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MULTI-VELOCITY SERBER-WILSON NEUTRON DIFFUSION CALCULATIONS

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MULTI-VELOCITY SERBER-WILSON

NEUTRON DIFFUSION CALCULATIONS

Certain types of neutron diffusion calculations were considerably simplified when the Serber-Wilson Method was introduced about eight years ago. This method, semi-empirical in nature and named after its co-discoverers,¹ was first formulated for the one-velocity isotropic theory and applied to spherical geometries. Within these limits it has in general proved to be a fairly accurate method. If restricted to the source-free case it has, in addition, turned out to be quite manageable both analytically and numerically.

The Serber-Wilson Method was, however, not extensively used here until about three years ago. At that time the computation techniques involved were systematized and somewhat improved.² A year later a set of special function tables were completed resulting in a considerable saving of computing time.³ The work involved was further shortened when the CPC calculator was brought into the picture about a year ago.

Let us consider neutron diffusion problems under the above restrictions for the moment. The corresponding mathematical description is then furnished by the integro-differential equation below:

(1) $\left[\mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} + \sigma\right] \mathcal{N}(r,\mu) = \frac{1}{2}\sigma c \mathcal{N}(r),$

LA-234 by R. Serber, BM-441 by A. H. Wilson.
 LA-756 by B. Carlson.
 LA-1364, 1365, 1366 by B. Carlson, M. Goldstein, and D. Sweeney.



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where $\mathcal{N}(r,\mu)$ denotes the neutron flux as a function of radius and direction cosine and $\mathcal{N}(r)$ the integral of $\mathcal{N}(r,\mu)$ over μ from -1 to +1. The quantities \neg and c in (1) represent known step-functions of r characterizing the assembly of media under consideration, the general medium being, in this case, a concentric spherical shell. Specifically, ς is the inverse mean free path for neutrons, and c the number of neutrons emerging per collision.

The following steps are involved in the Serber-Wilson Method and may, in fact, be regarded as a definition of the method:

(A) Prescribing $\mathcal{N}(\mathbf{r})$ for the general medium with an analytical expression involving two arbitrary constants.

(B) Defining and deriving two functionals of $\mathscr{N}(\mathbf{r})$, having the dimension of $\mathscr{N}(\mathbf{r})$, one depending perhaps on the geometry and the other being the net neutron flux.

(C) Applying a sufficient number of physical conditions, primarily continuity conditions, on the two functionals to determine the arbitrary constants.

An approximate or asymptotic expression for N(r) may be obtained either by applying the Spherical Harmonic transformation to (1) or by studying the integral equation equivalent to (1).¹ In either case we obtain:

(2) $\mathcal{N}(\mathbf{r}) \sim A\left[\frac{\mathrm{sinkr}}{\mathrm{kr}} + \overline{A} \frac{\mathrm{coskr}}{|\mathbf{k}|\mathbf{r}}\right] \equiv A^* \frac{\mathrm{sink}(\mathbf{r}+\mathbf{r}_0)}{\mathrm{kr}}$,

1. LA-247 by K. M. Case, LA-571 by B. Carlson. See also Appendix, p. 27.





with $A = A^* \operatorname{coskr}_{o}$, $\overline{A} = (|k|/k) \operatorname{tankr}_{o}$, and k from the transcendental equation $k/\sigma^- = c$ art (k/σ^-) . The quantity k, taken to be positive, may be either real $(c \ge 1)$ or pure imaginary $(1 > c \ge 0)$.

For the spherical geometry which is being considered the inward radial flux $\tilde{\mathcal{N}}(\mathbf{r}) = 2 \tilde{\mathcal{N}}(\mathbf{r}, -1)$ was chosen as one of the functionals, the net flux $\tilde{\mathcal{N}}(\mathbf{r}) = \int_{-1}^{1} \mu \tilde{\mathcal{N}}(\mathbf{r}, \mu) d\mu$ being the other. Differential equations for $\tilde{\mathcal{N}}(\mathbf{r})$ and $\tilde{\mathcal{N}}(\mathbf{r})$ are readily obtained from (1) and the solutions are immediate. For if we let $\mu = -1$ in (1) we have on the one hand:

(3)
$$\left[-\frac{d}{dr}+\sigma\right] \mathcal{N}(r) = \sigma c \mathcal{N}(r),$$

and hence:

(4)
$$\mathcal{N}(\mathbf{r}) = -\mathbf{e}^{\mathbf{\sigma}\mathbf{r}} \int^{\mathbf{r}} \mathbf{\sigma} \mathbf{c} \, \mathcal{N}(\mathbf{r}') \mathbf{e}^{-\mathbf{\sigma}\mathbf{r}'} \mathbf{d}\mathbf{r}'$$

On the other hand (1) may be written in the form:

(5)
$$\left[\mu\frac{\partial}{\partial r} + (\sigma + \frac{2\mu}{r}) + \frac{1}{r}\frac{\partial}{\partial \mu}(1-\mu^2)\right]\mathcal{N}(r,\mu) = \frac{1}{2}\sigma c \mathcal{N}(r),$$

which, integrated over μ from -1 to +1, gives

$$\left[\frac{\mathrm{d}}{\mathrm{d}\mathbf{r}}+\frac{2}{\mathrm{r}}\right]\widetilde{\mathcal{N}}(\mathbf{r})=\sigma(\mathbf{c}-1)\widetilde{\mathcal{N}}(\mathbf{r}),$$

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and hence:

(6)
$$\overline{N}(r) = \frac{1}{r^2} \int^r \sigma (c-1) r'^2 N(r') dr'$$

Substituting (2) in (4) we have:

(7)
$$\int (\mathbf{r}) = A \left[Q(|\mathbf{k}|\mathbf{r}, \phi) + \overline{A} R(|\mathbf{k}|\mathbf{r}, \phi) \right] = \frac{A\sigma c}{2k} \left\{ i \left[E_1((\sigma + i\mathbf{k})\mathbf{r}) - E_1((\sigma - i\mathbf{k})\mathbf{r}) \right] + \frac{k\overline{A}}{|\mathbf{k}|} \left[E_1((\sigma + i\mathbf{k})\mathbf{r}) + E_1((\sigma - i\mathbf{k})\mathbf{r}) \right] \right\} e^{\sigma \mathbf{r}},$$

where $\phi = \operatorname{art}(k/\sigma)$, $c \ge 1$, and $\phi = \operatorname{arth}[k]/\sigma$, $l > c_{2}0$. The functions Q and R are tabulated in LA-1364, LA-1365, and LA-1366, as are the functions S and T in the formula for $\mathcal{N}(r)$. The latter is obtained by substituting (2) in (6):

(8)
$$\overline{\mathcal{N}}(\mathbf{r}) = A \left[S(|\mathbf{k}|\mathbf{r}) + \overline{A} T(|\mathbf{k}|\mathbf{r}) \right] =$$

