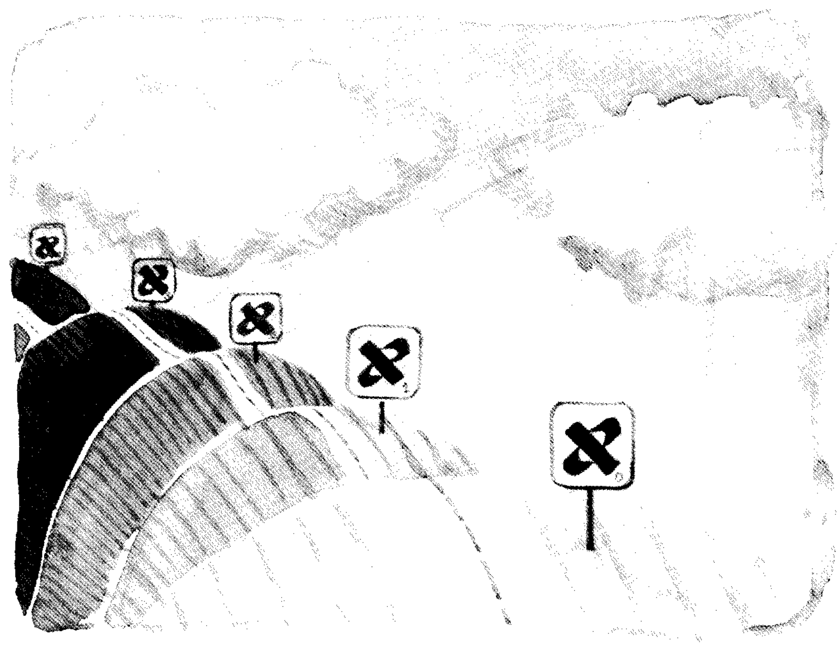
Learning *from* Ulam



·Measurable Cardinals ·Ergodicity ·Biomathematics

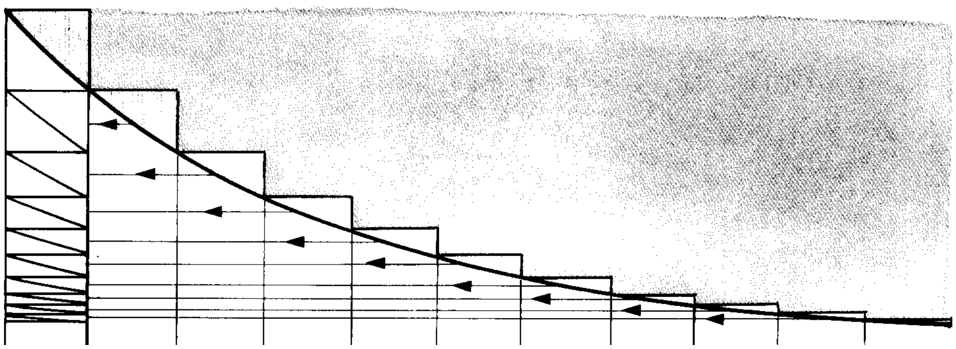
by Jan Mycielski

Stanislaw M. Ulam was one of a group of distinguished Polish mathematicians who immigrated to the United States around 1939. Other members of the group who come to my mind were Natan Aronszajn, Stefan Bergman, Samuel Eilenberg, Mark Kac, Otto Nikodym, Alfred Tarski, and Antoni Zygmund. Ulam was born in 1909 in Lwow, which was then within the boundaries of Poland. The society in which he grew up and was educated was almost completely obliterated during World War II; the surviving population was dispersed within the present boundaries of Poland. Lwow is now a Ukrainian city, and only its buildings remain to remind one that for several centuries it was an outpost of Polish culture.

Lwow was the birthplace of functional analysis, and Ulam, although he cannot be called a functional analyst, had strong connections to the group of mathematicians (Banach, Mazur, Steinhaus, and Schauder) who created it. Stan had a superb memory, and he beautifully described those times in his book *Adventures of a Mathematician,* in an article on the Scottish Cafe in the 1981 edition of *The Scottish Book,* and in an obituary for Stefan Banach. He received an excellent classical education at the Sixth Gymnasium in Lwow and throughout his life could quote Greek and Roman poetry. Encouraged by his parents to study "something useful," he went to the Lwow Polytechnic School to become an engineer, but he quickly became interested by a lively group of mathematicians and began to spend more and more time with them.

