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ON DISCONTINUOUS INITIAL VALUE PROBLEMS FOR NONLINEAR EQUATIONS AND FINITE DIFFERENCE SCHEMES

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PHYSICS

Abstract

This paper describes a new numerical scheme for calculating hydrodynamical flows with shocks. It is similar to a scheme promulgated some years ago by von Neumann, see [9], and modified more recently by him and R. Richtmyer, see [11], inasmuch as it is a straightforward numerical scheme which ignores the presence of discontinuities. It is more closely related to the scheme described in [9] since no viscosity term is used; what is new about the method is:

(a) The difference scheme used is based on the conservation form of the hydrodynamic equations.

(b) The difference scheme is unsymmetric in time.

Description of the difference equations: Write the hydrodynamic equations in the form of conservation laws (mass, momentum and energy); in this form each term in the equation is a perfect x or t derivative. Replace all x derivatives by centered difference quotients, all time derivatives f_t by a forward facing difference quotient of this sort:

$$\frac{f_{\ell}^{n+1} - \overline{f_{\ell}^{n}}}{\Delta t},$$

where $\overline{f_{\ell}^{n}}$ is taken as the arithmetic mean of the values of f at all neighboring space points at time cycle n.

This scheme uses a staggered lattice, i.e., at time cycle n we use all lattice vectors ℓ with, say, even components, at the next time cycle we use odd lattice vectors.

The conjecture is that if the meshsize tends to zero, and the stability condition of Courant-Friedrichs-Lewy is satisfied, the approximate solutions computed by this method will tend to the exact solution uniformly except in neighborhoods of discontinuity lines or surfaces.

The mathematical soundness of this proposition is discussed in detail, using as an example the equation $u_t + uu_x = 0$. Test calculations performed on this equation and on the hydrodynamic equations in one dimension, both Euler and Lagrange form, show fairly conclusively that the method works. Some of the numerical results are presented at the end of the report.

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Let $U_t + AU_x + B = 0$ be a quasilinear hyperbolic system of first order equations; U denotes a column vector of n unknown functions, A a coefficient matrix, and B a vector. A and B are assumed to be functions of x,t and U. The system is called hyperbolic if all eigenvalues of A are real and if A has n linearly independent eigenvectors.

The initial value problem for such a system is to find a solution with prescribed values on the x axis (or an interval of it), $U(x,0) = \Phi(x)$. According to the theory of hyperbolic equations this initial value problem has a (unique) solution if $\Phi(x)$ is differentiable, or is at least Lipschitz continuous (in this latter case the solution would not have continuous partial derivatives). The range of t for which the solution exists is at least as large as $c(\max|\Phi'|)^{-1}$, c being a constant depending on the coefficients A and B and their first derivatives.

The example of the simple equation $u_t + uu_x = 0$ shows that this estimate cannot be improved in general. In this case, namely, the solution of the initial value problem $u(x,0) = \varphi(x)$ is given by the implicit relation $u - \varphi(x - ut) = 0$. This relation defines u as a (differentiable) function of x and t as long as the derivative of the left hand side with respect to u, $1 + t\varphi$, does not vanish. The smallest value of t for which this quantity vanishes is

$t = (\max - \varphi')^{-1}$; this shows that the width, of the domain of existence in the t direction, does depend on a bound for the magnitude of φ' (although only on a one-sided bound).

Suppose we wish to solve an initial value problem where the initial values no longer satisfy a Lipschitz condition; say they are downright discontinuous, as in the Riemann shock tube problem. One could attempt to solve this problem by approximating the given differentiable initial values $\Phi_i(x)$, construct the corresponding solution U_i and take their limit - if it exists - in the sense of some norm or topology. This method works for linear equations but does not in general for quasi-linear equations; for if the sequence Φ_i approximates an initial vector that is not Lipschitz continuous, the first derivatives of Φ_i are not uniformly bounded, and so the range of t for which the solution of the i^{th} problem, U_i, exists shrinks to zero as i tends to infinity. This shows that the theory of discontinuous initial value-problems for nonlinear equations is not a mere appendix to the theory of differentiable initial value-problems but has to be developed independently.

There are several ways of developing such a theory. One is to generalize the concept of a function satisfying a differential equation. This leads to the notion of weak solutions and the initial value problem is to ascertain whether in the aggregate of all weak solutions there exists one with the prescribed initial data.

Another way is to define the solution of a discontinuous initial value problem directly by a limiting process of some kind. This limiting process would usually consist of approximating the equation by a sequence of equations for which the initial value problem can be solved. For the equations of hydrodynamics this is usually done by including viscous forces; what is proposed here is to use a straight-forward finite difference scheme; that such a method works is of interest for the theory and for practical computations.

It would be desirable to develop an abstract theory which would include these special methods. The appropriate class of abstract equations may possibly be the ones of the form

$$U_t = A\,N\,u$$

where A is an <u>unbounded</u> <u>linear</u>, N a <u>continuous</u> <u>nonlinear</u> operation.

We shall describe now the three methods mentioned, illustrating them on the equation $u_t + u\,u_x = 0$.

1. <u>Generalizing the concept of a solution</u>.

Let v be some <u>test</u> <u>function</u> which is zero on the boundary of some region G of the x,t plane; G is supposed to lie within the domain of definition of the solution u. Multiply the equation $u_t + u\,u_x = 0$ by v, integrate over G, and integrate by parts. The result is that the integral

$$\iint v_t u + \frac{1}{2} v_x u^2 \qquad (1)$$

is zero for all G test functions v and solutions u. <u>Conversely</u>: if u is a function <u>with</u> <u>continuous</u> <u>derivatives</u> for which the integral (1) vanishes for all test functions, then u is a solution of the original differential equation (this is easily seen by integrating (1) by parts and applying the so-called fundamental lemma of the calculus of variations).

We define u to be a generalized or weak solution if the integral (1) is zero for all test functions v. As stated before, a generalized solution which is differentiable is a bona fide solution. But amongst the class of non-differentiable functions we have a genuine extension of the notion of solution.

Weak solutions, for linear equations, are discussed briefly in Courant-Hilbert, vol. II, p. 469-470. They play an important role in Friedrich's work on differential operators; their theory was treated systematically by Sobolev, and L. Schwartz. In the nonlinear case - which interests us most - the concept of weak solutions is discussed, usually in connection with shock problems of hydrodynamics (see also E. Hopf, [7]).

Consider discontinuous solutions, i.e., functions u that suffer a jump discontinuity across a smooth arc C, on either side of which it has continuous derivatives and satisfies the equation. Straightforward application of the definition shows that a discontinuous solution is a weak solution if and only if U, the slope of the discontinuity line at any point on C is the arithmetic mean of the values of u on the two sides at this point (analogue of the shock relations).

This example shows (a) that there are weak solutions of our equation which are not genuine solutions

(b) that the class of weak solutions is associated not so much with an equation but with the form in which it is written. For had we written our equation in the form $u^{-1}u_t + u_x = 0$, the criterion for discontinuous solutions to be weak solutions would have been $\bar{U} = (u_1 - u_2)(\log u_1 - \log u_2)^{-1}$, which defines an entirely different class of weak solutions. The form of the equation to be used is dictated entirely by outside physical consideration. E.G., the equations of hydrodynamics in mass coordinates can be written as four different conservation laws; namely, conservation of mass, momentum, energy, and entropy. For physical reasons we would operate with the first three of these conservation laws.

The test of usefulness of the concept of weak solutions is whether weak solutions with arbitrarily prescribed initial data of a wide class (say, the class of all piecewise continuous or all bounded, measurable functions) exist, and whether the initial values determine the solutions uniquely (a weak solution having prescribed initial data can be defined either in an almost everywhere sense or in a weak sense). It turns out that the answer to the first query is affirmative, to the second, negative.

That for the equation $u_t + uu_x = 0$ weak solutions with arbitrarily prescribed initial data exist has been shown by E. Hopf in [7] as a corollary to the theory developed there. That the solution is not in general unique is well known; it can be seen from this example: Let the initial value be

$$u(x,0) = 0 \qquad \text{for } x < 0$$
$$= 1 \qquad \text{for } x > 0.$$

The function

$$u(x,t) = 0 \text{ for } t > 2x$$
$$= 1 \text{ for } t < 2x$$

is a weak solution of our problem since it assumes the initial value and satisfies the jump condition. But so is the function

$$u(x,t) = 0 \qquad \text{for } x < 0$$
$$= \frac{x}{t} \qquad \text{for } x > t$$
$$= 1 \qquad \text{for } x < t.$$

In analogy with hydrodynamics we would exclude the first solution since it represents a rarefaction shock; whether the exclusion of rarefaction shocks would leave only one weak solution of any initial value problem, is not known.

So the problem is to characterize the physically relevant weak solutions in some systematic way, and to prove that the initial value problem has a unique physically relevant weak solution for a wide class of initial values. In connection with this problem it should be remarked that whereas the class of regular solutions of our equation displays reversibility in time; i.e., if $u(x,t)$ is a regular solution, so is $u(-x,-t)$, and the class of all weak solutions likewise, the class of physically relevant weak solutions (i.e., the ones without rarefaction shocks) no longer share this property; e.g., the weak solution

$$u(x,t) = 1 \qquad \text{for } t > 2x$$
$$= 0 \qquad \text{for } t < 2x$$

is physically relevant for it represents a compression shock, whereas $u(-x, -t)$ represents a rarefaction shock.

One systematic method of introducing physically relevant weak solutions is to take those solutions which are limits of "viscous flows". I.e., consider the augmented equation

$$u_{,t} + u\, u_x = \lambda\, u_{xx} \qquad (2)$$

with some positive constant λ , solve the initial value problem $u_\lambda(x,t) = u_o$, and let λ tend to zero, Equation (2), and the above limiting process, was introduced into the literature, by Burgers; an especially elegant and rigorous treatment of it is due to E. Hopf [7]. This procedure was conceived as a simple analogue of the process of obtaining this discontinuous solution of the hydrodynamic equations as limits of viscous flows, see Becker [1], L. H. Thomas, Gilbarg [10], Grad [6], and Courant-Friedrichs [2], pp. 134-138.

