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A possible solution to the solar-neutrino problem

S. Peter Rosen

Ever since 1968, when Ray Davis and his colleagues reported their first observations of solar neutri-os, there have been continuing reports f a deficit in the number of electron eutrinos arriving at Earth compared with the number predicted by the stan-ard solar model. *In vacuo* neutrino scillations, in which electron neutrinos ave a certain probability of transform-ng into muon and/or tau neutrinos (or hange flavor) as they travel between he Sun and Earth, were often viewed s one way to explain that deficit.

To date, there have been four ifferent experiments with varying ensitivities to different parts of the olar-neutrino energy spectrum. All ave reported a significant reduction n the neutrino flux. The combined ata from them suggests that only eutrinos with energies between 1 and 0 million electron volts (MeV) are eriously depleted. It is unlikely that *in acuo* oscillations alone could produce hose results. Instead, the data suggest hat a resonance phenomenon might be t work, one in which the probability or a neutrino to transform into another flavor is high only over a limited region f the solar-neutrino energy spectrum.

The MSW effect provides us with ust such an energy-dependent phenom-non. Named after Lincoln Wolfenstein, who formulated the underlying physics, nd S. P. Mikheyev and Alexei mirnov, who recognized its impor-ance for the Sun, the MSW effect is an legant application of quantum mechan-cs that explains how neutrino oscilla-ons can be enhanced by the medium hrough which the neutrinos travel. At he right density of matter, an electron neutrino within a certain energy range can make a dramatic change to a different flavor even though its intrinsic *in vacuo* probability for doing so may be very small (Figure 1).

Neutrinos that Travel in a Vacuum

In vacuo neutrino oscillations can occur if the neutrino flavor states are "coherent," linear superpositions of neutrino mass states. Coherent means that the phases of two or more of the mass states are correlated, so that the relative phase between the states leads to an interference term in the calculation of quantum probabilities and expectation values. Quantum inter-ference between the mass states becomes an integral part of the neutrino's description.

In certain respects, this phase phe-nomenon parallels the relationship between circular and plane polarization for ordinary light. The left and right states of circular, polarized light emerge most naturally from Maxwell equations. All other states can be expressed as linear superpositions of those two states. In particular, plane, polarized light is a superposition of equal amounts of two circular states that have a constant rela-tive-phase difference. Changing the relative phase of the states rotates the plane of polarization.

Similarly, we think of the neutrino mass states $|\nu_1\rangle$, $|\nu_2\rangle$, and $|\nu_3\rangle$, with dis-tinct masses m_1, m_2, and m_3, as the analogues of the circularly polarized states. The weak-interaction states, or the flavor neutrinos, $|\nu_e\rangle$, $|\nu_\mu\rangle$, and $|\nu_\tau\rangle$, are created as coherent, linear superpo-sitions of the mass states and are the analogues of the independent planes of polarization. Because the phase of each mass state $|\nu_k\rangle$ depends on the mass m_k ($k = 1, 2, 3$), each state evolves with a different phase, and therefore the rela-tive phase between the states changes with time. Quite distinct from our analogy with ordinary light, this change can lead to the appearance of different neutrino flavors.

To keep things simple, in this article we shall consider only two neutrino mass states: $|\nu_1\rangle$ and $|\nu_2\rangle$. The electron- and muon-neutrino flavor states would then be written as

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle \ , \text{ and}$$
$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle \ .$$

The angle θ characterizes the amount of mixing between the mass states and is known as the *intrinsic mixing angle*. If θ is small, the electron neutrino consists primarily of the state $|\nu_1\rangle$ and has only a small admixture of $|\nu_2\rangle$, whereas the muon neutrino would be dominated by $|\nu_2\rangle$ and would have only a small amount of $|\nu_1\rangle$.

We assume that at $t = 0$, the neutrino is created in the distinct superposition that corresponds to the electron neutrino:

$$|\nu(0)\rangle = |\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle \ .$$

When the neutrino travels through a vacuum, each mass-state component $|\nu_k\rangle$ evolves with its own phase factor $\exp(-iE_kt)$. (We are working in units in which $\hbar = c = 1$.). We assume each mass state has the same momentum p, which is much greater than the masses

m_k. Thus,

$$E_k = \sqrt{p^2 + m_k^2} \approx p + \frac{m_k^2}{2p} \ ,$$

where $k = 1, 2$. Because $m_1 \neq m_2$, and hence $E_1 \neq E_2$, the relative phase between $|\nu_1\rangle$ and $|\nu_2\rangle$ will change as $|\nu(t)\rangle$ evolves with time. At an arbitrary time t, the neutrino has evolved to the state

$$|\nu(t)\rangle = \cos\theta\, e^{-iE_1t}\,|\nu_1\rangle + \sin\theta\, e^{-iE_2t}\,|\nu_2\rangle$$

