

## *Analytical Benchmark Test Set for Criticality Code Verification*

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*Avneet Sood\**

*R. A. Forster*

*D. K. Parsons*

*\*Graduate Research Assistant at Los Alamos. North Carolina State University,  
Nuclear Engineering Department, Burlington Labs, Box 7909, Raleigh, NC 27695-7909.*

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by

A. Sood (NCSU), R.A. Forster (LANL), and D.K. Parsons (LANL)

## ABSTRACT

A number of published numerical solutions to analytic eigenvalue ( $k_{eff}$ ) and eigenfunction equations are summarized for the purpose of creating a criticality verification benchmark test set. The 75-problem test set allows the user to verify the correctness of a criticality code for infinite medium and simple geometries in one- and two-energy groups, one- and two-media, and both isotropic and anisotropic neutron scattering. The problem specifications will produce both  $k_{eff}=1$  and the quoted  $k_{\infty}$  to at least five decimal places. Additional uses of the test set for code verification are also discussed. A list of 45 references and an appendix with  $k_{\infty}$  derivations is also included.



# I. INTRODUCTION

This report describes a set of benchmark problems with analytic eigenvalue ( $k_{eff}$ ) and eigenfunction (flux) solutions to the neutron transport equation from peer-reviewed journal articles. The purpose of the test set is to verify that transport algorithms and codes can correctly calculate the analytic  $k_{eff}$  and fluxes to at least five decimal places. These test set problems for infinite medium, slab, cylindrical, and spherical geometries in one- and two-energy groups, one- and two-media, and both isotropic and linearly anisotropic scattering are completely described using the listed references in this report. A three-group infinite medium and a six-group variant  $k_{\infty}$  problem (unpublished) are also included.

**Verification** is defined as “the process of evaluating a system or component to determine whether the products of a given development phase satisfy the conditions imposed at the start of the phase”<sup>1</sup> or as a “proof of correctness.” **Confirmation (proof) of correctness** is “a formal technique used to prove mathematically that a computer program satisfies its specified requirements.”<sup>1</sup> In contrast to verification, **validation** is defined as “the process of evaluating a system or component during or at the end of the development process to determine whether it satisfies specified requirements.”<sup>1</sup> Thus code verification checks that the intended calculations have been executed correctly, while code validation compares the calculated results with experimental data.

The objectives of this report are to define and document a set of analytic benchmarks for verifying criticality codes. **Benchmark** is defined as “a standard against which measurement or comparisons can be made.”<sup>1</sup> Available benchmarks for code verification do not focus on criticality problems.<sup>2</sup> Validation benchmarks from critical experiments do exist, but are not verification benchmarks.<sup>3</sup> Initial efforts to compile a benchmark test set for criticality calculation verification was begun, but not completed.<sup>4,5</sup> The analytic benchmarks described here can be used to verify computed numerical solutions for  $k_{eff}$  and the associated flux with virtually no uncertainty in the numerical benchmark values.

## II. WHY THESE SOLUTIONS SERVE AS A TEST SET

All critical dimensions,  $k_{eff}$ , and scalar neutron flux results quoted here are based on numerical computations using the analytic solutions to the  $k_{eff}$  eigenvalue (homogeneous) transport equation for “simple” problems. The analytic methods used include Case’s singular eigenfunction,<sup>6</sup>  $F_N$  and  $B_N$  methods,<sup>7,8</sup> and Green’s functions.<sup>9</sup> All of these test set problem specifications and results are from peer-reviewed journals, and have, in some cases, been solved numerically using more than one analytic solution. All calculated values for critical dimensions,  $k_{eff}$ , and the scalar neutron flux are believed to be accurate to at least five decimal places.

### III. SCOPE OF THE CRITICALITY VERIFICATION TEST SET

The verification test set was chosen to represent a “wide” range of problems from the relatively small number of published solutions. These problems include simple geometries, few neutron energy groups, and simplified (isotropic and linearly anisotropic) scattering models. The problems use neutron cross sections that are reasonable representations of the materials described. These cross sections are **not** general purpose multi-group values. The cross sections are used because they are extracted from the literature results and are intended to be used only to verify algorithm performance and **not** to predict criticality experiments.

The basic geometries include an infinite medium, slab, cylinder, and sphere with one- and two-energy group representations of uniform homogeneous materials. The slab and cylinder geometries are one-dimensional, as shown in Fig. 1; that is, each is finite in one dimension (thickness for slab and radius for the cylinders) and infinite elsewhere. The two-media problems surround each geometry with a specified thickness of reflector. Solutions for one-, two-, and three-group infinite medium problems are derived in Appendix A.

The critical dimension,  $r_c$ , is defined pictorially for the one-dimensional, one-medium problem geometries in Fig. 1, as well as the two-media infinite slab lattice cell. Reflector dimension(s) are provided for the reflected cases.

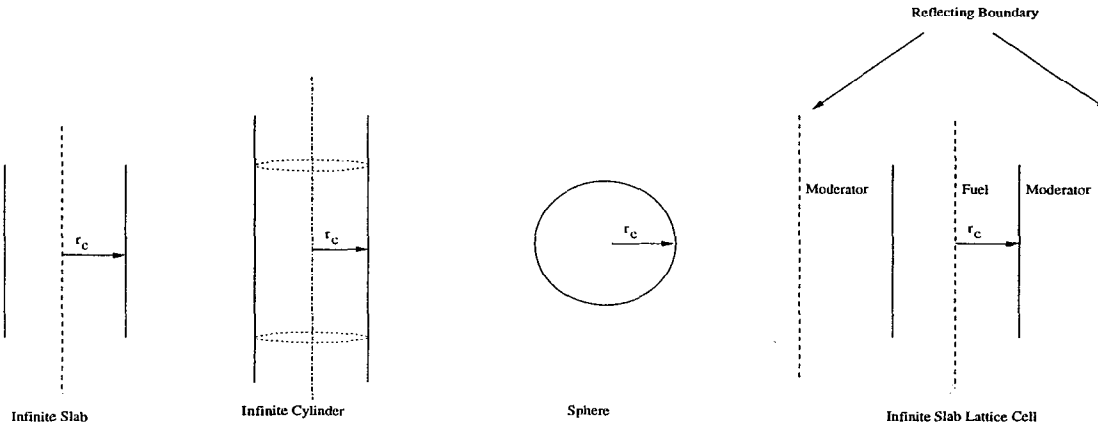


Figure 1: Critical Dimension,  $r_c$ , for Bare One-Dimensional Geometries and Infinite Slab Lattice Cell

The emphasis of the test set is on the fundamental eigenvalue,  $k_{eff}$ . All  $k_{eff}$  eigenvalues for finite fissile materials are unity to at least five decimal places. The  $k_{\infty}$  values for a uniform homogeneous infinite medium are greater than unity. Few numerical eigenfunction solutions are published; consequently, mainly one-group and uniform homogeneous infinite medium fluxes are included in the test set results.

To assist in verification, each problem has a unique problem number and identifier. Since the test set includes bare and multi-media problems, there are two forms of the identifier. The first form is for a bare geometry:

## Fissile Material - Energy Groups - Scattering - Geometry

The possible entries for each category are listed in Table 1. The fissile materials and identifier consist of Pu-239 (PU), U-235 (U), highly enriched uranium-aluminum-water assembly (UAL), low enrichment uranium and D<sub>2</sub>O reactor system (UD2O), and a highly enriched uranium research reactor (URR). The identifier may be followed by a letter to differentiate between different cross-section sets from nominally the same material. The table lists identifiers for the reflector material (if any), number of energy groups, scattering order, and geometry. The geometry is identified by the first two letters in the table. The exception is for the infinite slab lattice cell which uses ISLC. An example of the one material form of the identifier is:

### U-2-0-SP

which is the identifier for a bare U-235 reactor (no reflector), 2 energy groups, isotropically scattering, in spherical geometry.

The second form of the identifier includes the reflecting material. The reflectors are usually H<sub>2</sub>O with an exception of a three region Fe, Na, Fe reflector. Although many of the reflectors are identified as H<sub>2</sub>O, the reflector cross sections are often unique to each problem. Consequently, a letter may follow H<sub>2</sub>O indicating the H<sub>2</sub>O cross-section set used. The multi-media identifier form is:

### Fissile Material - Reflecting Material (thickness) - Energy Groups - Scattering - Geometry

To separate multiple reflector thicknesses for the same fissile material, the thickness is given in parenthesis in the title in units of mean free paths (mfp). For example,

### UD2O-H2O(10)-1-0-SL

is the identifier for a uranium and D<sub>2</sub>O reactor with a H<sub>2</sub>O reflector of 10 mean free path thickness, one-energy group, isotropically scattering, in slab geometry. An "IN" in parenthesis after the H<sub>2</sub>O means an infinite water reflector.

Table 1: Nomenclature for Problem Identifiers

Fissile Material	Reflector Material	Energy Groups	Scattering Order	Geometry
PU	bare	1 group	0 - P <sub>0</sub> Isotropic	<u>IN</u> finite
U	H <sub>2</sub> O	2 groups	1 - P <sub>1</sub> Anisotropic	<u>SL</u> ab
UD2O	Fe-Na	3 groups	2 - P <sub>2</sub> Anisotropic	<u>CY</u> linder
UAL		6 groups		<u>SP</u> here
URR				<u>IN</u> finite <u>SL</u> ab <u>LA</u> ttice <u>CE</u> ll

Tables 2, 3, and 4 summarize each of the 75 problems in the test set and give the page number in this report. An "x" in the "Flux" column appears if the associated normalized spatial neutron fluxes or energy group flux ratio are given. An asterisk in the "Scattering" column appears if the P<sub>1</sub> scattering expansion

becomes negative over part of the  $\mu$  interval from -1 to 1. The purpose of the layout of the test set problems is to increase user-friendliness. The user has all pertinent information to define any given problem in the verification test set on a single page.

There are 43 problems in the one-energy group case: 30 problems assume isotropic scattering and 13 have anisotropic scattering. For the two-energy group problems, there are 30 problems subdivided into 26 isotropic scattering problems and 4 linearly anisotropic problems. Also included for an infinite medium are a three-group and a six-group (2 coupled sets of three groups) isotropic problem. The test set includes 24 infinite medium problems, 24 slabs, 9 one-energy group cylinders, 14 spheres, and 4 infinite slab lattice cells.

Table 2: Overview and Page Location for One-Energy Group Problem Identifiers

Number	Problem Identifier	Flux	Page Number	Scattering
1	PUa-1-0-IN	x	16	Isotropic
2	PUa-1-0-SL		16	
3	PUa-H2O(1)-1-0-SL		17	
4	PUa-H2O(0.5)-1-0-SL		17	
5	PUB-1-0-IN	x	16	
6	PUB-1-0-SL	x	16	
7	PUB-1-0-CY	x	16	
8	PUB-1-0-SP	x	16	
9	PUB-H2O(1)-1-0-CY		17	
10	PUB-H2O(10)-1-0-CY		17	
11	Ua-1-0-IN	x	18	
12	Ua-1-0-SL	x	18	
13	Ua-1-0-CY		18	
14	Ua-1-0-SP	x	18	
15	Ub-1-0-IN	x	18	
16	Ub-H2O(1)-1-0-SP		19	
17	Uc-1-0-IN	x	18	
18	Uc-H2O(2)-1-0-SP		19	
19	Ud-1-0-IN	x	18	
20	Ud-H2O(3)-1-0-SP		19	
21	UD2O-1-0-IN	x	20	
22	UD2O-1-0-SL	x	20	
23	UD2O-1-0-CY		20	
24	UD2O-1-0-SP	x	20	
25	UD2O-H2O(1)-1-0-SL		20	
26	UD2O-H2O(10)-1-0-SL		20	
27	UD2O-H2O(1)-1-0-CY		20	
28	UD2O-H2O(10)-1-0-CY		20	
29	Ue-1-0-IN	x	21	
30	Ue-Fe-Na-1-0-SL	x	21	
31	PU-1-1-IN	x	22	
32	PUa-1-1-SL		22	Anisotropic
33	PUa-1-2-SL		22	
34	PUB-1-1-SL		22	
35	PUB-1-2-SL		22	
36	Ua-1-1-CY		23	
37	Ub-1-1-CY		23	
38	UD2Oa-1-1-IN		24	
39	UD2Oa-1-1-SP		24	
40	UD2Ob-1-1-IN		24	
41	UD2Ob-1-1-SP		24	
42	UD2Oc-1-1-IN		24	
43	UD2Oc-1-1-SP		24	

\*P<sub>1</sub> scattering expansion becomes negative over part of the  $\mu$  interval from -1 to 1.

Table 3: Overview and Page Location for Two-Energy Group Problem Identifiers

Number	Problem Identifier	Flux	Page Number	Scattering
44	PU-2-0-IN	x	27	Isotropic
45	PU-2-0-SL		27	
46	PU-2-0-SP		27	
47	U-2-0-IN	x	28	
48	U-2-0-SL		28	
49	U-2-0-SP		28	
50	UAL-2-0-IN	x	29	
51	UAL-2-0-SL		29	
52	UAL-2-0-SP		29	
53	URRa-2-0-IN	x	30	
54	URRa-2-0-SL	x	30	
55	URRa-2-0-SP		30	
56	URRb-2-0-IN	x	31	
57	URRc-2-0-IN	x	31	
58	URRb-H2Oa(1)-2-0-SL		31	
59	URRb-H2Oa(5)-2-0-SL		31	
60	URRb-H2Oa(IN)-2-0-SL		31	
61	URRc-H2Oa(IN)-2-0-SL		31	
62	URRd-2-0-IN	x	32	
63	URRd-H2Ob(1)-2-0-ISLC		32	
64	URRd-H2Ob(10)-2-0-ISLC		32	
65	URRd-H2Oc(1)-2-0-ISLC		32	
66	URRd-H2Oc(10)-2-0-ISLC		32	
67	UD2O-2-0-IN	x	33	
68	UD2O-2-0-SL		33	
69	UD2O-2-0-SP		33	
70	URRa-2-1-IN	x	34	
71	URRa-2-1-SL	x	34	
72	UD2O-2-1-IN	x	35	
73	UD2O-2-1-SL		35	

\*P<sub>1</sub> scattering expansion becomes negative over part of the  $\mu$  interval from -1 to 1.