= $\frac{A\sigma(\mathbf{c}-\mathbf{l})}{\mathbf{k}} \left[\frac{\sin k\mathbf{r} - k\mathbf{r} \cos k\mathbf{r}}{(k\mathbf{r})^2} + \frac{\mathbf{k}}{|\mathbf{k}|} \overline{A} \frac{\cos k\mathbf{r} + k\mathbf{r} \sin k\mathbf{r}}{(k\mathbf{r})^2} \right]$

The Serber-Wilson Method may be extended to the anisotropic case and to geometries such as plane and cylindrical.¹ For the anisotropic case the transcendental equation for k/r will be different. For other geometries a substitute for the functional $\mathcal{N}(r)$ may have to be found. And again, if a source function is present on the righthand side of (1) it may be difficult to find an asymptotic expression for $\mathcal{N}(r)$. Generalizations in the above directions have on the whole

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^{1.} Transport Theory of Neutrons (LT-18) by B. Davison

proved feasible, whereas, in the direction of more velocity groups serious difficulties have been encountered.

Let us then turn to the multi-velocity isotropic theory with G velocity groups. Instead of (1) we have the following equations where g is the group index, g = 1, 2, ... G:

(9)
$$\left[\mu \frac{\partial}{\partial \mathbf{r}} + \frac{1-\mu^2}{\mathbf{r}} \frac{\partial}{\partial \mu} + \sigma_{g}\right] \mathcal{N}_{g}(\mathbf{r},\mu) = \frac{1}{2} \sum_{h=1}^{G} \sigma_{h} c_{gh} \mathcal{N}_{h}(\mathbf{r}),$$

In the above expression \Im_g are the separate inverse mean free paths and c_{gh} the transfer coefficients. Denoting the group velocities by v_g , c_{gh} represents the number of neutrons of velocity v_g emerging per collision of neutron of velocity v_h . \Im_g as well as c_{gh} are calculated from measured cross-sections.

Applying the same principals to (9) as to the one-velocity case we find the following asymptotic form for the flux distributions:

(10)
$$\mathcal{N}_{g}(\mathbf{r}) \sim \sum_{i=1}^{G} \alpha_{g}^{i} \mathcal{N}_{i}(\mathbf{r}) = \sum_{i=1}^{G} \alpha_{g}^{i} A_{i} \left[\frac{\operatorname{sink}_{i}\mathbf{r}}{k_{i}\mathbf{r}} + \overline{A}_{i} \frac{\operatorname{cosk}_{i}\mathbf{r}}{|k_{i}|\mathbf{r}|} \right],$$

where k_i are the eigenvalues and $\left< \alpha_g^1 \right>$ the eigenvectors of the matrix equation: k_i/σ_1

$$(11) \begin{vmatrix} c_{11} - \frac{art(k_{1}/\sigma_{1})}{art(k_{1}/\sigma_{1})} & c_{12} \cdots c_{1G} \\ c_{21} & c_{22} - \frac{k_{1}/\sigma_{2}}{art(k_{1}/\sigma_{2})} \cdots c_{2G} \\ c_{G1} & c_{G2} \cdots c_{GG} - \frac{k_{1}/\sigma_{G}}{art(k_{1}/\sigma_{G})} \begin{vmatrix} \alpha_{1}^{i} \\ \alpha_{1}^{i} \\ \sigma_{2}^{i} \\ \sigma_{1}^{i} \\ \sigma_{2}^{i} \\ \sigma_{1}^{i} \\ \sigma_{1}^{i}$$

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After solving (11) for k_i and $\left\langle \begin{array}{c} \alpha \\ g \end{array}\right\rangle$ taking $\alpha \\ i \\ i \\ equal to unity,$ (10) is determined except for the arbitrary constants A_i and \overline{A}_i . This is remedied by introducing 2G functionals of (10), $\mathcal{N}_g(r)$ and $\mathcal{N}_g(r)$, and requiring these to be continuous at the boundaries. Applying the methods of pp 5-6 to equation (9), $\mathcal{N}_g(r)$ and $\mathcal{N}_g(r)$ are readily obtained. We have:

(12)
$$\begin{cases} \int_{g}^{g} (\mathbf{r}) = 2 \int_{g}^{g} (\mathbf{r}, -1) = -e^{-g^{2}} \int_{h=1}^{G} \sum_{h=1}^{G} \sigma_{h} c_{gh} \int_{g}^{g} (\mathbf{r}') e^{-g^{2}} d\mathbf{r}', \\ \int_{g}^{g} (\mathbf{r}) = \int_{-1}^{1} \mu \int_{g}^{g} (\mathbf{r}, \mu) d\mu = \frac{1}{r^{2}} \int_{r}^{r} r'^{2} \sum_{h=1}^{G} \sigma_{h} (c_{gh} - \delta_{gh}) \int_{g}^{g} (\mathbf{r}') d\mathbf{r}', \end{cases}$$

where \mathcal{S}_{gh} equal to unity if h=g and zero if h≠g.

The functionals (12) can be considerably simplified if we substitute (10) on the right-hand side and then make use of the following consequence of (11):

(13)
$$\sum_{h=1}^{G} \sigma_{h} c_{gh} \alpha_{h}^{i} = \sigma_{g} \alpha_{g}^{i} \frac{k_{i}/\sigma_{g}}{\operatorname{art}(k_{i}/\sigma_{g})} = \sigma_{g} \alpha_{g}^{i} c_{g}^{i}$$

where the last equality serves to define c_g^i . Performing these substitutions we obtain:

(14)
$$\begin{cases}
\int_{g}^{\pi} (\mathbf{r}) = -e^{-g} \int_{g}^{r} \sigma_{g} \sum_{i=1}^{G} c_{g}^{i} \alpha_{g}^{i} \int_{i}^{r} (\mathbf{r}') e^{-\sigma_{g} r'} \\
\int_{g}^{\pi} (\mathbf{r}) = \frac{1}{r^{2}} \int_{r}^{r} r'^{2} \sigma_{g} \sum_{i=1}^{G} (c_{g}^{i}-1) \alpha_{g}^{i} \int_{i}^{r} (\mathbf{r}') d\mathbf{r}',
\end{cases}$$

where the functions $\mathcal{N}_{i}(\mathbf{r})$ are those defined by (10).



If $c_{gh} = 0$ for g > h we have as a rule G distinct eigenvalues and consequently as many arbitrary constants as boundary conditions. We may, therefore, conclude that in this case the above generalization of the Serber-Wilson Method is valid. If, on the other hand, we are dealing with fissionable materials, for which $c_{gh} \neq 0$ for all g > h, we can, as a rule, not count on (11) to give as many as G eigenvalues. The result is that we are left with more boundary conditions than arbitrary constants.¹ A number of schemes have been proposed, mone of them entirely satisfactory, which in one way or another circumvent the above difficulty. A new method which may in the end prove satisfactory will be introduced below. In the very few applications made to date it has turned out to be both accurate and practical.