Equation (2) is a semi-linear parabolic equation; the introduction of a new unknown φ, related to u by $u = -2\lambda\, \varphi_x/\varphi$ reduces it, as E. Hopf has observed, to a linear parabolic equation $\varphi_t = \lambda\varphi_{xx}$ whose solution can be written down explicitly. This in turn gives an explicit representation of any solution of (2) in terms of its initial values; this representation enabled Hopf to prove that for fixed ini-

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tial values u_o the solution $u_\lambda(x,t)$ tends to a limit as λ tends to zero, for almost all x and t. This limit can be called the generalized solution of the initial value problem $u(x,0) = u_o$ of the original equation (1).

It is easy to show that these generalized solutions are weak solutions; just multiply equation (2) by any twice differentiable test function v and integrate by parts:

$$\iint v_t \; u + \frac{1}{2} v_x \; u^2 = \lambda \iint v_{xx} u \; ;$$

u remains uniformly bounded for λ, and so, v being held fixed, the right side tends to zero with λ.

This class of generalized solutions is irreversible in t; there is nothing surprising in this, for the process whereby they were defined is openly biased in favor of the positive t direction, i.e., the initial value problem for the parabolic equation (2) can be solved for positive t but not for negative t.

A different limiting procedure for constructing weak solutions is by a straightforward finite difference scheme; the conjecture is that this process furnishes the same class of physically relevant weak solutions as the viscosity method. Several arguments will be presented which make the conjecture plausible, or at least possible; the numerical evidence in favor of it is very strong but there is no rigorous proof for it yet.

First the description of the scheme itself: Since the concept of weak solutions is linked not to the equation itself but the form in which it is written, it is important that the difference scheme should be linked to the distinguished form of the equation. Secondly, the possibility of defining weak solutions rests on the fact that the given equation is in divergence form, i.e., each term is a pure x or t derivative. This feature should be preserved as much as possible in the difference scheme too. Both requirements are fulfilled by this scheme: replace space derivatives by difference quotients:

f_x by $\dfrac{f^n_{\ell+1} - f^n_{\ell-1}}{2\,\Delta x}$, and t derivatives u_t by a forward difference quotient of this kind:

$$\frac{1}{\Delta t}\left(u^{n+1}_{\ell} - \frac{u^n_{\ell+1} + u^n_{\ell-1}}{2}\right.\,.$$

Here superscripts refer to time cycle, subscripts to position in space.

This scheme, when applied to any hyperbolic system, is stable in the sense of von Neumann if $\frac{\Delta x}{\Delta t}$ satisfies the classical Courant-Friedrichs - Lewy condition, see [5], of being greater than the slope of the steepest characteristic. The equation $u_t + u\,u_x = 0$ has one characteristic, with slope u, so the stability condition is

$\frac{\Delta x}{\Delta t} \geq \max |u|$. Now if we choose $\frac{\Delta x}{\Delta t}$ so that this inequality is satisfied initially, the function generated by the difference scheme will never exceed its largest value initially, and so the stability condition is satisfied for all future times.

Solutions constructed by the difference scheme are defined only at the lattice points; imagine them extended to the whole relevant portion of the x,t plane by defining u inside any lattice square to have the same value as, say, at the upper left corner. Diminish the size of the lattice and suppose that the corresponding solutions, thus extended, converge in the $\mathcal{L}_2$ sense to some limit function u. This limit function u is a weak solution of the original differential equation as may be easily proved by multiplying the difference equation at each lattice point by the value of a test function v there, summing over all lattice points and summing by parts. A passage to the limit leads to an integral relation between u and v that characterizes u as a weak solution. What is not at all clear is

(i) Whether the sequence of solutions of the difference equations converges in the $\mathcal{L}_2$ sense.

(ii) Whether the sequence converges uniformly except in a neighborhood of the discontinuity lines.

(iii) Whether the weak solutions obtained in this manner are the physically relevant ones.

Experimental evidence, presented below, indicates that the answer to all three questions is yes. Concerning (iii) it should be pointed out that, just as in the case of the passage to the limit through viscous flows, the class of weak solutions obtainable by this finite difference method is not likely to be invariant under replacement of x by minus x and t by minus t, because the difference scheme distinguishes between the positive and negative t direction. I mention this as a possible guide to finding other adequate difference schemes.

In case of <u>regular</u> solutions, i.e., ones with continuous first derivatives, the difference scheme described here furnishes a uniformly convergent sequence of approximations to the true solutions. This has been proved, for arbitrary quasilinear hyperbolic systems, by Keller and Lax in [8] and for a slightly different scheme by Courant, Isaacson and Rees [4].

It should be pointed out that if the sequence of solutions of the difference equations or a subsequence of them converges only <u>weakly</u>, the weak limit is <u>not</u> a weak solution. For in this case the weak limit of u_n^2 is <u>not</u> the square of the weak limit of u_n and so the procedure of multiplying the difference equations by v, summing by parts and passing to the limit leads to an equation in which the role of u^2 is taken by the weak limit of u_n^2 .

Experimental calculations were performed using IBM Card Programmed Calculators; the problem was coded by Mr. Stewart Schlesinger. The first case considered was the initial values $u(x, 0) = 1$ for $x < 0$, $= 0$ for $x > 0$, taking $\Delta t/\Delta x$ to be one. The initial values were deliberately chosen to be homogeneous, so that carrying the calculations further in time would have the effect of refining the meshsize; the idea was to carry out the calculations until it became evident that the scheme was converging, diverging or oscillating. It turned out that the scheme was converging, and with astonishing rapidity. After 44 steps in time the calculated values of u were

x	u
17	1.00000
19	.99548
21	.76818
23	.21061
25	.02343
27	.00018
29	.00018

The values of u not listed differ from one or zero by at most 10^{-5}. The theoretical position of the discontinuity, propagating with speed 1/2, is at x = 22; this is precisely the center of zone of transition; the zone is, roughly speaking, spread over three intervals.

Four steps later, at t = 48, the calculated values of u were:

x	u
19	1.00000
21	.99548
23	.76817
25	.21061
27	.02344
29	.00210
31	.00018

The theoretical position of the discontinuity line is at x = 24; the figures show that relative to this discontinuity line the profile of the solution has changed by at most one figure in the last decimal; this suggests that not only does the solution of the difference scheme converge to the true discontinuous solution uniformly in every subset not containing the line of discontinuity, but that the shape of the transition tends to a definite limit. This limiting shape can be characterized as the steady state solution of the difference equations. The difference equation is

$$u_{\ell}^{n+1} = (u_{\ell+1}^{n} + u_{\ell-1}^{n})/2 + \frac{1}{4}\ ({u_{\ell-1}^{n}}^2 - {u_{\ell+1}^{n}}^2) \ ;$$

here the superscript n refers to time cycle, ℓ to space position. The equation satisfied by the steady state solution would be

$$\frac{f(x-1) + f(x+1)}{2} + \frac{f^2(x-1 - f^2(x+1)}{4} = f(x + \tfrac{1}{2}) \qquad (3)$$

and the boundary conditions are:

$$f(-\infty) = 1, \qquad f(\infty) = 0 \ . \qquad (4)$$

More precisely, the state of affairs is probably as follows: <u>The difference equation</u> (3), <u>subject to the boundary conditions</u> (4), <u>has a continuous, monotonic solution as function of the real variable x; this solution is unique except for an arbitrary phase shift.</u> Furthermore, starting with any function g(x) defined over the odd integers, repeated application of the transformation **T g = g'** defined by

$$\frac{g(x - 1) + g(x + 1)}{2} + \frac{g^2(x-1) - g^2(x+1)}{4} = g'(x + \tfrac{1}{2})$$

leads to the steady state solution f(x). I.E., if we denote $T^n{}_g$ by

$g_n(x)$, then $g_n(x)$ tends uniformly to $f(x + \alpha)$, where $f(x)$ is the steady state solution*; the phase shift α depends only on the initial distribution g.

Observe that the function $g_n(x)$ is defined only at points by n/2. Thus the $g_n(x)$ are defined either at the integers of halfway in between, and consequently we need the values of $f(x + \alpha)$ at these points only. This is however an exceptional situation which arose because $\Delta t/\Delta x$ was chosen to be commensurable to the speed of the propagation of the discontinuity.

The numerical evidence presented before for the verity of this theorem is very strong. The calculations cited refer to the initial values $g(x) = 1$ for x a negative odd integer, $= 0$ for x a positive odd integer; as a further check the values: $g(x) = 1$ for x an odd integer less than 0 minus one, $g(-1) = .9$, $g(x) = 0$ for x a positive odd integer were tried. The results were the same as with the original choice of initial $g(x)$; the tables below give the values of u at t = 44 and 48; these differ by less than one figure in the fifth decimal.

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*Fixed, say, uniquely by picking $f(0)$ to be 1/2.

$t = 44$		$t = 48$	
x	u	x	u
17	1.00000	19	1.00000
19	.99195	21	.99195
21	.71566	23	.71566
23	.17449	25	.17449
25	.01858	27	.01859
27	.00165	29	.00165
29	.00014	31	.00014

Table I, appended to this paper, gives the values of $g_{45}(x)$, $g_{46}(x)$, $g_{47}(x)$, $g_{48}(x)$ corresponding to the first choice of $g_o(x)$ over those values of x where the deviation from the constant values 0 or 1 is significant; (for all subsequent values of n, $g_n(x)$ coincides in the first five figures with one of the four listed). Table II contains the same information referring to the second choice for initial g.

Graphs I and II show a plot of these values; they lie on smooth curves, and these curves indeed appear to have the same shape.

Returning to the difference equation (3), it should be remarked that if the boundary values of f are switched, i.e. $f(-\infty) = 0$, $f(\infty) = 1$ or, more generally, are replaced by values for which $f(-\infty)$ is less than $f(\infty)$, then no solution would exist. This result, for which I have no proof at present, expresses the fact that the finite difference method furnishes solutions with compression shocks but not with rarefaction shocks. Mathematically, it is an analogue of a well-known result on steady viscous flows (see [1], [6], [10], [11]), which I shall present for the simplified equation $u_t + u\, u_x = \lambda\, u_{xx}$.