Table 4: Overview and Page Location for Three- and Six-Energy Group Problem Identifiers

Number	Problem Identifier	Flux	Page Number	Scattering
74	URR-3-0-IN	x	36	Isotropic
75	URR-6-0-IN	x	38	

## IV. USES OF THE CRITICALITY VERIFICATION TEST SET

This report provides all necessary problem definitions and published critical ( $k_{eff} = 1$ ) dimensions,  $k_{\infty}$ , and scalar neutron flux results to verify a criticality transport algorithm or code and associated numerics such as random number generation and round-off errors. All material cross sections provided are macroscopic, so the atom density used by the code should be unity. Not all of the analytic solutions from the references are used, however, because the number of problems in the test set becomes too large. For other solutions not included in this report, see the reference list.

The verification test set problems can be used in several ways. The user can choose to simply calculate the problems and compare forward and adjoint  $k_{eff}$  and neutron flux results with the benchmark solutions. However, there are several more verification processes that could be included. For example, in Monte Carlo codes, both multi-group and pointwise representation of multi-group data can be used. In multi-group problems, an alternative verification procedure is to change the energy group structure when up-scattering is allowed; that is, reverse the order of the fast and slow groups. To examine the alpha eigenvalue or time-dependent neutron decay or growth, the capture and total cross sections can be modified by  $\alpha/v$  to represent subcritical and supercritical systems.

Another part of code verification is testing different representations of the same geometry (e.g., reflecting boundaries and lattices). An example is an infinite one-dimensional slab (finite in one dimension and infinite in the other two dimensions) as shown in Fig. 1, which could be modeled as a three-dimensional cube with four reflective boundaries. Other geometry options can be tested by constructing several smaller cubes inside of the three-dimensional representation of a one-dimensional critical slab. The infinite medium problem can be represented by using large geometric boundaries, reflecting boundaries, or infinite lattices of finite shapes. Infinite medium problems can be used to verify constant scalar and angular flux in each energy group as well as scalar flux ratios for more than one energy group. Three-dimensional geometric representations of optically small objects can also be tested for  $k_{\infty}$  in infinite medium problems.<sup>10</sup> Purely absorbing one-group infinite medium problems can provide faster code verification since scattering does not alter the infinite medium  $k_{\infty}$  (see Appendix A).

Another use of this verification set includes testing of any flux tally approximations. This can be especially important at near tangential angles where some codes assume an average value for the cosine of the angle. This can also affect  $k_{eff}$  if it is estimated by same section of code that calculates the flux.

Different calculation capabilities of a code should be tested using these problems. For Monte Carlo codes, different variance reduction methods such as analog or implicit capture and geometric splitting or Russian roulette can be verified. Cycle-to-cycle correlations in the estimated  $k_{eff}$  standard deviation must be taken into account to form valid  $k_{eff}$  confidence intervals. Statistically independent runs can be made and analyzed if necessary. The magnitude of any negative bias in  $k_{eff}$ , which is a function of the number of neutron histories per fission generation, also needs to be considered and made smaller than 0.00001.<sup>11</sup>

Deterministic codes can assess convergence characteristics and correctness of  $k_{eff}$  and the flux as a function of space and angle representation. Various characteristics of discrete ordinates numerics can also be checked such as the effects of eigenvalue search algorithms, angular redistribution terms in curvilinear geometries, ray effects, and various alternative geometric descriptions.

## V. NEUTRON TRANSPORT EQUATION OVERVIEW

The neutron transport equation being solved in these benchmark problems is briefly described for one- and two-energy groups and the isotropic and linearly anisotropic cases. The infinite medium solutions for  $k_{\infty}$  and the flux ratios are described in Appendix A.

### A. General $k_{eff}$ Eigenvalue Equation

The steady state neutron transport equation can be written as a  $k_{eff}$  eigenvalue problem as:<sup>12</sup>

$$\begin{aligned} \vec{\Omega} \cdot \nabla \Psi(\vec{r}, E, \vec{\Omega}) + \Sigma_t(\vec{r}, E) \Psi(\vec{r}, E, \vec{\Omega}) &= \int_0^{\infty} dE' \int_{4\pi} d\vec{\Omega}' \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}) \Psi(\vec{r}, E', \vec{\Omega}') \\ &+ \chi(E) \int_0^{\infty} dE' \frac{\nu(\vec{r}, E')}{4\pi k_{eff}} \Sigma_f(\vec{r}, E') \int_{4\pi} \Psi(\vec{r}, E', \vec{\Omega}') d\vec{\Omega}' \end{aligned} \quad (1)$$

where:

$$\begin{aligned} \Psi(\vec{r}, E, \vec{\Omega}) &= \text{angular neutron flux as a function of space } \vec{r}, \text{ energy } E, \text{ and angle } \vec{\Omega} \\ \Sigma_t(\vec{r}, E) &= \text{total neutron macroscopic cross section} \\ \Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}) dE d\vec{\Omega} &= \text{neutron scattering macroscopic cross section} \\ &\quad \text{from } E' \text{ to } E+dE \text{ in direction } d\vec{\Omega}' \text{ about } \vec{\Omega} \\ &= \Sigma_{elastic} + \Sigma_{(n,n')} + 2\Sigma_{(n,2n)} + \dots \\ \Sigma_f(\vec{r}, E') &= \text{neutron fission macroscopic cross section} \\ \nu(\vec{r}, E') &= \text{number of neutrons emitted from each fission event} \\ \chi(E) &= \text{fission neutron energy distribution} \end{aligned}$$

For these test problems, there are no  $(n, xn')$  reactions,  $x > 1$ , included in  $\Sigma_s$ . The scattering cross section includes only isotropic scattering and no higher order scattering components. Therefore,  $\Sigma_c = \Sigma_t - \Sigma_s - \Sigma_f$ , where  $\Sigma_c$  is the neutron capture cross section (zero neutrons emitted). This report also provides values for the scalar neutron flux, which is defined as  $\phi(r, E) = \int_{\Omega} \Psi(\vec{r}, E, \vec{\Omega}) d\vec{\Omega}$ . The reported scalar neutron flux values are normalized to the flux at the center of the fissile material.

The correct solution for  $k_{eff}$  will provide equality or balance for Eq. 1: losses equal gains. The  $k_{eff}$  eigenvalue is only associated with the fission reaction and no other multiplying process such as  $(n, 2n)$ . The



fundamental eigenvalue,  $k_{eff}$ , is unity for a critical system, less than unity for a subcritical system, and greater than unity for a supercritical system. The steady state  $k_{eff}$  eigenvalue equation is physically correct only when  $k_{eff}$  is unity and there is no decay or growth in  $\Psi(r, E, \vec{\Omega})$ . A solution when  $k_{eff}$  is not unity is still a valuable indicator of the ability of a system to sustain a fission chain reaction. When an infinite medium is considered,  $k_{eff}$  will be referred to as  $k_{\infty}$ . This report gives results for the fundamental  $k_{eff}$  eigenvalue. For higher eigenvalue results, see Refs. 13,14,15,17,18,19,20.

Analytic solutions to this general form of the transport equation do not exist. If we begin to simplify the equation by considering a one-dimensional slab with azimuthally symmetric scattering, the transport equation reduces to:

$$\begin{aligned} \mu \frac{\partial \Psi(x, E, \mu)}{\partial x} + \Sigma_t(x, E) \Psi(x, E, \mu) &= 2\pi \int_0^\infty dE' \int_{-1}^1 \Sigma_s(x, E' \rightarrow E, \mu, \mu') \Psi(x, E', \mu') d\mu' \\ &+ \chi(E) \int_0^\infty dE' \frac{\nu(x, E')}{2k_{eff}} \Sigma_f(x, E') \int_{-1}^1 \Psi(x, E', \mu') d\mu' \end{aligned} \quad (2)$$

where  $\mu$  is the cosine of the angle relative to a given direction.

For the other finite geometries, the slab leakage term,  $\mu \frac{\partial \Psi}{\partial x}$  must be replaced with the appropriate leakage term for a cylinder or sphere.<sup>12</sup> This equation is still too difficult to solve analytically. Approximations are required in both the space, angle, and energy variables to allow an analytic solution.

## B. One-Energy Group in One-Dimensional Slab Geometry

### 1. Isotropic Scattering.

To simplify Eq. 2, isotropic scattering is assumed, which implies that the emission angle after scattering is equally likely in all directions, giving:

$$\Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega}) = \frac{1}{4\pi} \Sigma_s(r, E' \rightarrow E)$$

The transport equation can be further simplified by assuming a homogeneous medium and only one energy group. Using these additional assumptions to define one-group scalar flux weighted average cross sections, the transport equation becomes:<sup>12</sup>

$$\mu \frac{\partial \Psi(x, \mu)}{\partial x} + \Sigma_t \Psi(x, \mu) = \frac{\Sigma_s}{2} \int_{-1}^1 \Psi(x, \mu') d\mu' + \frac{\nu \Sigma_f}{2 k_{eff}} \int_{-1}^1 \Psi(x, \mu') d\mu' \quad (3)$$

A convenient transformation for this form of the transport equation is to change the position variable into a mean free path, or optical thickness,  $z$ . This change is done by using  $z = \Sigma_t x$ , thereby making  $\partial z / \partial x = \Sigma_t$ .

If we divide the transport equation by  $\Sigma_t$ , and use the change of variables, the neutron transport equation becomes:

$$\frac{\mu}{\Sigma_t} \frac{\partial \Psi(z, \mu)}{\partial z} \left( \frac{\partial z}{\partial x} \right) + \Psi(z, \mu) = \frac{\Sigma_s + \frac{\nu \Sigma_f}{k_{\text{eff}}}}{2\Sigma_t} \int_{-1}^1 \Psi(z, \mu') d\mu' \quad (4)$$

Notice that the dimensionless position variable,  $z$ , is now defined in terms of mean free paths (mfp) (i.e.,  $1/\Sigma_t$ ). The final simplified one-energy group transport equation for a slab can be written in a slightly more convenient form as:

$$\mu \frac{\partial \Psi(z, \mu)}{\partial z} + \Psi(z, \mu) = \frac{c}{2} \int_{-1}^1 \Psi(z, \mu') d\mu' \quad (5)$$

where:

$$c = \frac{\Sigma_s + \frac{\nu \Sigma_f}{k_{\text{eff}}}}{\Sigma_t} \quad (6)$$

The parameter  $c$  is defined as the mean number of secondary neutrons produced per neutron reaction and is also known as the secondaries ratio. This equation is the form of the one-medium, one-energy group transport equation for a slab. This equation still requires elaborate mathematics to solve as reported in the literature. Derivation of the one-energy group  $k_\infty$  solution for the infinite medium case is shown in Appendix A. A  $c$  of unity is equivalent to a  $k_\infty$  of unity. Thus,  $c$  must be greater than unity to have a finite critical system.

The literature uses this form of the neutron transport equation with a non-reentrant boundary condition to derive one-energy group, isotropic scattering analytic solutions for the critical ( $k_{\text{eff}}=1$ ) dimensional scalar neutron flux. It should be noted that  $c$  values for  $k_{\text{eff}} = 1$  in Eq. 6 are presented in the literature, thereby making  $c = \frac{\Sigma_s + \nu \Sigma_f}{\Sigma_t}$ . The typical range of  $c$  found in the literature for fissile materials is from 1.01 to 2.00. The one-group cross sections selected for the test set mimic the physical characteristics of the two-group problems and range from 1.02 to 1.50. A value of  $c$  of 1.5 is the upper limit for real fissile materials.

## 2. Linearly Anisotropic Scattering.

The scattering term,  $\Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega})$  can be a strong function of the cosine of the scattering angle,  $\mu_0 = \vec{\Omega}' \cdot \vec{\Omega}$ . The angular dependence can be analyzed by Legendre polynomial series expansion of  $\Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \cdot \vec{\Omega})$ .<sup>21</sup> Using the Legendre polynomial expansion, the one-energy group, one-dimensional slab, neutron transport equation can be written in a similar form to Eq. 5:<sup>22,23</sup>

$$\mu \frac{\partial \Psi(z, \mu)}{\partial z} + \Psi(z, \mu) = \frac{c}{2} \int_{-1}^1 \Psi(z, \mu') (1 + \mu' \mu) d\mu'$$

Solutions for this form of the neutron transport equation are often found in the literature. The solution of this equation includes linearly anisotropic scattering; however, it also includes a linearly anisotropic fission

source emission. For problems that include the anisotropic effect on the fission term, see Refs. 15,22,23. Solutions for the higher eigenvalues exist for this form of the transport equation.<sup>13,14,15,17,18,19,20</sup>

Numerical solutions exist that do not force the anisotropic effect on the fission term. This limitation on the different anisotropic behavior of scattering and fission can be removed by using different transfer functions for scattering and fission. The neutron transport equation can be written as:<sup>24</sup>

$$\mu \frac{\partial \Psi(z, \mu)}{\partial z} + \Psi(z, \mu) = \frac{c}{2} \int_{-1}^1 \Pi(\mu_o, \beta) \Psi(z, \mu') d\mu' \quad (7)$$

where:

$$\Pi(\mu_o, \beta) = \beta \Pi_s(\mu_o) + (1 - \beta) \Pi_f(\mu_o)$$

$$\beta = \frac{\Sigma_s}{\Sigma_s + \nu \Sigma_f}, \quad 0 \leq \beta \leq 1$$

The parameters,  $\Pi_s(\mu_o)$  and  $\Pi_f(\mu_o)$ , are the angular transfer functions for scattering and fission, respectively, and are dependent on  $\mu$  and  $\mu'$ . The weighting parameter,  $\beta$ , allows for the different anisotropic behavior for the scattering and fission terms. This is one of several ways found in the references of separating the anisotropic scattering and fission terms.