In general, this linear combination of $|\nu_1\rangle$ and $|\nu_2\rangle$ is neither a pure electron neutrino state $|\nu_e\rangle$ nor a pure muon neutrino state $|\nu_\mu\rangle$. Instead, it is a linear superposition of both states. Quantum mechanics then tells us that, after travelling a distance of x meters, $|\nu(t)\rangle$ could be detected as a muon neu-trino. That transformation probability, denoted as $P(\nu_e \rightarrow \nu_\mu)$, is given by

$$P(\nu_e \rightarrow \nu_\mu) = |\langle \nu_\mu | \nu(t)\rangle|^2$$
$$= \sin^2 2\theta \, \sin^2(\pi x/\lambda).$$

(Even massive neutrinos would be highly relativistic and would travel at nearly the speed of light c. We have made the approximation $t \approx x/c = x$ in our units. See the box "Derivation of Neutrino Oscillation Length" on page 161.) Because of the term $\sin^2(\pi x/\lambda)$, the probability oscillates with distance from the source. The parameter λ is called the oscillation length and is given in meters by

$$\lambda = \frac{\pi E_\nu}{1.27\Delta m^2} \ ,$$

where E_ν is the neutrino energy in million electron volts and

$$\Delta m^2 = m_2^2 - m_1^2$$

is approximately the mass difference between the muon neutrino and the electron neutrino measured in electron volts squared, assuming a small, intrinsic mixing angle. (The factor of 1.27 in the denominator allows λ to be expressed in meters. It derives in part from hidden factors of $\hbar$ and c.) Note that, if the

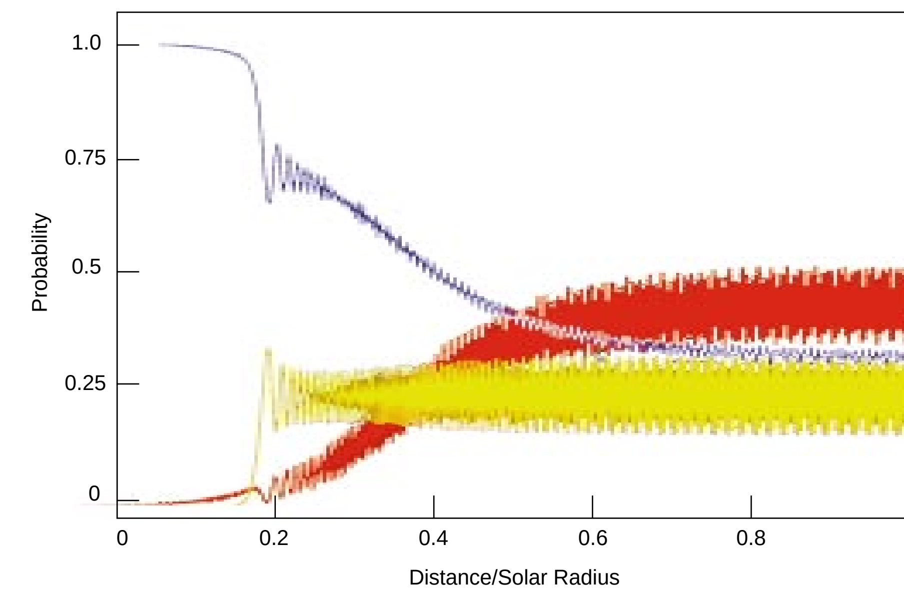


Figure 1. The Leopard Changes Its Spots

These curves represent the probability that a neutrino, after travelling through matter, would be detected as either an electron neutrino (blue), muon neutrino (red), or tau neutrino (yellow). A neutrino is "born" in the core of the sun as the superposition of mass states, in a combination that corresponds to an electron neutrino. In the specific model used to generate these curves, the superposition changes through the MSW resonance effect after the neutrino has traversed about 15 percent of the solar radius. The neutrino has only a 75 percent chance of being detected as an electron neutrino, and a 25 percent chance of being detected as a tau neutrino. By the time it flees the sun, the neutrino is most likely to be detected as a muon neutrino.

intrinsic mixing angle is small, then the oscillation probability will always be small, independent of neutrino energy.

