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STABILITY OF A STEADY PLANE SHOCK

WORK DONE BY:

A. E. Roberts

REPORT WRITTEN BY:

A. E. Roberts

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ABSTRACT OF LA REPORT 299

June 12, 1945

STABILITY OF A STEADY PLANE SHOCK - A. E. Roberts

In a steady plane shock in a two-dimensional perfect fluid having an equation of state which is independent of entropy, first-order perturbations are introduced into the shape of the shock and in the hydrodynamic quantities in the shocked material. For these perturbations a system of linear partial differential equations in two independent variables holds behind the shock, with boundary conditions at the shock given in terms of the perturbation in the shock position. For any given initial disturbance, the equations, when solved for conditions at a point on the shock at some later time, yield for the amplitude of each Fourier component of the perturbation in the shock position an integral equation which expresses the amplitude in terms of its past history and the initial conditions. The solution of this integral equation is discussed. It is shown that for initial disturbances which are finite in extent the shock is always stable, provided the Chapman-Jouguet point is not exceeded. In fact, for such disturbances, the shock undergoes damped oscillations which in general for large times die out at least as $t^{-\frac{1}{2}}$. In the case of an initial disturbance only in the immediate neighborhood of the shock front, the oscillations in fact die out asymptotically as $t^{-3/2}$. Numerical results are presented in graphical form for this latter type of initial disturbance for a shock with a compression ratio 2, and a value of $\gamma = 3$ in the equation of state.

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STABILITY OF A STEADY PLANE SHOCK

We wish to investigate the time behavior of small perturbations in a steady plane shock in a two-dimensional perfect compressible fluid having an equation of state

$$p = p(v)$$

which is independent of entropy. We have as the undisturbed state a plane shock advancing into material at rest, with pressure, specific volume, and material velocity all constant in space and time ahead of the shock, and behind the shock. First-order perturbations are then assumed in the position of the shock, and in the hydrodynamical quantities behind the shock. The time behavior of any initially specified disturbance of this sort is then determined.

I. Boundary Conditions at the Shock

Consider the problem in a reference system in which, in the undisturbed state, the shocked material is at rest. Let z be the coordinate normal to the undisturbed shock front, which we take to be traveling in the $-z$ direction; and let x be the coordinate along the shock front. If U and u_z are, respectively, the speed of the shock and the speed of the shocked material, both relative to the unshocked material, then in our reference system the shock travels with velocity $U-u_z$ in the $-z$ direction, and the unshocked material flows with velocity u_z in the $+z$ direction until it meets the shock which brings it to rest. Choosing as origin the position of the undisturbed shock at some initial time $t = 0$, we may draw the following z-t diagram for the undisturbed state.

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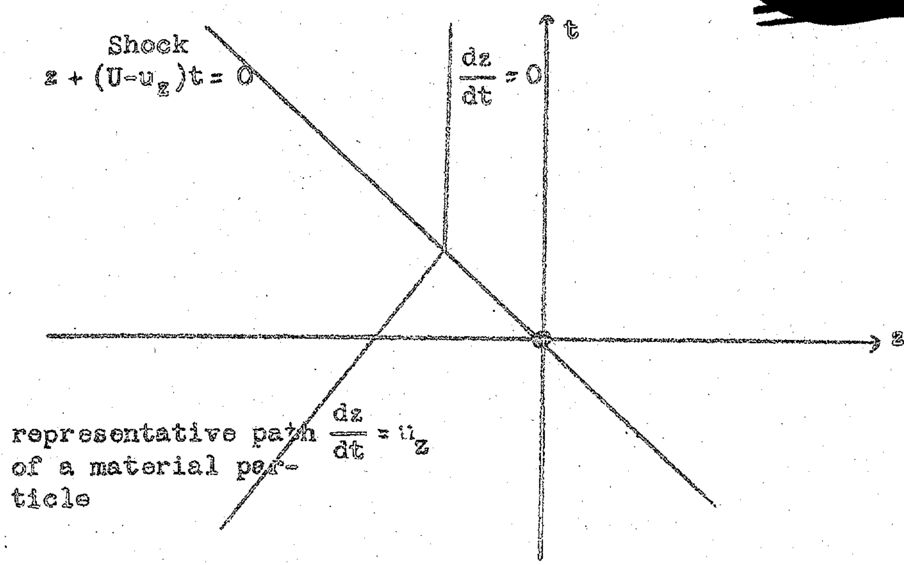

As the equations for first-order perturbations will be linear, it is sufficient to consider perturbations which depend on the coordinate x by the factor e^{ikx}. The general case will then be simply a superposition of such harmonic perturbations. Consequently we take as the position of the perturbed shock front:

$$z + (U-u_z)t = \epsilon h(t)e^{ikx} \tag{1.1}$$

the parameter ϵ being small of first order, of which powers greater than the first are neglected in the following.

Aheah of the shock, the material is at pressure zero, with normal specific volume v_o, and material velocity u_z in the +z direction. Behind the shock, pressure, specific volume and components of material velocity in the perturbed state are, respectively, of the forms:

$$\begin{aligned} &p + \epsilon p^1(z,t)e^{ikx} \\ &v + \epsilon v^1(z,t)e^{ikx} \qquad p, v \text{ constants} \\ &\epsilon u_x^1(z,t)e^{ikx} \\ &\epsilon u_z^1(z,t)e^{ikx} \end{aligned} \tag{1.2}$$

[redacted]

But, now:

$$p + \epsilon p^1 e^{ikx} = p(v + \epsilon v^1 e^{ikx}) = p(v) + \epsilon v^1 \frac{dp}{dv} e^{ikx}$$

and since

$$\frac{dp}{dv} = -\frac{1}{v^2}\frac{dp}{d\rho} = -\frac{c^2}{v^2}$$

where c is undisturbed sound velocity behind the shock, we have,

$$p^1(z,t) = -(c^2/v^2)\, v^1(z,t) \tag{1.3}$$

Unit vectors normal and tangent to the shock front are, respectively:

$$\begin{aligned} \underline{N}_1 &= -\epsilon ik\, h(t)\, e^{ikx}\, \underline{x}_1 + \underline{z}_1 \\ \underline{T}_1 &= \underline{x}_1 + \epsilon ik\, h(t)\, e^{ikx}\, \underline{z}_1 \end{aligned} \tag{1.4}$$

where $\underline{x}_1$, $\underline{z}_1$ are unit vectors in direction of increasing x, z.

Conservation of mass requires that $\rho(\underline{u}_{material} - \underline{u}_{shock})\cdot\underline{N}_1$ be equal on both sides of the shock, or:

at shock

$$\frac{u_z - \left[-U \quad u_z + \epsilon(dh/dt)e^{ikx}\right]}{v_o} = \frac{\epsilon u_z^1 e^{ikx} - \left[-U + u_z + \epsilon(dh/dt)e^{ikx}\right]}{v + \epsilon v^1 e^{ikx}}$$

$$\therefore \quad U/v_o = (U - u_z)/v$$

and

$$u_z^1 = (v^1/v_o)U + (1 - v/v_o)\, dh/dt$$

Let the undisturbed compression ratio be:

$$r = v/v_o \qquad 0 < r < 1 \tag{1.5}$$

then the above can be written:

at shock:

$$u_z = (1-r)U \tag{1.6}$$

$$u_z^1 = rU\, v^1/v + (1-r)\, dh/dt \tag{1.7}$$

[redacted]

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This is to be at the shock, $z + rUt = \epsilon h e^{ikx}$. However, as the undisturbed quantities are constants, we make no error in first-order terms if we take the shock conditions to hold at $z + rUt = 0$.

