + \frac{2m_e}{p^2} \frac{1}{\sqrt{E}}.$

Electron neutrinos, and only electron neutrinos, can interact with electrons through charged currents in a

Figure 1. The Leapard Changes Its Spots

These curves represent the probability that a neutrino, after travelling through matter, would be detected as either an electron neutrino (blue), muon neutrino (red), or tau neutrino (yellow). A neutrino is “born” in the core of the sun as the superposition of mass states, in a combination that corresponds to an electron neutrino. In the specific model used to generate these curves, the superposition changes through the MSW resonance effect after the neutrino has traversed about 15 percent of the solar radius. The neutrino has only a 75 percent chance of being detected as an electron neutrino, and a 25 percent chance of being detected as a tau neutrino. By the time it fuses the sun, the neutrino is most likely to be detected as a muon neutrino.
When this matching occurs, a phase \( \Delta \sin^2 \theta \) takes the form
\[
P_{\text{MSW}}(\nu_e \rightarrow \nu_x) = \sin^2 2\theta \frac{\sin W}{\frac{2}{\Delta m^2}} ,
\]
where
\[
\sin^2 2\theta = \frac{\sin^2 \theta}{\frac{1+\theta^2}{2}}, \quad W^2 = \sin^2 2\theta + (D-x \Delta m^2), \quad \text{and} \quad D = \frac{\sqrt{2} G N_f}{m_\nu} .
\]

Note the similarity between this expression for the MSW probability and the in vacuo oscillation probability. The in vacuo mixing angle \( \theta \) is replaced by an effective mixing angle \( \theta_{\text{eff}} \) that depends on the matter oscillation term (through the parameter \( D \)). When a neutrino travels in vacuum, the electron density is zero, and hence \( D \) is equal to zero, so that \( W^2 = 1 \). Thus, the MSW probability reduces to the in vacuo probability.

When a neutrino travels through matter, however, \( W^2 \) can become less than 1. This is the MSW resonance effect. The oscillation probability increases with the resonance condition given by \( D = \cos \Delta m^2 \). At that point, \( W^2 = \sin^2 2\theta + \sin^2 \theta = 1 \), and the oscillation probability reaches a maximum. The resonance condition is independent of the size of the intrinsic mixing angle \( \theta \), but it requires matching the properties of the material with the neutrino oscillation length \( \lambda \) through the relation:
\[
\sqrt{2} G N_f = \frac{1.2 \cos 2\theta}{\lambda} .
\]

We can estimate the range of \( \lambda \) or \( \Delta m^2 \) for which the MSW effect is important. We rewrite the enhancement condition in terms of the solar density \( \rho_s \) measured in grams per cubic centimeter and then multiply by Avogadro's number. For a range of neutrino energy measured in million electron volts and \( \Delta m^2 \) measured in electron

\[
0.10 \leq \lambda \leq 0.30 \text{ meters} ,
\]

mass differences are
\[
10^{-1} \leq \Delta m^2 \leq 10^{-3} \text{ eV}^2 .
\]

To study this range of mass differences with terrestrial neutrinos, we would need intense sources of extremely low energy neutrinos, well below 1 MeV. Those sources do not exist, and therefore solar neutrinos are the only means available to us.

The Most Favorable Solution to the Solar-Neutrino Problem

The value of \( \Delta m^2 \) determines how the curve for the survival probability overlays the spectrum of solar neutrinos. Decreasing \( \Delta m^2 \) moves the curve to the left, while increasing it moves...
we can write Equation (7) in the flavor basis:

\[ |\psi\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle , \quad \text{and} \quad |\psi\rangle = -\sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle . \tag{1a} \]

With the help of Equations (4a)

\[ a_1(t) \quad \text{and} \quad a_2(t) \]

we further assume that an electron neutrino is born at time \( t = 0 \). That neutrino will evolve in time as a superposition of states with time-dependent coefficients. The neutrino can be described by either mass states or flavor states:

\[ |\psi(0)\rangle = a_1(0) |\nu_1\rangle + a_2(0) |\nu_2\rangle = a_1(0) |\nu_1\rangle + a_2(0) |\nu_2\rangle . \tag{2} \]

\[ a_1(\theta) = a_1(\theta) \cos \theta + a_2(\theta) \sin \theta , \quad \text{and} \quad a_2(\theta) = a_1(\theta) \sin \theta - a_2(\theta) \cos \theta . \tag{3a} \]

We will also be using the inverse of Equations (1) and (3):

\[ |\psi(0)\rangle = \cos \theta |\nu_1\rangle - \sin \theta |\nu_2\rangle , \quad \text{and} \quad \sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle ; \tag{4a} \]