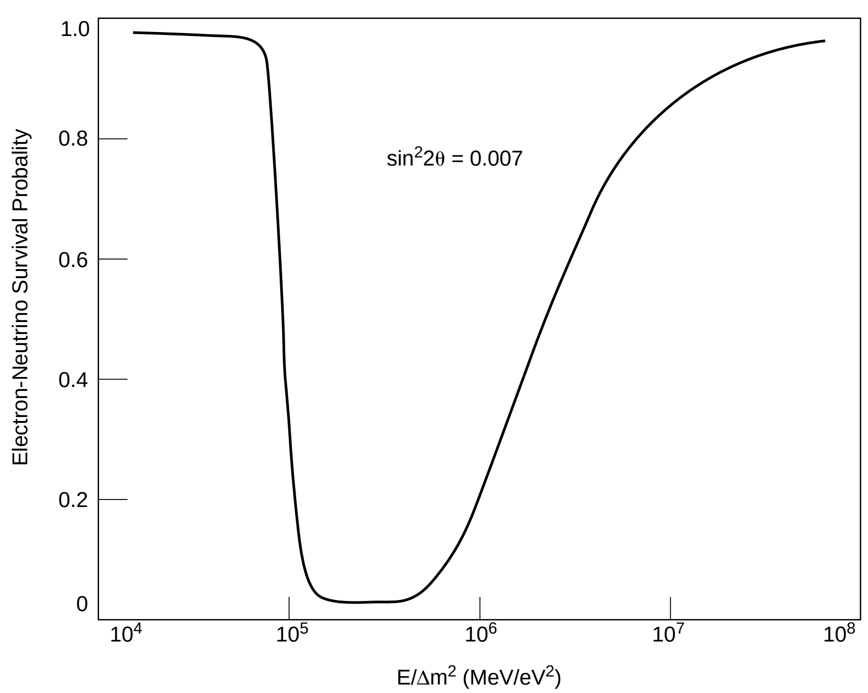
Neutrinos that Travel through Matter

The work of Wolfenstein, and then Mikheyev and Smirnov, showed that the oscillation probability could increase dramatically because of an additional phase shift that occurs when neutrinos travel through matter. The origin of this phase shift can be understood by analogy with a well-known phenome-non: When light travels through a material, it sees a refractive index because of coherent forward scattering from the constituents of the medium. Birefringent materials have different refractive indices for independent, linear polarizations of the light, and so the phase of each polarization component evolves differently. When polarized light passes through a birefringent material, the relative phase between the polarization states changes, and the plane of polarization rotates.

A similar phenomenon applies to the neutrino flavor states as they pass through matter. In the standard elec-troweak model, all neutrinos interact with up quarks, down quarks, and elec-trons through neutral currents (the exchange of neutral Z^0 bosons). All fla-vor states see a refractive index n that is a function of the neutral-current forward-scattering amplitude f_{nc}, the density of the electrons N_e, and the momentum p:

$$n_{nc} = 1 + \frac{2\pi N_e}{p^2} f_{nc} \ .$$

Electron neutrinos, and only electron neutrinos, can also interact with elec-trons through charged currents in a



igure 2. The MSW Survival-Probability Curve for the Sun
he probability that an electron neutrino born in the core of the Sun will emerge from he Sun as an electron neutrino is called the survival probability $P_s(\nu_e \rightarrow \nu_e)$. (The sur- val probability is equal to 1 − $P(\nu_e \rightarrow \nu_\mu)$.) It is plotted as a function of $E_\nu/\Delta m^2$, hich is essentially the *in vacuo* oscillation length λ. The mixing angle was chosen uch that $\sin^2 2\theta = 0.007$, and calculation of the curve takes into account the density rofile of the Sun. For a range of oscillation length values between 10^5 and 0^6 MeV/eV^2, the probability that an electron neutrino remains an electron neutrino is ery small. In our two-state model, the neutrino would oscillate into a muon neutrino.

harge-changing process mediated by he W^+ boson. The refractive index een by electron neutrinos, therefore, as an additional term n_{cc} given by

$$n_{cc} = \frac{2\pi N_e}{p^2} f_{cc} \quad .$$

he charged-current forward-scattering mplitude f_{cc} in this term is proportional o the weak-interaction coupling con- ant G_F (the Fermi constant) times the eutrino momentum

$$f_{cc} = \frac{G_F p}{\sqrt{2}\pi} \quad ,$$

o that n_{cc} takes the form

$$n_{cc} = \frac{\sqrt{2}G_F N_e}{p} \quad .$$

In travelling a distance x, each flavor state develops a phase $\exp[ip(n-1)x]$ due to the index of refraction. For muon or tau neutrinos, that phase is $\exp[ip(n_{nc}-1)x]$, while electron neutrinos have a phase given by $\exp[ip(n_{nc}+n_{cc}-1)x]$. Substituting in the expressions for n_{nc} and n_{cc} leads to the expressions

$$e^{i2\pi x N_e f_{nc}/p} \quad \text{for } \nu_\mu \text{ and}$$

$$e^{i(2\pi N_e f_{nc}/p + \sqrt{2}G_F N_e)x} \quad \text{for } \nu_e.$$

The additional term $\sqrt{2}G_F N_e$ in the phase of the electron neutrino is called the *matter oscillation term*. This term causes the relative phase between the electron and muon states to change with distance. Hence, the interference between the two states also changes with distance and, as always, is accompanied by interference phenomena. The interference can be totally destructive, totally constructive, or somewhere in between, depending upon relative conditions.