Conservation of momentum requires that

$$\rho(\underline{u}_m - \underline{u}_s) \cdot \underline{N}_1(\underline{u}_m - \underline{u}_s) + p\underline{N}_1$$

be equal on both sides of the shock. The tangential component of this, together with the conservation of mass, requires $\underline{u}_m \cdot \underline{T}_1$ to be continuous, or:

$$\epsilon u_z \, ik \, h(t) e^{ikx} = \epsilon u_x^1 \, e^{ikx}$$

or,

$$\text{at} \quad z + rUt = 0, \; u_x^1 = ik(1-r)U \, h(t) \qquad (1.8)$$

The normal component is:

$$\frac{U^2 - 2U\epsilon(dh/dt)e^{ikx}}{v_0} + 0 = \frac{r^2 U^2 + 2rU\epsilon(u_z^1 - dh/dt)e^{ikx}}{v + \epsilon v^1 e^{ikx}} + p - \epsilon(c^2/v^2) \, v^1 e^{ikx}$$

which gives:

$$p = \frac{r(1-r)}{v} U^2 \qquad (1.9)$$

and

$$u_z^1 = \frac{1}{2} rU\left(1 + \frac{c^2}{r^2 U^2}\right) \frac{v^1}{v} \quad \text{at} \quad z + rUt = 0 \qquad (1.10)$$

Let:

$$s = (rU/c)^2$$

Defining a constant γ in the shocked material by:

$$\gamma = -\frac{d \ln p}{d \ln v}$$

then

$$c^2 = \gamma \, pv$$

and

$$s = \frac{r^2 U^2}{\gamma p v} = \frac{r^2 U^2}{\gamma r U^2 (1-r)} = \frac{r}{\gamma(1-r)} \qquad (1.12)$$

 [redacted]

Collecting results, we have behind the shock front:

$$v = rv_0$$

$$p = \frac{r(1-r)}{v} U^2 \qquad (1.13)$$

with undisturbed material velocity zero; and at the shock front, namely for $z + rUt = 0$, the following boundary conditions for first-order terms:

$$u_x^1 = ik(1-r)Uh(t)$$

$$u_z^1 = rU \frac{(1+s)}{2s} \frac{v'}{v} \qquad (1.14)$$

$$u_z^1 = rU \frac{v^1}{v} + (1-r) \frac{dh(t)}{dt}$$

The latter two conditions can be rearranged:

$$u_z^1 = \frac{1+s}{1-s} (1-r) \frac{dh(t)}{dt}$$

$$\frac{v^1}{v} = \frac{2s}{1-s} \frac{(1-r)}{rU} \frac{dh(t)}{dt} \qquad (1.14')$$

If $s > 1$ the shock is clearly unstable, for then shock velocity decreases with increasing pressure. The Chapman-Jouguet condition is $s = 1$. In the following, then, we shall assume that:

$$0 < s < 1 \qquad (1.15)$$

II. Equations of Motion Behind the Shock

Behind the shock, namely for $z + rUt > 0$, the hydrodynamic equation of motion is:

$$\partial \underline{u}/\partial t + \underline{u} \cdot \nabla \underline{u} + v\nabla p = 0$$

or:

$$\partial \underline{u}/\partial t + \underline{u} \cdot \nabla \underline{u} - c^2 \nabla \ln v = 0$$

Hence:

[redacted]

[redacted]

$$\epsilon \frac{\partial u_x^1}{\partial t} e^{ikx} - c^2 \frac{\partial}{\partial x}\left(\epsilon \frac{v^1}{v} e^{ikx}\right) = 0$$

and

$$\epsilon \frac{\partial u_z^1}{\partial t} e^{ikx} - c^2 \frac{\partial}{\partial z}\left(\epsilon \frac{v^1}{v} e^{ikx}\right) = 0$$

or:

$$\frac{\partial u_x^1}{\partial t} - c^2\, ik \frac{v^1}{v} = 0 \tag{2.1}$$

and

$$\frac{\partial u_z^1}{\partial t} - c^2 \frac{\partial}{\partial z}\left(\frac{v^1}{v}\right) = 0 \tag{2.2}$$

Also for $z + rUt > 0$ the equation of continuity must hold:

$$\nabla \cdot (\underline{u}/v) - \frac{1}{v^2}\frac{\partial v}{\partial t} = 0$$

or

$$\nabla \cdot \underline{u} - \underline{u} \cdot \nabla \ln v - \frac{\partial}{\partial t}(\ln v) = 0$$

$$\therefore \quad \frac{\partial}{\partial x}\left(\epsilon\ u_x^1\ e^{ikx}\right) + \frac{\partial}{\partial z}\left(\epsilon\ u_z^1\ e^{ikx}\right) - \frac{\partial}{\partial t}\left(\epsilon \frac{v^1}{v} e^{ikx}\right) = 0$$

or

$$ik\ u_x^1 + \frac{\partial u_z^1}{\partial z} - \frac{\partial}{\partial t}\left(\frac{v^1}{v}\right) = 0 \tag{2.3}$$

III. Restatement of Problem

It is convenient to introduce the dimensionless quantities:

$$\chi(\rho,\tau) = \frac{u_x^1(z,t)}{rU}$$

$$\psi(\rho,\tau) = \frac{u_z^1(z,t)}{rU} \qquad\qquad (3.1)^*$$

$$\phi(\rho,\tau) = \frac{v^1(z,t)}{v}$$

$$g(\tau) = k(1-r)\ h(t)$$

as functions of the independent variables: [redacted]

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* From this point on $\rho = kz$, and not density.

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$$\rho = kz \qquad\qquad (3.2)$$
$$\tau = krUt$$

Eqs. (2.1), (2.2) and (2.3) are then:

$$\frac{\partial \chi}{\partial \tau} - \frac{i}{s}\phi = 0 \qquad\qquad (3.3)$$

$$\frac{\partial \psi}{\partial \tau} - \frac{1}{s}\frac{\partial \phi}{\partial \rho} = 0 \qquad\qquad (3.4)$$

$$-i\chi - \frac{\partial \psi}{\partial \rho} + \frac{\partial \phi}{\partial \tau} = 0 \qquad\qquad (3.5)$$

with boundary conditions at $\rho + \tau = 0$, from (1.14) and (1.14'):

$$\chi(-\tau, \tau) = \frac{i}{r}\, g(\tau) \qquad\qquad (3.6)$$

$$\psi(-\tau, \tau) = \frac{1+s}{1-s}\, g'(\tau) \qquad\qquad (3.7)$$

$$\phi(-\tau, \tau) = \frac{2s}{1-s}\, g'(\tau) \qquad\qquad (3.8)$$

where r and s are restricted parameters:

$$0 < r < 1 \qquad\qquad (3.9)$$
$$0 < s \leqslant 1$$

The problem is then to determine g as a function of τ, having given an initially specified disturbance; that is, given $\chi(\rho, 0) \equiv \chi_0(\rho)$, $\psi(\rho, 0) \equiv \psi_0(\rho)$ and $\phi(\rho, 0) \equiv \phi_0(\rho)$ for $\rho \geq 0$. This initial disturbance must develop in time in such a manner that the above differential equations for $\rho > 0$ and the boundary conditions at $\rho + \tau = 0$ remain satisfied at all times. $g(\tau)$ will then be uniquely determined for $\tau \geq 0$.