In general, the time development of the neutrino states described in Equation (2) has a phase that depends on both the momentum and the energy of the neutrino. For example, an electron neutrino evolves as

\[ |\psi(x, t)\rangle = \cos \theta e^{i\frac{p}{\hbar}t - \frac{E}{m}t} |\nu_1\rangle + \sin \theta e^{i\frac{p}{\hbar}t - \frac{E}{m}t} |\nu_2\rangle . \tag{5} \]

We work in units in which \( \hbar = c = 1 \). Let us first consider the evolution of \( |\psi(t)\rangle \) as a superposition of mass eigenstates during an infinitesimal time \( \Delta t \). We assume a common momentum for each mass state, so that only the difference between the energies of the mass states (due to the difference in the neutrino masses) governs the time development of the state. With \( p >> m_\nu \), we can approximate the energy as

\[ E = \sqrt{p^2 + m_\nu^2} \approx p + m_\nu^2 \frac{d\hat{p}}{dp} = p + M_\nu . \tag{6} \]

The neutrino evolves in time \( \Delta t \) as

\[ |\psi(t + \Delta t)\rangle = a_1(t) e^{i\frac{p}{\hbar}\Delta t - \frac{E}{m}\Delta t} |\nu_1\rangle + a_2(t) e^{i\frac{p}{\hbar}\Delta t - \frac{E}{m}\Delta t} |\nu_2\rangle \]

where

\[ M_\nu = m_\nu^2 \frac{d\hat{p}}{dp} (k = 1, 2) \]

The neutrino evolves in time \( \Delta t \) as

\[ |\psi(t + \Delta t)\rangle = a_1(t) e^{i\frac{p}{\hbar}\Delta t - \frac{E}{m}\Delta t} |\nu_1\rangle + a_2(t) e^{i\frac{p}{\hbar}\Delta t - \frac{E}{m}\Delta t} |\nu_2\rangle \]

We have dropped the overall phase factor of \( e^{i\frac{p}{\hbar}t} \) in Equation (7) because it has no bearing on the final result. With the help of Equations (4a) and (4b), we can write Equation (7) in the flavor basis:

\[ E_x = \sqrt{p^2 + m_\nu^2} \approx p + m_\nu^2 \frac{d\hat{p}}{dp} = p + M_\nu . \tag{6} \]
\[ |\psi(t+\Delta t)\rangle = [a_1(t+\Delta t)e^{-iM_1^\text{at}} \cos \theta + a_2(t+\Delta t)e^{-iM_2^\text{at}} \sin \theta] |\nu_1\rangle + [-a_1(t+\Delta t)e^{-iM_1^\text{at}} \sin \theta + a_2(t+\Delta t)e^{-iM_2^\text{at}} \cos \theta] |\nu_2\rangle. \] (8)

We next consider the neutrino as a superposition of flavor states:

\[ |\psi(t)\rangle = a_1(t) |\nu_1\rangle + a_2(t) |\nu_2\rangle. \] (9)

Because only electron neutrinos interact via charged currents, the two flavor states have different forward-scattering amplitudes, and each sees a different effective refractive index in matter. We assume that the change in the probability amplitudes \(a_1(t)\) and \(a_2(t)\) during an infinitesimal time \(\Delta t\) can be expressed as a simple phase shift that is proportional to the refractive index:

\[ a_1(t+\Delta t) \approx a_1(t) \exp \left[ \frac{i}{\hbar} (\xi n - \xi n - 1) \Delta t \right] = a_1(t) \exp \left[ \frac{i}{\hbar} \xi \Delta t \right], \text{ and } (10a) \]

\[ a_2(t+\Delta t) \approx a_2(t) \exp \left[ \frac{i}{\hbar} (\xi n - \xi n - 1) \Delta t \right] = a_2(t) \exp \left[ \frac{i}{\hbar} \xi \Delta t \right], \] (10b)

where \(\xi = 2 \sin \frac{\theta}{2} \sin \frac{\theta}{2} \). The latter relation is the mass oscillation term. We have also used \(\Delta t \approx \Delta t\). The state, therefore, evolves as

\[ |\psi(t)\rangle = a_1(t)|\nu_1\rangle + a_2(t)|\nu_2\rangle \approx |\nu_1\rangle + \frac{i}{\hbar} a_1(t)|\nu_2\rangle, \] (11)

where again we have dropped the overall phase factor of \(\exp(i\Delta t)\) because it does not affect the final result. Equations (8) and (11) are expressions for \(|\psi(t+\Delta t)\rangle\). Equating the coefficients of \(|\nu_1\rangle\) and \(|\nu_2\rangle\) results in a set of coupled equations:

\[ a_1(t+\Delta t) = a_1(t) \cos \theta + \frac{i}{\hbar} a_2(t) \sin \theta = a_1(t)|\nu_1\rangle + \frac{i}{\hbar} a_2(t) |\nu_2\rangle, \] (12a)

\[-a_1(t+\Delta t) = a_1(t) \sin \theta + \frac{i}{\hbar} a_2(t) \cos \theta = a_2(t)|\nu_1\rangle - \frac{i}{\hbar} a_1(t) |\nu_2\rangle. \]