One of the people who attracted him to mathematics was the famous topologist K. Kuratowski. Ulam told me about the following "little invention" he made while attending Kuratowski's course in calculus. Over a non-negative, decreasing function on the positive portion of the real line, build stairs of equal depth (Fig. 1). The problem is to prove that the shaded area (the sum of the areas between the function and the steps) is finite. Ulam said: "Shift the shaded pieces horizontally to the left until they all find themselves within the first column. " Since the area of the first column is finite, the sum of the shaded pieces is also finite. Kuratowski was very happy to hear this original thought from his student. Perhaps that was the pebble that turned Ulam into a mathematician.

At the age of twenty, Ulam published his first paper: "A Remark on the Gener-



A "LITTLE INVENTION"

Fig. 1. K. Kuratowski posed the following problem to his calculus class: Over a nonnegative, decreasing function on the positive part of the real line, construct a step function with steps of equal depth. Prove that the area of the shaded regions between the two functions is finite. Ulam's solution was to move each shaded region into the first column, the area of which is finite.

alized Bernstein's Theorem." That short paper solves a problem posed by Kuratowski. It belongs to the theory of jigsaw puzzles (also called the theory of equivalence by finite decomposition) and is one of the earliest applications of graphs in set theory. It appears in the 1974 volume *Stanislaw Ulam: Sets, Numbers, and Universes,* which contains more than half of Ulam's hundred or so then-published papers. We can learn a lot from that volume. I will try to describe some of what I have learned, but first let me record some memories from our numerous conversations over the years.

Ulam liked to consider amusing objects and processes. It didn't matter to him whether or not they were real or imaginary, but they had to be intrinsically interesting, not just tools. Consequently most of his work has a directness similar to the directness of an observation of nature. That distinguishes his work from the majority of mathematical papers, which elaborate existing theories. In fact, in his later life he became quite critical of such mathematical investigations, which he regarded as too abstruse or unimaginative. He would even remark that the study of specific subjects, such as advanced chapters of algebra, algebraic topology, or analysis, was motivated by the history of mathematics rather than by the interest or notoriety of their problems. I would reply that mathematics is also an art, motivated by its internal beauty, and that only persistent study may reveal that beauty. He would agree only that his opinion was not easy to interpret correctly. In the end I am sure that there is wisdom in what he said, if only because he discovered several facts that are fundamental to modem mathematical culture, and I can hardly imagine discoveries of that nature in the areas he was criticizing.

Measurable Cardinals

I will now try to tell you about one of Ulam's important discoveries. It pertains to the foundations of mathematics and to the theory of large cardinal numbers. To give it the proper perspective, let me recall that Euclid was the first to organize the mathematics of his time into an axiomatic theory. That means he started from certain basic principles called axioms that he accepted without proof, and from them he obtained by pure deduction all the mathematical knowledge of his time. The system of Euclid became the accepted definition of mathematics until the time of Newton and Leibnitz. After the discovery of calculus, it became apparent that the development of mathematics within the system of Euclid is very unwieldy, and the system had to be abandoned. For a few centuries mathematics was in a sense unruly. Axiomatic organization returned to it around the turn of this century with the discoveries of Frege, Cantor, and Zermelo. Frege developed logic, Cantor invented and developed set theory, and Zermelo gave axioms for Cantor's set theory. Soon it became clear that all modern mathematics can be smoothly developed within set theory. Gradually it also became apparent that there is a whole hierarchy of larger and larger set theories, and one of the best ways to classify them is to see how large are the infinite cardinal numbers that can be shown to exist in those theories. (By a famous definition of Cantor, two sets A and B, finite or infinite, have the same cardinal number if and only if there exists a one-to-one function mapping A onto B). One might think that very large cardinal numbers are rather exotic and abstract objects whose existence is not of great mathematical interest. But by a famous theorem Gödel proved in 1931 (the so-called second incompleteness theorem), it follows that the larger the cardinal number whose existence can be proven in a given set theory, the more theorems can be proved in that theory, even theorems pertaining to such elementary operations as the addition and multiplication of integers. This fact was not yet known at the time Ulam made his discovery. His motivation was different—he was attracted by the mystery of the very large cardinal numbers for its own sake. Let me try to explain his theorem.