While the solution was first carried out working directly with the above system of first-order linear partial differential equations, it is quite a bit simpler first to reduce the system to a single second-order equation, which in this case is possible[1]. Differentiating (3.4) with respect to ρ and (3.5) with respect to τ,

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1) This simpler derivation is due to J. W. Calkin. [redacted]

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[redacted]

and adding:

$$- i \frac{\partial x}{\partial \tau} - \frac{i}{s} \frac{\partial^2 \phi}{\partial \rho^2} + \frac{\partial^2 \phi}{\partial \tau^2} = 0 \tag{3.10}$$

But then using (3.3), we have:

$$s \frac{\partial^2 \phi}{\partial \tau^2} - \frac{\partial^2 \phi}{\partial \rho^2} + \phi = 0 \tag{3.11}$$

The boundary conditions at $\rho + \tau = 0$ can be expressed in terms of ϕ and a first derivative. Along $\rho + \tau = 0$, $d/dt \equiv \partial/\partial\tau - \partial/\partial\rho$, so from (3.7):

$$\frac{\partial \psi}{\partial \tau} - \frac{\partial \psi}{\partial \rho} = \frac{1+s}{1-s} g''(\tau) \quad \text{at} \quad \rho + \tau = 0 \tag{3.12}$$

But adding (3.4) and (3.5):

$$\frac{1}{s} \frac{\partial x}{\partial \rho} - \frac{\partial \phi}{\partial \tau} = \frac{\partial \psi}{\partial \tau} - \frac{\partial \psi}{\partial \rho} - ix \tag{3.13}$$

So from (3.13), (3.6) and (3.12), one boundary condition is:

$$\frac{1}{s} \frac{\partial \phi}{\partial \rho} - \frac{\partial \phi}{\partial \tau} = \frac{1+s}{1-s} g''(\tau) + \frac{1}{r} g(\tau), \quad \text{at} \quad \rho + \tau = 0 \tag{3.14}$$

and the other is (3.8):

$$\phi(-\tau, \tau) = \frac{2s}{1-s} g'(\tau) \tag{3.15}$$

Restating initial conditions:

$$\phi(\rho, 0) = \phi_0(\rho), \; 0 < 0 \tag{3.16}$$

and, from (3.5)

$$\frac{\partial \phi}{\partial \tau}(\rho, 0) = i x_0(\rho) + \psi_0(\rho) \tag{3.17}$$

IV. Reduction of Problem to an Integral Equation for $g(\tau)$

One proceeds by the method of Riemann for solving the initial value problem for a second-order linear hyperbolic partial differential equation. Now the Eq. (3.11) has characteristics: ($1/\sqrt{s}$ is the sound velocity in the units of the problem)

$$\rho - \rho_1 = (1/\sqrt{s})(\tau - \tau_1)$$

and

$$\rho - \rho_1 = -(1/\sqrt{s})(\tau - \tau_1) \tag{4.1}$$

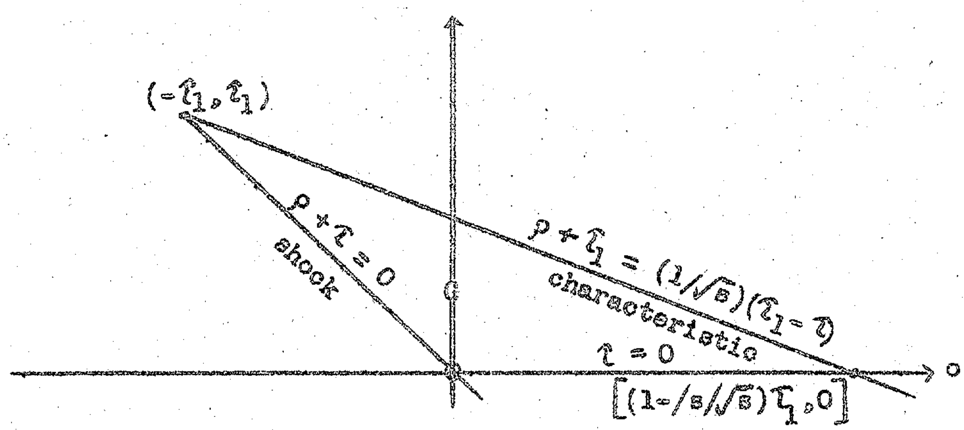
[redacted]

At a point $(-\tau_1, \tau_1)$ on the shock, just one characteristic comes from the shocked material, namely:

$$\rho + \tau_1 = (1/\sqrt{s})(\tau_1 - \tau) \tag{4.2}$$

Conditions at $(-\tau_1, \tau_1)$ are influenced only by what has happened in the region G defined by:

$$\begin{aligned} &\tau \geq 0 \\ &\rho + \tau \geq 0 \\ &\rho + \tau_1 \leq (1/\sqrt{s})(\tau_1 - \tau) \end{aligned} \tag{4.3}$$



One now finds an appropriate "Riemann function" for (3.11) and region G (analogous to the Green function for an elliptic differential equation). This should be a particular solution of the self-adjoint equation (3.11), say $f(\rho, \tau, \rho_0, \tau_0) \equiv f(\rho_0, \tau_0, \rho, \tau)$. Such a function can be found, assuming it depends only on the quantity.

$$R = \sqrt{(1/s)(\tau_0 - \tau)^2 - (\rho - \rho_0)^2} \equiv \sqrt{\left\{(1/\sqrt{s})(\tau_0 - \tau) - (\rho - \rho_0)\right\}\left\{(1/\sqrt{s})(\tau_0 - \tau) + (\rho - \rho_0)\right\}} \tag{4.4}$$

Substituting in (3.11)

$$f''(R)\,\frac{(\tau_0 - \tau)^2}{sR^2} + f'(R)\left\{\frac{1}{R} - \frac{(\tau - \tau_0)^2}{sR^3}\right\} - f''(R)\,\frac{(\rho - \rho_0)^2}{R^2} + f'(R)\left\{\frac{1}{R} + \frac{(\rho - \rho_0)^2}{R^3}\right\} + f(R) = 0$$

or:

[redacted]

$$f''(R) + (1/R)\, f'(R) + f(R) = 0 \tag{4.5}$$

This is the differential equation for the Bessel functions of order zero. The appropriate Riemann function is:

$$f(R) = J_0\left[\sqrt{(1/s)(\tau_0-\tau)^2 - (\rho - \rho_0)^2}\right] \tag{4.6}$$

For the region G we take $\tau_0 = \tau_1$, and $\rho_0 = -\tau_1$.