In this new method we replace the quantities c_{gh} in (11) by $\overline{c_{gh}}$ defined below, thus transforming (c_{gh}) into a right-triangular matrix:

(15)
$$\left(\begin{array}{c}
\overline{c_{gh}} = 0, \quad g \neq h \\
\overline{c_{gh}} = c_{gh} - c_{hg} \frac{N_g}{N_h}, \quad g \neq h, \\
\overline{c_{gg}} = \sum_{h=g}^{G} c_{hg} + \sum_{h=1}^{g-1} c_{gh} \frac{N_h}{N_g}.
\right)$$

This obviously eliminates the difficulties referred to but requires some explanations. Before turning to these, however, and defining N, g'

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^{1.} See <u>Transport Theory of Neutrons</u> (LT-18) by B. Davison, pp 180-185.

the following consequences of (5) may be noted. If $c_{gh} = 0$ for g > hthen $\overline{c_{gh}} = c_{gh}$. Equation (11) as modified by (15) will in general have G eigenvalues k_i , i = 1, 2, ..., G, $(k_i/\sigma_i) = \overline{c_{ii}} \cdot \operatorname{art}(k_i/\sigma_i)$. The elements of the eigenvectors $\langle \alpha_g^i \rangle$ can be obtained successively (starting with $\alpha_i^i = 1$) since the determinant is right-triangular, i.e., has zeros below the diagonal, and $\alpha_g^i = 0$, g > i.

For the purpose of illustrating the above formulae and notation we consider for the moment the three-velocity case: Consequently, $k_1/\sigma_1 = (c_{11}+c_{21}+c_{31})$ art k_1/σ_1 , $k_2/\sigma_2 = (c_{22}+c_{32}+c_{21}, \frac{N_1}{N_2})$ art k_2/σ_2 , and $k_3/\sigma_3 = (c_{33}+c_{31}, \frac{N_1}{N_3} + c_{32}, \frac{N_2}{N_3})$ art k_3/σ_3 . Also $(\alpha_1^1, \alpha_2^1, \alpha_3^1) =$ $= (1,0,0), (\alpha_1^2, \alpha_2^2, \alpha_3^2) = (\frac{\sigma_2 c_{12}}{\sigma_1 (c_1^2 - c_{11})}, 1,0),$ and $(\alpha_1^3, \alpha_2^3, \alpha_3^3) =$ $= (\frac{\sigma_3 \left[\overline{c_{12} c_{23}} + \overline{c_{13}} (c_2^3 - \overline{c_{22}}) \right]}{\sigma_1 (c_1^3 - \overline{c_{11}}) (c_2^3 - \overline{c_{22}})}, \frac{\sigma_3 \overline{c_{23}}}{\sigma_2 (c_2^3 - \overline{c_{22}})}, 1)$. The flux distributions

for a central core (for which $\overline{A}_i = 0$) are then given by:

$$(16) \begin{cases} \sqrt{1}(r) \sim A_{1} \frac{\sin k_{1}r}{k_{1}r} + \alpha_{1}^{2} A_{2} \frac{\sin k_{2}r}{k_{2}r} + \alpha_{1}^{3} A_{3} \frac{\sin k_{3}r}{k_{3}r} \\ \sqrt{2}(r) \sim A_{2} \frac{\sin k_{2}r}{k_{2}r} + \alpha_{2}^{3} A_{3} \frac{\sin k_{3}r}{k_{3}r} \\ \sqrt{3}(r) \sim A_{3} \frac{\sin k_{3}r}{k_{3}r} \end{cases}$$

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Due to the triangular character of the determinants and expressions above, the labor involved in finding eigenvalues and applying boundary conditions is considerably reduced. As an example, connect the six functionals of (16) with the corresponding expressions for an infinite shell. We find then that the resulting simultaneous equations can be grouped, in this case into three sets of two each, thus reducing the computational work.

Going back to(11), i.e., to the definition of $\overline{c_{gh}}$, it is evident that we are tampering with the interchange of neutrons. Studying groups #1 and #2, for instance, we find that for each collision in #1, c_{21} neutrons are given to #2. Hence, if these are given to #1 rather than #2, as is done in (15), then #2 should receive some compensation. Letting N_2/N_1 denote the number of collisions in groups #2 per collision in group #1, we should clearly reduce c_{12} (what #2 gives to #1) by $c_{21} \cdot N_1/N_2$. This ritual is performed for each pair of velocity groups and for each medium. However, since N_g is obtained as an integral over $\mathcal{N}_g(r)$ and $\mathcal{N}_g(r)$ is not available until the boundary conditions have been applied, we are faced with 2G simultaneous equations, transcendental in half of the unknowns involved. The method requires, therefore, a rather elaborate trial and error procedure. It is an exact method only if the ratios N_g/N_h are independent of r within each medium.

Using (14) we have, corresponding to (16), the following expressions for $M_g(r)$ and $M_g(r)$:

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and

$$(48) \begin{cases} \sqrt{1}(r) = A_1 \frac{\sigma_1}{k_1} (c_1^1 - 1) S(k_1 r) + \alpha_1^2 A_2 \frac{\sigma_1}{k_2} (c_1^2 - 1) S(k_2 r) + \alpha_1^3 A_3 \frac{\sigma_1}{k_3} (c_1^3 - 1) S(k_3 r) \\ \sqrt{2}(r) = A_2 \frac{\sigma_2}{k_2} (c_2^2 - 1) S(k_2 r) + \alpha_2^3 A_3 \frac{\sigma_2}{k_3} (c_2^3 - 1) S(k_3 r) \\ \sqrt{3}(r) = A_3 \frac{\sigma_3}{k_3} (c_3^3 - 1) S(k_3 r) \end{cases}$$

The above formulae can easily be extended to G groups and to the general spherical shell. In solving systems involving expressions like (14) and (15) and M separate spherical media, we start with the 2(M-1) equations involving A_{G} , solve for these unknowns and proceed to the 2(M-1) equations involving A_{G} and A_{G-1} , etc.