## C. Two-Energy Groups in One-Dimensional Slab Geometry

### 1. Isotropic Scattering.

Using the same procedures as in the one-group case, the two-energy group form of the transport equation for a slab can be written as:<sup>8</sup>

$$\mu \frac{\partial \Psi_1(x, \mu)}{\partial x} + \Sigma_1 \Psi_1(x, \mu) = \frac{\Sigma_{11}}{2} \int_{-1}^1 \Psi_1(x, \mu') d\mu' + \frac{\Sigma_{12}}{2} \int_{-1}^1 \Psi_2(x, \mu') d\mu' \quad (8)$$

$$\mu \frac{\partial \Psi_2(x, \mu)}{\partial x} + \Sigma_2 \Psi_2(x, \mu) = \frac{\Sigma_{21}}{2} \int_{-1}^1 \Psi_1(x, \mu') d\mu' + \frac{\Sigma_{22}}{2} \int_{-1}^1 \Psi_2(x, \mu') d\mu' \quad (9)$$

where:

$$\Sigma_i = \text{total neutron macroscopic cross section of group } i$$

$$\Sigma_{ij} = \text{total neutron group transfer macroscopic cross section from group } j \text{ to group } i$$

In this report, the fast energy group is group 2 to be consistent with most of the references. This notation is the reverse of most nuclear engineering textbooks.

Assuming group 2 is the fast group and no nonfission up-scatter for the slow group 1, the group transfer cross sections are given by:

$$\begin{aligned}
\Sigma_{11} &= \Sigma_{11s} + \chi_1 \nu_1 \Sigma_{1f} / k_{\text{eff}} \\
\Sigma_{21} &= \chi_2 \nu_1 \Sigma_{1f} / k_{\text{eff}} \\
\Sigma_{12} &= \Sigma_{12s} + \chi_1 \nu_2 \Sigma_{2f} / k_{\text{eff}} \\
\Sigma_{22} &= \Sigma_{22s} + \chi_2 \nu_2 \Sigma_{2f} / k_{\text{eff}}
\end{aligned}$$

Note that  $\Sigma_i = \Sigma_{ic} + \Sigma_{if} + \Sigma_{iis} + \Sigma_{jis}$ , where the  $\Sigma_{jis}$  represents nonfission scattering to group  $j \neq i$ . This equation for  $\Sigma_i$  again assumes that the  $\Sigma_{(n,2n)_i} + \dots$  components are zero.

The two-energy group form of the transport equation, which has solutions in the literature, can be written in a similar form to the one-group equations utilizing the optical thickness parameter,  $z = \Sigma_2 x$ , but in matrix-vector notation as seen below.

$$\mu \frac{\partial \bar{\Psi}(z, \mu)}{\partial z} + \bar{\Sigma} \bar{\Psi}(z, \mu) = \frac{\bar{C}}{2} \int_{-1}^1 \bar{\Psi}(z, \mu') d\mu' \quad (10)$$

where:

$$\bar{\Psi}(z, \mu) = \begin{bmatrix} \Psi_1(z, \mu) \\ \Psi_2(z, \mu) \end{bmatrix}, \bar{\Sigma} = \begin{bmatrix} \Sigma & 0 \\ 0 & 1 \end{bmatrix}, \bar{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \quad (11)$$

and

$$c_{ij} = \Sigma_{ij} / \Sigma_2. \quad (12)$$

The derivations for the infinite medium  $k_{\infty}$  and the group 2 to group 1 flux ratio are given in Appendix A.

## 2. Linearly Anisotropic Scattering.

One of the above simplifying assumptions to the steady state neutron transport equation is that neutron scattering is isotropic (no angular dependence). However, the scattering term,  $\Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \rightarrow \vec{\Omega})$  can be a strong function of the cosine of the scattering angle,  $\mu_0 = \vec{\Omega}' \cdot \vec{\Omega}$ . This angular dependence can be analyzed by Legendre polynomial series expansion of  $\Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \cdot \vec{\Omega})$ ,<sup>21</sup> giving

$$\Sigma_s(\vec{r}, E' \rightarrow E, \vec{\Omega}' \cdot \vec{\Omega}) = \sum_{l=0}^M \frac{2l+1}{4\pi} \Sigma_{sl}(\vec{r}, E' \rightarrow E) P_l(\vec{\Omega}' \cdot \vec{\Omega}) \quad (13)$$

where  $M$  indicates the degree of anisotropy. For  $M = 0$ , scattering in the lab system is isotropic and for  $M = 1$ , scattering is linearly anisotropic. A complete mathematical description is in Ref. 12 and 21. For linearly anisotropic scattering, the scattering cross section consists of two components,  $\Sigma_{s0}$  and  $\Sigma_{s1}$ , where

$\Sigma_{s_1}$  is the linear anisotropic scattering component and affects the scattering angular distribution for both in and out of group scattering. Anisotropic scattering can be forward or backward peaked and thus  $\Sigma_{s_1}$  can be positive or negative. For linear anisotropic scattering in the one-energy group case,  $\Sigma_s(\mu) = (\Sigma_{s_0} + 3\mu\Sigma_{s_1})/2$ . Thus,  $\Sigma_s(\mu)$  can become negative when  $|\Sigma_{s_1}| > \Sigma_{s_0}/3$ . Test set problems 34, 37, 43, and 71 exhibit this behavior. The total scattering cross section is not dependent on  $\Sigma_{s_1}$ . The anisotropic cross section only affects the angular distribution. Infinite medium  $k_\infty$  and neutron flux results are independent of the anisotropic cross section.

Using the Legendre polynomial expansion, the neutron transport equation for a one-dimensional slab with azimuthally symmetric scattering can be written in a form similar to Eq. 2:

$$\begin{aligned} \mu \frac{\partial \Psi(x, E, \mu)}{\partial x} + \Sigma_t(x, E) \Psi(x, E, \mu) &= 2\pi \int_0^\infty dE' \sum_{l=0}^M \frac{2l+1}{4\pi} \Sigma_{sl}(E' \rightarrow E) P_l(\mu) \int_{-1}^1 P_l(\mu') \Psi(x, \mu') d\mu' \\ &+ \chi(E) \int_0^\infty dE' \frac{\nu(x, E')}{2k_{\text{eff}}} \Sigma_f(x, E') \int_{-1}^1 \Psi(r, E', \mu') d\mu' \end{aligned} \quad (14)$$

Following the same procedures as in Eq. 5, the general two-speed linearly anisotropically scattering analogue to Eq. 10 which also has numerical solutions is:<sup>25</sup>

$$\mu \frac{\partial \bar{\Psi}(z, \mu)}{\partial z} + \bar{\Sigma} \bar{\Psi}(z, \mu) = \frac{1}{2} \sum_{l=0}^1 \bar{C}_l P_l(\mu) \int_{-1}^1 \bar{\Psi}(z, \mu') P_l(\mu') d\mu' \quad (15)$$

where:

$$\bar{\Psi}(z, \mu) = \begin{bmatrix} \Psi_1(z, \mu) \\ \Psi_2(z, \mu) \end{bmatrix}, \bar{\Sigma} = \begin{bmatrix} \Sigma & 0 \\ 0 & 1 \end{bmatrix}, \bar{C}_l = \begin{bmatrix} c_{11l} & c_{12l} \\ c_{21l} & c_{22l} \end{bmatrix} \quad (16)$$

and

$$c_{ijl} = (2l+1)\Sigma_{ijl}/\Sigma_2. \quad (17)$$

The  $\Sigma_{ijl}$  term is the linearly anisotropic scattering cross section and is given in the problem descriptions without the  $(2l+1)$  term.

## VI. ONE-ENERGY GROUP PROBLEM DEFINITIONS AND RESULTS

For the one-energy group cases, the critical dimension(s) for each geometry depends upon the  $c$  value chosen from the literature and not specific cross-section sets. To use the literature results, the cross sections were selected to match published  $c$  values with  $k_{eff}=1$  at low, middle, and high  $c$  values listed. Values ranging from 1.02, 1.30, 1.40, and 1.50 were chosen because they are similar to the physical systems in the two-group cases: uranium-D<sub>2</sub>O reactor, U-235, and Pu-239. These problems use cross sections that are reasonable representations of these materials; however, these cross sections are **not** general purpose one-group values. The cross sections are used because they define the  $c$  values used in the literature and are intended to be used only to verify algorithm performance and **not** to predict any actual criticality experiments.

The isotropic neutron macroscopic cross sections provided for each case are: the total cross section,  $\Sigma_t$ , the capture (no neutrons emitted) cross section,  $\Sigma_c$ , the scattering cross section,  $\Sigma_s$ , the fission cross section,  $\Sigma_f$ , and the number of neutrons,  $\nu$ , emitted for each fission. The  $(n,2n)$ ,  $(n,3n)$ , ... cross sections are assumed to be zero (but need not be). Thus the total cross section equals the sum of  $\Sigma_c$ ,  $\Sigma_s$ , and  $\Sigma_f$ , thereby providing a consistency check on the cross-section set. Many references give  $(\nu\Sigma_f)$  instead of  $\nu$  and  $\Sigma_f$ . Since both parameters (not the product) may be required by a code for the problem solution, the product  $(\nu\Sigma_f)$  has been split into  $\nu$  and  $\Sigma_f$  preserving their product and  $\Sigma_t$ . The value of  $c$  for  $k_{eff}=1$  in Eq. 6 is also included in each cross-section table. For the reflected spheres, different secondary ratios,  $c$ , are reported with the critical dimension for  $k_{eff}=1$  for various combinations of core and reflector thicknesses. To maintain consistent cross sections with the U-235 set, the parameter,  $\nu$ , was modified to match  $c$  to the literature values.

When anisotropic scattering cross sections are provided, the anisotropic components are designated by  $\Sigma_{s1}$  and  $\Sigma_{s2}$ , respectively. Similarly, the isotropic scattering component is designated by  $\Sigma_{s0}$ .

The value of  $k_\infty$ , as defined in Appendix A, is given for each cross-section set. For finite problems where  $k_{eff}$  is unity, the critical dimension,  $r_c$ , is listed for each geometry in both mean free paths (to indicate the neutron optical thickness) and in centimeters for the one-dimensional geometries. When available in the literature, the scalar flux values, normalized to the flux at the center of the fissile material, are also provided. The two-media problems have  $c_1 > 1$  for the core region 1 and  $c_2 < 1$  for the surrounding reflector region 2. The two-media problems use the cross sections for the nonmultiplying reflector. The critical dimensions for the multiplying medium and reflector thickness are given in both mean free paths and centimeters.

A comparison of the critical dimensions for the different geometries behave as expected; that is, the critical dimension is smallest for the one-dimensional slab and increases for the cylinder and sphere. This behavior is to be expected due to the increased leakage with the curvi-linear geometries. For the reflected geometries, the critical dimension decreases with increasing reflector thickness.

## A. One-Energy Group Isotropic Scattering

### 1. One-Group Pu-239.

#### One-Energy Group Isotropic Cross Sections

Table 5 gives the one-group, isotropic cross sections for two case of Pu-239 ( $c=1.50$  and  $c=1.40$ ) and a H<sub>2</sub>O ( $c=0.90$ ) reflector. The total cross sections are the same for both Pu-239 cases and H<sub>2</sub>O as required by the reference for the two-media solutions.

Table 5: One-Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Pu-239 ( $c=1.40, 1.50$ ) and H<sub>2</sub>O ( $c=0.90$ )

Material	$\nu$	$\Sigma_f$	$\Sigma_c$	$\Sigma_s$	$\Sigma_t$	$c$
Pu-239 (a)	3.24	0.081600	0.019584	0.225216	0.32640	1.50
Pu-239 (b)	2.84	0.081600	0.019584	0.225216	0.32640	1.40
H <sub>2</sub> O (refl)	0.0	0.0	0.032640	0.293760	0.32640	0.90

#### Infinite Medium (PUa-1-0-IN and PUB-1-0-IN)

Using the cross sections for Pu-239 (a) (problem 1) in Table 5,  $k_\infty = 2.612903$  with a constant angular and scalar flux everywhere. Using the cross sections for Pu-239 (b) (problem 5) in Table 5,  $k_\infty = 2.290323$  with a constant angular and scalar flux everywhere.

#### One-Medium Slab, Cylinder, and Sphere Critical Dimensions

The Pu-239 (a) critical dimension,  $r_c$ , is listed in Table 6.

Table 6: Critical Dimensions,  $r_c$ , for One-Group Bare Pu-239 ( $c=1.50$ )

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
2	PUa-1-0-SL	Slab	0.605055	1.853722	9

The Pu-239 (b) critical dimensions,  $r_c$ , are listed in Table 7. The normalized scalar flux for four spatial positions are given in Table 8 using the same references. The flux ratios for PUB-1-0-CY are only available to four decimal places.

Table 7: Critical Dimensions,  $r_c$ , for One-Group Bare Pu-239 ( $c=1.40$ )

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
6	PUB-1-0-SL	Slab	0.73660355	2.256751	26
7	PUB-1-0-CY	Cylinder	1.396979	4.279960	27,28
8	PUB-1-0-SP	Sphere	1.9853434324	6.082547	26

Table 8: Normalized Scalar Fluxes for One-Group Bare Pu-239 ( $c=1.40$ )

Problem	Identifier	Geometry	$r/r_c = 0.25$	$r/r_c = 0.5$	$r/r_c = 0.75$	$r/r_c = 1.0$
6	PUB-1-0-SL	Slab	0.9701734	0.8810540	0.7318131	0.4902592
7	PUB-1-0-CY	Cylinder	—	0.8093	—	0.2926
8	PUB-1-0-SP	Sphere	0.93538006	0.75575352	0.49884364	0.19222603

## Two-Media Slab and Cylinder Critical Dimensions

The literature values in Tables 9 and 10, give the critical dimensions for Pu-239 (a) for two H<sub>2</sub>O reflector thicknesses. The first two-media problem (problem 3) in Table 9 is a special nonsymmetric two-region, Pu-239 and H<sub>2</sub>O, problem. The second two-media problem (problem 4) in Table 10 is a symmetric three-region problem with the reflector on both sides of the fissile medium.