L denoting the operator $s(\partial^2/\partial\tau^2) - (\partial^2/\partial\rho^2)$, then from (3.11):

$$fL\phi - \phi Lf = -f\phi + \phi f = 0 \tag{4.7}$$

Also:

$$fL\phi - \phi Lf = s\,\frac{\partial}{\partial\tau}\left\{f\,\frac{\partial\phi}{\partial\tau} - \phi\,\frac{\partial f}{\partial\tau}\right\} - \frac{\partial}{\partial\rho}\left\{f\,\frac{\partial\phi}{\partial\rho} - \phi\,\frac{\partial f}{\partial\rho}\right\} \tag{4.8}$$

Integrating this divergence expression over G and employing Gauss' theorem to express the surface integral as a line integral around the boundary:

$$\begin{aligned} 0 = \iint_G (fL\phi - \phi Lf)\, d\sigma = &- s\int_0^{[(1-\sqrt{s})/\sqrt{s}]\tau_1} \left\{f\,\frac{\partial\phi}{\partial\tau} - \phi\,\frac{\partial f}{\partial\tau}\right\}_{\tau=0} d\rho \\ &+ \int_0^{\tau_1}\left[\sqrt{s}\left\{f\,\frac{\partial\phi}{\partial\tau} - \phi\,\frac{\partial f}{\partial\tau}\right\} - \left\{f\,\frac{\partial\phi}{\partial\rho} - \phi\,\frac{\partial f}{\partial\rho}\right\}\right]_{\rho+\tau_1=(\tau_1-\tau)/\sqrt{s}} d\tau \\ &- \int_0^{\tau_1}\left[s\left\{f\,\frac{\partial\phi}{\partial\tau} - \phi\,\frac{\partial f}{\partial\tau}\right\} - \left\{f\,\frac{\partial\phi}{\partial\rho} - \phi\,\frac{\partial f}{\partial\rho}\right\}\right]_{\rho+\tau=0} d\tau \end{aligned} \tag{4.9}$$

Now, at $\tau = 0$:

$$f = J_0\left[\sqrt{\tau_1^2/s - (\rho+\tau_1)^2}\right]$$

and

$$\frac{\partial f}{\partial\tau} = \frac{\tau_1\, J_1\left[\sqrt{\tau_1^2/s - (\rho+\tau_1)^2}\right]}{s\sqrt{\tau_1^2/s - (\rho+\tau_1)^2}}$$

at $\rho + \tau_1 = \dfrac{\tau_1-\tau}{\sqrt{s}}$:

$$f = 1,\quad \frac{\partial f}{\partial\tau} = \frac{\tau_1-\tau}{2s},\ \text{and}\ \frac{\partial f}{\partial\rho} = \frac{\rho+\tau_1}{2} = \sqrt{s}\,\frac{\partial f}{\partial\tau} \tag{4.10}$$

[redacted]

and at $\rho+\tau=0$:

$$f = J_0\left\{\sqrt{\frac{1-s}{s}}\,(\tau_1-\tau)\right\},\quad \frac{\partial f}{\partial \tau} = \frac{J_1\left\{\sqrt{(1-s)/s}\,(\tau_1-\tau)\right\}}{\sqrt{s(1-s)}}$$

$$\text{and}\quad \frac{\partial f}{\partial \rho} \quad \frac{\sqrt{s}\,J_1\left\{\sqrt{(1-s)/s}\,(\tau_1-\tau)\right\}}{\sqrt{1-s}} = s\,\frac{\partial f}{\partial \tau}$$

And for ϕ, there are the initial conditions (3.16), (3.17) and the boundary conditions (3.14), (3.15) at $\rho+\tau=0$. Hence (4.9) becomes:

$$0 = -s\int_0^{[(1-\sqrt{s})/\sqrt{s}]\tau_1}\left[\left\{i\chi_0(\rho)+\psi_0'(\rho)\right\} J_0\left(\sqrt{\tau_1^2/s-(\rho+\tau_1)^2}\right) - \phi_0(\rho)\,\frac{\tau_1\,J_1\left(\sqrt{\tau_1^2/s-(\rho+\tau_1)^2}\right)}{s\sqrt{\tau_1^2/s-(\rho+\tau_1)^2}}\right]d\rho$$

$$+\int_0^{\tau_1}\left\{\sqrt{s}\,\frac{\partial\phi}{\partial\tau}-\frac{\partial\phi}{\partial\rho}\right\}_{\rho+\tau_1=(\tau_1-\tau)/\sqrt{s}}\,d\tau \qquad (4.11)$$

$$+s\int_0^{\tau_1}\left[\frac{1+s}{1-s}\,g''(\tau)+\frac{1}{r}\,g(\tau)\right] J_0\left\{\sqrt{\frac{1-s}{s}}\,(\tau_1-\tau)\right\}$$

Along

$$\rho+\tau_1 = (\tau_1-\tau)/\sqrt{s},\quad \frac{d}{d\tau} = \frac{\partial}{\partial\tau}-\frac{1}{\sqrt{s}}\,\frac{\partial}{\partial\xi},$$

so the integration along the characteristic can be carried out. If this is done, and if furthermore we write τ for τ_1, and σ for τ as variable of integration, we then have the following integro-differential equation for $g(\tau)$:

$$\frac{2\sqrt{s}}{1-s}\,g'(\tau)+\int_0^{\tau}\left[\frac{1+s}{1-s}\,g''(\sigma)+\frac{1}{r}\,g(\sigma)\right] J_0\left\{\sqrt{\frac{1-s}{s}}\,(\tau-\sigma)\right\}d\sigma = \ell(\tau) \qquad (4.12)$$

where $\ell(\tau)$ is a known function in terms of the initial values:

[redacted]

[redacted]

$$\ell(\tau) = \frac{1}{\sqrt{s}}\,\phi_0\left(\frac{1-\sqrt{s}}{\sqrt{s}}\,\tau\right)$$

$$+\int_0^{[(1-\sqrt{s})/\sqrt{s}]\tau}\left[\left\{i\chi_0(\rho)+\psi_0'(\rho)\right\}J_0\left(\sqrt{\tau^2/s-(\rho+\tau)^2}\right)-\phi_0(\rho)\,\frac{\tau J_1\left(\sqrt{\tau^2/s-(\rho+\tau)^2}\right)}{s\sqrt{\tau^2/s-(\rho+\tau)^2}}\right]d\rho \tag{4.13}$$

or

$$\ell(\tau) = \frac{1}{\sqrt{s}}\,\phi_0\left(\frac{1-\sqrt{s}}{\sqrt{s}}\,\tau\right)+\psi_0\left(\frac{1-\sqrt{s}}{\sqrt{s}}\,\tau\right)-\frac{1+s}{1-s}\,g'(0)\,J_0\left(\sqrt{\frac{1-s}{s}}\,\tau\right)$$

$$+\int_0^{[(1-\sqrt{s})/\sqrt{s}]\tau}\left[i\chi_0(\rho)J_0\left(\sqrt{\tau^2/s-(\rho+\tau)^2}\right)-\left\{(\rho+\tau)\psi_0(\rho)+\frac{\tau}{s}\,\phi_0(\rho)\right\}\frac{J_1\left(\sqrt{\tau^2/s-(\rho+\tau)^2}\right)}{\sqrt{\tau^2/s-(\rho+\tau)^2}}\right]d\rho$$

In general in physical cases in which we are interested, the initial disturbance will be limited to some finite region. Say, for example, that $\chi_0(\rho)$, $\psi_0(\rho)$, $\phi_0(\rho)$ all vanish for $\rho > \rho_1$, for some fixed ρ_1, then for $\tau > \left[\sqrt{s}/(1-\sqrt{s})\right]\rho_1$ we have:

$$\ell(\tau) = -\frac{1+s}{1-s}\,g'(0)\,J_0\left\{\sqrt{\frac{1-s}{s}}\,\tau\right\}+\int_0^{\rho_1}\left[i\chi_0(\rho)\,J_0\left(\sqrt{\tau^2/s-(\rho+\tau)^2}\right)\right.$$

$$\left.-\left\{(\rho+\tau)\,\psi_0(\rho)+\frac{\tau}{s}\,\phi_0(\rho)\right\}\frac{J_1\left(\sqrt{\tau^2/s-(\rho+\tau)^2}\right)}{\sqrt{\tau^2/s-(\rho+\tau)^2}}\right]d\rho$$

and since for $x \longrightarrow \infty$, both $J_0(x) \longrightarrow 0$ and $J_1(x) \longrightarrow 0$, we then have:

for $$\tau \longrightarrow \infty, \qquad \ell(\tau) \longrightarrow 0 \tag{4.14}$$

Moreover, it is seen that in such a case $\int_0^\infty \ell(\tau)d\tau$ exists, as both $\int_0^\infty J_0(x)dx$ and $\int_0^\infty J_1(x)dx$ exist.