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Let $u_o(x,t)$ be a steady state solution of the equation $u_t + u\, u_x = \lambda\, u_{xx}$, i.e. u_o is a function of $x - c\, t$ only, $u_o = u(x - ct)$. Then $u(\xi)$ satisfies the ordinary differential equation

$$c\, u' + u\, u' = \lambda\, u'' .$$

Integrate both sides with respect to ξ :

$$K + c\, u + \frac{1}{2} u^2 = \lambda\, u' ,$$

so

$$\frac{d\xi}{du} = \frac{2\lambda}{u^2 + 2c\, u + 2K} . \qquad (5)$$

We are interested in solutions which at $\xi = -\infty$ and $\xi = \infty$ have prescribed values u_i (initial) and u_f (final). From equation (5) it is clear that ξ will approach infinity only if u approaches one of the roots of the quadratic function in the denomination of the right hand side; these roots must be then just the initial and final values of u, u_i and u_f, and c, the propagation speed, must be their arithmetic mean. Furthermore, $u^2 + 2\, cu + 2K$ is <u>negative</u> between the two roots u_i and u_f, and so, λ being positive, $\frac{d\xi}{du}$ is negative, i.e., u is a decreasing function of ξ . So we conclude that an initial and final state can be connected through a solution of (5) only if $u_i \gg u_f$. If this inequality is fulfilled, then they can be connected and the explicit formula

$$\xi = \frac{2\lambda}{u_i - u_f} \log \frac{u_i - u}{u - u_f}$$

gives the shape of the connecting curve.

Numerical calculations were carried out for the initial values $u(x,0) = 0$ for $x < 0$, $u(x,0) = 1$ for $x > 0$, using the same difference scheme as before; the results after 48 steps, are tabulated in Table III and plotted in Graph III. The dashed line in Graph III refers to the exact solution.

The same problem was run with $\Delta t/\Delta x = 1/2$; the results after steps in time, are tabulated in Table IV, plotted in Graph IV.

So far, only the equation $u_t + u\, u_x = 0$ has been discussed; the question is, how much of what was said before can be generalized to quasilinear systems. The first observation is that <u>weak solutions</u> are defined only for systems in which all first order terms are perfect x or t derivatives (or at most combinations of such terms with coefficients which are functions of independent variables only); for such systems I propose the same finite difference scheme, i.e. replace all x derivatives by centered difference quotients, and replace all t derivatives v_t by $\left(v_\ell^{n+1} - \frac{v_{\ell+1}^n + v_{\ell-1}^n}{2}\right/ \Delta t$.

This was tried on the hydrodynamic equations of one dimensional time dependent flow; the equations were written in the form of conservation laws. They are, in Eulerian coordinates,

$$\rho_t + (u\rho)_x = 0, \qquad \text{Cons. mass}$$

$$(u\rho)_t + (u^2\rho)_x + p_x = 0, \qquad \text{Cons. of momentum}$$

$$\left(\rho e + \frac{u^2}{2}\right)_t + \left(\rho\, e\, u + \frac{u^3\rho}{2}\right)_x + (up)_x = 0, \qquad \text{Cons. of energy.}$$

Here ρ, u,p and e denote density, velocity, pressure and energy per unit mass. The equation of state expresses e as a function of p and ρ, e.g. for an ideal gas $e = \frac{p}{\rho(\lambda-1)}$.

In the computations we will operate with the quantities ρ, $u\rho = m$ and $e\rho + u^2\rho/2 = E$, the mass, momentum and total energy per unit volume. In terms of these the equations are:

$$\rho_t + m_x = 0$$

$$m_t + \left[(\gamma-1)E + \frac{3-\gamma}{2}\frac{m^2}{\rho}\right]_x = 0$$

$$E_t + \left[\gamma\frac{m}{\rho}E - \frac{\gamma-1}{2}\frac{m^3}{\rho^2}\right]_x = 0$$

To these equations the difference scheme described before was applied. Several calculations were made, with different choice of the initial values and γ, and in all cases the answer agreed fairly well with the theoretically calculated flow. The calculations were performed on the Los Alamos MANIAC. The flow diagram for the calculations was prepared by Stewart Schlesinger and the problem was coded by Lois Cook.

In the first problem γ was chosen equal to 1.5, and

$u = 2$	for $x < 0$,
$= 0$	for $x > 0$
$p = 50$	for $x < 0$
0	for $x > 0$
$\rho = 50$	for $x < 0$
$= 10$	for $x > 0$.

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The two constant states chosen can be connected by a shock (notice that compression is five-fold at the shock, the value corresponding to $\gamma = 1.5$): $\Delta t/\Delta x$ was chosen to be .25.

The results after 49 time cycles are given in Table V. The fourth column, ν, gives the label of the lattice point in hexadecimal notation; the Eulerian position x is related to the label ν by $x - 4(2\nu - 52)$ (taking t to be one). There is a rapid transition from one state to another around $\nu = 41$; this corresponds to $x - 124$, and gives for the speed of propagation of the discontinuity $\frac{122}{49} = 2.48$; this agrees pretty well with the theoretical value of the shock speed which is 2.5.

The values of ρ, u and p after 99 time cycles are given in Table VI; the position of the ν^{th} subdivision now is given by $x = 4\,(2\nu - 102)$. Again there is a rapid transition from one set of values to the other, around $\nu = 82$; so the speed of propagation is $\frac{248}{99} = 2.50$.

Notice that the width of the zone of transition is approximately the same in both calculations.

The stability constant, i.e. the reciprocal of the ratio of $\Delta x/\Delta t$ to the maximum of the true propagation speed is .863.

A second calculation started with the initial states $u = 2$, $p = 50$, $\rho = 50$ to the left, $u = 0$, $p = 0$, $\rho = 10$ to the right of $x = 0$. $\Delta t/\Delta x$ was taken to be .25. These two constant states can be connected to each other through a rarefaction wave, a contact discontinuity, a constant state and a shock (going from left to right). According to theory, the constant state behind the shock is $u = 1.47$, $p = 27.1$, $\rho = 50$, and shock speed is $U = 1.84$.

The results after 49 time cycles are given in Table VII, after 99 time cycles in Table VIII. In Table VII there is a rapid transition around $\nu = 37$ which corresponds to a shock speed of $\frac{88}{49} = 1.79$, which is in fair agreement with the calculated value. In Table VIII the transition occurs around $\nu = 74$ which gives for shockspeed $184/99 = 1.86$, in even better agreement with the calculated value.

In Table VIII, u and p appear to be fairly constant for a while behind the shock, the value of p being around $27 \pm .3$, and of u around $(.184 \pm .001)8 = 1.47 \pm .01$. These are in fair agreement with the theoretically calculated values, in spite of the fact that the value of ρ is way off (only around 39 at the shock front, whereas the correct value is 50).

A third calculation was done for the case $\gamma = 2$, and initial states $u = 2$, $\rho = 50$, $p = 100$ to the left of $x = 0$, $u = 0$, $\rho = 10$, $p = 0$ to the right. $\frac{\Delta t}{\Delta x}$ was chosen as .25 which turned out larger than permissible by the Courant-Friedrichs-Lewy criterion. Consequently, instability occurred near the shock front, but not enough to make the calculations meaningless, as the listings in Table IX and X show; these present the calculated values of the unknowns after 49, resp. 99 steps.

The exact solution connects the two states through a rarefaction wave, a contact discontinuity, a constant state and a shock. The theoretically calculated value of u, ρ and p behind the shock front are: u = 2.26, ρ = 30, p = 76.5; these compare favorably with the calculated values of u and p.

Two general features of these calculations are:

(i) The width of the transition shock in the shock is narrowest if $\Delta t / \Delta x$ is chosen as large as possible.

(ii) The values of u and p converge to the exact value more rapidly than the value of ρ.

The method can be set up in Lagrange coordinates as well. Denoting specific volume by V and by ξ unit mass along the x axis, the conservation equations are:

$$V_t = u_\xi \qquad \text{Conservation of mass}$$

$$u_t = p_\xi \qquad \text{Conservation of momentum}$$

$$(e + {}^1/_2 u^2) = -(up)_\xi \qquad \text{Conservation of energy}$$

Introduce as unknowns V, u and $E = e + 1/2\, u^2$, mass, momentum and energy per unit volume. In terms of these, the equations for a perfect gas $(e = \frac{pV}{\gamma - 1})$ can be written as

$$V_t = u_\xi$$

$$u_t = \left[(\gamma - 1) \frac{E - 1/2\, u^2}{V} \right]$$

$$E_t = \left[(\gamma - 1) \frac{uE - 1/2 u^3}{V} \right]$$

Experimental calculations in this setup are being carried out by Lester Baumhoff. Results so far are encouraging.

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x	u
-8	.99991
-7	.99921
-6	.99548
-5	.98205
-4	.94713
-3	.87779
-2	.76816
-1	.62581
0	.47071
1	.32661
2	.21061
3	.12798
4	.07450
5	.04216
6	.02344
7	.01291
8	.00707
9	.00386
10	.00210
11	.00114
12	.000621

TABLE I

x	u
-9	.99998
-8	.99978
-7	.99836
-6	.99195
-5	.97176
-4	.92476
-3	.83976
-2	.71566
-1	.56551
0	.41203
1	.27747
2	.17449
3	.10407
4	.05982
5	.03359
6	.01859
7	.01021
8	.00558
9	.00304
10	.00166
11	.00090
12	.00049

TABLE II

x	u
47	.92695
45	.88187
43	.83994
41	.79948
39	.7599
37	.7209
35	.6825
33	.6444
31	.6066
29	.5692
27	.5321
25	.4954
23	.4590
21	.4229
19	.3873
17	.3523
15	.3177
13	.2839
11	.2509
9	.2189
7	.1881
5	.1587
3	.1310
1	.1055
-1	.0823
-3	.0619
-5	.0447
-7	.0306
-9	.0198
-11	.0120

TABLE III

Rarefaction wave, $t = 48$, $\frac{\Delta t}{\Delta x} = 1$

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x	u
64	.8553
60	.8170
56	.7758
52	.7322
48	.6869
44	.6405
40	.5933
36	.5457
32	.4980
28	.4506
24	.4039
20	.3580
16	.3134
12	.2704
8	.2295
4	.1911
0	.1555
-4	.1234
-8	.0949
-12	.0706
-16	.0505
-20	.0345
-24	.0225
-28	.0139