The smallest infinite cardinal number is called $\aleph_0$ (aleph zero). It is the cardinality of the set of integers. Clearly $\aleph_0$

has the following property: If we multiply less than $\aleph_0$ cardinal numbers each of which is less than $\aleph_0$, then the product is also less than $\aleph_0$. Well, of course, this tells only that the product of finitely many integers is finite. Thus we can say that $\aleph_0$ is inaccessible by products. This property is also called *strong inaccessibility*: $\aleph_0$ is strongly inaccessible. Are there any cardinal numbers larger than $\aleph_0$ (such cardinals are called uncountable) that are also strongly inaccessible? It turns out that this problem cannot be solved. The axioms of set theory do not imply the existence of such cardinals, and one can only postulate their existence as an axiom, which is what Felix Hausdorff did. Indeed, a set theory in which we accept this axiom is stronger (in the sense that it gives rise to more theorems of arithmetic) than the original set theory of Cantor and Zermelo.

To explain the work of Ulam we need the concept of a measure. For a set of points on the plane, area is a measure, and for a set of points in three-dimensional space, volume is a measure. In general given any set X, a measure is a function μ that attaches to subsets of X some non-negative numbers in such a way that the following condition is satisfied:

C_0: If A and B are disjoint subsets of X, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

There are many variants of the concept of measure. The version that is the most important for mathematical analysis says that $\mu(A)$ must be defined for all subsets A of X that are in a collection Ω of subsets such that Ω is closed under countable unions and complementations. That is, if $A_i \in \Omega$ for $i = 1, 2, \ldots$, then the union of the A_i's is in Ω, $\bigcup_{i=1}^{\infty} A_i \in \Omega$, and if $A_i \in \Omega$, then its complement, or $X - A_i$, is in Ω. Moreover the measure μ must be countably additive; that is, if $A_i \in \Omega$, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$, provided the A_i's are disjoint.

Ulam considered special measures that satisfy some additional conditions:

C_1: $\mu(A) = 0$ whenever A consists of just one element of X;

C_2: $\mu(X) = 1$ (that is, the measure of the whole space is 1); and

C_3: Ω is the set of *all* subsets of X.

Measures that satisfy C_0, C_1, C_2, and C_3 are called *universal* measures. Readers familiar with Lebesgue's measure may recall that it is not a universal measure since the collection Ω on which it is defined is not the set of *all* subsets of $[0,1]$. On the other hand, Lebesgue's measure is invariant under translations, whereas the set X is just an abstract set without any transformations upon which μ could be assumed to be invariant. Even in this abstract setting it is very difficult to construct a universal measure. For example, if X is countable, no such measure exists since condition C_1 plus countable additivity forces $\mu(X)$ to equal 0, contrary to C_2.

Ulam proved two fundamental results about universal measures. The first tells that no universal measure exists for many uncountable sets. In particular, for many consecutive cardinals larger than $\aleph_0$ (for example, $\aleph_1, \aleph_2, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots$), sets of those cardinalities do not have universal measures.

To explain Ulam's second result, we restrict the concept of universal measure still further by adding the following condition:

C_4: $\mu(A) = 0$ or $\mu(A) = 1$ for all subsets A of X.

The cardinal number of a set that has a countably additive measure satisfying C_1, C_2, C_3, and C_4 is called an Ulam cardinal. Again, we can ask whether any Ulam cardinals exist. Ulam's famous theorem is that if such a cardinal does exist, then it is strongly inaccessible. This result implies that if we consider two set theories, one in which we assume the existence of uncountable strongly inaccessible cardinals and the second in which we assume the existence of Ulam cardinals, then the second theory is at least as strong as the first. Today we know many interesting theorems that follow from postulating the existence of Ulam cardinals. In particular, thirty years after Ulam's paper on measurable cardinals, William Hanf and Alfred Tarski proved that the least uncountable strongly inaccessible cardinal is smaller than the least Ulam cardinal. Thus a set theory in which Ulam cardinals exist is strictly stronger than one in which only uncountable strongly inaccessible cardinals exist. Many more results of this sort have been discovered since. The theory of large cardinals has become very rich, but Ulam's paper remains one of its keystones.