Table 9: Critical Dimensions for One-Group Pu-239 Slab ( $c=1.50$ ) with Non-Symmetric H<sub>2</sub>O Reflector ( $c=0.90$ )

Problem	Identifier	Geometry	Pu $r_c$	H <sub>2</sub> O thickness	Pu+H <sub>2</sub> O Radius	Reference
3	PUa-H2O(1)-1-0-SL	Slab (mfp) (cm)	0.48255 1.478401	1 3.063725	4.542126	9

Table 10: Critical Dimensions for One-Group Pu-239 Slab ( $c=1.50$ ) with H<sub>2</sub>O Reflector ( $c=0.90$ )

Problem	Identifier	Geometry	Pu $r_c$	H <sub>2</sub> O thickness	Pu+H <sub>2</sub> O radius	Reference
4	PUa-H2O(0.5)-1-0-SL	Slab (mfp) (cm)	0.43014 1.317831	0.5 1.531863	2.849694	9

The literature values in Table 11, give the critical dimensions for Pu-239 (b) with two H<sub>2</sub>O reflector thicknesses.

Table 11: Critical Dimensions for One-Group Pu-239 Cylinder ( $c=1.40$ ) with H<sub>2</sub>O Reflector ( $c=0.90$ )

Problem	Identifier	Geometry	Pu $r_c$	H <sub>2</sub> O thickness	Pu+H <sub>2</sub> O Radius	Reference
9	PUB-H2O(1)-1-0-CY	Cylinder (mfp) (cm)	1.10898 3.397610	1 3.063725	6.461335	29
10	PUB-H2O(10)-1-0-CY	Cylinder (mfp) (cm)	1.00452 3.077574	10 30.637255	33.714829	29



## 2. One-Group U-235.

### One-Group Isotropic Cross Sections

Table 12 gives the one-group, isotropic cross sections for two cases of U-235 and a H<sub>2</sub>O reflector. Notice that one-group  $\Sigma_t$  for Pu-239 and U-235 are the same as given in reference 32, but the secondaries ratio,  $c$ , differs.

Table 12: One-Group Macroscopic Cross Sections (cm<sup>-1</sup>) for U-235 ( $c=1.30$ )

Material	$\nu$	$\Sigma_f$	$\Sigma_c$	$\Sigma_s$	$\Sigma_t$	$c$
U-235 (a)	2.70	0.065280	0.013056	0.248064	0.32640	1.30
U-235 (b)	2.797101	0.065280	0.013056	0.248064	0.32640	1.3194202
U-235 (c)	2.707308	0.065280	0.013056	0.248064	0.32640	1.3014616
U-235 (d)	2.679198	0.065280	0.013056	0.248064	0.32640	1.2958396
H <sub>2</sub> O (refl)	0.0	0.0	0.032640	0.293760	0.32640	0.90

### Infinite Medium (Ua-1-0-IN , Ub-1-0-IN , Uc-1-0-IN , and Ud-1-0-IN )

Using the cross sections for U-235 (a) in Table 12,  $k_\infty = 2.25$  (problem 11) with a constant angular and scalar flux everywhere. Using the cross sections for U-235 (b), U-235 (c), and U-235 (d) in Table 12,  $k_\infty = 2.330917$  (problem 15), 2.256083 (problem 17), and 2.232667 (problem 19) with a constant angular and scalar flux everywhere, respectively.

### One-Medium Slab, Cylinder, and Sphere Critical Dimensions

The critical dimension,  $r_c$ , and spatial flux ratios are given in Table 13 and 14 for U-235 (a). The references are the same for both tables.

Table 13: Critical Dimensions,  $r_c$ , for One-Group Bare U-235 ( $c=1.30$ )

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
12	Ua-1-0-SL	Slab	0.93772556	2.872934	26
13	Ua-1-0-CY	Cylinder	1.72500292	5.284935	27,28
14	Ua-1-0-SP	Sphere	2.4248249802	7.428998	26

Table 14: Normalized Scalar Fluxes for One-Group Bare U-235 ( $c=1.30$ )

Problem	Identifier	Geometry	$r/r_c = 0.25$	$r/r_c = 0.5$	$r/r_c = 0.75$	$r/r_c = 1.0$
12	Ua-1-0-SL	Slab	0.9669506	0.8686259	0.7055218	0.4461912
14	Ua-1-0-SP	Sphere	0.93244907	0.74553332	0.48095413	0.17177706

### Two-Media Sphere Critical Dimensions

The literature values in Table 15, give the critical dimensions for U-235 (b), U-235 (c), and U-235 (d) for three spherical H<sub>2</sub>O reflector thicknesses.

Table 15: Critical Dimensions for One-Group U-235 Sphere with H<sub>2</sub>O Reflector ( $c=0.90$ )

Problem	Identifier	Geometry	U $r_c$	H <sub>2</sub> O thickness	U+H <sub>2</sub> O Radius	Reference
16	Ub-H <sub>2</sub> O(1)-1-0-SP	Sphere (mfp) (cm)	2 6.12745	1 3.063725	9.191176	17,20
18	Uc-H <sub>2</sub> O(2)-1-0-SP	Sphere (mfp) (cm)	2 6.12745	2 6.12745	12.2549	17,20
20	Ud-H <sub>2</sub> O(3)-1-0-SP	Sphere (mfp) (cm)	2 6.12745	3 9.191176	15.318626	17,20

### 3. One-Group U-D<sub>2</sub>O Reactor.

#### One-Group Isotropic Cross Sections

Table 16 gives the one-group, isotropic cross sections for the uranium-D<sub>2</sub>O reactor and H<sub>2</sub>O reflector. Note that the uranium-D<sub>2</sub>O reactor and H<sub>2</sub>O reflector have the same total cross section as required by the references for the reflected cylindrical solutions.

Table 16: One-Group Macroscopic Cross Sections (cm<sup>-1</sup>) for U-D<sub>2</sub>O Reactor (c=1.02) and H<sub>2</sub>O (c=0.90)

Material	$\nu$	$\Sigma_f$	$\Sigma_c$	$\Sigma_s$	$\Sigma_t$	c
U-D <sub>2</sub> O	1.70	0.054628	0.027314	0.464338	0.54628	1.02
H <sub>2</sub> O (refl)	0.0	0.0	0.054628	0.491652	0.54628	0.90

#### Infinite Medium (UD2O-1-0-IN)

Using the cross sections for U-D<sub>2</sub>O in Table 16,  $k_{\infty} = 1.133333$  (problem 21) with a constant angular and scalar flux everywhere.

#### One-Medium Slab, Cylinder, and Sphere Critical Dimensions

The critical dimension,  $r_c$ , and spatial flux ratios are listed in Table 17 and 18.

Table 17: Critical Dimensions,  $r_c$ , for One-Group Bare U-D<sub>2</sub>O Reactor (c=1.02)

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
22	UD2O-1-0-SL	Slab	5.6655054562	10.371065	26
23	UD2O-1-0-CY	Cylinder	9.043255	16.554249	27,28
24	UD2O-1-0-SP	Sphere	12.0275320980	22.017156	26

Table 18: Normalized Scalar Fluxes for One-Group Bare U-D<sub>2</sub>O Reactor (c=1.02)

Problem	Identifier	Geometry	$r/r_c = 0.25$	$r/r_c = 0.5$	$r/r_c = 0.75$	$r/r_c = 1.0$
22	UD2O-1-0-SL	Slab	0.93945236	0.76504084	0.49690627	0.13893858
24	UD2O-1-0-SP	Sphere	0.91063756	0.67099621	0.35561622	0.04678614

#### Two-Media Slabs and Cylinders Critical Dimensions

Table 19 gives the U-D<sub>2</sub>O critical dimension,  $r_c$ , for two H<sub>2</sub>O reflector thicknesses.

Table 19: Critical Dimensions for One-Group U-D<sub>2</sub>O (c=1.02) Slab and Cylinder with H<sub>2</sub>O (c=0.90) Reflector

Problem	Identifier	Geometry	UD <sub>2</sub> O $r_c$	H <sub>2</sub> O thickness	UD <sub>2</sub> O + H <sub>2</sub> O radius	Reference
25	UD2O-H2O(1)-1-0-SL	Slab (mfp)	5.0335	1		33,34
		(cm)	9.214139	1.830563	11.044702	
26	UD2O-H2O(10)-1-0-SL	Slab (mfp)	4.6041	10		33,34
		(cm)	8.428096	18.30563	26.733726	
27	UD2O-H2O(1)-1-0-CY	Cylinder (mfp)	8.411027	1		29
		(cm)	15.396916	1.830563	17.227479	
28	UD2O-H2O(10)-1-0-CY	Cylinder (mfp)	7.979325	10		29
		(cm)	14.606658	18.30563	32.912288	

#### 4. One-Group U-235 Reactor.

##### One-Group Isotropic Cross Sections

Table 20 gives the one-group, isotropic cross sections for the U-235 reactor with a Fe reflector and Na moderator.

Table 20: One-Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for U-235 Reactor, Fe reflector, and Na Moderator

Material	$\nu$	$\Sigma_f$	$\Sigma_c$	$\Sigma_s$	$\Sigma_t$	c
U-235 (e)	2.50	0.06922744	0.01013756	0.328042	0.407407	1.230
Fe (refl)	0.0	0.0	0.00046512	0.23209488	0.23256	0.9980
Na (mod)	0.0	0.0	0.0	0.086368032	0.086368032	1.00

##### Infinite Medium (Ue-1-0-IN)

Using the cross sections for the U-235 reactor in Table 20,  $k_\infty = 2.1806667$  (problem 29) with a constant angular and scalar flux everywhere.

##### One-Medium Slab Critical Dimensions

Note that this problem is a nonsymmetric four-region problem. The U-235 is surrounded by a Fe cladding on two sides but moderated by Na on one side. The critical dimension,  $r_c$ , is listed in Tables 21 and 22.

Table 21: Critical Dimensions,  $r_c$ , for One-Group U-235 Reactor

Problem	Identifier	Geometry	Fe thickness	U-235 thickness	Fe thickness	Na thickness	Reference
30	Ue-Fe-Na-1-0-SL	Slab (mfp) (cm)	0.0738	2.0858098	0.0738	0.173	45
			0.317337461	5.119720083	0.317337461	2.002771002	

Table 22: Critical Dimensions,  $r_c$ , for One-Group U-235 Reactor

Problem	Identifier	Geometry	Fe thickness	Fe+U	Fe+U+Fe	Fe+U+Fe+Na
30	Ue-Fe-Na-1-0-SL	Slab (cm)	0.317337461	5.437057544	5.754395005	7.757166007

The U-235 (e) critical dimensions,  $r_c$ , are listed in Tables 21 and 22. The normalized scalar flux for four spatial positions are given in Table 23 using the same references. These positions correspond to the material boundaries and are normalized by the scalar neutron flux at the left boundary.

Table 23: Normalized Scalar Fluxes for One-Group U-235 Reactor

Problem	Identifier	Geometry	Fe-U	U-Fe	Fe-Na	Na
30	Ue-Fe-Na-1-0-SL	Slab	1.229538	1.49712	1.324899	0.912273

## B. One-Group Anisotropic Scattering

### 1. One-Group Pu-239.

#### One-Energy Group Anisotropic Cross Sections

Table 24 gives the one-group, anisotropic cross sections for two cases of anisotropic scattering. The first cross-section set, Pu-239 (a), includes  $P_1$  and  $P_2$  scattering cross sections, where  $|\mu| < 1/3$ . The second cross-section set, Pu-239 (b), includes the  $P_1$  and  $P_2$  scattering cross sections where  $|\mu| > 1/3$ . **Care must be used to correctly solve benchmark problem 34 because of the negative scattering for  $\mu$  near -1.**

Table 24: One-Group Macroscopic Anisotropic Cross Sections ( $\text{cm}^{-1}$ ) for Pu-239 ( $c=1.40$ )

Material	$\nu$	$\Sigma_f$	$\Sigma_c$	$\Sigma_{s_0}$	$\Sigma_{s_1}$	$\Sigma_{s_2}$	$\Sigma_t$	c
Pu-239 (a)	2.5	0.266667	0.0	0.733333	0.20	0.075	1.0	1.40
Pu-239 (b)	2.5	0.266667	0.0	0.733333	0.333333	0.125	1.0	1.40

#### Infinite Medium (PU-1-1-IN)

Using the cross sections for Pu-239 (a) and Pu-239 (b) in Table 24,  $k_\infty = 2.5$  (problem 31) with a constant angular and scalar flux everywhere. The anisotropic scattering cross sections do not change  $k_\infty$ .

#### One-Medium Slab Critical Dimensions

The Pu-239 critical dimensions,  $r_c$ , for both  $P_1$  and  $P_2$  problems are listed in Table 25.

Table 25: Critical Dimensions,  $r_c$ , for One-Group Bare Pu-239 ( $c=1.40$ )

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
32	PUa-1-1-SL	Slab	0.77032	0.77032	30
33	PUa-1-2-SL	Slab	0.76378	0.76378	30
34	PUb-1-1-SL	Slab	0.79606	0.79606	30
35	PUb-1-2-SL	Slab	0.78396	0.78396	30

## 2. One-Group U-235

### One-Energy Group Anisotropic Cross Sections

Table 26 gives the two sets of one-group, anisotropic cross sections for U-235. Notice that the cross sections are the same as in Table 12 with the addition of  $P_1$  scattering cross sections. The first cross-section set, U-235 (a), includes  $P_1$  scattering cross sections, where  $|\mu| < 1/3$ . The second cross-section set, U-235 (b), includes the  $P_1$  scattering cross sections where  $|\mu| > 1/3$ . **Care must be used to correctly solve benchmark problem 37 because of the negative scattering for  $\mu$  near -1.**

Table 26: One-Group Macroscopic Anisotropic Cross Sections ( $\text{cm}^{-1}$ ) for U-235 ( $c=1.30$ )

Material	$\nu$	$\Sigma_f$	$\Sigma_c$	$\Sigma_{s0}$	$\Sigma_{s1}$	$\Sigma_t$	c
U-235 (a)	2.70	0.065280	0.013056	0.248064	0.042432	0.32640	1.30
U-235 (b)	2.70	0.065280	0.013056	0.248064	0.212160	0.32640	1.30

### Infinite Medium (U-1-1-IN)

Using the cross sections for U-235 (a) and U-235 (b) in Table 26,  $k_\infty = 2.25$  (problem 11) with a constant angular and scalar flux everywhere. The anisotropic scattering cross sections do not change  $k_\infty$ .