TABLE IV

Rarefaction wave, $t = 63$, $\frac{\Delta t}{\Delta x} = 1/2$

ρ	u/8	p	ν
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØ1ØØØØ1
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØ2ØØØØ2
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØ3ØØØØ3
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØ4ØØØØ4
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØ5ØØØØ5
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØ6ØØØØ6
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØØ7ØØØØ7
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØØ8ØØØØ8
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØØ9ØØØØ9
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØØAØØØØA
ØØØ5ØØØØØØ	Ø249999995	ØØØ5ØØØØØØ	ØØØØBØØØØB
ØØØ5ØØØØØØ	Ø249999994	ØØØ5ØØØØØØ	ØØØØCØØØØC
ØØØ5ØØØØØØ	Ø249999992	ØØØ5ØØØØØØ	ØØØØDØØØØD
ØØØ4999999	Ø249999997	ØØØ5ØØØØØØ	ØØØØEØØØØE
ØØØ4999999	Ø25ØØØØØØ3	ØØØ4999999	ØØØØFØØØØF
ØØØ4999998	Ø25ØØØØØ34	ØØØ4999998	ØØØ1ØØØØ1Ø
ØØØ4999995	Ø25ØØØØ13Ø	ØØØ4999993	ØØØ11ØØØ11
ØØØ4999985	Ø25ØØØØ448	ØØØ4999977	ØØØ12ØØØ12
ØØØ4999954	Ø25ØØØ1372	ØØØ4999932	ØØØ13ØØØ13
ØØØ4999874	Ø25ØØØ3843	ØØØ4999811	ØØØ14ØØØ14
ØØØ4999676	Ø25ØØØ9892	ØØØ4999515	ØØØ15ØØØ15
ØØØ4999233	Ø25ØØ23472	ØØØ499885Ø	ØØØ16ØØØ16
ØØØ4998317	Ø25ØØ51481	ØØØ4997478	ØØØ17ØØØ17
ØØØ4996581	Ø25Ø1Ø4557	ØØØ4994879	ØØØ18ØØØ18
ØØØ4993556	Ø25Ø19695Ø	ØØØ499Ø358	ØØØ19ØØØ19
ØØØ4988711	Ø25Ø344587	ØØØ4983141	ØØØ1AØØØ1A
ØØØ4981579	Ø25Ø56Ø8Ø9	ØØØ4972585	ØØØ1BØØØ1B
ØØØ4971936	Ø25Ø85Ø137	ØØØ4958489	ØØØ1CØØØ1C
ØØØ4959944	Ø2512Ø1791	ØØØ49414ØØ	ØØØ1DØØØ1D
ØØØ4946159	Ø251585849	ØØØ4922792	ØØØ1EØØØ1E
ØØØ4931292	Ø251954895	ØØØ49Ø4964	ØØØ1FØØØ1F
ØØØ49157Ø3	Ø2522527Ø3	ØØØ489Ø617	ØØØ2ØØØØ2Ø
ØØØ4898791	Ø252428575	ØØØ4882157	ØØØ21ØØØ21
ØØØ4878665	Ø252452643	ØØØ488Ø985	ØØØ22ØØØ22
ØØØ48526Ø1	Ø252325585	ØØØ4887Ø19	ØØØ23ØØØ23
ØØØ4818585	Ø252Ø77313	ØØØ4898683	ØØØ24ØØØ24
ØØØ4777518	Ø251751784	ØØØ491322Ø	ØØØ25ØØØ25
ØØØ4734457	Ø251369295	ØØØ4926485	ØØØ26ØØØ26
ØØØ4695222	Ø25Ø8Ø5127	ØØØ4928633	ØØØ27ØØØ27
ØØØ4648115	Ø249273378	ØØØ488Ø761	ØØØ28ØØØ28
ØØØ44973Ø7	Ø243178521	ØØØ462Ø551	ØØØ29ØØØ29
ØØØ3919992	Ø219682262	ØØØ36729Ø8	ØØØ2AØØØ2A
ØØØ2622113	Ø15241632Ø	ØØØ17539ØØ	ØØØ2BØØØ2B
ØØØ14240888	ØØ49Ø6ØØ26	ØØØØ289912	ØØØ2CØØØ2C
ØØØ1Ø47ØØ3	ØØØ42956Ø7	ØØØØØ12999	ØØØ2DØØØ2D
ØØØ1ØØ29ØØ	ØØØØ1795Ø6	ØØØØØØØ347	ØØØ2EØØØ2E
ØØØ1ØØØ129	ØØØØØØ5885	ØØØØØØØØØ7	ØØØ2FØØØ2F
ØØØ1ØØØØØ4	ØØØØØØØ161	FØØØØØØØØØ	ØØØ3ØØØØ3Ø
ØØØ1ØØØØØØ	ØØØØØØØØØ1	FØØØØØØØØØ	ØØØ31ØØØ31
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ32ØØØ32

TABLE V $\gamma = 1.5$

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ρ	u/8	p	ν
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØ1ØØØØ1
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØ2ØØØØ2
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØ3ØØØØ3
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØ4ØØØØ4
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØ5ØØØØ5
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØ6ØØØØ6
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØØ7ØØØØ7
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØØ8ØØØØ8
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØØ9ØØØØ9
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØØAØØØØA
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØØBØØØØB
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØØCØØØØC
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØØDØØØØD
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØØEØØØØE
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØØFØØØØF
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØ1ØØØØ1Ø
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØ11ØØØ11
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØ12ØØØ12
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØ13ØØØ13
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØ14ØØØ14
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØ15ØØØ15
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØ16ØØØ16
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØ17ØØØ17
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØ18ØØØ18
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØ19ØØØ19
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØ1AØØØ1A
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØ1BØØØ1B
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØ1CØØØ1C
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØ1DØØØ1D
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØ1EØØØ1E
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØ1FØØØ1F
ØØØ4999999	Ø249999995	ØØØ5ØØØØØØ	ØØØ2ØØØØ2Ø
ØØØ5ØØØØØØ	Ø249999993	ØØØ5ØØØØØØ	ØØØ21ØØØ21
ØØØ5ØØØØØØ	Ø249999992	ØØØ5ØØØØØØ	ØØØ22ØØØ22
ØØØ4999999	Ø249999992	ØØØ5ØØØØØØ	ØØØ23ØØØ23
ØØØ4999999	Ø249999996	ØØØ5ØØØØØØ	ØØØ24ØØØ24
ØØØ4999999	Ø249999999	ØØØ4999999	ØØØ25ØØØ25
ØØØ4999999	Ø25ØØØØØØ4	ØØØ4999999	ØØØ26ØØØ26
ØØØ4999999	Ø25ØØØØØ19	ØØØ4999999	ØØØ27ØØØ27
ØØØ4999997	Ø25ØØØØØ58	ØØØ4999997	ØØØ28ØØØ28
ØØØ4999995	Ø25ØØØØ141	ØØØ4999993	ØØØ29ØØØ29
ØØØ4999989	Ø25ØØØØ327	ØØØ4999983	ØØØ2AØØØ2A
ØØØ4999975	Ø25ØØØØ728	ØØØ4999964	ØØØ2BØØØ2B
ØØØ4999949	Ø25ØØØ155Ø	ØØØ4999924	ØØØ2CØØØ2C
ØØØ4999896	Ø25ØØØ3166	ØØØ4999844	ØØØ2DØØØ2D
ØØØ4999797	Ø25ØØØ62Ø7	ØØØ4999695	ØØØ2EØØØ2E
ØØØ4999617	Ø25ØØ117Ø2	ØØØ4999426	ØØØ2FØØØ2F
ØØØ49993Ø6	Ø25ØØ21217	ØØØ499896Ø	ØØØ3ØØØØ3Ø
ØØØ4998791	Ø25ØØ37Ø14	ØØØ4998186	ØØØ31ØØØ31
ØØØ4997975	Ø25ØØ62145	ØØØ4996956	ØØØ32ØØØ32

TABLE VI - $\gamma = 1.5$

ρ	u/8	p	ν
0004996719	0250100453	0004995080	0003300033
0004994892	0250156385	0004992343	0003400034
0004992341	0250234525	0004988521	0003500035
0004988933	0250338921	0004983418	0003600036
0004984586	0250472142	0004976912	0003700037
0004979298	0250634225	0004969007	0003800038
0004973182	0250821754	0004959873	0003900039
0004966478	0251027272	0004949879	0003A0003A
0004959555	0251239273	0004939588	0003B0003B
0004952874	0251443000	0004929714	0003C0003C
0004946936	0251621990	0004921054	0003D0003D
0004942193	0251760277	0004914371	0003E0003E
0004938950	0251844846	0004910288	0003F0003F
0004937269	0251867837	0004909178	0004000040
0004936891	0251827979	0004911101	0004100041
0004937197	0251730811	0004915790	0004200042
0004937229	0251587657	0004922704	0004300043
0004935771	0251413526	0004931121	0004400044
0004931526	0251224559	0004940261	0004500045
0004923350	0251035561	0004949404	0004600046
0004910556	0250858225	0004957977	0004700047
0004893217	0250700262	0004965597	0004800048
0004872400	0250565399	0004972077	0004900049
0004850238	0250454031	0004977390	0004A0004A
0004829740	0250364137	0004981618	0004B0004B
0004814306	0250291814	0004984875	0004C0004C
0004806966	0250229836	0004987155	0004D0004D
0004809290	0250157723	0004987796	0004E0004E
0004819311	0249994894	0004983197	0004F0004F
0004824868	0249395725	0004956963	0005000050
0004777634	0246902908	0004840299	0005100051
0004508862	0236906638	0004388264	0005200052
0003646730	0202778528	0003094194	0005300053
0002208321	0120726590	0001150186	0005400054
0001250264	0027848932	0000131838	0005500055
0001024238	0001956039	0000005078	0005600056
0001001503	0000084023	0000000146	0005700057
0001000074	0000003133	0000000003	0005800058
0001000003	0000000107	F000000000	0005900059
0001000000	0000000001	F000000000	0005A0005A
0001000000	0000000000	F000000000	0005B0005B
0001000000	0000000000	F000000000	0005C0005C
0001000000	0000000000	F000000000	0005D0005D
0001000000	0000000000	F000000000	0005E0005E
0001000000	0000000000	F000000000	0005F0005F
0001000000	0000000000	F000000000	0006000060
0001000000	0000000000	F000000000	0006100061
0001000000	0000000000	F000000000	0006200062
0001000000	0000000000	F000000000	0006300063
0001000000	0000000000	F000000000	0006400064

TABLE VI - $\gamma = 1.5$ (Continued)