With the above method G-velocity problems are essentially reduced to G one-velocity problems each of which (for the proper values of N_g) must give the same result for the required critical parameter. The

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sine part of the formulae for N_g for three velocity groups and a spherical shell of inner and outer radii a_1 and a_2 are given by:

$$(19) \begin{cases} N_{1}(r) = A_{1} \frac{\sigma_{1}}{k_{1}} S(k_{1}r) + \alpha_{1}^{2}A_{2} \frac{\sigma_{1}}{k_{2}} S(k_{2}r) + \alpha_{1}^{3}A_{3} \frac{\sigma_{1}}{k_{3}} S(k_{3}r) \end{bmatrix}_{a_{1}}^{a_{2}} \\ N_{2}(r) = A_{2} \frac{\sigma_{2}}{k_{2}} S(k_{2}r) + \alpha_{2}^{3}A_{3} \frac{\sigma_{2}}{k_{3}} S(k_{3}r) \end{bmatrix}_{a_{1}}^{a_{2}} \\ N_{3}(r) = A_{3} \frac{\sigma_{3}}{k_{3}} S(k_{3}r) \end{bmatrix}_{a_{1}}^{a_{2}} \end{cases}$$







EXAMPLE I

Consider an untamped Oralloy sphere of density 18.8 gr/cm³ described by the 3-velocity parameters given in LA-1276:

$$(20) \quad \mathbf{v}_{\mathbf{g}} = \left\{ \begin{array}{c} 6 \\ 12 \\ 24 \end{array} \right\}, \quad \mathbf{\sigma}_{\mathbf{g}} = \left\{ \begin{array}{c} .3853 \\ .2408 \\ .1879 \end{array} \right\}, \quad \mathbf{c}_{\mathbf{gh}} = \left\{ \begin{array}{c} .8412 & .3682 & .3902 \\ .1862 & .6925 & .4725 \\ .2081 & .2783 & .5804 \end{array} \right\}.$$

We propose to calculate the critical radius of the sphere and the three flux distributions using the method described above. These are then to be compared with the results in LA-1272, obtained by the Integral Theory Method.

We take as a first trial, $N_1/N_3 = 2.6$ and $N_2/N_3 = 1.4$; as a second trial 2.6 and 1.45; and as a final trial 2.7 and 1.4. By calculation the following table is obtained:

CASE	I	II	III
Calculated Quantities	N ₁ /N ₃ =2.6 N ₂ /N ₃ =1.4	N ₁ /N ₃ =2.6 N ₂ /N ₃ =1.45	N ₁ /N ₃ =2.7 N ₂ /N ₃ =1.4
<u>,</u> ,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,	1.2355, .0224,1509	1.2355, .0343,1509	1.2355, .0091,1717
<u>,</u> <u></u> 22, <u></u> 23	1.3166, .0829	1.3047, .0690	1.3299, .0829
	1.5111	1.5250	1.5319
^k 1, ^k 2, ^k 3	.35319,.26290,.27656	.35319,. 25692, .28142	.35319,.26952,.28381
c_1^1, c_1^2, c_1^3	1.2355,1.1396,1.1530	1.2355,1.1339,1.1578	1.2355,1.1460,1.1602
c ₂ ² c ₂ ³	0 1.3166,1.3442	0 1.3047,1.3542	0 1.3299,1.3591
c3 3	0 0 1.5111	0 0 1.5250	0 0 1.5319



Table (continued)

CASE		I			II			III	
∝ ¹ ,∝ ² ,∝ ³	1	146	0.4943	1	211	0.6470	1	063	5 .9 ⁴⁴ 7
∞ ² ,∞ ²	0	1	2.3438	0	l	1.0877	0	1	2.2153
∞ ³ ₃	0	0	1	0	0	1	0	0	l

Taking A_3 =1 and solving $N_3(r) = Q(k_3a_1, k_3/\sigma_3c_3^3) = 0$ for a_1 we find: Case I: $a_1=8.489$, Case II: $a_1=8.323$, and Case III: $a_1=8.243$. In solving $N_3(r) = 0$ and calculating the quantities below we make use of one of the Serber-Wilson Tables, in this case LA-1364.

Continuing the work, denoting $k_i / \Im_g c_g^i$ by ϕ_g^i , likewise $Q(k_i a_1, \phi_g^i)$ by Q_{ig} , and $S(k_i a_1)$ by S_i , we have:

CASE	I	II	III
$\phi_1^1, \phi_1^2, \phi_1^3$.7419, .5987, .6225	.7419, .5881, .6308	.7419, .6104, .6349
φ ² ₂ , φ ³ ₂	8292, .8544	8178, .8630	8416, .8672
¢ _3^3	9740	9821	9860
^k 1 ^a 1, ^k 2 ^a 1, ^k 3 ^a 1	2.9982,2.2318,2.3477	2.9396,2.1383,2.3422	2.9113,2.2217,2.3394
^s 1 ^s 2 ^s 3	.34600,.43355,.42800	.35648,.43580,.42833	.36134,.43389,.42849
911 921 931	12417,.13861,.08812	11204,.17882,.08781	10588,.13907,.08768
Q22 Q32	.07418,.02780	.11167,.02757	.07454,.02749
Q ₃₃	.00011	.00002	.00002

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CASE	I	II	III
$\frac{\sigma_3}{\sigma_2}\frac{N_2}{N_3}-\alpha_2^3$	-1.25136	.04376	-1.12286
$\frac{\sigma_3}{\sigma_1}\frac{N_1}{N_3}-\alpha_1^3$.77364	.62094	.37201
A ₂	A ₂ -1.1743		-1.0531
A_1	•9335	.9503	.4438
	04860	05115	.04514
	02196	.03438	01760

Table (continued)

We use the above results for \tilde{M}_1 and \tilde{M}_2 (which should be equal to zero for the correct trial combination) to interpolate for N_1/N_3 and N_2/N_3 . Linear interpolation is in this case equivalent to solving the equations:

(21)
$$\begin{cases} \sqrt{\frac{1}{1}} \approx -.04860 + \frac{.09374}{.10} (\frac{N_1}{N_3} - 2.6) - \frac{.00255}{.05} (\frac{N_2}{N_3} - 1.4) = 0, \\ \sqrt{\frac{1}{2}} \approx -.02196 + \frac{.00436}{.10} (\frac{N_1}{N_3} - 2.6) + \frac{.05634}{.05} (\frac{N_2}{N_3} - 1.4) = 0, \end{cases}$$

simultaneously. The solution of (21) gives $N_1/N_3=2.653$ and $N_2/N_3=1.417$ from which, by calculation, we find $k_2=.26430$, $k_3=.28207$, $a_1=8.301$,

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 α_1^2 =-.1295, α_1^3 =.7568, α_2^3 =1.6882, A₁=.6918, and A₂=-.5378.

This problem was also solved using the two-velocity parameters of Example II. Result: $a_1 = 8.315$. Furthermore, a variation of the method was tried, making \overline{c}_{12} rather than \overline{c}_{21} equal to zero. The result in this case: $a_1 = 8.320$.

The following table gives a comparison of the Serber-Wilson Method and the Integral Theory Method:

magazi	al			N ₁ /N ₃	N ₂ /N ₃
Theory	l-vel.*	2-vel.	3-vel.	3-vel.	3-vel.
S.W. I.T.	8.39 8.72	8.32 8.70	8.30 8.70	2.653 2.630	1.417 1.416
				Ū.	