### One-Medium Slab Critical Dimensions

The U-235 critical dimensions,  $r_c$ , for both  $P_1$  problems are listed in Table 27.

Table 27: Critical Dimensions,  $r_c$ , for One-Group Bare U-235 ( $c=1.30$ )

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
36	Ua-1-1-CY	Cylinder	1.799866479	5.514296811	31
37	Ub-1-1-CY	Cylinder	2.265283130	6.940205668	31

### 3. One-Group U-D<sub>2</sub>O

#### One-Energy Group Anisotropic Cross Sections

Table 28 gives the two sets of one-group, anisotropic cross sections for U-D<sub>2</sub>O reactor. Notice that the cross sections are the same as in Table 16 with the addition of P<sub>1</sub> scattering cross sections. The cross-sections set for two U-D<sub>2</sub>O cases include P<sub>1</sub> scattering cross sections, where  $|\mu| < 1/3$ , and a P<sub>1</sub> case where  $\mu < 0$  and the scattering cross section is negative. Care must be used to correctly solve benchmark problem 43 because of the negative scattering for  $\mu$  near 1.

Table 28: One-Group Macroscopic Anisotropic Cross Sections (cm<sup>-1</sup>) for U-D<sub>2</sub>O Reactor

Material	$\nu$	$\Sigma_f$	$\Sigma_c$	$\Sigma_{s0}$	$\Sigma_{s1}$	$\Sigma_t$	c
U-D <sub>2</sub> O (a)	1.808381	0.054628	0.027314	0.464338	0.056312624	0.54628	1.0308381
U-D <sub>2</sub> O (b)	1.841086	0.054628	0.027314	0.464338	0.112982569	0.54628	1.0341086
U-D <sub>2</sub> O (c)	1.6964	0.054628	0.027314	0.464338	-0.27850447	0.54628	1.01964

#### Infinite Medium UD2Oa-1-1-IN , UD2Ob-1-1-IN , and UD2Oc-1-1-IN

Using the cross sections for U-D<sub>2</sub>O (a), U-D<sub>2</sub>O (b), and U-D<sub>2</sub>O (c) in Table 28,  $k_\infty = 1.205587$  (problem 38), 1.227391 (problem 40), and 1.130933 (problem 42), respectively, with a constant angular and scalar flux everywhere. The anisotropic scattering cross sections do not change  $k_\infty$ .

#### One-Medium Slab Critical Dimensions

The U-D<sub>2</sub>O critical dimensions,  $r_c$ , for the P<sub>1</sub> problems are listed in Table 29.

Table 29: Critical Dimensions,  $r_c$ , for One-Group Bare U-D<sub>2</sub>O

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
39	UD2Oa-1-1-SP	Sphere	10	18.30563081	15
41	UD2Ob-1-1-SP	Sphere	10	18.30563081	15
43	UD2Oc-1-1-SP	Sphere	10	18.30563081	16

## VII. TWO-ENERGY GROUP PROBLEM DEFINITIONS AND RESULTS

The isotropic two-energy group cross sections for five bare and two water reflected cases are listed in this section. There are also two linearly anisotropic scattering cross-sections sets provided for bare and infinite medium reactors. Unlike the one-group case, there is no flexibility in choosing these values since they are used throughout the literature. The cross sections listed here are similar to Pu-239,<sup>32</sup> U-235,<sup>32</sup> a realistic enriched uranium-aluminum-water assembly,<sup>8</sup> a 93% enriched U-235 model of a university research reactor,<sup>8,35,36</sup> and a typical large size D<sub>2</sub>O reactor with low enrichment of U-235.<sup>8,35,36</sup> Also included are critical dimensions for a similar uranium research reactor with a water reflector in an infinite lattice.<sup>34</sup> Again, these problems use cross sections that are reasonable representations of the materials described. These cross sections are **not** general purpose two-group values. The cross sections are used because they are defined in the literature and are intended to be used only to verify algorithm performance and **not** to predict any actual criticality experiments.

The isotropic neutron cross macroscopic sections ( $\text{cm}^{-1}$ ) provided for these problems are the total cross section of group  $i$ ,  $\Sigma_i$ , the capture (no neutrons emitted) cross section,  $\Sigma_{ci}$ , the within group scattering cross section,  $\Sigma_{iis}$ , the group-to-group scattering cross sections,  $\Sigma_{ijs}$  and  $\Sigma_{jis}$ , the fission cross section,  $\Sigma_{if}$ , the number of neutrons,  $\nu_i$ , emitted from each fission in a group, and the fission distribution,  $\chi_i$ .

**In this report, the fast energy group is group 2 to be consistent with most of the references. This notation is the reverse of most nuclear engineering textbooks.**

The literature solutions are often based on the group transfer cross sections,  $\Sigma_{ij}$ , given in the references; therefore, the individual cross sections may not be unique. Most references give  $(\nu\Sigma_f)_i$  instead of  $\nu_i$  and  $\Sigma_{fi}$ . Since both parameters (not the product) may be required by a code for the problem solution, the product  $(\nu\Sigma_f)_i$  has been split into  $\nu_i$  and  $\Sigma_{fi}$  preserving their product and  $\Sigma_i$ . The infinite slab lattice problems use a slightly unphysical set of cross sections to possibly stress code verification.

The two sets of linearly anisotropic cross sections provided are extensions of the university research reactor and D<sub>2</sub>O cases.<sup>25</sup> The anisotropic scattering component is designated for the in-group and group-to-group scattering cross section by  $\Sigma_{iis_1}$  and  $\Sigma_{jis_1}$ , respectively. Similarly, the isotropic scattering component is designated by  $\Sigma_{iis_0}$  and  $\Sigma_{jis_0}$ .

The value for  $k_\infty$  is given for each cross-section set. The in-group scattering cross sections do not affect  $k_\infty$ . For finite problems, the critical dimension,  $r_c$ , is listed in both fast group mean free paths (to indicate the neutron optical thickness) and in centimeters. For two-media problems, critical dimensions for the inner multiplying medium and outer reflector thickness are given in both fast mean free paths and centimeters. The critical dimensions,  $r_c$ , and reflector half thicknesses are also given for a water reflected infinite slab



lattice cell. Flux values are given for the university research reactor (a) (problems 54 and 71) at four spatial points. Angular fluxes can be found in the dissertation references.

To distinguish between the different URR fissile material cross-section sets, each is labeled with a letter "a," "b," "c," or "d", respectively. The infinite slab lattice cell cross sections are similar to the other three cross-section sets for the university research reactor and are labeled with URRd identifiers. The literature also uses three different H<sub>2</sub>O reflectors. Their cross sections are also labeled with a letter "a," "b," or "c" in the identifier. URR cross-section sets "b" and "c" have thermal upscattering. All other two-group cross sections have no thermal upscattering.

A comparison of the critical dimensions for the different geometries behave as expected; that is, the critical dimension is smallest for the one-dimensional slab and increases for the cylinder and sphere. This behavior is to be expected due to the increased leakage with the curvi-linear geometries. The effect of increased leakage on the critical dimension can be also be seen for the forward peaked linear anisotropically scattering cases. For the reflected geometries, the critical dimension decreases with increasing reflector thickness. However, the critical dimension for the infinite lattice cell *increases* with the increasing moderator thickness. Even though this may seem counter-intuitive, it should be expected because the amount of interaction between the fissile medium and adjacent cells decreases with increasing moderator half thickness.<sup>34</sup>

## A. Isotropic Scattering

### 1. Two-Group Pu-239.

#### Two-Group Isotropic Cross Sections

Tables 30 and 31 give the two-group, isotropic cross sections for Pu-239.

Table 30: Fast Energy Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Pu-239

Material	$\nu_2$	$\Sigma_{2f}$	$\Sigma_{2c}$	$\Sigma_{22s}$	$\Sigma_{12s}$	$\Sigma_2$	$\chi_2$
Pu-239	3.10	0.0936	0.00480	0.0792	0.0432	0.2208	0.575

Table 31: Slow Energy Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Pu-239

Material	$\nu_1$	$\Sigma_{1f}$	$\Sigma_{1c}$	$\Sigma_{11s}$	$\Sigma_{21s}$	$\Sigma_1$	$\chi_1$
Pu-239	2.93	0.08544	0.0144	0.23616	0.0	0.3360	0.425

#### Infinite Medium (PU-2-0-IN)

Using the two-group isotropic Pu-239 cross-section set from Tables 30 and 31,  $k_\infty = 2.683767$  (problem 44) with a constant group angular and scalar flux and a group 2 to group 1 flux ratio = 0.675229.

#### One-Medium Slab and Sphere Critical Dimensions

The critical dimensions,  $r_c$ , are listed in Table 32.

Table 32: Critical Dimensions,  $r_c$ , for Two-Group Bare Pu-239

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
45	PU-2-0-SL	Slab	0.396469	1.795602	8,35,36
46	PU-2-0-SP	Sphere	1.15513	5.231567	8

## 2. Two-Group U-235.

### Two-Group Cross Sections

Tables 33 and 34 give the two-group, isotropic cross sections for U-235.

Table 33: Fast Energy Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for U-235

Material	$\nu_2$	$\Sigma_{2f}$	$\Sigma_{2c}$	$\Sigma_{22s}$	$\Sigma_{12s}$	$\Sigma_2$	$\chi_2$
U-235	2.70	0.06192	0.00384	0.078240	0.0720	0.2160	0.575

Table 34: Slow Energy Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for U-235

Material	$\nu_1$	$\Sigma_{1f}$	$\Sigma_{1c}$	$\Sigma_{11s}$	$\Sigma_{21s}$	$\Sigma_1$	$\chi_1$
U-235	2.50	0.06912	0.01344	0.26304	0.0	0.3456	0.425

### Infinite Medium (U-2-0-IN)

Using the two-group U-235 cross-section set from Tables 33 and 34,  $k_\infty = 2.216349$  (problem 47) with a constant group angular and scalar flux and the group 2 to group 1 flux ratio = 0.474967.

### One-Medium Slab and Sphere Critical Dimensions

The critical dimensions,  $r_c$ , are listed in Table 35.

Table 35: Critical Dimension,  $r_c$ , for Two-Group Bare U-235

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
48	U-2-0-SL	Slab	0.649377	3.006375	8,35,36
49	U-2-0-SP	Sphere	1.70844	7.909444	8

### 3. Two-Group Uranium-Aluminum-Water Assembly.

#### Two-Group Isotropic Cross Sections

Tables 36 and 37 gives the two-group, isotropic cross sections for the uranium, aluminum, and water assembly.

Table 36: Fast Energy Group Macroscopic Cross Sections( $\text{cm}^{-1}$ ) for U-Al

Material	$\nu_2$	$\Sigma_{2f}$	$\Sigma_{2c}$	$\Sigma_{22s}$	$\Sigma_{12s}$	$\Sigma_2$	$\chi_2$
U-Al	0.0	0.0	0.000222	0.247516	0.020432	0.26817	1.0

Table 37: Slow Energy Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for U-Al

Material	$\nu_1$	$\Sigma_{1f}$	$\Sigma_{1c}$	$\Sigma_{11s}$	$\Sigma_{21s}$	$\Sigma_1$	$\chi_1$
U-Al	2.83	0.06070636042	0.00314363958	1.21313	0.0	1.27698	0.0

#### Infinite Medium (UAL-2-0-IN)

With the two-group cross-section set from Tables 36 and 37,  $k_\infty = 2.661745$  (problem 50) and the group 2 to group 1 flux ratio = 3.1250.

#### One-Medium Slab and Sphere Critical Dimensions

Using the cross sections given in Tables 36 and 37, the critical dimensions,  $r_c$ , are given in Table 38.

Table 38: Critical Dimensions,  $r_c$ , for Two-Group Uranium-Aluminum-Water Assembly

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
51	UAL-2-0-SL	Slab	2.09994	7.830630	8,35,36
52	UAL-2-0-SP	Sphere	4.73786	17.66738	8

#### 4. Two-Group Uranium Research Reactor.

##### Two-Group Isotropic Cross Sections

The cross sections for the one-medium (a), two-media (b and c), and infinite slab lattice (d) cases are different and are therefore listed separately. Tables 39 and 40 gives the two-group, one-medium, isotropic cross sections for the 93% enriched uranium bare university research reactor.

Table 39: Fast Energy Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor (a)

Material	$\nu_2$	$\Sigma_{2f}$	$\Sigma_{2c}$	$\Sigma_{22s}$	$\Sigma_{12s}$	$\Sigma_2$	$\chi_2$
Research Reactor (a)	2.50	0.0010484	0.0010046	0.62568	0.029227	0.65696	1.0

Table 40: Slow Energy Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor (a)

Material	$\nu_1$	$\Sigma_{1f}$	$\Sigma_{1c}$	$\Sigma_{11s}$	$\Sigma_{21s}$	$\Sigma_1$	$\chi_1$
Research Reactor (a)	2.50	0.050632	0.025788	2.44383	0.0	2.52025	0.0

##### Infinite Medium (URRa-2-0-IN)

The test set uses the two-group enriched U-235 cross-section set for the research reactor in Tables 39 and 40 with  $k_\infty = 1.631452$  (problem 53) and the group 2 to group 1 flux ratio = 2.614706.

##### One-Medium Slab and Sphere Critical Dimensions

The critical dimensions,  $r_c$ , are listed in Table 41.