Ergodic Theory and Topology

Between 1929 and 1938 Ulam published about twenty papers. What distinguishes those from the papers of other members of the Polish school before 1939 was his interest in topological groups, especially the groups of homeomorphisms of spheres.

A homeomorphism of a space X is a transformation of X onto itself that is one-to-one and continuous and whose inverse is also continuous. Of course such transformations constitute a group under composition. It is not obvious how to introduce a natural topology or even metrization into such a group. The following formula was often proposed (for example, it appears in Banach's classic book *Theorie des Operations Lineaires):* The distance between two homeomorphisms,f and *g* of a compact space X, *Dist(f, g)*, is given by

$$\mathrm{Dist}(f, g) = \max_{x \in X} \mathrm{dist}\big(f(x), g(x)\big) + \max_{x \in X} \mathrm{dist}\big(f^{-1}(x), g^{-1}(x)\big),$$

where dist denotes the distance in X.

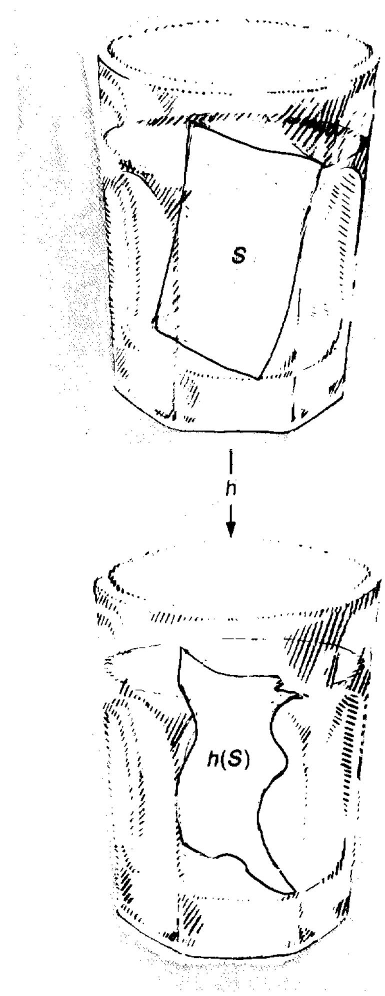
The surprising property of this formula is that it converts the space of homomorphisms of X into a complete metric space. In other words, if a sequence of homeomorphisms satisfies the condition of Cauchy, then it has a limit that is a homeomorphism. The fact that the space of homeomorphisms can be treated as a complete metric space is very important because for such spaces there exist very natural definitions of largeness or smallness of subsets. The small ones are called meager (or of the first category) and the large ones comeager (or complements of meager). These topological concepts were invented by Baire. Several brilliant applications of these notions were made by Banach and Mazur. A very famous one was made by John C. Oxtoby and Ulam around 1941. Let me try to describe it here.

Take a glass of water, gently stir its contents, and let the water stop moving. Each particle of water has an initial and final position. The operation has thus defined a transformation of the interior of the glass into itself. Since water is viscous, this transformation is continuous and its inverse is also continuous. So we have here a homomorphism. Moreover, since water is incompressible, the homomorphism is volume-preserving. Homomorphisms with that property constitute a complete subspace of the space of all homeomorphisms. If our transformation had been a simple rotation, then the altitudes of the particles of water and their distances from the central axis of the glass would not have changed. Many parts of the water would have remained invariant; that is, such parts would have been mapped into themselves. Even if we had applied the rotation many times, the water would never have been mixed. Are there any volume-preserving homomorphisms that do mix? Such transformations, which are called ergodic, or metrically transitive. must exist if the ergodic hypothesis of statistical mechanics is correct. However, the existence of such transformations had remained an open question since the work of Poincare and G. D. Birkhoff. Oxtoby and Ulam, in their paper entitled "Measure-Preserving Homeomorphisms and Metrical Transitivity," showed not only that such homomorphisms exist but also that the set of ergodic homomorphisms is comeager, that is, large in the sense of category. More precisely, any homomorphism of that comeager set has the property that its application to any proper part of our glass of water deflects its boundary (Fig. 2). Thus the homeomorphism mixes the water in the sense that no part returns to its initial position. The Oxtoby - Ulam theorem remains one of the high points of the mathematical theory concerning ergodic properties of dynamical systems, The introduction to their paper, excerpted on the following page, explains the connection to the ergodic hypothesis. (These excerpts may be better understood after reading "The Ergodic Hypothesis: A Complicated Problem of Mathematics and Physics," as well as the section entitled Problem 2. Geometry, Invariant Measures, and Dynamical Systems in the article "Probability and Nonlinear Systems," all in this **issue.)**