*Parameters (IA-1276): - 2821, c = 1.2936.

The flux densities (as functions of r) do not agree nearly as well. Cf. graph on page 24 and Table VII (Second Set) in LA-1272.



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EXAMPLE II

We consider next an Oralloy (Oy) sphere of density 18.8 gr/cm^3 tamped by an infinite Tuballoy (Tu) shell of density 19.0 and look for the critical radius and the flux distributions. To simplify the work here we content ourselves with a two-velocity calculation. In LA-1276 we find the following parameters for Oy and Tu:

(22)
$$\begin{cases} \text{Oy: } \mathbf{v}_{g} = \left\{ \begin{array}{c} 6.43 \\ 19.6 \end{array} \right\}, \quad \mathbf{\sigma}_{g} = \left\{ \begin{array}{c} .365 \\ .197 \end{array} \right\}, \quad \mathbf{c}_{gh} = \left\{ \begin{array}{c} .897 \\ .347 \end{array}, \begin{array}{c} .553 \\ .863 \end{array} \right\}, \\ \text{Tu: } \mathbf{v}_{g} = \left\{ \begin{array}{c} 6.43 \\ 19.6 \end{array} \right\}, \quad \mathbf{\sigma}_{g} = \left\{ \begin{array}{c} .365 \\ .197 \end{array} \right\}, \quad \mathbf{c}_{gh} = \left\{ \begin{array}{c} .98 \\ .00 \end{array}, \begin{array}{c} .62 \\ .50 \end{array} \right\}. \end{cases}$$

For the flux distributions we write according to (10):

(23) Oy:

$$\begin{pmatrix}
\int_{1}^{r} (r) = A_{1} \frac{\sin k_{1}r}{k_{1}r} + \alpha_{1}^{2} A_{2} \frac{\sin k_{2}r}{k_{2}r}, \\
\int_{2}^{r} (r) = A_{2} \frac{\sin k_{2}r}{k_{2}r}, \\
\int_{1}^{r} (r) = B_{1} \frac{e^{-k_{1}r}}{k_{1}r} + \alpha_{1}^{2} B_{2} \frac{e^{-k_{2}r}}{k_{2}r}, \\
(2^{4}) Tu: \\
\int_{2}^{r} (r) = B_{2} \frac{e^{-k_{2}r}}{k_{2}r}.$$

Note that \mathbf{v}_g , \mathbf{c}_g , \mathbf{c}_{gh} , \mathbf{k}_i , \mathbf{c}_g^i , and $\boldsymbol{\alpha}_g^i$ in general are functions of the medium, although for the sake of simplicity, we omit notations to this effect.

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The boundary conditions are described by the following equations, where the left-hand side refers to the Oy and the right-hand side to the Tu:

$$(25) \begin{cases} (a) A_{1}Q(k_{1}a_{1},k_{1}/5_{1}c_{1}^{1}) + \alpha_{1}^{2}A_{2}Q(k_{2}a_{1},k_{2}/5_{1}c_{1}^{2}) = B_{1}R_{t}(5_{1}a_{1},k_{1}/5_{1}) + \alpha_{1}^{2}B_{2}R_{t}(5_{1}a_{1},k_{2}/5_{1}), \\ (b) A_{1}\frac{\sigma_{1}}{k_{1}}(c_{1}^{1}-1)S(k_{1}a_{1}) + \alpha_{1}^{2}A_{2}\frac{\sigma_{1}}{k_{2}}(c_{1}^{2}-1)S(k_{2}a_{1}) = B_{1}\frac{\sigma_{1}}{k_{1}}(c_{1}^{1}-1)T_{t}(k_{1}a_{1}) + \alpha_{1}^{2}B_{2}\frac{\sigma_{1}}{k_{2}}(c_{1}^{2}-1)T_{t}(k_{2}a_{1}), \\ (c) A_{2}Q(k_{2}a_{1},k_{2}/5_{2}c_{2}^{2}) = B_{2}R_{t}(5_{2}a_{1},k_{2}/5_{2}), \\ (d) A_{2}\frac{\sigma_{2}}{k_{2}}(c_{2}^{2}-1)S(k_{2}a_{1}) = B_{2}\frac{\sigma_{2}}{k_{2}}(c_{2}^{2}-1)T_{t}(k_{2}a_{1}). \end{cases}$$

The procedure for solving (25) will be the following: Equations (c) and (d) define a one-velocity problem which we solve by the methods of LA-756, taking A_2 =1 and obtaining a_1 and B_2 (as functions of N_1/N_2) by calculation. Next we calculate A_1 , as in previous example, from:

(26)
$$A_{1} = \frac{k_{1}}{k_{2}} \frac{S(k_{2}a_{1})}{S(k_{1}a_{1})} \left[\frac{\sigma_{2}}{\sigma_{1}} \frac{N_{1}}{N_{2}} - \alpha_{1}^{2} \right],$$

and then B_1 from equation (b) above. Finally, equation (a) is used as a test equation for the trial quantity N_1/N_2 .

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The functionals $R_t(\sigma r, k/\sigma)$ and $T_t(kr)$ correspond to the flux densities (24). Note that the latter have a special form, different from (10), due to the requirement that the neutron flux vanish at infinity. Hence:

(27)
$$R_t(\sigma r, k/\sigma) = c \frac{\sigma}{k} e^{\sigma r} E_1((1+\frac{k}{\sigma})\sigma r); T_t(kr) = -\frac{1+kr}{(kr)^2} e^{-kr},$$

with c from $k/\sigma = c \operatorname{arth}(k/\sigma)$.

Turning to the first part of the computation, taking 2.4, 2.5, and 2.6 as successive trials for N_1/N_2 , we calculate:

		Tu		
•	N ₁ /N ₂ =2.4	N ₁ /N ₂ =2.5	N ₁ /N ₂ =2.6	
^c 11, ^c 12 ^c 21, ^c 22	1.2440,2798 0 1.6958	1.2440,314 0 1.730	5 1.2440,34 <i>9</i> 2 5 0 1.7652	.98 .62 0 .50
^k 1' ^k 2	.34155, .35585	.34155, .367	90 .34155 .37987	.08869 .18863
c ¹ , c ²	1.2440, 1.2617	1.2440, 1.276	3 1.2440, 1.2923	.98, .9035
c ¹ ₂ , c ² ₂	- 1.6958	- 1.730	5 - 1.7652	50
∝ ¹ ,∝ ²	1 -8.532	1 -5.255	1 -3.902	1 -4.374
∞ ¹ ₂ , ∞ ² ₂	0 1	0 1	0 1	0 1

These calculations are followed by three one-velocity calculations and the computation of A_1 and B_1 . The results are given in the table below:

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|--|

N ₁ /N ₂	al	^B 2	Al	A ₁ +α ²	^B l
2.4	6.129	.6322	9.408	.876	1.345
2.5	5.905	.5827	6.125	.870	1.237
2.6	5.696	.5403	4.785	. 883	1.144

Finally, testing N_1/N_2 by calculating D where D = left-hand side of (25,a) minus the right-hand side, we obtain:

 $N_1/N_2=2.4$, D=-.0518; $N_1/N_2=2.5$, D=.0387; $N_1/N_2=2.6$, D=.1122.