Table 41: Critical Dimensions,  $r_c$ , for Two-Group Bare Research Reactor (a)

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
54	URRa-2-0-SL	Slab	4.97112	7.566853	8,35,36
55	URRa-2-0-SP	Sphere	10.5441	16.049836	8

##### One-Medium Slab Scalar Neutron Fluxes

Table 42 gives the normalized scalar neutron flux for the two-group bare research reactor (a) at four spatial points 36. All values are normalized with the fast group flux at the center.

Table 42: Normalized Scalar Fluxes for Two-Group Bare Research Reactor (a)

Problem	Identifier	Geometry	Energy Group	$r/r_c = 0.241394$	$r/r_c = 0.502905$	$r/r_c = 0.744300$	$r/r_c = 1.0$
54	URRa-2-0-SL	Slab	Fast	0.943363	0.761973	0.504012	0.147598
			Slow	0.340124	0.273056	0.173845	0.0212324

##### Two-Media Cross Sections for Slab Geometry

The cross sections for the two-media problems for the uranium research reactor are given for two different multiplying media with one nonmultiplying reflector. The two multiplying materials are labeled (b) and (c),

respectively. The multiplying region consists of an  $\text{H}_2\text{O} + \text{U-235}$  mixture surrounded by a  $\text{H}_2\text{O}$  reflector. The results in the literature for case (c) only include the infinite water reflector. The cross sections are given in Tables 43 and 44. Notice that this problem allows thermal upscattering in both multiplying and nonmultiplying regions.

Table 43: Fast Energy Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor (b), (c) and  $\text{H}_2\text{O}$  Reflector (a)

Material	$\nu_2$	$\Sigma_{2f}$	$\Sigma_{2c}$	$\Sigma_{22s}$	$\Sigma_{12s}$	$\Sigma_2$	$\chi_2$
Research Reactor (b)	2.50	0.000836	0.001104	0.83892	0.04635	0.88721	1.0
Research Reactor (c)	2.50	0.001648	0.001472	0.83807	0.04536	0.88655	1.0
$\text{H}_2\text{O}$ (a) (refl)	0.0	0.0	0.00074	0.83975	0.04749	0.88798	0.0

Table 44: Slow Energy Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor (b), (c) and  $\text{H}_2\text{O}$  Reflector (a)

Material	$\nu_1$	$\Sigma_{1f}$	$\Sigma_{1c}$	$\Sigma_{11s}$	$\Sigma_{21s}$	$\Sigma_1$	$\chi_1$
Research Reactor (b)	2.50	0.029564	0.024069	2.9183	0.000767	2.9727	0.0
Research Reactor (c)	2.50	0.057296	0.029244	2.8751	0.00116	2.9628	0.0
$\text{H}_2\text{O}$ (a) (refl)	0.0	0.0	0.018564	2.9676	0.000336	2.9865	0.0

#### Infinite Medium (URRb-2-0-IN and URRc-2-0-IN )

Using the two-group cross-section set for the research reactor (b) from Tables 43 and 44,  $k_\infty = 1.365821$  (problem 56) with a constant group angular and scalar flux and the group 2 to group 1 flux ratio = 1.173679. Using the two-group cross-section set for research reactor (c) from Tables 43 and 44,  $k_\infty = 1.633380$  (problem 57) with a constant group angular and scalar flux and the group 2 to group 1 flux ratio = 1.933422.

#### Two-Media Slab Critical Dimensions

Using the cross sections in Tables 43 and 44 for the  $\text{H}_2\text{O} + \text{U-235}$  research reactor and the  $\text{H}_2\text{O}$  reflector, the critical dimensions are given in Table 45. The mfp results use the group 2 total macroscopic cross section of region  $i$  to obtain the dimensions in cm.

Table 45: Critical Dimensions for Two-Group Research Reactor (b),(c) with  $\text{H}_2\text{O}$  Reflector (a)

Problem	Identifier	Geometry	U-235, $r_c$	$\text{H}_2\text{O}$ Width	U-235 + $\text{H}_2\text{O}$ Width	Ref.
58	URRb-H2Oa(1)-2-0-SL	Slab (mfp)	5.94147	1		37
		(cm)	6.696802	1.126152	7.822954	
59	URRb-H2Oa(5)-2-0-SL	Slab (mfp)	4.31485	5		37
		(cm)	4.863392	5.630757	10.494149	
60	URRb-H2Oa(IN)-2-0-SL	Slab (mfp)	4.15767	$\infty$	$\infty$	37
		(cm)	4.686230	$\infty$	$\infty$	
61	URRc-H2Oa(IN)-2-0-SL	Slab (mfp)	2.1826	$\infty$	$\infty$	37
		(cm)	2.461903	$\infty$	$\infty$	

## Two-Media Cross Sections for Infinite Slab Lattice Cell

The two-media cross sections are given in Tables 46 and 47 for a similar uranium enriched university research reactor. The  $\nu_2$  is slightly unphysical to stress criticality codes. Two different reflector materials are also given in Tables 46 and 47. The problems that use these cross sections are for an infinite slab lattice cell.

Table 46: Fast Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor(d) and H<sub>2</sub>O Reflector (b), (c)

Material	$\nu_2$	$\Sigma_{2f}$	$\Sigma_{2c}$	$\Sigma_{22s}$	$\Sigma_{12s}$	$\Sigma_2$	$\chi_2$
Research Reactor (d)	1.004	0.61475	0.0019662	0.0	0.0342008	0.650917	1.0
H <sub>2</sub> O (b) (refl)	0.0	0.0	$8.480293 \times 10^{-6}$	0.1096742149	0.001000595707	0.1106832906	0.0
H <sub>2</sub> O (c) (refl)	0.0	0.0	$4.97229 \times 10^{-4}$	1.226381244	0.1046395340	1.331518007	0.0

Table 47: Slow Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor(d) and H<sub>2</sub>O Reflector (b), (c)

Material	$\nu_1$	$\Sigma_{1f}$	$\Sigma_{1c}$	$\Sigma_{11s}$	$\Sigma_{21s}$	$\Sigma_1$	$\chi_1$
Research Reactor (d)	2.50	0.045704	0.023496	2.06880	0.0	2.13800	0.0
H <sub>2</sub> O (b) (refl)	0.0	0.0	0.00016	0.36339	0.0	0.36355	0.0
H <sub>2</sub> O (c) (refl)	0.0	0.0	0.0188	4.35470	0.0	4.37350	0.0

## Infinite Medium (URRd-2-0-IN )

The test set uses the two-group cross-section set for the research reactor in Tables 46 and 47 with  $k_\infty = 1.034970$  (problem 62) and the group 2 to group 1 flux ratio = 2.023344.

## Two-Media Infinite Slab Lattice Cell Critical Dimensions

Using the cross sections in Tables 46 and 47 for the enriched uranium research reactor with a H<sub>2</sub>O reflector, the critical dimensions for an infinite slab lattice cell as shown in Fig. 1 are given in Table 48. Because this is an infinite slab lattice cell with reflecting outer boundaries, notice that the moderator half thickness is given.

Table 48: Critical Dimensions for Two-Group Infinite Slab Lattice Cell and H<sub>2</sub>O Reflector (b), (c)

Problem	Identifier	Geometry	U-235, $r_c$	H <sub>2</sub> O Width	U-235+H <sub>2</sub> O Width	Ref.
63	URRd-H2Ob(1)-2-0-ISLC	Inf. Slab (mfp)	0.02142	1		34
		Lat. Cell (cm)	0.0329074	9.034787	9.067695	
64	URRd-H2Ob(10)-2-0-ISLC	Inf. Slab (mfp)	0.29951	10		34
		Lat. Cell (cm)	0.460135	90.347875	90.808010	
65	URRd-H2Oc(1)-2-0-ISLC	Inf. Slab (mfp)	0.22197	1		34
		Lat. Cell (cm)	0.341011	0.751023	1.092034	
66	URRd-H2Oc(10)-2-0-ISLC	Inf. Slab (mfp)	1.7699	10		34
		Lat. Cell (cm)	2.719087	7.510225	10.229312	

## 5. Two-Group U-D<sub>2</sub>O Reactor.

### Two-Group Isotropic Cross Sections

Tables 49 and 50 give the two-group, isotropic cross sections for the uranium-D<sub>2</sub>O system.

Table 49: Fast Energy Group Macroscopic Cross Sections (cm<sup>-1</sup>) for U-D<sub>2</sub>O

Material	$\nu_2$	$\Sigma_{2f}$	$\Sigma_{2c}$	$\Sigma_{22s}$	$\Sigma_{12s}$	$\Sigma_2$	$\chi_2$
U-D <sub>2</sub> O	2.50	0.002817	0.008708	0.31980	0.004555	0.33588	1.0

Table 50: Slow Energy Group Macroscopic Cross Sections (cm<sup>-1</sup>) for U-D<sub>2</sub>O

Material	$\nu_1$	$\Sigma_{1f}$	$\Sigma_{1c}$	$\Sigma_{11s}$	$\Sigma_{21s}$	$\Sigma_1$	$\chi_1$
U-D <sub>2</sub> O	2.50	0.097	0.02518	0.42410	0.0	0.54628	0.0

### Infinite Medium (UD2O-2-0-IN )

The test set uses the two-group U-D<sub>2</sub>O cross-section set from Tables 49 and 50 with  $k_\infty = 1.000196$  (problem 67) and the group 2 to group 1 flux ratio = 26.823271.

### One-Medium Slab and Sphere Critical Dimensions

The critical dimensions,  $r_c$ , are listed in Table 51.

Table 51: Critical Dimension,  $r_c$ , for Two-Group D<sub>2</sub>O System

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
68	UD2O-2-0-SL	Slab	284.367	846.632726	8,35,36
69	UD2O-2-0-SP	Sphere	569.430	1695.337621	8



## B. Linearly Anisotropic Scattering

The anisotropic scattering cross sections for the enriched U-235 research reactor and U-D<sub>2</sub>O reactor cases are the same as the isotropic set with the addition of the anisotropic cross sections,  $\Sigma_{22s_1}$ ,  $\Sigma_{12s_1}$ , and  $\Sigma_{11s_1}$ .

### 1. Two-Group Uranium Research Reactor.

#### Two-Group Anisotropic Macroscopic Cross Sections

Tables 52 and 53 gives the two-group, linearly anisotropic cross sections for the research reactor. Care must be used to correctly solve benchmark problem 71 because of the negative scattering for  $\mu$  near -1.

Table 52: Fast Group Cross Sections for Linearly Anisotropic Scattering (cm<sup>-1</sup>) Research Reactor (a)

Material	$\nu_2$	$\Sigma_{2f}$	$\Sigma_{2c}$	$\Sigma_{22s_0}$	$\Sigma_{22s_1}$	$\Sigma_{12s_0}$	$\Sigma_{12s_1}$	$\Sigma_2$	$\chi_2$
Research Reactor (a)	2.50	0.0010484	0.0010046	0.62568	0.27459	0.029227	0.0075737	0.65696	1.0

Table 53: Slow Group Cross Sections for Linearly Anisotropic Scattering (cm<sup>-1</sup>) Research Reactor (a)

Material	$\nu_1$	$\Sigma_{1f}$	$\Sigma_{1c}$	$\Sigma_{11s_0}$	$\Sigma_{11s_1}$	$\Sigma_{21s}$	$\Sigma_1$	$\chi_1$
Research Reactor (a)	2.50	0.050632	0.025788	2.44383	0.83318	0.0	2.52025	0.0

#### Infinite Medium (URRa-2-1-IN)

The test set uses the two-group enriched U-235 cross-section set from Tables 52 and 53 with  $k_\infty = 1.631452$  (problem 70) and the group 2 to group 1 flux ratio = 2.614706.

#### One-Medium Slab Critical Dimension

The critical dimensions are listed in Table 54.

Table 54: Critical Dimension,  $r_c$ , for Two-Group Linearly Anisotropic Scattering Research Reactor (a)

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
71	URRa-2-1-SL	Slab	6.2356	9.491600	25

#### One-Medium Slab Scalar Neutron Fluxes

Table 55 gives the normalized scalar neutron flux for the two-group bare research reactor (a) with linearly anisotropic scattering at four spatial points.<sup>25</sup> All values are normalized with the fast group flux at the center.

Table 55: Normalized Scalar Fluxes for Two-Group Bare Research Reactor (a)

Problem	Identifier	Geometry	Energy Group	$r/r_c = 0.20$	$r/r_c = 0.50$	$r/r_c = 0.80$	$r/r_c = 1.0$
71	URRa-2-1-SL	Slab	Fast	0.963873	0.781389	0.472787	0.189578
			Slow	0.349006	0.280870	0.157376	0.0277639

## 2. Two-Group U-D<sub>2</sub>O Reactor.

### Two-Group Anisotropic Macroscopic Cross Sections

Tables 56 and 57 gives the two-group, linearly anisotropic cross sections for the U-D<sub>2</sub>O system.

Table 56: Fast Energy Group Cross Sections for Linearly Anisotropic Scattering (cm<sup>-1</sup>) for U-D<sub>2</sub>O

Material	$\nu_2$	$\Sigma_{2f}$	$\Sigma_{2c}$	$\Sigma_{22s_0}$	$\Sigma_{22s_1}$	$\Sigma_{12s_0}$	$\Sigma_{12s_1}$	$\Sigma_2$	$\chi_2$
D <sub>2</sub> O	2.50	0.002817	0.008708	0.31980	0.06694	0.004555	-0.0003972	0.33588	1.0

Table 57: Slow Energy Group Cross Sections for Linearly Anisotropic Scattering (cm<sup>-1</sup>) for U-D<sub>2</sub>O

Material	$\nu_1$	$\Sigma_{1f}$	$\Sigma_{1c}$	$\Sigma_{11s_0}$	$\Sigma_{11s_1}$	$\Sigma_{21s}$	$\Sigma_1$	$\chi_1$
D <sub>2</sub> O	2.50	0.097	0.02518	0.42410	0.05439	0.0	0.54628	0.0

### Infinite Medium (UD2O-2-1-IN )

The test set uses the two-group linearly anisotropic D<sub>2</sub>O cross-section set from Tables 56 and 57 with  $k_\infty = 1.000196$  (problem 72) and the group 2 to group 1 flux ratio = 26.823271.