ρ	u/8	p	ν
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ1ØØØØ1
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ2ØØØØ2
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ3ØØØØ3
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ4ØØØØ4
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ5ØØØØ5
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ6ØØØØ6
ØØØ4999999	Ø125ØØØØØØ	ØØØ4999999	ØØØØ7ØØØØ7
ØØØ4999999	Ø125ØØØØØ5	ØØØ4999999	ØØØØ8ØØØØ8
ØØØ4999999	Ø125ØØØØ25	ØØØ4999998	ØØØØ9ØØØØ9
ØØØ4999995	Ø125ØØØ128	ØØØ4999993	ØØØØAØØØØA
ØØØ4999981	Ø125ØØØ558	ØØØ4999972	ØØØØBØØØØB
ØØØ499993Ø	Ø125ØØ2124	ØØØ4999895	ØØØØCØØØØC
ØØØ4999766	Ø125ØØ7154	ØØØ4999649	ØØØØDØØØØD
ØØØ4999293	Ø125Ø21613	ØØØ4998941	ØØØØEØØØØE
ØØØ4998075	Ø125Ø589Ø9	ØØØ4997114	ØØØØFØØØØF
ØØØ4995235	Ø125145787	ØØØ4992861	ØØØ1ØØØØ1Ø
ØØØ4989231	Ø125329351	ØØØ4983884	ØØØ11ØØØ11
ØØØ4977661	Ø125682875	ØØØ4966631	ØØØ12ØØØ12
ØØØ4957242	Ø1263Ø6583	ØØØ49363Ø6	ØØØ13ØØØ13
ØØØ4924Ø62	Ø12732Ø271	ØØØ4887333	ØØØ14ØØØ14
ØØØ4874146	Ø128847595	ØØØ4814278	ØØØ15ØØØ15
ØØØ48Ø4189	Ø13Ø995577	ØØØ4713Ø26	ØØØ16ØØØ16
ØØØ4712274	Ø133835291	ØØØ4581827	ØØØ17ØØØ17
ØØØ4598364	Ø137388475	ØØØ4421882	ØØØ18ØØØ18
ØØØ4464475	Ø141621783	ØØØ4237352	ØØØ19ØØØ1S
ØØØ4314571	Ø146447161	ØØØ4Ø3485Ø	ØØØ1AØØØ1A
ØØØ4154283	Ø151725428	ØØØ3822668	ØØØ1BØØØ1B
ØØØ399Ø574	Ø157271Ø68	ØØØ36Ø994Ø	ØØØ1CØØØ1C
ØØØ3831415	Ø162858252	ØØØ34Ø5844	ØØØ1DØØØ1D
ØØØ3685479	Ø16823Ø818	ØØØ3218856	ØØØ1EØØØ1E
ØØØ3561819	Ø173120097	ØØØ3Ø55951	ØØØ1FØØØ1F
ØØØ3469376	Ø17727Ø73Ø	ØØØ2921594	ØØØ2ØØØØ2Ø
ØØØ3415949	Ø18Ø459794	ØØØ281626Ø	ØØØ21ØØØ21
ØØØ34Ø5562	Ø182462672	ØØØ2733818	ØØØ22ØØØ22
ØØØ3431719	Ø18287Ø1Ø4	ØØØ2655878	ØØØ23ØØØ23
ØØØ3461991	Ø18Ø617433	ØØØ2539879	ØØØ24ØØØ24
ØØØ3412297	Ø173138931	ØØØ23Ø29Ø8	ØØØ25ØØØ25
ØØØ3138476	Ø155484943	ØØØ183536Ø	ØØØ26ØØØ26
ØØØ2541275	Ø121241Ø99	ØØØ1127898	ØØØ27ØØØ27
ØØØ1795728	ØØ7Ø47ØØ25	ØØØØ444845	ØØØ28ØØØ28
ØØØ127Ø824	ØØ244Ø33Ø5	ØØØØØ95873	ØØØ29ØØØ29
ØØØ1Ø62314	ØØØ4718241	ØØØØØ12112	ØØØ2AØØØ2A
ØØØ1Ø1Ø716	ØØØØ632853	ØØØØØØ1182	ØØØ2BØØØ2B
ØØØ1ØØ1496	ØØØØØ7Ø634	ØØØØØØØ1Ø4	ØØØ2CØØØ2C
ØØØ1ØØØ176	ØØØØØØ6932	ØØØØØØØØØ8	ØØØ2DØØØ2D
ØØØ1ØØØØ17	ØØØØØØØ596	ØØØØØØØØØØ	ØØØ2EØØØ2E
ØØØ1ØØØØØ1	ØØØØØØØØ41	FØØØØØØØØØ	ØØØ2FØØØ2F
ØØØ1ØØØØØØ	ØØØØØØØØØ1	FØØØØØØØØØ	ØØØ3ØØØØ3Ø
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ31ØØØ31
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ32ØØØ32

TABLE VII - $\gamma = 1.5$ $\qquad \frac{\Delta t}{\Delta x} = .25$

ρ	u/8	p	ν
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ1ØØØØ1
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ2ØØØØ2
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ3ØØØØ3
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ4ØØØØ4
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ5ØØØØ5
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØØ6ØØØØ6
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ7ØØØØ7
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ8ØØØØ8
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ9ØØØØ9
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØAØØØØA
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØBØØØØB
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØCØØØØC
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØDØØØØD
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØØEØØØØE
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØØFØØØØF
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØ1ØØØØ1Ø
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØ11ØØØ11
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØ12ØØØ12
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØ13ØØØ13
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØ14ØØØ14
ØØØ4999999	Ø125ØØØØØØ	ØØØ4999999	ØØØ15ØØØ15
ØØØ4999999	Ø125ØØØØØ1	ØØØ4999999	ØØØ16ØØØ16
ØØØ4999999	Ø125ØØØØØ3	ØØØ4999999	ØØØ17ØØØ17
ØØØ4999999	Ø125ØØØØØ7	ØØØ4999999	ØØØ18ØØØ18
ØØØ4999999	Ø125ØØØØ17	ØØØ4999999	ØØØ19ØØØ19
ØØØ4999998	Ø125ØØØØ47	ØØØ4999997	ØØØ1AØØØ1A
ØØØ4999993	Ø125ØØØ143	ØØØ4999993	ØØØ1BØØØ1B
ØØØ4999987	Ø125ØØØ37Ø	ØØØ4999981	ØØØ1CØØØ1C
ØØØ4999969	Ø125ØØØ935	ØØØ4999954	ØØØ1DØØØ1D
ØØØ4999927	Ø125ØØ223Ø	ØØØ499989Ø	ØØØ1EØØØ1E
ØØØ4999833	Ø125ØØ5Ø81	ØØØ4999751	ØØØ1FØØØ1F
ØØØ4999639	Ø125Ø11Ø43	ØØØ4999458	ØØØ2ØØØØ2Ø
ØØØ499925Ø	Ø125Ø22938	ØØØ4998876	ØØØ21ØØØ21
ØØØ4998511	Ø125Ø45559	ØØØ4997768	ØØØ22ØØØ22
ØØØ4997169	Ø125Ø86644	ØØØ4995756	ØØØ23ØØØ23
ØØØ4994841	Ø125157913	ØØØ4992268	ØØØ24ØØØ24
ØØØ499Ø98Ø	Ø125276131	ØØØ4986486	ØØØ25ØØØ25
ØØØ4984851	Ø1254638Ø9	ØØØ4977318	ØØØ26ØØØ26
ØØØ497553Ø	Ø1257493Ø3	ØØØ4963398	ØØØ27ØØØ27
ØØØ4961933	Ø126166Ø37	ØØØ4943134	ØØØ28ØØØ28
ØØØ4942878	Ø12675Ø718	ØØØ4914817	ØØØ29ØØØ29
ØØØ4917178	Ø12754Ø732	ØØØ4876765	ØØØ2AØØØ2A
ØØØ4883749	Ø128571Ø28	ØØØ4827499	ØØØ2BØØØ2B
ØØØ4841726	Ø129871Ø56	ØØØ4765915	ØØØ2CØØØ2C
ØØØ479Ø548	Ø131462178	ØØØ469142Ø	ØØØ2DØØØ2D
ØØØ473ØØ22	Ø133355935	ØØØ46Ø4ØØ5	ØØØ2EØØØ2E
ØØØ466Ø345	Ø135553271	ØØØ45Ø4264	ØØØ2FØØØ2F
ØØØ4582Ø87	Ø138Ø44579	ØØØ4393347	ØØØ3ØØØØ3Ø
ØØØ4496156	Ø14Ø81Ø4ØØ	ØØØ4272869	ØØØ31ØØØ31
ØØØ44Ø374Ø	Ø143822382	ØØØ41448Ø6	ØØØ32ØØØ32