Hence, by interpolation $N_1/N_2=2.457$, $a_1=6.001$, $B_2=.6040$, $A_1+\alpha_2^1=.872$, $B_1=1.283$, and by calculation $k_2(0y)=.36274$, $\alpha_1^2(0y)=-6.219$, and

A_=7.091.

The average velocity $\overline{\mathbf{v}}$ in the core is then given by: $\overline{\mathbf{v}} = 3.457 \div \left[(2.457/\sigma_1 \mathbf{v}_1) + (1.000/\sigma_2 \mathbf{v}_2) \right] = 9.04$ cm/shake to be compared with the Integral Theory result of 8.98, (estimated from the table of $\overline{\mathbf{v}}$ vs tamper thickness given in LA-1276).



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EXAMPLE III

As a final example let us consider the same problem as in Example II but with a finite rather than an infinite tamper. We take the outer radius a₂ equal to 2a₁. The parameters are the same as in Example II but the Tu flux distributions will be different and the boundary conditions more complicated. We have:

(28) Oy:
$$\begin{cases} \sqrt{1}(r) = A_{1} \frac{\sin k_{1}r}{k_{1}r} + \alpha_{1}^{2} A_{2} \frac{\sin k_{2}r}{k_{2}r} \\ \sqrt{2}(r) = A_{2} \frac{\sin k_{2}r}{k_{2}r} \\ \\ \sqrt{2}(r) = B_{1} \left[\frac{\sinh k_{1}r}{k_{1}r} + \overline{B}_{1} \frac{\cosh k_{1}r}{k_{1}r} \right] + \\ \\ (29) Tu: \begin{cases} \sqrt{1}(r) = B_{1} \left[\frac{\sinh k_{2}r}{k_{2}r} + \overline{B}_{2} \frac{\cosh k_{2}r}{k_{2}r} \right] \\ + \alpha_{1}^{2} B_{2} \left[\frac{\sinh k_{2}r}{k_{2}r} + \overline{B}_{2} \frac{\cosh k_{2}r}{k_{2}r} \right] \\ \\ \sqrt{2}(r) = B_{2} \left[\frac{\sinh k_{2}r}{k_{2}r} + \overline{B}_{2} \frac{\cosh k_{2}r}{k_{2}r} \right] \end{cases}$$

Before writing down the boundary conditions we introduce the following abbreviated notation: We denote $Q(k_i r, k_i / \sigma_g c_g^i)$ by $Q_{ig}(r)$, $\frac{\sigma_g}{k_i}(c_g^i-1)S(k_i r)$ by $S_{ig}(r)$, and similarly $R(k_i r, k_i / \sigma_g c_g^i)$ by $R_{ig}(r)$ and $\frac{\sigma_g}{k_i}(c_g^i-1)T(k_i r)$ by $T_{ig}(r)$: The boundary conditions can then be written as:

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$$(30) \begin{cases} (a) A_{1}Q_{11}(a_{1}) + \alpha_{1}^{2}A_{2}Q_{21}(a_{1}) = B_{1}Q_{11}(a_{1}) + \alpha_{1}^{2}B_{2}Q_{21}(a_{1}) + B_{1}\overline{B}_{1}R_{11}(a_{1}) + \alpha_{1}^{2}B_{2}\overline{B}_{2}R_{21}(a_{1}) \\ (b) A_{1}S_{11}(a_{1}) + \alpha_{1}^{2}A_{2}S_{21}(a_{1}) = B_{1}S_{11}(a_{1}) + \alpha_{1}^{2}B_{2}S_{21}(a_{1}) + B_{1}\overline{B}_{1}T_{11}(a_{1}) + \alpha_{1}^{2}B_{2}\overline{B}_{2}T_{21}(a_{1}) \\ (c) A_{2}Q_{22}(a_{1}) = B_{2}Q_{22}(a_{1}) + B_{2}\overline{B}_{2}R_{22}(a_{1}) \\ (d) A_{2}S_{22}(a_{1}) = B_{2}S_{22}(a_{1}) + B_{2}\overline{B}_{2}T_{22}(a_{1}) \\ (e) B_{1}Q_{11}(a_{2}) + \alpha_{1}^{2}B_{2}Q_{21}(a_{2}) + B_{1}\overline{B}_{1}R_{11}(a_{2}) + \alpha_{1}^{2}B_{2}\overline{B}_{2}R_{21}(a_{2}) = 0 \\ (f) B_{2}Q_{22}(a_{2}) + B_{2}\overline{B}_{2}R_{22}(a_{2}) = 0 \end{cases}$$

The procedure for solving the above system is very similar to the one outlined in Example II. We take $A_2 = 1$ and solve for a_1 , B_2 , and \overline{B}_2 by one-velocity methods using (c), (d), and (f). We obtain A_1 from (26), then B_1 and \overline{B}_1 by solving (b) and (e) simultaneously, and finally a check on N_1/N_2 by calculating D, where D = left-hand side of (2) minus the right-hand side.

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CASE	I	II	III
Calculated Quantities	N ₁ /N ₂ =2.15	N ₁ /N ₂ =2.20	N ₁ /N ₂ =2.25
k_1, k_2 a_1 $\alpha_1^2, \alpha_1^2 + A_1$ B_2, \overline{B}_2 B_1, \overline{B}_1 D	.34155, .32532 6.800 5.297, .906 7968,9990 -1.8652,9033 0903	.34155, .33148 6.658 9.247, .895 7599,9989 -1.7851,8998 0447	.34155, .33761 6.522 25.439, .887 7257,9987 -1.7117,8954 .0035

The results of the calculations are summarized in the table below:

Hence, by interpolation $N_1/N_2=2.246$, $a_1 = 6.532$, $\alpha_1^2 + A_1 = .888$, $B_2 = ..7282$, $\overline{B}_2 = ..9987$, $B_1 = -1.7170$, $\overline{B}_1 = ..8957$, and by calculation $k_2 = .33712$, $\alpha_1^2 = 22.541$, and $A_1 = .21.653$. For the tamper we have as in Example II $k_1 = .08869$, $k_2 = .18863$, and $\alpha_1^2 = .4.374$.