### One-Medium Slab Critical Dimension

The critical dimension,  $r_c$ , is listed in Table 58.

Table 58: Critical Dimension,  $r_c$ , for Two-Group Linearly Anisotropic Scattering for U-D<sub>2</sub>O Reactor

Problem	Identifier	Geometry	$r_c$ (mfp)	$r_c$ (cm)	Reference
73	UD2O-2-1-SL	Slab	336.05	1000.506133	25

## VIII. THREE-ENERGY GROUP PROBLEM DEFINITIONS AND RESULTS

A three-energy group isotropic infinite medium problem is defined in this section. The derivation appears in Appendix A.<sup>38,43</sup> This problem assumes no thermal upscattering and no fission neutrons born in the slowest energy group.

**The fast energy group is group 3 to be consistent with most of the references. Again, this notation is the reverse of most nuclear engineering textbooks.**

The cross sections listed here are similar to the uranium university research reactors. Again, this problem uses cross sections that are reasonable representations of the materials described and are **not** general purpose values. The cross sections are intended to be used to verify algorithm performance and **not** to predict criticality experiments. The cross sections are from Ref. 38 and are derived in Appendix A.

Table 59: Fast Energy Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor

Material	$\nu_3$	$\Sigma_{3f}$	$\Sigma_{3c}$	$\Sigma_{33s}$	$\Sigma_{23s}$	$\Sigma_{13s}$	$\Sigma_3$	$\chi_3$
Research Reactor	3.0	0.006	0.006	0.024	0.171	0.033	0.240	0.96

Table 60: Middle Energy Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor

Material	$\nu_2$	$\Sigma_{2f}$	$\Sigma_{2c}$	$\Sigma_{22s}$	$\Sigma_{32s}$	$\Sigma_{12s}$	$\Sigma_2$	$\chi_2$
Research Reactor	2.50	0.060	0.040	0.60	0.0	0.275	0.975	0.04

Table 61: Slow Energy Group Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor

Material	$\nu_1$	$\Sigma_{1f}$	$\Sigma_{1c}$	$\Sigma_{11s}$	$\Sigma_{21s}$	$\Sigma_{31s}$	$\Sigma_1$	$\chi_1$
Research Reactor	2.0	0.90	0.20	2.0	0.0	0.0	3.10	0.0

### Infinite Medium (URR-3-0-IN )

Using the three-group cross-section set from Tables 59, 60, and 61,  $k_\infty = 1.60$  (problem 74) with a constant group angular and scalar flux and the group 2 to group 3 flux ratio = 0.480, the group 1 to group 2 flux ratio = 0.3125, and the group 1 to group 3 flux ratio = 0.150.

# IX. SIX-ENERGY GROUP PROBLEM DEFINITIONS AND RESULTS

A six-energy group isotropic infinite medium problem comprised of two coupled three-energy group cross sections used in URR-3-0-IN is defined in this section. This test problem defines a six group cross-section set<sup>39</sup> such that energy groups 6 and 1, 5 and 2, and 4 and 3 are equivalent. The top three groups are decoupled from the lower three groups except for the fission distribution,  $\chi_i$ , which affects energy groups 6, 5, 2, and 1. Energy group 6 (group 1) scatters to groups 5 and 4 (groups 2 and 3). Energy group 5 (group 2) scatters to group 4 (group 3). Energy group 4 (group 3) self-scatters only. Since groups 1,2, and 3 are upscatter equivalents of groups 6, 5, and 4, respectively, this problem should only be used with codes that allow for thermal upscattering.

Table 62: Fast Energy Group 6 Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor

Material	$\nu_6$	$\Sigma_{6f}$	$\Sigma_{6c}$	$\Sigma_{66s}$	$\Sigma_{56s}$	$\Sigma_{46s}$	$\Sigma_6$	$\chi_6$
Research Reactor	3.0	0.006	0.006	0.024	0.171	0.033	0.240	0.48

Table 63: Energy Group 5 Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor

Material	$\nu_5$	$\Sigma_{5f}$	$\Sigma_{5c}$	$\Sigma_{55s}$	$\Sigma_{65s}$	$\Sigma_{45s}$	$\Sigma_5$	$\chi_5$
Research Reactor	2.50	0.060	0.040	0.60	0.0	0.275	0.975	0.02

Table 64: Energy Group 4 Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor

Material	$\nu_4$	$\Sigma_{4f}$	$\Sigma_{4c}$	$\Sigma_{44s}$	$\Sigma_{54s}$	$\Sigma_{64s}$	$\Sigma_4$	$\chi_4$
Research Reactor	2.0	0.90	0.20	2.0	0.0	0.0	3.10	0.0

Table 65: Energy Group 3 Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor

Material	$\nu_3$	$\Sigma_{3f}$	$\Sigma_{3c}$	$\Sigma_{33s}$	$\Sigma_{23s}$	$\Sigma_{13s}$	$\Sigma_3$	$\chi_3$
Research Reactor	2.0	0.90	0.20	2.0	0.0	0.0	3.10	0.0

Table 66: Energy Group 2 Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor

Material	$\nu_2$	$\Sigma_{2f}$	$\Sigma_{2c}$	$\Sigma_{22s}$	$\Sigma_{12s}$	$\Sigma_{32s}$	$\Sigma_2$	$\chi_2$
Research Reactor	2.50	0.060	0.040	0.60	0.0	0.275	0.975	0.02

Table 67: Slow Energy Group 1 Macroscopic Cross Sections ( $\text{cm}^{-1}$ ) for Research Reactor

Material	$\nu_1$	$\Sigma_{1f}$	$\Sigma_{1c}$	$\Sigma_{11s}$	$\Sigma_{21s}$	$\Sigma_{31s}$	$\Sigma_1$	$\chi_1$
Research Reactor	3.0	0.006	0.006	0.024	0.171	0.033	0.240	0.48

### Infinite Medium (URR-6-0-IN )

Since this problem is comprised of problem URR-3-0-IN cross sections with modified  $\chi_i$  values, the final  $k_\infty$  value and flux ratios will not change. Using the six-group cross-section set from Tables 62, 63, 64 65, 66, and 67,  $k_\infty = 1.60$  (problem 75) with a constant angular and scalar flux in each group. The group 5 to group 6 and group 2 to group 1 flux ratio  $= 0.480$ , the group 4 to group 5 and group 3 to group 2 flux ratio  $= 0.3125$ , and a group 4 to group 6 and group 3 to group 1 flux ratio  $= 0.150$ . These ratios are the same as in the three-group problem.

## X. SUMMARY

In this report, we have documented 75 problem descriptions with precise results for the critical dimensions,  $k_{eff}$  eigenvalue, and some eigenfunction (scalar neutron flux) results for infinite, slab, cylindrical, and spherical geometries for one- and two-energy group, multiple-media, and both isotropic and linearly anisotropic scattering using the listed references. We have not given a complete listing of every referenced result that has been published. Instead, we have included the references that provide both true transport solutions and enough information to reproduce the results. Several other references are included for reference completeness. All test set problems specifications and results are from peer reviewed journals, and have, in many cases, been solved numerically by more than one analytic method. These calculated values for  $k_{eff}$  and the scalar neutron flux are believed to be accurate to at least five decimal places. Criticality codes can be verified using these analytic benchmark test problems.

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## APPENDIX A

### Derivation of One-, Two-, and Three-Group $k_\infty$

To follow the benchmark referenced literature for the multi-group problems, the lowest energy group is group 1. This notation is the reverse from most nuclear engineering textbooks.

#### I. One-Energy Group Infinite Medium $k_\infty$

For an infinite, isotropic, homogeneous medium, the neutron leakage term,  $\Omega \cdot \nabla \Psi = 0$ , and the angular and scalar neutron flux is constant everywhere. Integrating the one-energy group infinite medium form of the transport equation over angle produces:

$$\Sigma_t \phi = \Sigma_s \phi + \frac{\nu \Sigma_f}{k_\infty} \phi \quad (18)$$

where  $\phi$  is the scalar neutron flux. The equation can be directly solved for  $k_\infty$ .

$$k_\infty = \frac{\nu \Sigma_f}{\Sigma_t - \Sigma_s} \quad (19)$$

or, in terms of mean number of secondaries,  $c$ :

$$k_\infty = c \left[ \frac{\nu \Sigma_f \Sigma_t}{(\Sigma_t - \Sigma_s)(\Sigma_s + \nu \Sigma_f)} \right] \quad (20)$$

#### II. Two-Energy Group Infinite Medium $k_\infty$

The two-group infinite medium form of the neutron transport equation reduces to:

$$\Sigma_2 \phi_2 = \Sigma_{22s} \phi_2 + \Sigma_{21s} \phi_1 + \frac{\chi_2}{k_\infty} [\nu_2 \Sigma_{2f} \phi_2 + \nu_1 \Sigma_{1f} \phi_1] \quad (21)$$

$$\Sigma_1 \phi_1 = \Sigma_{11s} \phi_1 + \Sigma_{12s} \phi_2 + \frac{\chi_1}{k_\infty} [\nu_1 \Sigma_{1f} \phi_1 + \nu_2 \Sigma_{2f} \phi_2] \quad (22)$$

Rearranging the equations in terms of  $\phi_1$  and  $\phi_2$ :

$$\left[ \Sigma_2 - \Sigma_{22s} - \frac{\chi_2}{k_\infty} \nu_2 \Sigma_{2f} \right] \phi_2 - \left[ \Sigma_{21s} + \frac{\chi_2}{k_\infty} \nu_1 \Sigma_{1f} \right] \phi_1 = 0 \quad (23)$$

$$\left[ \Sigma_1 - \Sigma_{11s} - \frac{\chi_1}{k_\infty} \nu_1 \Sigma_{1f} \right] \phi_1 - \left[ \Sigma_{12s} + \frac{\chi_1}{k_\infty} \nu_2 \Sigma_{2f} \right] \phi_2 = 0 \quad (24)$$

This equation can be written in matrix form as:<sup>42</sup>

$$\begin{bmatrix} -\left(\Sigma_{21s} + \frac{\chi_2}{k_\infty} \nu_1 \Sigma_{1f}\right) & \left(\Sigma_2 - \Sigma_{22s} - \frac{\chi_2}{k_\infty} \nu_2 \Sigma_{2f}\right) \\ \left(\Sigma_1 - \Sigma_{11s} - \frac{\chi_1}{k_\infty} \nu_1 \Sigma_{1f}\right) & -\left(\Sigma_{12s} + \frac{\chi_1}{k_\infty} \nu_2 \Sigma_{2f}\right) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (25)$$

To simplify the matrix elements, it is useful to define a total removal cross section,  $\Sigma_g$ , for each energy group  $g$  as the difference between the total cross section and in-group scattering or:<sup>40</sup>

$$\Sigma_2^{rem} = \Sigma_2 - \Sigma_{22s} \quad (26)$$

$$\Sigma_1^{rem} = \Sigma_1 - \Sigma_{11s} \quad (27)$$

Setting the determinant of this matrix equal to zero will give an equation that can be solved for  $k_\infty$ . One solution is  $k_\infty = 0$ . The other solution is:

$$k_\infty = \frac{\chi_1(\nu_2 \Sigma_{2f} \Sigma_{21s} + \Sigma_2^{rem} \nu_1 \Sigma_{1f}) + \chi_2(\nu_1 \Sigma_{1f} \Sigma_{12s} + \Sigma_1^{rem} \nu_2 \Sigma_{2f})}{\Sigma_1^{rem} \Sigma_2^{rem} - \Sigma_{12s} \Sigma_{21s}} \quad (28)$$

If there is no thermal upscattering, the equation reduces to:

$$k_\infty = \frac{\chi_1 \nu_1 \Sigma_{1f}}{\Sigma_1^{rem}} + \chi_2 \left( \frac{\nu_1 \Sigma_{1f} \Sigma_{12s}}{\Sigma_1^{rem} \Sigma_2^{rem}} + \frac{\nu_2 \Sigma_{2f}}{\Sigma_2^{rem}} \right) \quad (29)$$

To obtain the flux ratio, Eqs. 23 and 24 are added to eliminate  $\chi_1$  and  $\chi_2$  to give:

$$\left[ \Sigma_1^{rem} - \Sigma_{21s} - \frac{\nu_1 \Sigma_{1f}}{k_\infty} \right] \phi_1 + \left[ \Sigma_2^{rem} - \Sigma_{12s} - \frac{\nu_2 \Sigma_{2f}}{k_\infty} \right] \phi_2 = 0 \quad (30)$$

where  $\chi_1 + \chi_2 = 1$ .