The equation for the MSW probability is derived in a heuristic fashion in the box on page 161. Here, we simply state the results. The MSW probability for an electron neutrino to transform into a muon neutrino is

$$P_{MSW}(\nu_e \rightarrow \nu_\mu) = \sin^2 2\theta_m \sin^2\left(\frac{\pi x W}{\lambda}\right) \; ,$$

where

$$\sin^2 2\theta_m = \frac{\sin^2 2\theta}{W^2} \quad ,$$

$$W^2 = \sin^2 2\theta + (D - \cos 2\theta)^2 \text{ , and}$$

$$D = \sqrt{2}G_F N_e \frac{2E_\nu}{\Delta m^2} \quad .$$

Note the similarity between this expression for the MSW probability and the *in vacuo* oscillation probability. The *in vacuo* mixing angle θ is replaced by an effective mixing angle θ_m that depends on the matter oscillation term (through the parameter D). When a neutrino travels in vacuum, the electron density is zero, and hence D is equal to zero, so that $W^2 = 1$. Thus, the MSW probability reduces to the *in vacuo* probability.

When a neutrino travels through matter, however, W^2 can become less than 1. This is the MSW resonance effect. The oscillation probability increases with the resonance condition given by $D = \cos 2\theta$. At that point, $W^2 = \sin^2 2\theta$ and $\sin^2 2\theta_m = 1$, and the oscillation probability reaches a maximum. This resonance condition is independent of the size of the intrinsic mixing angle θ, but it requires matching the properties of the material with the neutrino oscillation length λ through the relation

$$\sqrt{2}G_F N_e = 1.2 \frac{\cos 2\theta}{\lambda} \quad .$$

When this matching occurs, a neutrino will have a high probability

of transforming into a neutrino of a different flavor, even if its intrinsic *in vacuo* probability for doing so might be very small. This enhancement is called the MSW effect.

Oscillation Enhancement in the Sun

The density of the Sun is not constant but decreases monotonically from about 150 grams per cubic centimeter (g/cm^3) at the center to 0.1 g/cm^3 at a radius of 700,000 kilometers. In other words, the density decreases from 50 times the density of terrestrial rocks to one-tenth the density of water. For this range of densities, the value of the parameter D varies from a few tens near the core to a few hundredths near the edge. It passes through 1 at some intermediate point. Large values of D lead to very small values for the effective mixing angle and damp out neutrino oscillations, whereas small values essentially leave the *in vacuo* oscillation unchanged. Values near 1 give rise to the maximum enhancement (we are assuming that θ is small).

An electron neutrino born in the core of the Sun will start its journey to the edge without oscillating—until it reaches the region where D is of the order of 1. In this region, it will undergo the oscillation enhancement, and the probability that it will remain an electron neutrino will decrease significantly. In Figure 1, this region is observed at a distance of 0.2 solar radius. At larger solar radii, D is much smaller, and the electron neutrino's survival probability will oscillate around a relatively small value with an amplitude corresponding to the *in vacuo* mixing angle.

We can estimate the range of λ or Δm^2 for which the MSW effect is important. We rewrite the enhancement condition in terms of the solar density ρ_e measured in grams per cubic centimeter and then multiply by Avogadro's number. For the neutrino energy measured in million electron volts and Δm^2 measured in electron