In the special case of a disturbance limited initially to the immediate neighborhood of the shock front, we have for $\tau > 0$,

$$\ell(\tau) = \ell_0(\tau) = -\frac{1+s}{1-s}\,g'(0)\,J_0\left\{\sqrt{\frac{1-s}{s}}\,\tau\right\} \tag{4.15}$$

In obtaining an analytical expression for $g(\tau)$ by, [illegible] Laplace transforms, it is only slightly more convenient to reduce (4.12) to a Volterra integral equation. In any case, if it is desired to calculate $g(\tau)$ numerically for some particular $\ell(\tau)$, it is definitely more convenient to reduce (4.12) to such a form. This is done by integrating by parts under the integral sign on the left of (4.12), and then integrating the entire equation once with respect to τ. There results the following equation

$$g(\tau) = f(\tau) + \int_0^\tau g(\sigma)\, K(\tau-\sigma)\, d\sigma \tag{4.16}$$

where the kernel is:

$$K(x) = \frac{1-\sqrt{s}}{1+\sqrt{s}}\sqrt{\frac{s}{1-s}}\left\{\frac{1+s}{s}\, J_1\left(\sqrt{\frac{1-s}{s}}\, x\right) - \frac{1}{r}\int_0^{\sqrt{(1-s)/s}\, x} J_o(w)\, dw\right\} \tag{4.17}$$

and where

$$f(\tau) = \frac{1-\sqrt{s}}{1+\sqrt{s}}\int_0^\tau \left\{\ell(\tau) + \frac{1+s}{1-s}\, g'(0)\, J_o\left(\sqrt{\frac{1-s}{s}}\,\tau\right)\right\} d\tau + \frac{g(0)}{(1+\sqrt{s})^2}\left\{2\sqrt{s} + (1+s)\, J_o\left(\frac{1-s}{s}\,\tau\right)\right\} \tag{4.18}$$

From (4.18) and the above-discussed properties of $\ell(\tau)$, it is seen that for initial disturbances limited to a finite region, that:

for $$\tau \to \infty,\ f(\tau) \Rightarrow C,\ C \text{ some finite constant} \tag{4.19}$$

For the special case of a disturbance initially only in the immediate neighborhood of the shock:

$$f(\tau) = f_o(\tau) = \frac{g(0)}{(1+\sqrt{s})^2}\left\{2\sqrt{s} + (1+s)\, J_o\left(\sqrt{(1-s)/s}\,\tau\right)\right\} \tag{4.20}$$

V. Solution of the Integral Equation

Taking the Laplace transform of (4.16), making use of the fact that

$$\int_0^\infty e^{-\alpha\tau}\left\{\int_0^\tau f_1(\sigma) f_2(\tau-\sigma)\, d\sigma\right\} d\tau = \int_0^\infty\int_\sigma^\infty e^{-\alpha\sigma} f_1(\sigma) e^{-\alpha(\tau-\sigma)} f_2(\tau-\sigma)\, d\tau\, d\sigma$$

$$= \int_0^\infty e^{-\alpha\sigma} f_1(\sigma)\, d\sigma \cdot \int_0^\infty e^{-\alpha\xi} f_2(\xi)\, d\xi,\quad \xi = \tau - \sigma \tag{5.1}$$

$$= f_{1L}(\alpha) \cdot f_{2L}(\alpha)$$

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we have:

$$g_L(\alpha) = f_L(\alpha) + g_L(\alpha) \cdot K_L(\alpha)$$

or

$$g_L(\alpha) = \frac{f_L(\alpha)}{1 - K_L(\alpha)} \qquad (5.2)$$

The Laplace transforms of the Bessel functions are known:

$$\int_0^\infty e^{-\alpha x}\, J_1\left(\sqrt{\frac{1-s}{s}}\, x\right) dx = \sqrt{\frac{s}{1-s}} \left\{1 - \frac{\alpha}{\sqrt{\alpha^2 + (1-s)/s}}\right\}$$

$$\int_0^\infty e^{-\alpha x} \int_0^{\sqrt{1-s/s}\, x} J_o(w)\, dw\, dx = \frac{\sqrt{(1-s)/s}}{\alpha\sqrt{\alpha^2 + (1-s)/s}} \qquad (5.3)$$

So for the kernel $K(x)$:

$$K_L(\alpha) = \frac{1-\sqrt{s}}{1+\sqrt{s}} \left\{\frac{1+s}{1-s} - \frac{(1+s)\alpha/(1-s) + 1/\alpha r}{\sqrt{\alpha^2 + (1-s)/s}}\right\} \qquad (5.4)$$

and

$$\frac{1}{1 - K_L(\alpha)} = \frac{(1+\sqrt{s})^2\, \alpha\sqrt{\alpha^2 + (1-s)/s}}{2\sqrt{s}\, \alpha\sqrt{\alpha^2 + (1-s)/s} + (1+s)\,\alpha^2 + (1-s)/r} \qquad (5.5)$$

For $\alpha \longrightarrow \infty$, $1/\left[1 - K_L(\alpha)\right] \longrightarrow 1$, so that the coefficient of $f_L(\alpha)$ in (5.2) contains a term which is the transform of the delta function. This leads us to write (5.2) in the form:

$$g_L(\alpha) = f_L(\alpha) + \frac{(1+s)\alpha\left\{\sqrt{\alpha^2 + (1-s)/s} - \alpha\right\} - (1-s)/r}{2\sqrt{s}\, \alpha\sqrt{\alpha^2 + (1-s)/s} + (1+s)\,\alpha^2 + (1-s)/r} \cdot f_L(\alpha) \qquad (5.6)$$

and the solution in the form:

$$g(\tau) = f(\tau) + \int_0^\tau f(\sigma)\, \bar{K}(\tau - \sigma)\, d\sigma \qquad (5.7)$$

where the resolving kernel is the function whose Laplace transform is:

$$\bar{K}_L(\alpha) = \frac{(1+s)\alpha\left\{\sqrt{\alpha^2 + (1-s)/s} - \alpha\right\} - (1-s)/r}{2\sqrt{s}\, \alpha\sqrt{\alpha^2 + (1-s)/s} + (1+s)\,\alpha^2 + (1-s)/r} \qquad (5.8)$$