Solving for  $\phi_2/\phi_1$ :

$$\frac{\phi_2}{\phi_1} = \frac{\left[ \Sigma_1^{rem} - \Sigma_{21s} - \frac{\nu_1 \Sigma_{1f}}{k_\infty} \right]}{\left[ \frac{\nu_2 \Sigma_{2f}}{k_\infty} - \Sigma_2^{rem} + \Sigma_{12s} \right]} \quad (31)$$

If there is no thermal upscattering, the equation reduces to:

$$\frac{\phi_2}{\phi_1} = \frac{\left[ \Sigma_1^{rem} - \frac{\nu_1 \Sigma_{1f}}{k_\infty} \right]}{\left[ \frac{\nu_2 \Sigma_{2f}}{k_\infty} - \Sigma_2^{rem} + \Sigma_{12s} \right]} \quad (32)$$

### III. Three-Energy Group Infinite Medium $k_\infty$ (a)

To make the three-energy group problem simpler, the following assumptions are made:<sup>40</sup>

- No thermal upscattering from group  $j$  to group  $i$ ,  $j < i$ ,  $\Sigma_{ij,s} = 0$
- No fission neutrons are born in the lowest energy group, i.e.,  $\chi_1 = 0$

The neutron transport equation can be written as:

$$\Sigma_3 \phi_3 = \Sigma_{33s} \phi_3 + \frac{\chi_3}{k_\infty} [\nu_3 \Sigma_{3f} \phi_3 + \nu_2 \Sigma_{2f} \phi_2 + \nu_1 \Sigma_{1f} \phi_1] \quad (33)$$

$$\Sigma_2 \phi_2 = \Sigma_{22s} \phi_2 + \Sigma_{23s} \phi_3 + \frac{\chi_2}{k_\infty} [\nu_3 \Sigma_{3f} \phi_3 + \nu_2 \Sigma_{2f} \phi_2 + \nu_1 \Sigma_{1f} \phi_1] \quad (34)$$

$$\Sigma_1 \phi_1 = \Sigma_{11s} \phi_1 + \Sigma_{12s} \phi_2 + \Sigma_{13s} \phi_3 \quad (35)$$

Rearranging the equations in terms of  $\phi_1$ ,  $\phi_2$  and  $\phi_3$ :

$$\left[ \Sigma_3 - \Sigma_{33s} - \frac{\chi_3}{k_\infty} \nu_3 \Sigma_{3f} \right] \phi_3 - \left[ \frac{\chi_3}{k_\infty} \nu_2 \Sigma_{2f} \right] \phi_2 - \left[ \frac{\chi_3}{k_\infty} \nu_1 \Sigma_{1f} \right] \phi_1 = 0 \quad (36)$$

$$-\left[ \Sigma_{23s} + \frac{\chi_2}{k_\infty} \nu_3 \Sigma_{3f} \right] \phi_3 + \left[ \Sigma_2 - \Sigma_{22s} - \frac{\chi_2}{k_\infty} \nu_2 \Sigma_{2f} \right] \phi_2 - \left[ \frac{\chi_2}{k_\infty} \nu_1 \Sigma_{1f} \right] \phi_1 = 0 \quad (37)$$

$$-\Sigma_{13s} \phi_3 - \Sigma_{12s} \phi_2 + [\Sigma_1 - \Sigma_{11s}] \phi_1 = 0 \quad (38)$$

This equation can be written in matrix form as:<sup>40</sup>

$$\begin{bmatrix} \left( \Sigma_3 - \Sigma_{33s} - \frac{\chi_3}{k_\infty} \nu_3 \Sigma_{3f} \right) & -\left( \frac{\chi_3}{k_\infty} \nu_2 \Sigma_{2f} \right) & -\left( \frac{\chi_3}{k_\infty} \nu_1 \Sigma_{1f} \right) \\ -\left( \Sigma_{23s} + \frac{\chi_2}{k_\infty} \nu_3 \Sigma_{3f} \right) & \left( \Sigma_2 - \Sigma_{22s} - \frac{\chi_2}{k_\infty} \nu_2 \Sigma_{2f} \right) & -\left( \frac{\chi_2}{k_\infty} \nu_1 \Sigma_{1f} \right) \\ -\Sigma_{13s} & -\Sigma_{12s} & (\Sigma_1 - \Sigma_{11s}) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (39)$$

Using the total removal cross sections defined in Eqs. 26 and 27, the determinant of the matrix then becomes:

$$\begin{bmatrix} \left( \Sigma_3^{rem} - \frac{\chi_3}{k_\infty} \nu_3 \Sigma_{3f} \right) & -\left( \frac{\chi_3}{k_\infty} \nu_2 \Sigma_{2f} \right) & -\left( \frac{\chi_3}{k_\infty} \nu_1 \Sigma_{1f} \right) \\ -\left( \Sigma_{23s} + \frac{\chi_2}{k_\infty} \nu_3 \Sigma_{3f} \right) & \left( \Sigma_2^{rem} - \frac{\chi_2}{k_\infty} \nu_2 \Sigma_{2f} \right) & -\left( \frac{\chi_2}{k_\infty} \nu_1 \Sigma_{1f} \right) \\ -\Sigma_{13s} & -\Sigma_{12s} & \Sigma_1^{rem} \end{bmatrix} \quad (40)$$

If we multiply the second line by  $\chi_3$ , multiply the first line by  $\chi_2$ , and subtract the results, and multiply the first line by  $k_\infty$ , the determinant becomes:

$$\begin{bmatrix} (\Sigma_3^{rem} k_\infty - \chi_3 \nu_3 \Sigma_{3f}) & -(\chi_3 \nu_2 \Sigma_{2f}) & -(\chi_3 \nu_1 \Sigma_{1f}) \\ -(\chi_3 \Sigma_{23s} + \chi_2 \Sigma_3^{rem}) & (\chi_3 \Sigma_2^{rem}) & 0 \\ -\Sigma_{13s} & -\Sigma_{12s} & \Sigma_1^{rem} \end{bmatrix} \quad (41)$$

Two of the  $k_\infty$  solutions are zero. The other solution is:

$$k_\infty = \frac{(\chi_3 \Sigma_{23s} + \chi_2 \Sigma_3^{rem}) (\nu_1 \Sigma_{1f} \Sigma_{12s} + \nu_2 \Sigma_{2f} \Sigma_1^{rem}) + \chi_3 \Sigma_2^{rem} (\nu_1 \Sigma_{1f} \Sigma_{13s} + \nu_3 \Sigma_{3f} \Sigma_1^{rem})}{\Sigma_1^{rem} \Sigma_2^{rem} \Sigma_3^{rem}} \quad (42)$$

#### IV. Three-Energy Group Infinite Medium $k_\infty$ (b)

An alternative method for solving the three-group  $k_\infty$  problem<sup>38</sup> is to rearrange the three-group transport equations in Eqs. 33, 34, 35:

$$(\Sigma_3 - \Sigma_{33s}) \phi_3 = \frac{\chi_3}{k_\infty} [\nu_3 \Sigma_{3f} \phi_3 + \nu_2 \Sigma_{2f} \phi_2 + \nu_1 \Sigma_{1f} \phi_1] \quad (43)$$

$$(\Sigma_2 - \Sigma_{22s}) \phi_2 = \Sigma_{23s} \phi_3 + \frac{\chi_2}{k_\infty} [\nu_3 \Sigma_{3f} \phi_3 + \nu_2 \Sigma_{2f} \phi_2 + \nu_1 \Sigma_{1f} \phi_1] \quad (44)$$

$$(\Sigma_1 - \Sigma_{11s}) \phi_1 = \Sigma_{12s} \phi_2 + \Sigma_{13s} \phi_3 \quad (45)$$

Divide each equation by  $\phi_3$  and define:

$$\phi_{23} = \frac{\phi_2}{\phi_3} \quad (46)$$

$$\phi_{13} = \frac{\phi_1}{\phi_3} \quad (47)$$

$$\Sigma_i^{rem} = \Sigma_i - \Sigma_{iis} \quad (48)$$

The result gives:

$$\Sigma_3^{rem} = \frac{\chi_3}{k_\infty} [\nu_3 \Sigma_{3f} + \nu_2 \Sigma_{2f} \phi_{23} + \nu_1 \Sigma_{1f} \phi_{13}] \quad (49)$$

$$\phi_{23} \Sigma_2^{rem} = \Sigma_{23s} + \frac{\chi_2}{k_\infty} [\nu_3 \Sigma_{3f} + \nu_2 \Sigma_{2f} \phi_{23} + \nu_1 \Sigma_{1f} \phi_{13}] \quad (50)$$

$$\phi_{13} \Sigma_1^{rem} = \Sigma_{13s} + \Sigma_{12s} \phi_{23} \quad (51)$$

If we divide Eqs. 50 and 51 by  $\nu_3 \Sigma_{3f}$  and define:

$$f_{23} = \frac{\nu_2 \Sigma_{2f}}{\nu_3 \Sigma_{3f}} \phi_{23} \quad (52)$$

$$f_{13} = \frac{\nu_1 \Sigma_{1f}}{\nu_3 \Sigma_{3f}} \phi_{13} \quad (53)$$

Then we get:

$$\frac{\Sigma_3^{rem}}{\chi_3 \nu_3 \Sigma_{3f}} = \frac{1}{k_\infty} [1 + f_{23} + f_{13}] \quad (54)$$

$$\frac{1}{\chi_2 \nu_3 \Sigma_{3f}} [\Sigma_2^{rem} \phi_{23} - \Sigma_{23s}] = \frac{1}{k_\infty} [1 + f_{23} + f_{13}] \quad (55)$$

$$\phi_{13} \Sigma_1^{rem} = \Sigma_{13s} + \Sigma_{12s} \phi_{23} \quad (56)$$

If we substitute Eq. 56 into 57 and rearranging,

$$\frac{\Sigma_2^{rem}}{\Sigma_3^{rem}} \phi_{23} = \frac{\Sigma_{23s}}{\Sigma_3^{rem}} + \frac{\chi_2}{\chi_3} \quad (57)$$

$$\frac{\Sigma_1^{rem}}{\Sigma_3^{rem}} \phi_{13} = \frac{\Sigma_{13s}}{\Sigma_3^{rem}} + \frac{\Sigma_{12s}}{\Sigma_3^{rem}} \phi_{23} \quad (58)$$

$$k_\infty = \frac{\chi_3 \nu_3 \Sigma_{3f}}{\Sigma_3^{rem}} [1 + f_{23} + f_{13}] \quad (59)$$

Equations 57, 58, and 59 give the flux ratios and  $k_\infty$ . Tables 43, 44, and 45 give the cross sections used for the three-group problem.<sup>38</sup>

There are numerous cross sections involved in these equations, implying that there are numerous arbitrary choices we can make that will yield solutions to these equations. We show one set of cross sections that will satisfy a set of chosen conditions.<sup>38</sup>

If we make our basic choices as:

- $k_\infty = 1.600$
- $\chi_3=0.96, \chi_2=0.04, \chi_1=0.0$
- 5% of fission production occurs in group 3
- 20% of fission production occurs in group 2
- 75% of fission production occurs in group 1

With these choices and the definitions of  $f_{23}$  and  $f_{13}$ , we get:

$$f_{23} = \frac{\nu_2 \Sigma_{2f} \phi_2}{\nu_3 \Sigma_{3f} \phi_3} = 4 \quad (60)$$

$$f_{13} = \frac{\nu_1 \Sigma_{1f} \phi_1}{\nu_3 \Sigma_{3f} \phi_3} = 15 \quad (61)$$

Using this gives:

$$\frac{\Sigma_3^{rem}}{\nu_3 \Sigma_{3f}} = \frac{\chi_3}{k_\infty} [1 + f_{23} + f_{13}] \quad (62)$$

$$\frac{\Sigma_3^{rem}}{\nu_3 \Sigma_{3f}} = 12.0 \quad (63)$$

We can now make more arbitrary choices. If we choose:

- $\nu_3=3.0, \Sigma_{3f}=0.006$
- $\nu_2=2.5, \Sigma_{2f}=0.060$
- $\nu_1=2.0, \Sigma_{1f}=0.900$

Then we get:

$$\phi_{23} = 0.480 \quad (64)$$

$$\phi_{13} = 0.150 \quad (65)$$

making  $\Sigma_3^{rem}=0.216$  from Eq. 63. If we make more choices:

$$\Sigma_{33s} = 0.024 \quad (66)$$

$$\Sigma_{3c} = 0.006 \quad (67)$$

$$\Sigma_{13s} = 0.033 \quad (68)$$

making  $\Sigma_3=0.240$  and  $\Sigma_{23s}=0.171$ . Using Eq. 57,  $\Sigma_2^{rem}=0.375$ . This result now gives:

$$\Sigma_{22s} = 0.600 \quad (69)$$

$$\Sigma_{2c} = 0.040 \quad (70)$$

making  $\Sigma_2=0.975$  and  $\Sigma_{12s}=0.275$ . Using Eq. 58,  $\Sigma_1^{rem}=1.10$ . One last arbitrary choice is:

$$\Sigma_{11s} = 2.00 \quad (71)$$

making  $\Sigma_1=3.10$  and  $\Sigma_{1c}=0.20$ .

## V. General Multigroup Infinite Medium $k_\infty$

More than three-group  $k_\infty$  derivations have been done (see Ref. 43). A general multigroup  $k_\infty$  derivation is included in this section for completeness.<sup>44</sup>

Given

$$\overline{\overline{\Sigma_t}} \overline{\phi} = \overline{\overline{\Sigma_s}} \overline{\phi} + \frac{\overline{\chi} \overline{\nu \Sigma_f} \overline{\phi}}{k} \quad (72)$$

where:

$$\overline{\overline{\Sigma_t}} = GxG \text{ matrix}$$

$$\overline{\overline{\Sigma_s}} = GxG \text{ matrix}$$

$$\overline{\chi} = Gx1 \text{ vector}$$

$$\overline{\nu \Sigma_f} = 1xG \text{ vector}$$

$$\overline{\phi} = Gx1 \text{ vector}$$

$$k = \text{scalar}$$

then:

$$\left( \overline{\overline{\Sigma_t}} - \overline{\overline{\Sigma_s}} \right) \overline{\phi} = \frac{1}{k} \overline{\chi} \overline{\nu \Sigma_f} \overline{\phi} \quad (73)$$

$$\overline{\phi} = \frac{1}{k} \left( \overline{\overline{\Sigma_t}} - \overline{\overline{\Sigma_s}} \right)^{-1} \overline{\chi} \overline{\nu \Sigma_f} \overline{\phi} \quad (74)$$

$$\overline{\nu \Sigma_f} \overline{\phi} = \frac{1}{k} \overline{\nu \Sigma_f} \left( \overline{\overline{\Sigma_t}} - \overline{\overline{\Sigma_s}} \right)^{-1} \overline{\chi} \overline{\nu \Sigma_f} \overline{\phi} \quad (75)$$

Since  $\nu \Sigma_f \phi$  is a scalar, it can be cancelled out and we get the following explicit result:

$$k = \overline{\nu \Sigma_f} \left( \overline{\Sigma_t} - \overline{\Sigma_s} \right)^{-1} \overline{\phi} \quad (76)$$

The right hand side of this equation is a scalar, equal to  $k$ . Only one matrix inversion is necessary.



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