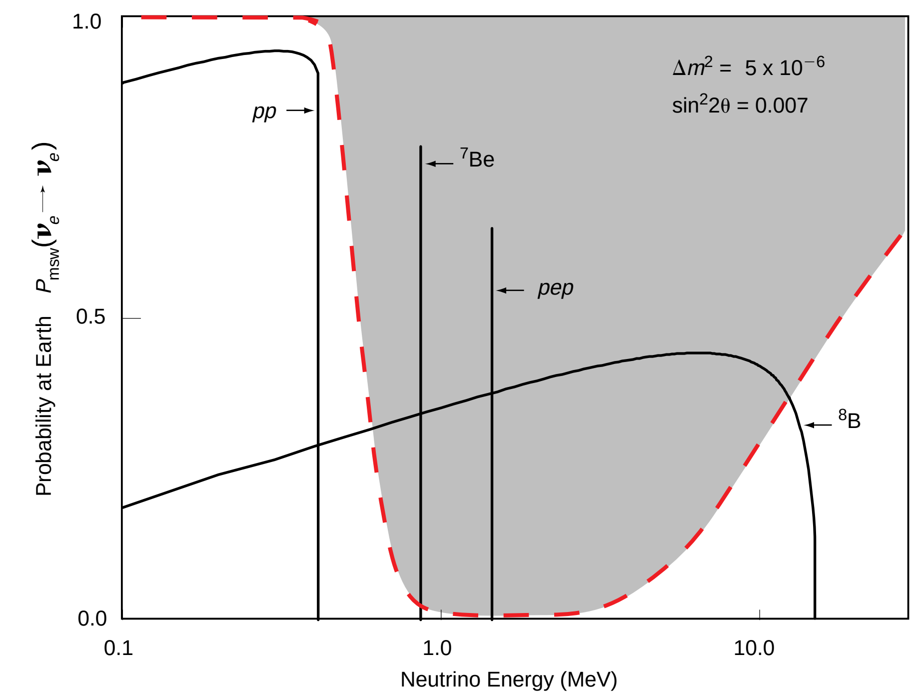


Figure 3. MSW Solution to the Solar-Neutrino Problem
The MSW survival probability $P_{MSW}(\nu_e \rightarrow \nu_e)$ (red dashed line) has been plotted as a function of neutrino energy and superimposed over a simplified picture of the solar-neutrino spectrum. Solar neutrinos with energies that fall within the gray-shaded region (those from the ^{7}Be, *pep*, and ^{8}B reactions) have a low survival probability, and thus have a high probability of oscillating into muon and/or tau neutrinos as they flee the solar core. If the MSW effect occurs in the Sun, experiments that detect only electron neutrinos would measure a reduced flux from those reactions. An experiment such as the one planned at the Sudbury Neutrino Observatory, which is sensitive to electron, muon, and tau flavors, should detect the full flux.

volts squared (eV^2),

$$2.5\ \lambda = \frac{7 \times 10^6}{\rho_e} \cos 2\theta \quad .$$

The *in vacuo* oscillation length λ is measured in meters for a neutrino with given momentum and mass parameters. The scale of the right-hand side is 1.8×10^4 kilometers, which is comparable with Earth's diameter. Given the range of solar densities, the MSW effect could occur if λ were in the range

$$10^4 \leq \lambda \leq 10^8 \text{ meters} \quad ,$$

as seen in Figure 2.

Because $\lambda = \pi E_\nu / 1.27\Delta m^2$ and because solar-neutrino energies vary from a fraction of 1 MeV to about 10 MeV, the MSW effect can occur in the Sun if the squared mass differences are

$$10^{-4} \geq \Delta m^2 \geq 10^{-9}\ \text{eV}^2 \quad .$$

To study this range of mass differences with terrestrial neutrinos, we would need intense sources of extremely low energy neutrinos, well below 1 MeV. Those sources do not exist, and therefore solar neutrinos are the only means available to us.

The Most Favored Solution to the Solar-Neutrino Problem

The value of Δm^2 determines how the curve for the survival probability overlays the spectrum of solar neutrinos. Decreasing Δm^2 moves the curve to the left, while increasing it moves he curve to the right. For values of oughly $\Delta m^2 = 5 \times 10^{-6}$ eV^2 and n$^2 2\theta = 0.007$, we find that *pp* neutri-os survive as electron neutrinos most f the time, while ^{7}Be neutrinos are lmost completely converted to muon eutrinos (Figure 3). This appears to e a good description of data measured y the current generation of solar-eutrino experiments. (See the article Exorcising Ghosts" on page 136.) Next-generation experiments, such s the one planned at the Sudbury Neutrino Observatory, are designed o determine whether oscillations o other neutrino states do indeed ccur and whether the MSW effect or *n vacuo* oscillations solve the solar-eutrino problem.

There is one final, but very interest-ng, comment. Our planet Earth may lay a unique role in the study of the MSW effect. There turns out to be a well-defined range of mixing angles nd mass differences for which the nhancement density is less than 5 g/cm^3. This density occurs in both he Sun and Earth, and thus neutrinos hat are converted from electron neutri-os to muon neutrinos in the Sun may e reconverted to electron neutrinos when they pass through Earth.