We **must caution,** however, that application of mathematical theorems to the real world is sometimes a delicate problem. As you know, a sequence of heads and tails obtained by consecutive tosses of a fair coin has the property that the frequency of heads converges to 1/2 as the number of tosses becomes large. One can say (and prove in precise mathematical terms) that if we choose a sequence at random from the space of all such sequences, then, with probability 1, the limiting frequency of heads in this sequence is **1/2.** Unfortunately, in another sense, namely that of category, almost all sequences (namely a comeager set) do *not* have any limiting value for the frequency of heads! So the very sense in which almost all volume-preserving homeomor-

EFFECT **OF AN ERGODIC TRANSFORMATION**

Fig. 2. If *h* is an ergodic transformation, every surface S separating the water is deflected by *h* from its original position.



phisms of a cube are ergodic suggests the physically false result that almost all sequences of heads and tails lack a well-defined frequency of heads. Can we then trust the theorem of Ulam and Oxtoby as an expression of the truth of the ergodic hypothesis in physics? Stan and I often discussed this question. We thought that the answer is yes, but what is really needed is a new theorem in which almost all, in the sense of category, is replaced by some other more reliable sense. (I have outlined an idea of such a new theorem or conjecture in two papers in *Journal of Symbolic Logic,* one in volume 46 (1981) and the other in volume 51 (1986), but I do not know how to prove it.

Ulam and Josef Schreier obtained another interesting result about the group of homomorphisms of a spherical surface. They proved that there exist two special homomorphisms such that every homomorphism in the group can be approximated with arbitrary accuracy (relative to the distance defined above) by appropriate iterative compositions of those two homeomorphisms and their inverses.