The average velocity \overline{v} in the core is given by: $\overline{v} = 3.246 - \left[(2.246/\sigma_1 v_1) + (1.000/\sigma_2 v_2) \right] = 9.24$ compared to the Integral Theory result of 9.14 given in LA-1276. The critical radii also agree well (S.W.: 6.53, I.T.: 6.68), the error being in the direction and of the magnitude one is accustomed to in one-velocity calculations.

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APPENDIX

THE SPHERICAL HARMONIC METHOD

We would like to compare the Serber-Wilson Method with another important method, the Spherical Harmonic Method, both with regard to the complexity of the computations involved, and with regard to the accuracy one may expect. At present, the latter comparison cannot be made since very few multi-velocity calculations have been carried out with either method. In the one-velocity case, however, it is generally held that the Serber-Wilson Method and the P_3 -Approximation (the Spherical Harmonic Approximation of order three) are, as far as accuracy is concerned, about equivalent.

For the purpose of comparing the two methods with regard to complexity (computational as well as mathematical) the following brief outline of the Spherical Harmonic Method will probably suffice. We consider again the G-velocity isotropic theory and expand the flux distributions $\mathcal{N}_{g}(\mathbf{r},\mu)$ in Legendre series:

where n denotes the degree of approximation and $P_k(\mu)$ the Legendre

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Polynomials. We substitute the above expansion in (9), multiply by $P_{\ell}(\mu)$, $\ell = 0, 1, ..., n$, on both sides and integrate over μ from -1 to +1. Before performing these integrations, $\mu P_{k}(\mu)$ and $(1-\mu^{2})P_{k}'(\mu)$ are conveniently replaced by:

(32)

$$(2k+1)(1-\mu^{2})P_{k}'(\mu) = \left[(k+1)P_{k+1}(\mu) + kP_{k-1}(\mu) \right],$$

$$(32)$$

$$(2k+1)(1-\mu^{2})P_{k}'(\mu) = k(k+1) \left[P_{k-1}(\mu) - P_{k+1}(\mu) \right],$$

respectively. Carrying out the above steps we arrive at the following system of differential equations:

(33)
$$(k+1)(D_{r} + \frac{k+2}{r}) \psi_{g,k+1} + k(D_{r} - \frac{k-1}{r}) \psi_{g,k-1} + (2k+1)\sigma_{g} \psi_{g,k} = \\ = \begin{cases} \sum_{h=1}^{G} \sigma_{h}c_{gh} \psi_{h,0}; \ k = 0, \\ 0; \ k = 1, 2, ..., n, \end{cases}$$

usually referred to as the P_n -transform of (9). For reasons which will not be discussed here, n is usually taken to be odd, n = 1, 3, 5, ...

Equations (33) can be written in several alternate forms. For instance, if we let $\gamma_{g,k} = \phi_{g,k}/r^{k+1}$, we obtain:

(34)
$$\frac{k+1}{r} D_{r} \phi_{g,k+1}^{++k} \left[r D_{r}^{-(2k-1)} \right] \phi_{g,k-1}^{++(2k+1)} \sigma_{g} \phi_{g,k}^{+} = \begin{cases} \sum_{h=1}^{G} \sigma_{h}^{-} c_{gh} \phi_{h,0}^{+}; & k = 0, \\ 0; & k = 1, 2, ..., n, \end{cases}$$

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and replacing r^2 by x, denoting the right-hand side of (34) by RHS (34), we have:

(35)
$$2(k+1)D_x \phi_{g,k+1} + k \left[2xD_x - (2k-1) \right] \phi_{g,k-1} + (2k+1)\sigma_g \phi_{g,k} = RHS$$
 (34).

To derive the differential equation for $\phi_{g,0}$ for a particular n, the following formula, obtained from (35) by differentiation, is very useful:

(36)
$$2(k+1)D_x^{k+1}\phi_{g,k+1} + \frac{1}{2}kD_r^2D_x^{k-1}\phi_{g,k-1} + (2k+1)\sigma_g D_x^k\phi_{g,k} = RHS$$
 (34)

For with the aid of (36) we can eliminate the higher order $\phi_{g,k}$'s and be left with a (n+1)-order differential equation in $\phi_{g,0}$ alone. We obtain, omitting the subscript g for the moment:

$$\begin{cases} 2^{2}2:D_{x}^{2}\phi_{2} = -D_{r}^{2}\phi_{0}-3\sigma(2D_{x}\phi_{1}), \\ 2^{3}3:D_{x}^{3}\phi_{3} = 5\sigma D_{r}^{2}\phi_{0}-(4D_{r}^{2}-15\sigma^{2})(2D_{x}\phi_{1}), \\ 2^{4}4:D_{x}^{4}\phi_{4} = (9D_{r}^{2}-35\sigma^{2})D_{r}^{2}\phi_{0}+\sigma(55D_{r}^{2}-105\sigma^{2})(2D_{x}\phi_{1}), \\ 2^{5}5:D_{x}^{5}\phi_{5} = -\sigma(161D_{r}^{2}-315\sigma^{2})D_{r}^{2}\phi_{0}+(64D_{r}^{4}-735\sigma^{2}D_{r}^{2}+945\sigma^{4})(2D_{x}\phi_{1}), \\ 2^{6}6:D_{x}^{6}\phi_{6} = -(25D_{r}^{4}-294\sigma^{2}D_{r}^{2}+385\sigma^{4})D_{r}^{2}\phi_{0}-\sigma(231D_{r}^{4}-1190\sigma^{2}D_{r}^{2}+1155\sigma^{4})(2D_{x}\phi_{1}), \\ 2^{6}e:D_{x}\phi_{6} = -(25D_{r}^{4}-294\sigma^{2}D_{r}^{2}+385\sigma^{4})D_{r}^{2}\phi_{0}-\sigma(231D_{r}^{4}-1190\sigma^{2}D_{r}^{2}+1155\sigma^{4})(2D_{x}\phi_{1}), \\ where 2D_{x}\phi_{g,1} = (\sum_{h=1}^{G}\sigma_{h}c_{gh}\phi_{h,0}) - \sigma_{g}\phi_{g,0}. \text{ Since } \phi_{g,n+1} = 0 \text{ in } \\ P_{n}\text{-approximation we have the following differential equations for } \\ n = 1, 3, \text{ and } 5; \end{cases}$$

$$\left\{ \begin{array}{l} P_{1} : \left[\frac{(3 \sigma_{g}^{2} - D_{r}^{2})}{3 \sigma_{g}^{2}} \right] \phi_{g,0} = \frac{1}{\sigma_{g}} \sum_{h=1}^{G} \sigma_{h} c_{gh} \phi_{h,0} \\ P_{3} : \left[\frac{105 \sigma_{g}^{4} - 90 \sigma_{g}^{2} D_{r}^{2} + 9D_{r}^{4}}{\sigma_{g}^{2} (105 \sigma_{g}^{2} - 55D_{r}^{2})} \right] \phi_{g,0} = \frac{1}{\sigma_{g}} \sum_{h=1}^{G} \sigma_{h} c_{gh} \phi_{h,0} \\ P_{5} : \left[\frac{1150 \sigma_{g}^{6} - 1575 \sigma_{g}^{4} D_{r}^{2} + 525 \sigma_{g}^{2} - 25D_{r}^{6}}{\sigma_{g}^{2} (1155 \sigma_{g}^{4} - 1190 \sigma_{g}^{2} D_{r}^{2} + 231) D_{r}^{4}} \right] \phi_{g,0} = \frac{1}{\sigma_{g}} \sum_{h=1}^{G} \sigma_{h} c_{gh} \phi_{h,0}$$

where it is understood that the denominator in the brackets operates on the right-hand side and the numerator on $\phi_{g,0}$.