This effect would be seen as a sig-ificant increase in the solar-neutrino gnal at night, when Earth is between he Sun and the detector. Observation f such a "day-night" effect would be n unambiguous and definitive proof of he MSW effect and of neutrino mass. would also be nature's tip-of-the-hat o the insightful Lincoln Wolfenstein, who once observed that "…for neutri-os, the Sun shines at night!" ■

S. Peter Rosen, a native of London, earned his B.S. in mathematics and an M.A. and Ph.D. in theoretical physics from Merton College, Oxford, in 1954 and 1957, respectively. Rosen came to the United States as a research associate at Washington University, and from 1959 through 1961 he worked as scientist for the Midwestern Universities Research Association at Madison, Wisconsin. In 1961, he was awarded a NATO Fellowship to the Clarendon Laboratory at Oxford. Rosen returned to the United States as a Professor of Physics at Purdue University and later served as senior theoretical physicist for the High Energy Physics Program of the U.S. Energy Research and Development Administration. Some of Rosen's additional appointments include program associate for theoretical physics with the National Science Foundation, chairman of the U.S. Department of Energy's Technical Assessment Panel for Proton Decay and of Universities Research Association, Inc., Board of Overseers at Fermi National Accelerator Laboratory. Rosen first came to Los Alamos in 1977 as a visiting staff member and later became associate division leader for Nuclear and Particle Physics of the Theoretical Division. In 1990, he accepted the position of dean of science and professor of physics at the University of Texas, Arlington, and remained in that position until January 1997, when he was appointed associate director for High Energy and Nuclear Physics at the Department of Energy in Washington, D.C. He is a fellow of the American Physical Society and of the American Association for the Advancement of Science.

Further Reading

Rosen, S. P., and J. M. Gelb. 1986. Mikheyev-Smirnov-Wolfenstein Enhancement of Oscillations as a Possible Solution to the Solar-Neutrino Problem. *Physical Review D* **34** (4): 969.

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Heuristic Derivation of the MSW Effect
(for the students in us all)

For simplicity, we shall consider only oscillations between electron and muon neutrinos. The neutrino mass states $|\nu_1\rangle$ and $|\nu_2\rangle$ are assumed to have distinct masses m_1 and m_2, respectively. We define the neutrino flavor states $|\nu_e\rangle$ and $|\nu_\mu\rangle$ in terms of two mass states:

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle \ , \text{ and} \tag{1a}$$
$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle \ . \tag{1b}$$

We further assume that an electron neutrino is born at time $t = 0$. That neutrino will evolve in time as a superposition of states with time-dependent coefficients. The neutrino can be described by either mass states or flavor states:

$$|\nu(t)\rangle = a_1(t)\,|\nu_1\rangle + a_2(t)\,|\nu_2\rangle = a_e(t)\,|\nu_e\rangle + a_\mu(t)\,|\nu_\mu\rangle \ , \tag{2}$$

where

$$a_e(t) = a_1(t)\cos\theta + a_2(t)\sin\theta \ , \text{ and} \tag{3a}$$
$$a_\mu(t) = -a_1(t)\sin\theta + a_2(t)\cos\theta \ . \tag{3b}$$

We will also be using the inverse of Equations (1) and (3):

$$|\nu_1\rangle = \cos\theta\,|\nu_e\rangle - \sin\theta\,|\nu_\mu\rangle \ , \text{ and} \tag{4a}$$
$$|\nu_2\rangle = \sin\theta\,|\nu_e\rangle + \cos\theta\,|\nu_\mu\rangle \ ; \tag{4b}$$

$$a_1(t) = a_e(t)\cos\theta - a_\mu(t)\sin\theta \ , \text{ and} \tag{4c}$$
$$a_2(t) = a_e(t)\sin\theta + a_\mu(t)\cos\theta \ . \tag{4d}$$

In general, the time development of the neutrino states described in Equation (2) has a phase that depends on both the momentum and the energy of the neutrino. For example, an electron neutrino evolves as

$$|\nu_e(t)\rangle = \cos\theta\, e^{ip\cdot x - iE_1 t}\,|\nu_1\rangle + \sin\theta\, e^{ip\cdot x - iE_2 t}\,|\nu_2\rangle \ . \tag{5}$$

We work in units in which $\hbar = c = 1$. Let us first consider the evolution of $|\nu(t)\rangle$ as a superposition of mass eigenstates during an infinitesimal time Δt. We assume a common momentum for each mass state, so that only the difference between the energies of the mass states (due to the difference in the neutrino masses) governs the time development of the state. With $p >> m_k$, we can approximate the energy as

$$E_k = \sqrt{p^2 + m_k^2} \approx p + m_k^2/2p = p + M_k \ , \tag{6}$$

where $M_k = m_k^2/2p$ ($k = 1, 2$). The neutrino evolves in time Δt as

$$\begin{aligned} |\nu(t+\Delta t)\rangle &= a_1(t+\Delta t)e^{-iE_1\Delta t}\,|\nu_1\rangle + a_2(t+\Delta t)e^{-iE_2\Delta t}\,|\nu_2\rangle \\ &\approx a_1(t+\Delta t)\,e^{-iM_1\Delta t}\,|\nu_1\rangle + a_2(t+\Delta t)e^{-iM_2\Delta t}\,|\nu_2\rangle \ . \end{aligned} \tag{7}$$