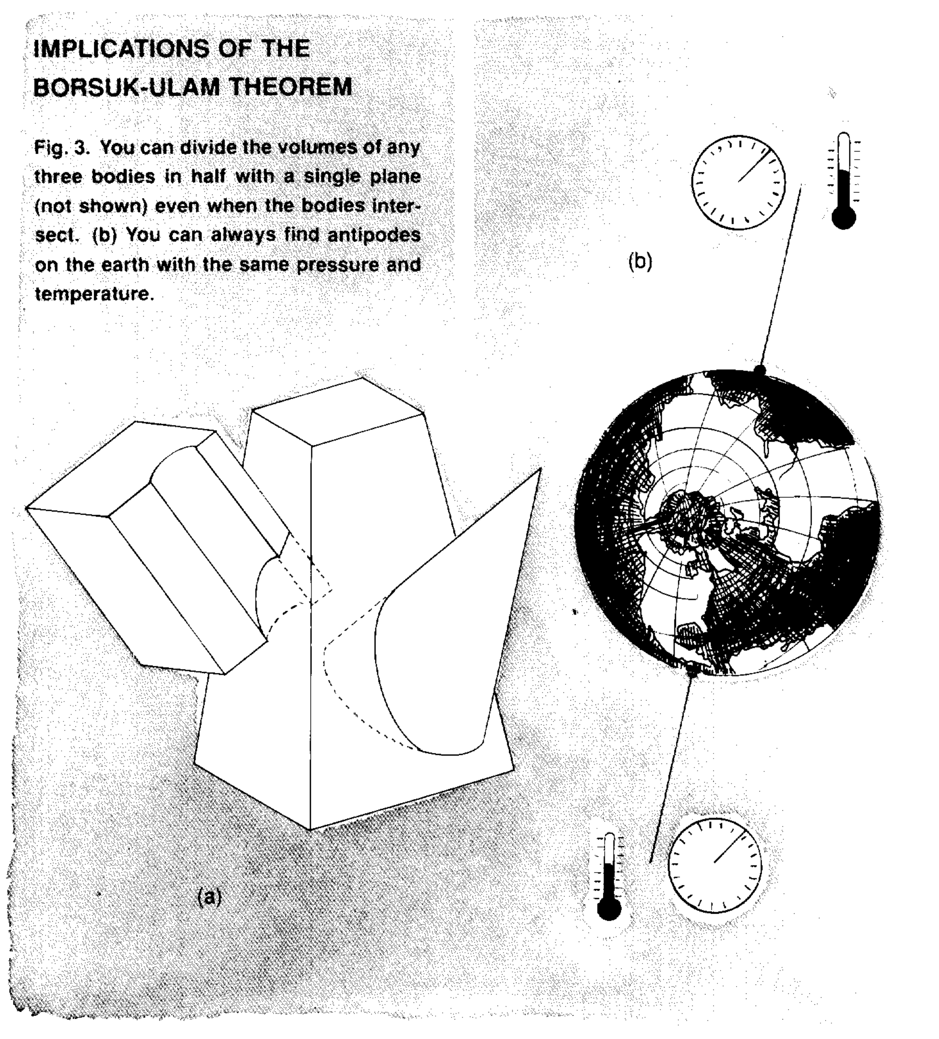
Kuratowski and Ulam proved an extension of the theorem of Fubini to the context of Baire's category that is often very useful.

An interesting feature of Ulam's work followed from his great ability to collaborate with others. Almost all of his papers are co-authored with other mathematicians or physicists. He had many ideas, and he was very successful in stirring the imagination and enthusiasm of others. His most important collaborators were Josef Schreier, John C. Oxtoby, and C. J. Everett. He invented a large number of original problems, some of which were solved by other mathematicians and even became famous theorems. One such conjecture was proved by K. Borsuk and is known today as the theorem on antipodes (the two points at the opposite ends of a diameter of a sphere are called antipodes of each other). It is sometimes called the ham-and-cheese sandwich theorem. It tells the following: For every continuous mapping of the spherical surface into the plane, there exist antipodes that are mapped into the same point on the plane. This theorem is equivalent to the following statement. Given three bodies (say ham, cheese, and bread), one can find a single plane that divides each body into two parts of equal volume. (Each body may consist of disjoint pieces, as does the bread in a sandwich, and the bodies may overlap, as shown in Fig. 3a.) Another equivalent statement is that at any time antipodal points can be found on the earth where the temperature and the barometric pressure are the same (Fig. 3b).

Topics in Biology and Some Applications of Computers

I began to collaborate with Stan Ulam in 1969 when he invited me to the University of Colorado in Boulder. We spoke frequently about the problems of the organization and function of the human brain and the structure of memory. He presented his ideas on this subject in the talk "Reflections on the Brain's Attempts to Understand Itself," which is posthumously published in this issue.

We also talked often about the problem of accumulation of mutations in a given species. As a result of our discussions I



IMPLICATIONS OF THE BORSUK-ULAM THEOREM

Fig. 3. You can divide the volumes of any three bodies in half with a single plane (not shown) even when the bodies intersect. (b) You can always find antipodes on the earth with the same pressure and temperature.

The Existence and Significance of Ergodic Transformations

Excerpts from the Introduction to Oxtoby and Ulam's "Measure-Preserving Homomorphisms and Metrical Transitivity"

In the study of dynamical systems one is led naturally to the consideration of measure-preserving transformations. A Hamiltonian system of $2n$ differential equations induces in the phase space of the system a measure-preserving flow, that is, a one-parameter group of transformations that leave invariant the $2n$-dimensional measure. . . . If the differential equations are sufficiently regular the flow will have corresponding properties of continuity and differentiability. Thus the study of one-parameter continuous groups of measure-preserving automorphisms of finite dimensional spaces has an immediate bearing on dynamics and the theory of differential equations.