Note that the operators in (38) are continued fraction approximations of z/art z with $z = iD_r/\sigma_g$. For we have:

(39)
$$z/art z = 1 + z^2/3+4z^2/5+9z^2/7+...$$

Denoting the $(n+1)^{\text{th}}$ approximation (the first (n+1) terms) of (39) by $\left[z/\operatorname{art} z\right]_n$, we have in the general case:

(40)
$$P_n: \left[(iD_r/\sigma_g) \div art (iD_r/\sigma_g) \right]_n \phi_{g,0} = \frac{1}{\sigma_g} \sum_{h=1}^G \sigma_h c_{gh} \phi_{h,0}.$$

It can now readily be verified that the general solution of (40) is given by:

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(41)
$$\phi_{g,0} \equiv r \psi_{g,0} \equiv r \mathcal{N}_g(r) = r \sum_{i=1}^{\frac{1}{2}(n+1)G} \alpha_g^i A_i \left[\frac{\sinh_i r}{\frac{k_i r}{k_i r}} + \overline{A}_i \frac{\cosh_i r}{|k_i|r} \right],$$

provided the k_i 's and α_g^i 's satisfy the following matrix equation:

which obviously approaches (11) as n approaches -. The eigenvalues k_i come in this case from an algebraic equation in k_i^2 of degree $\frac{1}{2}(n+1)G$. We can, therefore, expect $\frac{1}{2}(n+1)G$ eigenvalues in the right half of the complex plane.

In comparing the Serber-Wilson and the Spherical Harmonic Methods we observe that the diffiuclty of having a sufficient number of k_i 's is replaced by the hardships involved in having to deal with complex ones, and that the Spherical Harmonic Method has $\frac{1}{2}(n+1)$ times as many



 k_i 's as the Serber-Wilson Method, which implies (for $n \ge 3$) more terms in (41), more α_g^i 's to calculate, etc. Finally, the boundary conditions associated with the Spherical Harmonic Method are more difficult to apply, not only because there are more of them $(n \ge 3)$, but also because the remaining angular moments, i.e., $\psi_{g,k}$, k = 1, 2, ..., n, must be computed. For the conditions usually imposed on the $\psi_{g,k}$'s (or $\phi_{g,k}$'s) are that they be continuous at each boundary.

In P_1 -Approximation we obtain from the first equation in (37):

(43)
$$2D_x \phi_{gl} = \frac{1}{r} D_r \phi_{gl} = -\frac{1}{3\sigma_g} D_r^2 \phi_{g0}$$
,

and hence:

(44)
$$\gamma_{gl} = \sum_{i=1}^{G} \frac{k_{i}}{3\sigma_{g}} \sigma_{g}^{i} A_{i} \left[S_{1}(k_{i}r) + \frac{k_{i}\overline{A}_{i}}{|k_{i}|} T_{1}(k_{i}r) \right] ,$$

where $S_1(x) = (\sin x - x \cos x)/x^2$ and $T_1(x) = (\cos x + x \sin x)/x^2$.

In P₃-Approximation we first solve for $2D_x \phi_{gl}$ in the third equation of (37), then obtaining $2^2 2! D_x^2 \phi_{g2}$ and $2^3 3! D_x^3 \phi_{g3}$ from the first and second equation:

$$(45) \begin{cases} 2D_{x} \phi_{g1} \equiv \frac{1}{r} D_{r} \phi_{g1} = -\frac{1}{\sigma_{g}} \frac{9D_{r}^{2} - 35\sigma_{g}^{2}}{55D_{r}^{2} - 105\sigma_{g}^{2}} D_{r}^{2} \phi_{g0} , \\ 2^{2}2! D_{x}^{2} \phi_{g2} \equiv 2! \frac{1}{r} D_{r} \frac{1}{r} D_{r} \phi_{g2} = -\frac{28}{55D_{r}^{2} - 105\sigma_{g}^{2}} D_{r}^{4} \phi_{g0} , \\ 2^{3}3! D_{x}^{3} \phi_{g3} \equiv 3! \frac{1}{r} D_{r} \frac{1}{r} D_{r} \frac{1}{r} D_{r} \phi_{g3} = \frac{36/\sigma_{g}}{55D_{r}^{2} - 105\sigma_{g}^{2}} D_{r}^{6} \phi_{g0} . \\ -33- \frac{33}{2} - \frac{36}{2} -$$



Solving these differential equations we have:

$$(46) \begin{cases} \psi_{g1} = \sum_{i=1}^{2G} \frac{k_i}{\sigma_g} \left[\frac{35 + 9k_1^2 / \sigma_g^2}{105 + 55k_1^2 / \sigma_g^2} \right] \propto \frac{i}{g} A_i \left[S_1(k_1 r) + \frac{k_1 \overline{A}_1}{|k_1|} T_1(k_1 r) \right], \\ \psi_{g2} = \sum_{i=1}^{2G} \frac{14k_1^2 / \sigma_g^2}{105 + 55k_1^2 / \sigma_g^2} \propto \frac{i}{g} A_i \left[S_2(k_1 r) + \frac{k_1 \overline{A}_1}{|k_1|} T_2(k_1 r) \right], \\ \psi_{g3} = \sum_{i=1}^{2G} \frac{6k_1^3 / \sigma_g^3}{105 + 55k_1^2 / \sigma_g^2} \propto \frac{i}{g} A_i \left[S_3(k_1 r) + \frac{k_1 \overline{A}_1}{|k_1|} T_3(k_1 r) \right], \end{cases}$$

where
$$S_2(x) = \left[(3-x^2)\sin x - 3x \cos x \right] / x^3$$
,
 $T_2(x) = \left[(3-x^2)\cos x + 3x \sin x \right] / x^3$,
 $S_3(x) = \left[(15-6x^2)\sin x - (15-x^2)x \cos x \right] / x^4$, and
 $T_3(x) = \left[(15-6x^2)\cos x + (15-x^2)x \sin x \right] / x^4$.

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