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or:

$$\bar{K}_L(\alpha) = \frac{\frac{1+\sqrt{s}}{r(1-\sqrt{s})}\,\alpha\left(1+\frac{1+s}{1-s}\,r\alpha^2\right)\left\{\sqrt{\alpha^2+(1-s)/s}-\alpha\right\}-1/r^2-\frac{1}{r(1-s)}\left\{(1-\sqrt{s})^2+\frac{2r}{\sqrt{s}}(1+s)\right\}\alpha^2}{\alpha^4+2(1+s-2r)\,\alpha^2/r(1-s)+1/r^2}$$

In order to construct K, one breaks this down into appropriate factors which are known known transforms. In the case s r, this can be written in real form:

$$\begin{aligned}\bar{K}_L(\alpha) = {} & \frac{1}{\alpha^2+\left\{\sqrt{1-r}+\sqrt{s-r}\right\}^2/r(1-s)}\left\{\frac{C_1\alpha}{\alpha+\sqrt{\alpha^2+(1-s)/s}}+C_2\right\} \\ & +\frac{1}{\alpha^2+\left\{\sqrt{1-r}-\sqrt{s-r}\right\}^2/r(1-s)}\left\{\frac{C_3\alpha}{\alpha+\sqrt{\alpha^2+(1-s)/s}}+C_4\right\}\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}\left.\begin{matrix}C_1\\ C_3\end{matrix}\right\} &= \frac{1+\sqrt{s}}{2(1-\sqrt{s})}\left[\frac{1+s}{s}\pm\frac{2-r(1+s)/s}{\sqrt{(1-r)(s-r)}}\right] \\ \left.\begin{matrix}C_2\\ C_4\end{matrix}\right\} &= \frac{1}{2r(1-s)}\left[-(1-\sqrt{s})^2-2r(1+s)/\sqrt{s}\pm\frac{\sqrt{s}(1-\sqrt{s})^2+r\left\{(1-\sqrt{s})^2-(1+s)^2/r\right\}+2r^2(1+s)/\sqrt{s}}{\sqrt{(1-r)(s-r)}}\right]\end{aligned} \tag{5.10}$$

Similarly in the case $s < r$, it is convenient to write:

$$\begin{aligned}\bar{K}_L(\alpha) = {} & \frac{1}{\alpha^2+2\alpha\sqrt{r-s}/\sqrt{r(1-s)}+1/r}\left\{\frac{B_1\alpha+D_1}{\alpha+\sqrt{\alpha^2+(1-s)/s}}+B_2\alpha+D_2\right\} \\ & +\frac{1}{\alpha^2-2\alpha\sqrt{r-s}/\sqrt{r(1-s)}+1/r}\left\{\frac{B_3\alpha+D_3}{\alpha+\sqrt{\alpha^2+(1-s)/s}}\quad B_4\alpha+D_4\right\}\end{aligned} \tag{5.11}$$

where

$$\begin{aligned} B_1 &= B_3 = \frac{1+\sqrt{s}}{2(1-\sqrt{s})}\cdot\frac{1+s}{s} \\ B_2 &= -B_4 = \frac{r(1+s)/\sqrt{s}-\sqrt{s}(1-\sqrt{s})}{2\sqrt{r(1-s)(r-s)}} \\ D_1 &= -D_3 = \frac{(1+\sqrt{s})^2}{2\sqrt{r(1-s)(r-s)}} \\ D_2 &= D_4 = -1/2r \end{aligned} \tag{5.12}$$

[redacted]

We make use of the following known transforms:

$$\frac{1}{(\alpha-\beta)^2+\gamma^2} \text{ is the transform of } e^{\beta x}\,\sin\gamma x \tag{5.13}$$

$$\frac{\alpha}{(\alpha-\beta)^2+\gamma^2} \quad " \quad " \quad " \quad " \quad e^{\beta x}\left\{\cos\gamma x+\frac{\beta}{\gamma}\sin\gamma x\right\} \tag{5.14}$$

and

$$\frac{1}{\alpha+\sqrt{\alpha^2+\gamma^2}} \quad " \quad " \quad " \quad " \quad \frac{J_1(\gamma x)}{\gamma x} \tag{5.15}$$

In the case $s > r$, we use (5.14) taking $\beta=0$ and $\gamma=(\sqrt{1-r}+\sqrt{s-r})/\sqrt{r(1-s)}$, and (5.15) taking $\gamma=\sqrt{(1-s)/s}$. Then:

$$\begin{aligned}\overline{K}(x) = {} & \frac{C_2\sqrt{r(1-s)}}{\sqrt{1-r}+\sqrt{s-r}}\sin\left\{\frac{\sqrt{1-r}+\sqrt{s-r}}{\sqrt{r(1-s)}}\,x\right\}+\frac{C_4\sqrt{r(1-s)}}{\sqrt{1-r}-\sqrt{s-r}}\sin\left\{\frac{\sqrt{1-r}-\sqrt{s-r}}{\sqrt{r(1-s)}}\,x\right\} \\ & + C_1\int_0^x \cos\left\{\frac{\sqrt{1-r}+\sqrt{s-r}}{\sqrt{r(1-s)}}\,(x-y)\right\}\frac{J_1\left(\sqrt{(1-s)/s}\;y\right)}{\sqrt{(1-s)/s}\;y}\,dy \\ & + C_3\int_0^x \cos\left\{\frac{\sqrt{1-r}-\sqrt{s-r}}{\sqrt{r(1-s)}}\,(x-y)\right\}\frac{J_1\left(\sqrt{(1-s)/s}\;y\right)}{\sqrt{(1-s)/s}\;y}\,dy\end{aligned} \tag{5.16}$$

This can be put into different form which will be more convenient for seeing the asymptotic behavior of $R(x)$ for large x. The following definite integrals will be useful:

$$\int_0^\infty e^{-\alpha y}\cos\beta y\,\frac{J_1(\gamma y)}{\gamma y}\,dy = -\alpha/\gamma^2+2^{-\frac{1}{2}}\gamma^{-2}\sqrt{\sqrt{\{\alpha^2+(\beta-\gamma)^2\}\{\alpha^2+(\beta+\gamma)^2\}}-\beta^2+\alpha^2+\gamma^2} \tag{5.17}$$

and

$$\int_0^\infty e^{-\alpha y}\sin\beta y\,\frac{J_1(\gamma y)}{\gamma y}\,dy = \beta/\gamma^2-2^{-\frac{1}{2}}\gamma^{-2}\sqrt{\sqrt{\{\alpha^2+(\beta-\gamma)^2\}\{\alpha^2+(\beta+\gamma)^2\}}+\beta^2-\alpha^2-\gamma^2} \tag{5.18}$$

For immediate purposes we require the special cases where $\alpha=0$. Here we must distinguish cases, as $\sqrt{(\beta^2-\gamma^2)^2}=\beta^2-\gamma^2$ or $\gamma^2-\beta^2$ according as $\beta^2>\gamma^2$ or $\beta^2<\gamma^2$

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$$\int_0^\infty \cos \beta y \, \frac{J_1(\gamma y)}{\gamma y} \, dy = \begin{cases} 0 & \beta^2 \geq \gamma^2 \\ \dfrac{\sqrt{\gamma^2-\beta^2}}{\gamma^2} & \beta^2 \leq \gamma^2 \end{cases} \tag{5.19}$$

$$\int_0^\infty \sin \beta y \, \frac{J_1(\gamma y)}{\gamma y} \, dy = \begin{cases} \dfrac{\beta - \sqrt{\beta^2-\gamma^2}}{\gamma^2} & \beta^2 \geq \gamma^2 \\ \dfrac{\beta}{\gamma^2} & \beta^2 \leq \gamma^2 \end{cases} \tag{5.20}$$