TABLE VIII - $\gamma = 1.5$ $\qquad \frac{\Delta t}{\Delta x} = .25$

ρ	u/8	p	ν
ØØØ43Ø6247	Ø147Ø44333	ØØØ4Ø11372	ØØØ33ØØØ33
ØØØ42Ø5248	Ø15Ø433162	ØØØ3874919	ØØØ34ØØØ34
ØØØ41Ø2431	Ø153939577	ØØØ3737847	ØØØ35ØØØ35
ØØØ3999555	Ø1575Ø87Ø1	ØØØ36Ø253Ø	ØØØ36ØØØ36
ØØØ3898419	Ø161Ø8Ø611	ØØØ3471265	ØØØ37ØØØ37
ØØØ38ØØ835	Ø1645912Ø3	ØØØ3346216	ØØØ38ØØØ38
ØØØ37Ø859Ø	Ø167973545	ØØØ322937Ø	ØØØ39ØØØ39
ØØØ362342Ø	Ø17116Ø248	ØØØ3122482	ØØØ3AØØØ3A
ØØØ3546974	Ø174Ø86953	ØØØ3Ø26994	ØØØ3BØØØ3B
ØØØ348Ø794	Ø17669Ø7Ø87	ØØØ2943962	ØØØ3CØØØ3C
ØØØ34253Ø4	Ø17894737Ø	ØØØ2873963	ØØØ3DØØØ3D
ØØØ3384857	Ø18Ø813Ø99	ØØØ28170Ø2Ø	ØØØ3EØØØ3E
ØØØ33578Ø5	Ø182291846	ØØØ2772572	ØØØ3FØØØ3F
ØØØ3346618	Ø1834Ø4Ø6Ø	ØØØ27395Ø6	ØØØ4ØØØØ4Ø
ØØØ3352972	Ø18418979Ø	ØØØ271626Ø	ØØØ41ØØØ41
ØØØ3378734	Ø1847Ø1446	ØØØ27ØØ968	ØØØ42ØØØ42
ØØØ3425735	Ø184993Ø77	ØØØ269159Ø	ØØØ43ØØØ43
ØØØ349519Ø	Ø1851Ø5149	ØØØ2685864	ØØØ44ØØØ44
ØØØ358656Ø	Ø185Ø3943Ø	ØØØ268Ø821	ØØØ45ØØØ45
ØØØ3695391	Ø1847Ø9371	ØØØ2671317	ØØØ46ØØØ46
ØØØ38Ø9114	Ø183835998	ØØØ26466Ø9	ØØØ47ØØØ47
ØØØ3898991	Ø181737768	ØØØ2583639	ØØØ48ØØØ48
ØØØ39Ø7Ø3Ø	Ø176945998	ØØØ24372Ø55	ØØØ49ØØØ49
ØØØ37352Ø2	Ø16662 4Ø61	ØØØ2134915	ØØØ4AØØØ4A
ØØØ3273272	Ø146121181	ØØØ1613665	ØØØ4BØØØ4B
ØØØ2525437	Ø11Ø3819Ø8	ØØØØ932564	ØØØ4CØØØ4C
ØØØ1747644	ØØ61828936	ØØØØ35Ø267	ØØØ4DØØØ4D
ØØØ1253448	ØØ21147218	ØØØØØ756Ø5	ØØØ4EØØØ4E
ØØØ1Ø619Ø6	ØØØ4342123	ØØØØØ1Ø365	ØØØ4FØØØ4F
ØØØ1Ø12Ø28	ØØØØ663114	ØØØØØØ1167	ØØØ5ØØØØ5Ø
ØØØ1ØØ2Ø18	ØØØØØ89157	ØØØØØØØ124	ØØØ51ØØØ51
ØØØ1ØØØ3Ø7	ØØØØØ11252	ØØØØØØØØ12	ØØØ52ØØØ52
ØØØ1ØØØØ43	ØØØØØØ1358	ØØØØØØØØØØ	ØØØ53ØØØ53
ØØØ1ØØØØØ5	ØØØØØØØ156	FØØØØØØØØØ	ØØØ54ØØØ54
ØØØ1ØØØØØØ	ØØØØØØØØ14	FØØØØØØØØØ	ØØØ55ØØØ55
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ56ØØØ56
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ57ØØØ57
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ58ØØØ58
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ59ØØØ59
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ5AØØØ5A
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ5BØØØ5B
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ5CØØØ5C
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ5DØØØ5D
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ5EØØØ5E
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ5FØØØ5F
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ6ØØØØ6Ø
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ61ØØØ61
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ62ØØØ62
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ63ØØØ63
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ64ØØØ64

TABLE VIII - $\gamma = 1.5$ $\frac{\Delta t}{\Delta x} = .25$ (Continued)

ρ	u/8	p	ν
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ1ØØØØ1
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ2ØØØØ2
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ3ØØØØ3
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ4ØØØØ4
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ5ØØØØ5
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ6ØØØØ6
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ7ØØØØ7
ØØØ4999999	Ø249999997	ØØ1ØØØØØØØ	ØØØØ8ØØØØ8
ØØØ4999999	Ø249999999	ØØ1ØØØØØØØ	ØØØØ9ØØØØ9
ØØØ4999999	Ø25ØØØØØ1Ø	ØØØ9999999	ØØØØAØØØØA
ØØØ4999998	Ø25ØØØØØ63	ØØØ9999994	ØØØØBØØØØB
ØØØ4999994	Ø25ØØØØ28Ø	ØØØ9999977	ØØØØCØØØØC
ØØØ4999978	Ø25ØØØ1Ø64	ØØØ9999914	ØØØØDØØØØD
ØØØ4999927	Ø25ØØØ3597	ØØØ9999712	ØØØØEØØØØE
ØØØ499978Ø	Ø25ØØ1Ø946	ØØØ9999124	ØØØØFØØØØF
ØØØ4999396	Ø25ØØ3Ø185	ØØØ9997585	ØØØ1ØØØØ1Ø
ØØØ4998482	Ø25ØØ75868	ØØØ9993932	ØØØ11ØØØ11
ØØØ49965Ø6	Ø25Ø174632	ØØØ9986Ø37	ØØØ12ØØØ12
ØØØ49926Ø1	Ø25Ø369757	ØØØ997Ø454	ØØØ13ØØØ13
ØØØ4985525	Ø25Ø7233Ø7	ØØØ9942269	ØØØ14ØØØ14
ØØØ4973717	Ø251313185	ØØØ9895382	ØØØ15ØØØ15
ØØØ4955497	Ø252223496	ØØØ9823365	ØØØ16ØØØ16
ØØØ4929374	Ø2535293Ø9	ØØØ972Ø769	ØØØ17ØØØ17
ØØØ48944ØØ	Ø255279613	ØØØ9584547	ØØØ18ØØØ18
ØØØ485Ø461	Ø257483122	ØØØ9415144	ØØØ19ØØØ19
ØØØ4798426	Ø26Ø1ØØ471	ØØØ9216911	ØØØ1AØØØ1A
ØØØ474Ø133	Ø263Ø43837	ØØØ89978Ø2	ØØØ1BØØØ1B
ØØØ4678215	Ø266183326	ØØØ8768477	ØØØ1CØØØ1C
ØØØ4615789	Ø26935 9Ø18	ØØØ854 1Ø38	ØØØ1DØØØ1D
ØØØ4556Ø14	Ø272398224	ØØØ8327573	ØØØ1EØØØ1E
ØØØ45Ø15Ø7	Ø275137689	ØØØ8138632	ØØØ1FØØØ1F
ØØØ4453626	Ø277448822	ØØØ7981774	ØØØ2ØØØØ2Ø
ØØØ4411678	Ø27926Ø357	ØØØ786Ø467	ØØØ21ØØØ21
ØØØ4372245	Ø28Ø569345	ØØØ7773773	ØØØ22ØØØ22
ØØØ4328883	Ø281433279	ØØØ77171Ø1	ØØØ23ØØØ23
ØØØ4272568	Ø281945537	ØØØ7683856	ØØØ24ØØØ24
ØØØ41932Ø1	Ø2822Ø7565	ØØØ7667157	ØØØ25ØØØ25
ØØØ4Ø82348	Ø282312942	ØØØ766Ø756	ØØØ26ØØØ26
ØØØ3936881	Ø282344899	ØØØ7659168	ØØØ27ØØØ27
ØØØ3762335	Ø282371915	ØØØ7657992	ØØØ28ØØØ28
ØØØ35739ØØ	Ø28242 7Ø81	ØØØ7655241	ØØØ29ØØØ29
ØØØ3393169	Ø282482978	ØØØ7652372	ØØØ2AØØØ2A
ØØØ3241468	Ø28249 8Ø72	ØØØ7654535	ØØØ2BØØØ2B
ØØØ313ØØ88	Ø2823155Ø5	ØØØ7656295	ØØØ2CØØØ2C
ØØØ3Ø64247	Ø282478443	ØØØ7679252	ØØØ2DØØØ2D
ØØØ3ØØ4559	Ø28Ø53934Ø	ØØØ757356Ø	ØØØ2EØØØ2E
ØØØ3Ø64186	Ø28822Ø456	ØØØ7971564	ØØØ2FØØØ2F
ØØØ1741213	Ø137682163	ØØØ2197261	ØØØ3ØØØØ3Ø
ØØØ1ØØ7659	ØØØØ74722Ø	ØØØØØØ2379	ØØØ31ØØØ31
ØØØ1ØØØØØ4	ØØØØØØØ194	FØØØØØØØØØ	ØØØ32ØØØ32

TABLE IX - $\gamma = 2 \quad \frac{\Delta t}{\Delta x} = .25$

ρ	u/8	p	ν
0004999999	0249999996	0010000000	0000100001
0004999999	0249999996	0010000000	0000200002
0004999999	0249999996	0010000000	0000300003
0004999999	0249999996	0010000000	0000400004
0004999999	0249999996	0010000000	0000500005
0004999999	0249999996	0010000000	0000600006
0004999999	0249999996	0010000000	0000700007
0004999999	0249999996	0010000000	0000800008
0004999999	0249999996	0010000000	0000900009
0004999999	0249999996	0010000000	0000A0000A
0004999999	0249999996	0010000000	0000B0000B
0004999999	0249999997	0010000000	0000C0000C
0004999999	0249999997	0010000000	0000D0000D
0004999999	0249999997	0010000000	0000E0000E
0004999999	0249999996	0010000000	0000F0000F
0004999999	0249999997	0010000000	0001000010
0004999999	0249999997	0010000000	0001100011
0004999999	0249999997	0010000000	0001200012
0004999999	0249999996	0010000000	0001300013
0004999999	0249999997	0010000000	0001400014
0004999999	0249999996	0010000000	0001500015
0004999999	0249999997	0010000000	0001600016
0004999999	0249999995	0009999999	0001700017
0004999999	0249999998	0010000000	0001800018
0004999999	0249999997	0009999999	0001900019
0004999999	0250000000	0009999999	0001A0001A
0004999999	0250000002	0009999999	0001B0001B
0004999999	0250000018	0009999998	0001C0001C
0004999998	0250000050	0009999996	0001D0001D
0004999997	0250000143	0009999988	0001E0001E
0004999992	0250000365	0009999970	0001F0001F
0004999982	0250000891	0009999928	0002000020
0004999958	0250002068	0009999834	0002100021
0004999908	0250004596	0009999633	0002200022
0004999804	0250009742	0009999220	0002300023
0004999603	0250019802	0009998416	0002400024
0004999228	0250038570	0009996914	0002500025
0004998558	0250072083	0009994235	0002600026
0004997412	0250129346	0009989656	0002700027
0004995537	0250223095	0009982165	0002800028
0004992595	0250370174	0009970420	0002900029
0004988167	0250591600	0009952760	0002A0002A
0004981764	0250911746	0009927267	0002B0002B
0004972863	0251356947	0009891902	0002C0002C
0004960946	0251953263	0009844684	0002D0002D
0004945553	0252723999	0009783913	0002E0002E
0004926338	0253687058	0009708380	0002F0002F
0004903110	0254852820	0009617547	0003000030
0004875866	0256222551	0009511649	0003100031
0004844803	0257787762	0009391729	0003200032