In statistical mechanics one is especially interested in time-averaging properties of a system. In the classical theory the assumption was made that the average time spent in any region of phase space is proportional to the volume of the region in terms of the invariant measure, more generally, that time-averages may be replaced by space-averages. To justify this interchange, a number of hypotheses were proposed, variously known as ergodic or quasi-ergodic hypotheses, but a rigorous discussion of the precise conditions under which the interchange is permissible was only made possible in 1931 by the ergodic theorem of Birkhoff. This established the *existence* of the time-averages in question, for almost all initial conditions, and showed that if we neglect sets of measure zero, the interchange of time- and space-averages is permissible if and only if the flow in the phase space is *metrically transitive.* A transformation or a flow is said to be metrically transitive if there do not exist two disjoint invariant sets both having positive measure. Thus the effect of the ergodic theorem was to replace the ergodic hypothesis by the hypothesis of metrical transitivity.

Nevertheless, in spite of the simplification introduced by the ergodic theorem, the problem of deciding whether particular systems are metrically transitive or not has proved to be very difficult. . . .

. . . The known examples of metrically transitive continuous flows are all in manifolds, indeed in manifolds of restricted topological type, either toruses or manifolds of direction elements over surfaces of negative curvature. An outstanding problem in ergodic theory has been the existence question-can a metrically transitive continuous flow exist in an arbitrary manifold, or in any space that is not a manifold? In the present paper we shall obtain a complete answer to this question, at least on the topological level, for polyhedra of dimension three or more. It will appear that the only condition that needs to be imposed is a trivially necessary kind of connectedness. In particular, there exists a metrically transitive continuous flow in the cube, in the solid torus, and in any pseudo-manifold of dimension at least three. Since the phase spaces of dynamical systems have the required kind of connectedness, it follows that the hypothesis of metrical transitivity in dynamics involves no *topological* contradiction. More precisely, in any phase space there can exist a continuous flow metrically transitive with respect to the invariant measure associated with the system.

It must be emphasized, however, that our investigation is on the topological level. The flows we construct are continuous groups of measure-preserving automorphisms, but not necessarily differentiable or derivable from differential equations. Thus they correspond to dynamical systems only in a generalized sense.

It may be recalled that the original ergodic hypothesis of Boltzmann—that a single streamline passes through all points of phase space—had to be abandoned because it involved a topological impossibility. It was replaced by a quasi-ergodic hypothesis—that some streamline passes arbitrarily close to all points of phase space. But it is not obvious that even this weak hypothesis is topologically reasonable in general phase spaces, and in any case it is not sufficient to justify the interchange of time- and space-averages. It is therefore of some interest to know that the ergodic hypothesis in its modem form of metrical transitivity is at least free from any objection on topological grounds.... ■

Editor's note: Despite the existence of ergodic transformations, the ergodicity of many familiar dynamical systems remains an open and thought-provoking question. For details see Patrascioiu's article, "The Ergodic Hypothesis: A Complicated Problem of Mathematics and Physics."

proposed to study the "genealogical distance" $d(a, b)$ between two individuals a and b, which is defined as follows. Count the number of ancestors of a that are not ancestors of b, and add to it the number of ancestors of b that are not ancestors of a. Assume that the size of the population is constant in time, that mating is random, and that a and b belong to the same generation. Ulam soon discovered by experimenting on a computer that under those conditions the expected value of $d(a, b)$ is twice the size of the population. Later Joseph Kahane and Robert Marr proved this conjecture (Journal of *Combinatorial Theory,* Series A, volume 13 (1972)). The smallness of this expected distance suggests that all profitable mutations are soon present in all individuals of subsequent generations.