Now, for $\beta = (\sqrt{1-r} \pm \sqrt{s-r})/\sqrt{r(1-s)}$ and $\gamma = \sqrt{(1-s)/s}$;

$$\beta^2 - \gamma^2 = \frac{s - r \pm 2s\sqrt{(1-r)(s-r)} + s^2(1-r)}{rs(1-s)} = \frac{\left\{s\sqrt{1-r} \pm \sqrt{s-r}\right\}^2}{rs(1-s)} \geq 0$$

Moreover, $s\sqrt{1-r} > \sqrt{s-r}$, for

$$s^2(1-r) - (s-r) = s^2 - s + r(1-s^2) = (1-s^2)\left(r - s/(1+s)\right)$$

and, from (1.12), where the γ in these equations is now the γ of the equation of state, so that $\gamma > 1$:

$$r - \frac{s}{1+s} = r - \frac{r}{\gamma(1-r)+r} = \frac{r}{\gamma(1-r)+r}(\gamma-1)(1-r) > 0$$

Hence, in any case:

$$\sqrt{\beta^2-\gamma^2} = \frac{s\sqrt{1-r} \pm \sqrt{s-r}}{\sqrt{rs(1-s)}}$$

So using (5.19) and (5.20) with these values of β and γ:

$$\begin{aligned} \int_0^\infty \cos \beta(x-y) \, \frac{J_1(\gamma y)}{\gamma y} \, dy &= \cos \beta x \int_0^\infty \cos \beta y \, \frac{J_1(\gamma y)}{\gamma y} \, dy \\ &\quad + \sin \beta x \int_0^\infty \sin \beta y \, \frac{J_1(\gamma y)}{\gamma y} \, dy \\ &= \frac{\beta - \sqrt{\beta^2-\gamma^2}}{\gamma^2} \sin \beta x \end{aligned} \tag{5.21}$$

or

$$\int_0^\infty \cos\left\{\frac{\sqrt{1-r} \pm \sqrt{s-r}}{\sqrt{r(1-s)}}(x-y)\right\} \frac{J_1\left(\sqrt{(1-s)/s}\; y\right)}{\sqrt{(1-s)/s}\; y} \, dy = \frac{s\sqrt{1-r} \mp \sqrt{s(s-r)}}{(1+\sqrt{s})\sqrt{r(1-s)}} \sin\left\{\frac{\sqrt{1-r} \pm \sqrt{s-r}}{\sqrt{r(1-s)}}(x-y)\right\}$$

Performing the appropriate algebra shows that:

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$$C_1\,\frac{s\sqrt{1-r}-\sqrt{s(s-r)}}{\sqrt{r(1-s)}\,(1+\sqrt{s})} = -\frac{C_2\sqrt{r(1-s)}}{\sqrt{1-r}+\sqrt{s-r}} \qquad (5.22)$$

with similar relation between C_3 and C_4, hence:

$$\frac{C_2\sqrt{r(1-s)}}{\sqrt{1-r}+\sqrt{s-r}}\,\sin\left\{\frac{\sqrt{1-r}+\sqrt{s-r}}{\sqrt{r(1-s)}}\,x\right\} = -C_1\int_0^\infty \cos\left\{\frac{\sqrt{1-r}+\sqrt{s-r}}{\sqrt{r(1-s)}}\,(x-y)\right\}\frac{J_1\left(\sqrt{(1-s)/s}\;y\right)}{\sqrt{(1-s)/s}}\,dy \qquad (5.23)$$

and the similar relation with C_3 and C_4. But since $-\int_0^\infty + \int_0^x = -\int_x^\infty$, we can by (5.23) write $\overline{K}(x)$ in the form:

$$\text{for } s > r,\ \overline{K}(x) = -C_1\int_x^\infty \cos\left\{\frac{\sqrt{1-r}+\sqrt{s-r}}{\sqrt{r(1-s)}}\,(x-y)\right\}\frac{J_1\left(\sqrt{(1-s)/s}\;y\right)}{\sqrt{(1-s)/s}\;y}\,dy - C_3\int_x^\infty \cos\left\{\frac{\sqrt{1-r}-\sqrt{s-r}}{\sqrt{r(1-s)}}\,(x-y)\right\}\frac{J_1\left(\sqrt{(1-s)/s}\;y\right)}{\sqrt{(1-s)/s}\;y}\,dy$$

or using the trigonometric addition formula, making a change of variable: $y = z + x$, and using (5.10):

$$\text{for } s > r,\quad \overline{K}(x) = \frac{1+\sqrt{s}}{1-\sqrt{s}}\int_0^\infty \left[-\frac{1+s}{s}\cos\sqrt{\frac{1-r}{r(1-s)}}\,z\,\cos\sqrt{\frac{s-r}{r(1-s)}}\,z + \frac{2-r(1+s)/s}{\sqrt{(1-r)(s-r)}}\sin\sqrt{\frac{1-r}{r(1-s)}}\,z\,\sin\sqrt{\frac{s-r}{r(1-s)}}\,z\right]\frac{J_1\left\{\sqrt{(1-s)/s}\;(x+z)\right\}}{\sqrt{(1-s)/s}\;(x+z)}\,dz \qquad (5.24)$$

As a limiting case of this, we can obtain the expression for $\overline{K}(x)$ in the case $s = r$. Namely:

$$\text{for } s = r,\ \overline{K}(x) = \frac{1+\sqrt{r}}{1-\sqrt{r}}\int_0^\infty \left[-\frac{1+r}{r}\cos\frac{z}{\sqrt{r}} + \frac{z}{\sqrt{r}}\sin\frac{z}{\sqrt{r}}\right]\frac{J_1\left\{\sqrt{(1-r)/r}\;(x+z)\right\}}{\sqrt{(1-r)/r}\;(x+z)}\,dz \qquad (5.25)$$

Proceeding in similar manner for the case $s < r$, taking in (5.13) and (5.14) $\beta = \mp\sqrt{(r-s)/r(1-s)}$ and $\gamma = \sqrt{(1-r)/r(1-s)}$, and in (5.15) $\gamma = \sqrt{(1-s)/s}$, we have from (5.11):

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$$\overline{K}(x) = e^{-\sqrt{(r-s)/r(1-s)}\;x}\left\{B_2 \cos\sqrt{\frac{1-r}{r(1-s)}}\,x + \left(-B_2\sqrt{\frac{r-s}{1-r}} + D_2\sqrt{\frac{r(1-s)}{1-r}}\right)\sin\sqrt{\frac{1-r}{r(1-s)}}\,x\right\}$$

$$+ e^{+\sqrt{(r-s)/r(1-s)}\;x}\left\{B_4 \cos\sqrt{\frac{1-r}{r(1-s)}} + \left(B_4\sqrt{\frac{r-s}{1-r}} + D_4\sqrt{\frac{r(1-s)}{1-r}}\right)\sin\sqrt{\frac{1-r}{r(1-s)}}\,x\right\} \tag{5.26}$$

$$+ \int_0^x e^{-\sqrt{(r-s)/r(1-s)}\;(x-y)}\left\{B_1 \cos\sqrt{\frac{1-r}{r(1-s)}}\,(x-y) + \left(-B_1\sqrt{\frac{r-s}{1-r}} + D_1\sqrt{\frac{r(1-s)}{1-r}}\right)\sin\sqrt{\frac{1-r}{r(1-s)}}\,(x-y)\right\}\frac{J_1\left(\sqrt{(1-s)/s}\;y\right)}{\sqrt{(1-s)/s}\;\;y}\,dy$$

$$+ \int_0^x e^{+\sqrt{(r-s)/r(1-s)}\;(x-y)}\left\{B_3 \cos\sqrt{\frac{1-r}{r(1-s)}}\,(x-y) + \left(B_3\sqrt{\frac{r-s}{1-r}} + D_3\sqrt{\frac{r(1-s)}{1-r}}\right)\sin\sqrt{\frac{1-r}{r(1-s)}}\,(x-y)\right\}\frac{J_1\left(\sqrt{(1-s)/s}\;y\right)}{\sqrt{(1-s)/s}\;\;y}\,dy$$