TABLE X - $\gamma = 2$ $\quad \frac{\Delta t}{\Delta x} = .25$

ρ	u/8	p	ν
ØØØ4810314	Ø259530190	ØØØ9259583	ØØØ33ØØØ33
ØØØ4772972	Ø261422580	ØØØ9117667	ØØØ34ØØØ34
ØØØ47335Ø3	Ø263429792	ØØØ8968945	ØØØ35ØØØ35
ØØØ4692748	Ø26551Ø342	ØØØ8816732	ØØØ36ØØØ36
ØØØ465163Ø	Ø267618123	ØØØ8664523	ØØØ37ØØØ37
ØØØ46111Ø3	Ø2697Ø44Ø9	ØØØ8515824	ØØØ38ØØØ38
ØØØ4572114	Ø27172Ø151	ØØØ8373984	ØØØ39ØØØ39
ØØØ4535546	Ø273618584	ØØØ8242Ø28	ØØØ3AØØØ3A
ØØØ45Ø2164	Ø275358149	ØØØ8122489	ØØØ3BØØØ3B
ØØØ4472566	Ø2769Ø5522	ØØØ8Ø17254	ØØØ3CØØØ3C
ØØØ4447127	Ø278238225	ØØØ7927441	ØØØ3DØØØ3D
ØØØ4425973	Ø27934659Ø	ØØØ7853323	ØØØ3EØØØ3E
ØØØ44Ø897Ø	Ø28Ø234256	ØØØ7794344	ØØØ3FØØØ3F
ØØØ4395731	Ø28Ø917Ø19	ØØØ7749211	ØØØ4ØØØØ4Ø
ØØØ4385657	Ø28142Ø129	ØØØ771609Ø	ØØØ41ØØØ41
ØØØ4377964	Ø281774514	ØØØ7692839	ØØØ42ØØØ42
ØØØ43717Ø3	Ø282Ø12618	ØØØ7677262	ØØØ43ØØØ43
ØØØ4365751	Ø282164977	ØØØ7667326	ØØØ44ØØØ44
ØØØ4358748	Ø282257657	ØØØ7661312	ØØØ45ØØØ45
ØØØ43490Ø8	Ø28231101Ø2	ØØØ7657876	ØØØ46ØØØ46
ØØØ4334432	Ø282339725	ØØØ76560Ø59	ØØØ47ØØØ47
ØØØ4312454	Ø282353673	ØØØ7655211	ØØØ48ØØØ48
ØØØ428Ø111	Ø282359599	ØØØ7654892	ØØØ49ØØØ49
ØØØ4234284	Ø282362486	ØØØ7654776	ØØØ4AØØØ4A
ØØØ417217Ø	Ø282366262	ØØØ765461Ø	ØØØ4BØØØ4B
ØØØ4Ø91933	Ø282373164	ØØØ7654251	ØØØ4CØØØ4C
ØØØ39934Ø4	Ø282382916	ØØØ7653725	ØØØ4DØØØ4D
ØØØ3878614	Ø282393174	ØØØ7653185	ØØØ4EØØØ4E
ØØØ3751937	Ø2824Ø2Ø78	ØØØ7652749	ØØØ4FØØØ4F
ØØØ3619728	Ø28241Ø796	ØØØ7652348	ØØØ5ØØØØ5Ø
ØØØ3489498	Ø282422Ø71	ØØØ7651814	ØØØ51ØØØ51
ØØØ3368764	Ø282434529	ØØØ7651223	ØØØ52ØØØ52
ØØØ326385Ø	Ø282442141	ØØØ7651192	ØØØ53ØØØ53
ØØØ3178571	Ø282418127	ØØØ765Ø968	ØØØ54ØØØ54
ØØØ3115717	Ø2825Ø6165	ØØØ7656845	ØØØ55ØØØ55
ØØØ3Ø66646	Ø282Ø77Ø19	ØØØ762848Ø	ØØØ56ØØØ56
ØØØ3Ø61654	Ø28416Ø967	ØØØ7723382	ØØØ57ØØØ57
ØØØ2981Ø44	Ø277856Ø67	ØØØ7342687	ØØØ58ØØØ58
ØØØ3175166	Ø294981876	ØØØ8228542	ØØØ59ØØØ59
ØØØ2734976	Ø251Ø6Ø594	ØØØ58910Ø79	ØØØ5AØØØ5A
ØØØ3497571	Ø3137536ØØ	ØØØ927Ø922	ØØØ5BØØØ5B
ØØØ1993561	Ø151592162	ØØØ2139653	ØØØ5CØØØ5C
ØØØ38922Ø1	Ø3514890Ø25	ØØ1216Ø273	ØØØ5DØØØ5D
ØØØ1853143	Ø148Ø33Ø33	ØØØ2426583	ØØØ5EØØØ5E
ØØØ1Ø35241	ØØØ4553368	ØØØØØ2Ø656	ØØØ5FØØØ5F
ØØØ1ØØØØ16	ØØØØØØØ8Ø2	ØØØØØØØØØØ	ØØØ6ØØØØ6Ø
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ61ØØØ61
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ62ØØØ62
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ63ØØØ63
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ64ØØØ64

TABLE X - $\gamma = 2$ $\frac{\Delta t}{\Delta x} = .25$ (Continued)