Ulam liked to invent problems that could-be studied by means of electronic computers. He was the first to realize that computers are ideal tools for watching the evolution of patterns governed by simple laws. He proposed many experiments of this type, the most famous of which is reported in the paper of Fermi, Pasta, and Ulam on dynamical evolution governed by nonlinear laws. Later he invented various simple rules to produce crystal-like growths in space. He also observed simple cases of "wars" between growing populations of crystals or cells. Nowadays many such processes are being investigated; Conway's "game of life" is a popular example. It is hoped that this approach will help us to understand certain qualitative features of natural evolution. For example, one can replace the complicated rules of chemistry governing real life by simpler rules and, through numerical simulation, watch the ways in which the patterns (objects) yielded by these rules grow and compete in complicated and surprising ways. (In my own work 1 am trying to explain human thought and learning, which we so often discussed together, by applying local rules of interaction that may define interesting processes. It is already known that the computations going on in the cerebral cortex are local in some sense.)

I have tried to give you glimpses of certain works of Stan Ulam. Of course, in this short article I have discussed only those that seem to me the most important or with which I was the most familiar.

Every creative mathematician must allow his imagination to flow in a free way. I think that Ulam did this more than others. He was drawn to work upon problems that suggested essentially new ideas and avoided the attractive pull of well-developed mathematics. Few mathematicians have the intelligence or the courage that Ulam had to think about important problems irrespective of whether their solutions are in sight. But this is the only course that can lead to outstanding achievements. ■

Further Reading

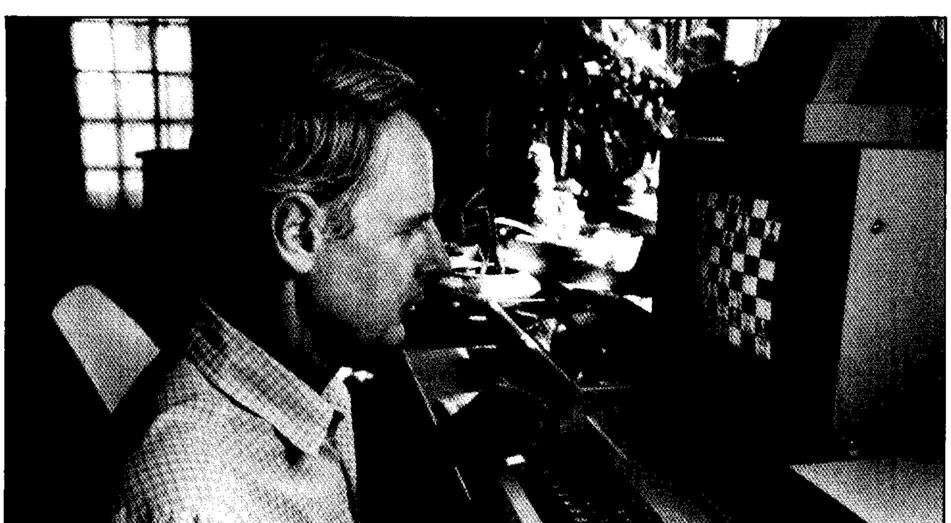
W. A, Beyer, J. Mycielski, and G.-C. Rota, editors. 1974. Stanislaw *Ulam: Sets, Numbers. and Univcrses* Cambridge, Massachusetts: The MIT Press.

A collection of declassified Los Alamos technical reports by Ulam and his collaborators is to be published soon by University of California Press.

Frank R, Drake. 1974. *Set Theory: An Introduction to Large Cardinals.* Amsterdam: North-Holland Publishing Company.

Jan Mycielski, 1985. Can mathematics explain natural intelligence? Los Alamos National Laboratory report LA–10492–MS. Also in *Physica D* 22(1986):366-375.

Jan Mycielski and S. Swierczkowski. A model of the neocortex. Los Alamos National Laboratory report. To be published.



Jan Mycielski was born in Poland in *1932.* After receiving his Ph.D, from the University of Wroclaw, he worked at the Institute of Mathematics of tbe Polish Academy of Science. In 1969 he immigrated to the United States and obtained a professorship at the University of Colorado, a position he has retained to the present. He has held visiting positions at the Centre National de la Recherche Scientifique. Case Western Reserve University, and the University of California, Berkeley. His research has resulted in over one hundred papers pertaining to mathematical logic, set theory game theory, geometry, algebra, and learning systems. He is a member of the American Mathematical Society, the Mathematical Association of America, the Association for Symbolic Logic, and the Polish Mathematical Society,