We have dropped the overall phase factor of $\exp(-ip\Delta t)$ in Equation (7) because it has no bearing on the final result. With the help of Equations (4a) and (4b), we can write Equation (7) in the flavor basis:

$$|\nu(t+\Delta t)\rangle = [a_1(t+\Delta t)e^{-iM_1\,\Delta t}\cos\theta + a_2(t+\Delta t)e^{-iM_2\,\Delta t}\sin\theta]\,|\nu_e\rangle + [-a_1(t+\Delta t)e^{-iM_1\,\Delta t}\sin\theta + a_2(t+\Delta t)e^{-iM_2\,\Delta t}\cos\theta]\,|\nu_\mu\rangle \quad . \tag{8}$$

We next consider the neutrino as a superposition of flavor states:

$$|\nu(t)\rangle = a_e(t)\,|\nu_e\rangle + a_\mu(t)\,|\nu_\mu\rangle \quad . \tag{9}$$

Because only electron neutrinos interact via charged currents, the two flavor states have different forward-scattering amplitudes, and each sees a different effective refractive index in matter. We assume that the change in the probability amplitudes $a_e(t)$ and $a_\mu(t)$ during an infinitesimal time Δt can be expressed as a simple phase shift that is proportional to the refractive index:

$$a_e(t+\Delta t) \approx a_e(t)\exp[ip\,(n_{nc} + n_{cc} - 1)\Delta t] = a_e(t)\exp[i(\xi+\eta)\Delta t] \text{ , and} \tag{10a}$$

$$a_\mu(t+\Delta t) \approx a_\mu(t)\exp[ip\,(n_{nc} - 1)\Delta t] = a_\mu(t)\exp[i\xi\,\Delta t] \text{ ,} \tag{10b}$$

where $\xi = (2\pi N_e/p)f_{nc}$ and $\eta = \sqrt{2}G_F N_e$. The latter relation is the matter oscillation term. We have also used $\Delta x \approx \Delta t$. The neutrino state, therefore, evolves as

$$\begin{aligned}|\nu(t+\Delta t)\rangle &= a_e(t)e^{i(\xi+\eta)\Delta t}\,|\nu_e\rangle + a_\mu(t)e^{\,i\xi\,\Delta t}\,|\nu_\mu\rangle \\ &= a_e(t)e^{i\eta\Delta t}\,|\nu_e\rangle + a_\mu(t)\,|\nu_\mu\rangle \text{ ,}\end{aligned} \tag{11}$$

where again we have dropped the overall phase factor of $\exp(i\xi\Delta t)$ because it does not affect the final result. Equations (8) and (11) are expressions for $|\nu(t+\Delta t)\rangle$. Equating the coefficients of $|\nu_e\rangle$ and $|\nu_\mu\rangle$ results in a set of coupled equations:

$$a_1(t+\Delta t)e^{-iM_1\,\Delta t}\cos\theta + a_2(t+\Delta t)e^{-iM_2\,\Delta t}\sin\theta = a_e(t)e^{i(\xi+\eta)\Delta t} \text{ ,} \tag{12a}$$

$$-a_1(t+\Delta t)e^{-iM_1\,\Delta t}\sin\theta + a_2(t+\Delta t)e^{-iM_2\,\Delta t}\cos\theta = a_\mu(t)e^{i\xi\,\Delta t} \text{ .} \tag{12b}$$

Both sides of Equations (12a) and (12b) are expanded to first order in Δt,

$$[a_1(t) + \dot{a}_1(t)\Delta t - ia_1(t)M_1\Delta t]\cos\theta + [a_2(t) + \dot{a}_2(t)\Delta t - ia_2(t)M_2\Delta t]\sin\theta = a_e(t)(1+i\eta\Delta t) \tag{13a}$$

$$-[a_1(t) + \dot{a}_1(t)\Delta t - ia_1(t)M_1\Delta t]\sin\theta + [a_2(t) + \dot{a}_2(t)\Delta t - ia_2(t)M_2\Delta t]\cos\theta = a_\mu(t) \text{ ,} \tag{13b}$$

where a dot indicates the time derivative. Equations (4c) and (4d) are used to express $a_1(t)$, $\dot{a}_1(t)$, $a_2(t)$, and $\dot{a}_2(t)$ in terms of $a_e(t)$, $\dot{a}_e(t)$, $a_\mu(t)$, and $\dot{a}_\mu(t)$. Following more algebraic operations,

$$-i\dot{a}_e(t) = [M_1\cos^2\theta + M_2\sin^2\theta + \eta\,]a_e(t) + (M_2 - M_1)\cos\theta\sin\theta\, a_\mu(t) \text{ ,} \tag{14a}$$

$$-i\dot{a}_\mu(t) = (M_2 - M_1)\cos\theta\sin\theta\, a_e(t) + [M_1\sin^2\theta + M_2\cos^2\theta]a_\mu(t) \text{ .} \tag{14b}$$