Both sides of Equations (12a) and (12b) are expanded to first order in \(\Delta t\),

\[ [a_1(t) + a_2(t)\Delta t - i\hbar (M_1^\text{at} \cos \theta + M_2^\text{at} \sin \theta)] |\nu_1\rangle + [i\hbar (M_1^\text{at} \sin \theta - M_2^\text{at} \cos \theta)] |\nu_2\rangle, \]

\[ -[a_1(t) + a_2(t)\Delta t - i\hbar (M_1^\text{at} \sin \theta - M_2^\text{at} \cos \theta)] |\nu_1\rangle - [i\hbar (M_1^\text{at} \cos \theta + M_2^\text{at} \sin \theta)] |\nu_2\rangle, \] (13a)

where a distinct time derivative. Equations (4c) and (4d) are used to express \(a_1(t), \dot{a}_1(t), \dot{a}_2(t),\) and \(a_2(t)\) in terms of \(a_1(t), \dot{a}_1(t), \dot{a}_2(t),\) and \(a_2(t)\). Following more algebraic operations,

\[ -\dot{a}_1(t) = (M_2 \cos \theta + M_2 \sin \theta) a_1(t) + (M_1 \cos \theta - M_2 \cos \theta) a_2(t), \]

\[ -\dot{a}_2(t) = (M_2 \cos \theta - M_1 \cos \theta) a_1(t) + (M_1 \cos \theta + M_2 \cos \theta) a_2(t). \] (14a)

These expressions can be cast in a Schrödinger-like equation for a column matrix \(A\) consisting of the probability amplitudes \(a_1(t)\) and \(a_2(t)\):

\[ -\frac{i}{\hbar} \frac{dA}{dt} = HA, \] (15)

where \(H = \begin{pmatrix} M_2 \cos \theta + M_2 \sin \theta & (M_2 - M_1 \cos \theta) \sin \theta \\ (M_1 - M_2 \cos \theta) \sin \theta & M_1 \cos \theta + M_2 \cos \theta \end{pmatrix}\) and \(A = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}\).

The eigenvalues of the matrix \(H\) are given by

\[ \chi_{1,2} = \frac{\gamma + M_1 + M_2}{2} \pm \frac{\sqrt{(M_1 + M_2)^2 - 2\gamma(M_1 - M_2)\cos 2\theta}}{2}. \] (16)

Equation (15) can then be solved:

\[ A(t) = A(0) \exp \left[ \frac{i}{\hbar} \chi_t (\gamma x - \gamma x) \right] \] (17)

where \(\gamma = \sqrt{2}\alpha\gamma\). At time \(t = 0\), the beam consists only of electron neutrinos. Thus, \(a_1(0) = 1, \) and \(a_2(0) = 0\) so that

\[ a_1(t) = \frac{\cos \theta}{M_2 - M_1} \sin \theta. \]

The probability of detecting a muon neutrino after a time \(t\) is given by

\[ P_{MSW}(\nu_e \rightarrow \nu_\mu)(\theta) = |a_2(t)|^2 = |a_1(t)|^2 \sin^2 \theta. \] (19)

so that

\[ P_{MSW}(\nu_e \rightarrow \nu_\mu) = \frac{(M_2 - M_1)^2 \sin^2 \theta}{4(\gamma x - \gamma x)^2} \] (20)

By substituting in the expressions for \(x, x, x, M_1, M_2\) we have

\[ (x - x)^2 = (M_2 - M_1)^2 \sin^2 \theta \left( \frac{(x - x)^2}{(x - x)^2} \right) \] (21a)

\[ (M_2 - M_1) = \Delta m^2 \approx \frac{\sin 2\theta}{4\hbar}. \] (21b)

Recalling that \(x = \cos \theta\) and \(\gamma = \sqrt{2}\alpha\gamma\), we arrive at the MSW probability for an electron neutrino to oscillate into a muon neutrino:

\[ P_{MSW}(\nu_e \rightarrow \nu_\mu) = \frac{\sin^2 \theta}{4\hbar} \sin \left( \frac{\pi W}{\lambda} \right), \] (22)

\[ W = \sin^2 \theta + \sqrt{2}\alpha\gamma \frac{\Delta m^2}{2
\cos 2\theta}, \] (23)

where \(\lambda\) is the in vacuo oscillation length.

\[ \lambda = 2\pi \frac{\Delta m^2}{2
\cos 2\theta}, \] (24)

and \(\Delta m^2 = m_2^2 - m_1^2\) is required to be nonzero. If the numerical values of the hidden factors of \(\gamma, \) \(\alpha, \) and \(c\) are included, the expression for the oscillation length becomes \(\lambda = \sin 1.273\Delta m^2.\)