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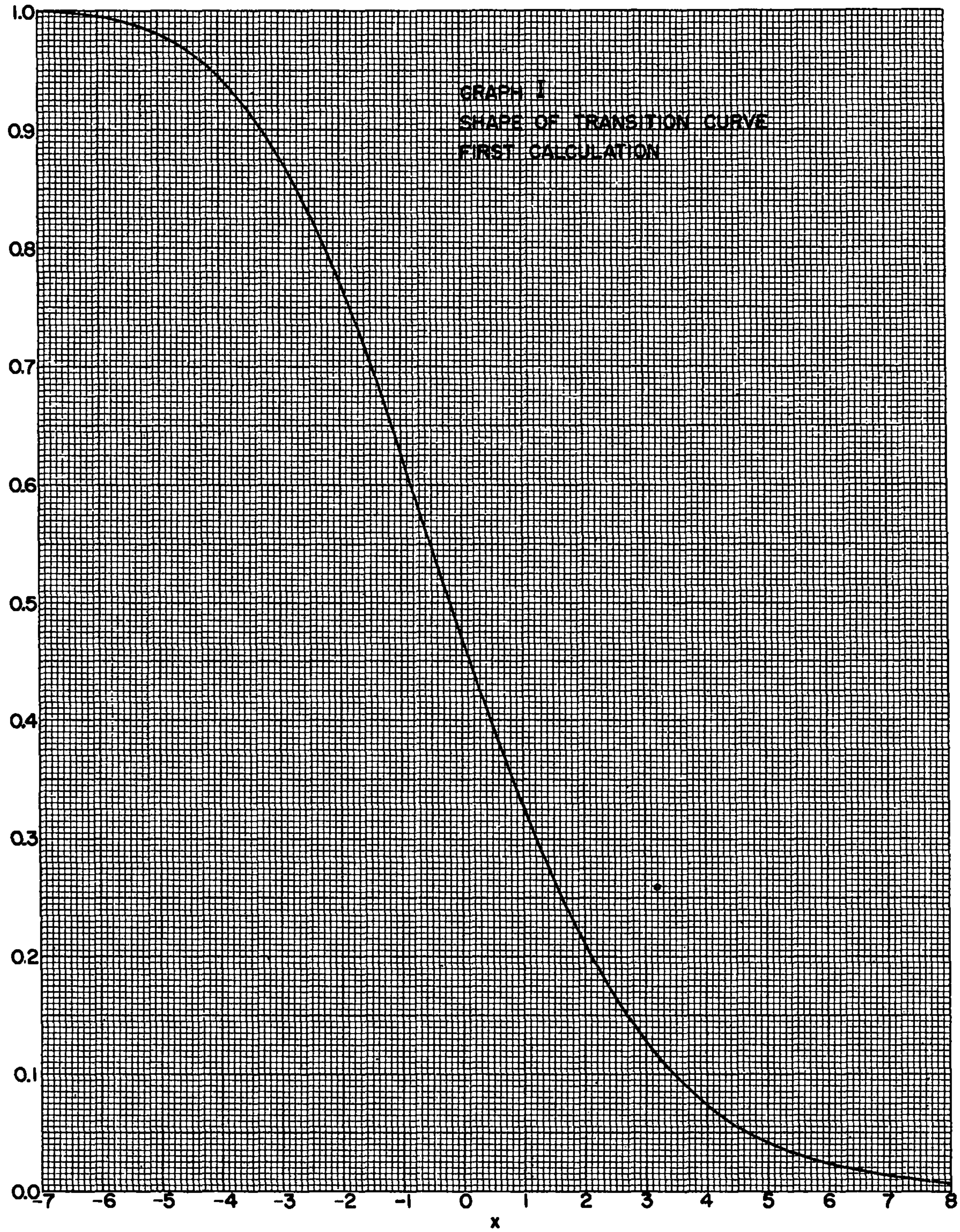
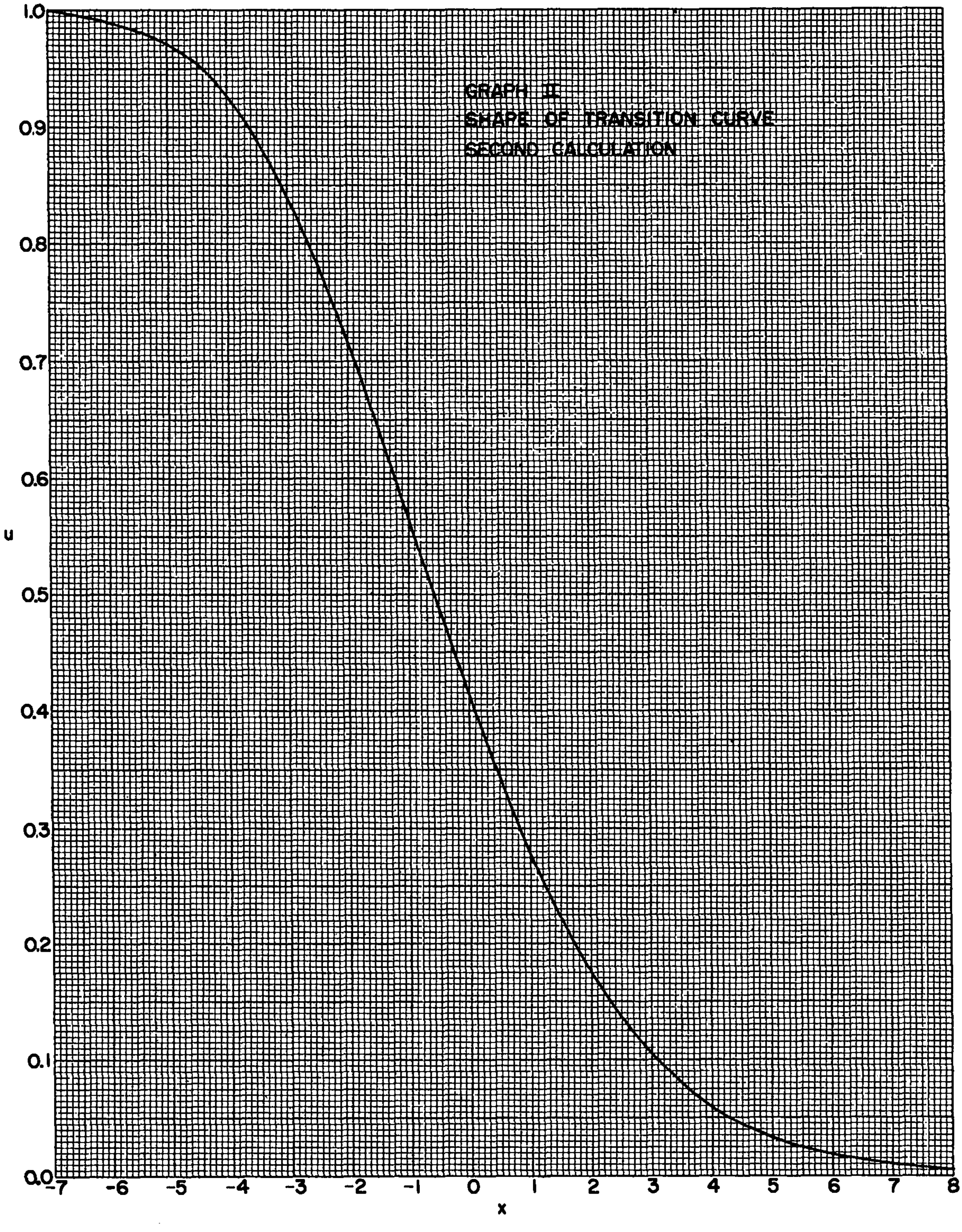
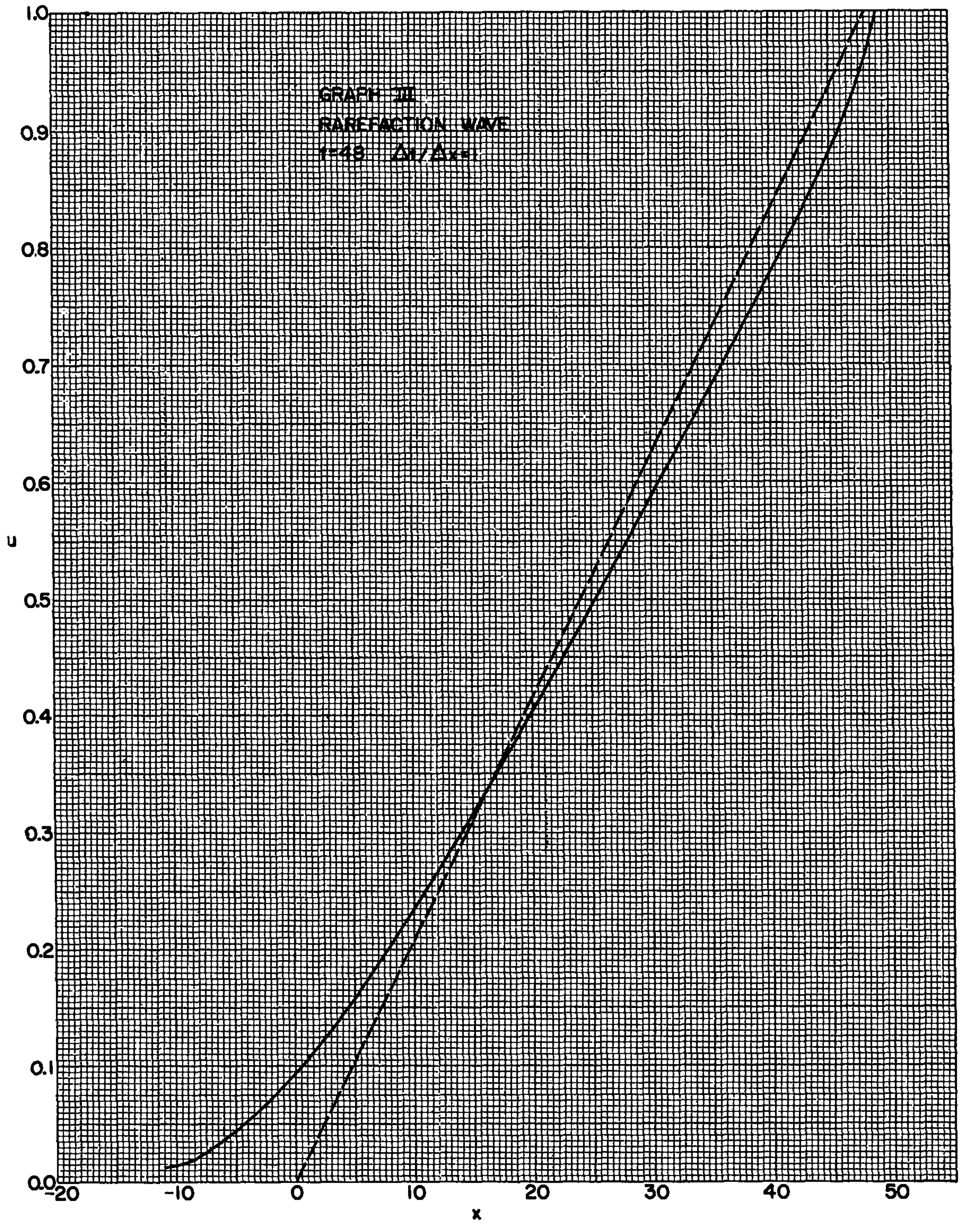
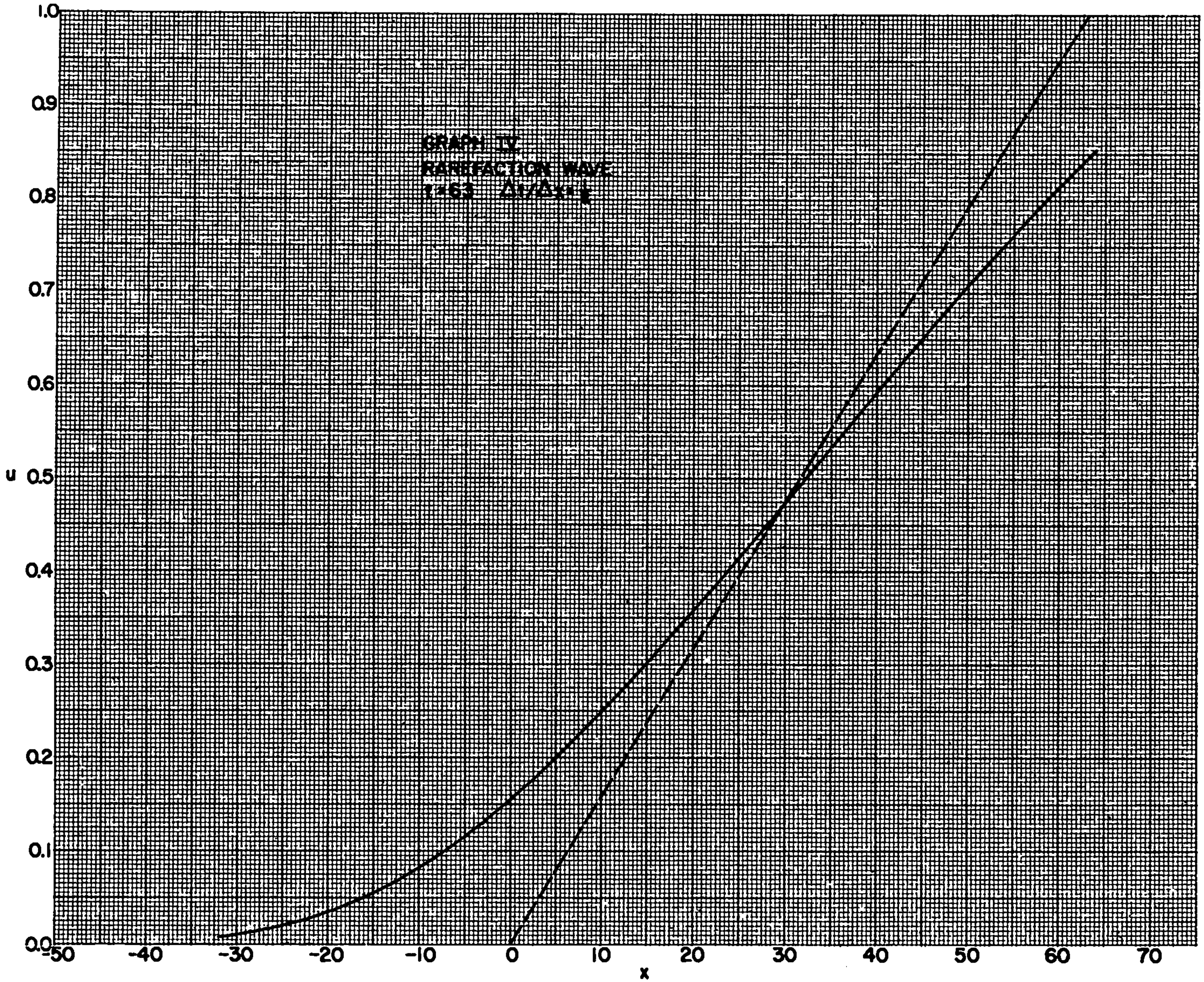

GRAPH I
SHAPE OF TRANSITION CURVE
FIRST CALCULATION
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
x


GRAPH II
SHAPE OF TRANSITION CURVE
SECOND CALCULATION
u
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
x


GRAPH III
RAREFACTION WAVE
t=48 Δt/Δx=1
u
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-20
-10
0
10
20
30
40
50
x



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