These expressions can be cast in a Schrödinger-like equation for a column matrix A consisting of the probability amplitudes $a_e(t)$ and $a_\mu(t)$:

$$-i\,\frac{dA}{dt} = HA \text{ ,} \tag{15}$$

where $H = \begin{pmatrix} M_1\cos^2\theta + M_2\sin^2\theta + \eta & (M_2 - M_1)\cos\theta\sin\theta \\ (M_2 - M_1)\cos\theta\sin\theta & M_1\sin^2\theta + M_2\cos^2\theta \end{pmatrix}$ and $A = \begin{pmatrix} a_e(t) \\ a_\mu(t) \end{pmatrix}$

The eigenvalues of the matrix H are given by

$$\chi_{1,2} = \frac{\eta + M_1 + M_2}{2} \mp \frac{\sqrt{\eta^2 + (M_2 - M_1)^2 - 2\eta(M_2 - M_1)\cos 2\theta}}{2} \quad . \tag{16}$$

Equation (15) can then be solved:

$$A(t) = \left[\frac{\chi_2 e^{i\chi_1 t} - \chi_1 e^{i\chi_2 t}}{\chi_2 - \chi_1} I + \frac{e^{i\chi_2 t} - e^{i\chi_1 t}}{\chi_2 - \chi_1} H\right] A(0) \text{ ,} \tag{17}$$

where I is the identity matrix. At time $t = 0$, the beam consists only of electron neutrinos. Thus, $a_e(0) = 1$, and $a_\mu(0) = 0$ so that

$$a_e(t) = \frac{\chi_2 e^{i\chi_1 t} - \chi_1 e^{i\chi_2 t}}{\chi_2 - \chi_1} + \frac{e^{i\chi_2 t} - e^{i\chi_1 t}}{\chi_2 - \chi_1}(M_1\cos^2\theta + M_2\sin^2\theta + \eta) \text{ ,} \tag{18a}$$

$$a_\mu(t) = \frac{e^{i\chi_2 t} - e^{i\chi_1 t}}{\chi_2 - \chi_1}(M_2 - M_1)\cos\theta\sin\theta \text{ .} \tag{18b}$$

The probability of detecting a muon neutrino after a time t is given by

$$P_{MSW}(\nu_e \to \nu_\mu) = |\langle\nu_\mu|\nu(t)\rangle|^2 = |a_e(t)\langle\nu_\mu|\nu_e\rangle + a_\mu(t)\langle\nu_\mu|\nu_\mu\rangle|^2 = |a_\mu(t)|^2 \tag{19}$$

so that

$$\begin{aligned}P_{MSW}(\nu_e \to \nu_\mu) &= \frac{(M_2 - M_1)^2\cos^2\theta\sin^2\theta}{(\chi_2 - \chi_1)^2} 2(1 - \cos(\chi_2 - \chi_1)t) \\ &= \frac{(M_2 - M_1)^2\sin^2 2\theta}{(\chi_2 - \chi_1)^2}\sin^2\frac{(\chi_2 - \chi_1)}{2}t \text{ .}\end{aligned} \tag{20}$$

By substituting in the expressions for χ_1, χ_2, M_1, M_2, we have

$$(\chi_2 - \chi_1)^2 = (M_2 - M_1)^2\left[\sin^2 2\theta + \left(\frac{\eta}{M_2 - M_1} - \cos 2\theta\right)^2\right] \tag{21a}$$

$$(M_2 - M_1) = \frac{\Delta m^2}{2p} \approx \frac{\Delta m^2}{2E_\nu} \quad . \tag{21b}$$

Recalling that $x = t$, and $\eta = \sqrt{2}G_F N_e$, we arrive at the MSW probability for an electron neutrino to oscillate into a muon neutrino:

$$P_{MSW}(\nu_e \to \nu_\mu) = \frac{\sin^2 2\theta}{W^2}\sin^2\left(\frac{\pi x W}{\lambda}\right) \text{ ,} \tag{22}$$

$$W^2 = \sin^2 2\theta + \left(\sqrt{2}G_F N_e\,\frac{2E_\nu}{\Delta m^2} - \cos 2\theta\right)^2 \text{ ,} \tag{23}$$

where λ is the *in vacuo* oscillation length,

$$\lambda = 2\pi\,\frac{2E_\nu}{\Delta m^2} \text{ ,} \tag{24}$$

and $\Delta m^2 = m_2^2 - m_1^2$ is required to be nonzero. If the numerical values of the hidden factors of $\hbar$ and c are included, the expression for the oscillation length becomes $\lambda = \pi E/1.27\Delta m^2$. ■