Putting in the values of the B's and D's from (5.12); and noting that (5.17) and (5.18) give, on taking $\alpha = \sqrt{(r-s)/r(1-s)}$, $\beta = \sqrt{(1-r)/r(1-s)}$, and $\gamma = \sqrt{(1-s)/s}$ and performing some algebra, the following:

$$\int_0^\infty e^{\sqrt{(r-s)/r(1-s)}\;(x-y)}\begin{pmatrix}\cos\\ \sin\end{pmatrix}\sqrt{\frac{1-r}{r(1-s)}}\,(x-y)\;\frac{J_1\left(\sqrt{(1-s)/s}\;y\right)}{\sqrt{(1-s)/s}\;y}\,dy$$

$$= \frac{1}{(1+\sqrt{s})\sqrt{r(1-s)}}\,e^{\sqrt{(r-s)/r(1-s)}\;x}\left\{\sqrt{s(r-s)}\begin{pmatrix}\cos\\ \sin\end{pmatrix}\sqrt{\frac{1-r}{r(1-s)}}\,x \pm s\sqrt{1-r}\begin{pmatrix}\sin\\ \cos\end{pmatrix}\sqrt{\frac{1-r}{r(1-s)}}\,x\right\} \tag{5.27}$$

it is seen that $\overline{K}(x)$ can be written:

for $s < r$:

$$\overline{K}(x) = \frac{1}{2\sqrt{r(1-s)}}\,e^{-\sqrt{(r-s)/r(1-s)}\;x}\left\{\frac{r(1+s)/\sqrt{s} - \sqrt{s}(1-\sqrt{s})}{\sqrt{r-s}}\cos\sqrt{\frac{1-r}{r(1-s)}}\,x - \frac{r(1+s)/\sqrt{s} + 1 - \sqrt{s}}{\sqrt{1-r}}\sin\sqrt{\frac{1-r}{r(1-s)}}\,x\right\}$$

$$+ \frac{1+\sqrt{s}}{2(1-\sqrt{s})}\int_0^x e^{-\sqrt{(r-s)/r(1-s)}\;(x-y)}\left[\frac{1+s}{s}\cos\sqrt{\frac{1-r}{r(1-s)}}\,(x-y) + \frac{2-r(1+s)/s}{\sqrt{(1-r)(r-s)}}\sin\sqrt{\frac{1-r}{r(1-s)}}\,(x-y)\right]\frac{J_1\left(\sqrt{(1-s)/s}\;y\right)}{\sqrt{(1-s)/s}\;y}\,dy \tag{5.28}$$

$$- \frac{1+\sqrt{s}}{2(1-\sqrt{s})}\int_0^\infty e^{-\sqrt{(r-s)/r(1-s)}\;z}\left[\frac{1+s}{s}\cos\sqrt{\frac{1-r}{r(1-s)}}\,z + \frac{2-r(1+s)/s}{\sqrt{(1-r)(r-s)}}\sin\sqrt{\frac{1-r}{r(1-s)}}\,z\right]\frac{J_1\left\{\sqrt{(1-s)/s}\;(x+z)\right\}}{\sqrt{(1-s)/s}\;(x+z)}\,dz$$

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Having the explicit expression for the resolving kernel $\overline{K}(x)$ in any of the forms (5.24), (5.25) or (5.28), the problem is now in principle solved for any given initial disturbance, for by (5.7)

$$g(\tau) = f(\tau) + \int_0^{\tau} f(\sigma)\, \overline{K}(\tau-\sigma)\, d\sigma \qquad (5.29)$$

In any practical case the numerical integrations that would have to be carried out in these cases would be prohibitive. We should, nevertheless, like to get some idea of the behavior of $g(\tau)$ for large values of τ, and in the general case of an initial disturbance which is finite in extent. That is, for an $f(\tau)$ having the property (4.19). Now we know the asymptotic behavior of $J_1(x)$:

for large x, $$J_1(x) \sim \sqrt{(2/\pi x)}\, \sin(x - \pi/4) \qquad (5.30)$$

From this it is seen from the above expressions for $\overline{K}(x)$, that:

$$\text{as } x \to \infty,\ \overline{K}(x) \to 0 \text{ at least as rapidly as } x^{-3/2} \qquad (5.31)$$

In all cases, then $\int_0^{\infty} \overline{K}(x)dx$ exists and can then in fact be evaluated as the limit of (5.8) for $\alpha \to 0$. It is seen from this that:

$$\int_0^{\infty} \overline{K}(x)dx = -1 \qquad (5.32)$$

Hence:

$$\text{as} \quad \tau \to \infty \quad g(\tau) \to C + \int_0^{\infty} C\, \overline{K}(\sigma)d\sigma = C - C$$

$$\text{or} \quad g(\tau) \to 0 \qquad (5.33)$$

As a general result, we can consequently state that, for an initial disturbance finite in extent, the disturbance on the shock front will eventually be damped out. While the rate of damping depends on the particular $f(\tau)$, it appears from the above that it is at least as $\tau^{-\frac{1}{2}}$.

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VI. Further Analysis of the Special Case of an Initial Disturbance Limited to the Immediate Neighborhood of the Shock Front

We shall derive the analytical expressions for g( ) in the special case mentioned in (4.20) above. Namely, for:

$$f(\tau) = f_o(\tau) = \frac{g(0)}{(1+\sqrt{s})^2}\left\{2\sqrt{s} + (1+s)\, J_o\left(\sqrt{(1-s)/s}\ \tau\right)\right\} \tag{6.1}$$

Instead of doing this by integration in the form (5.7) it is perhaps simpler to proceed again by Laplace transforms. For this:

$$g_L(\alpha) = \frac{f_{oL}(\alpha)}{1 - K_L(\alpha)} \tag{6.2}$$

Knowing the transform of the Bessel function:

$$\int_0^\infty e^{-\alpha\tau} J_o\left(\sqrt{(1-s)/s}\ \tau\right) d\tau = \frac{1}{\sqrt{\alpha^2 + (1-s)/s}} \tag{6.3}$$

we have:

$$f_{oL}(\alpha) = \frac{1}{(1+\sqrt{s})^2}\left[\frac{2\sqrt{s}}{\alpha} + \frac{1+s}{\sqrt{\alpha^2 + (1-s)/s}}\right] \tag{6.4}$$

where g(0) has been taken as 1, for convenience. Using, then, (5.5), we have:

$$g_L(\alpha) = \frac{1}{r(1-s)} \cdot \frac{2\sqrt{s}\left\{\sqrt{\alpha^2 + (1-s)/s} - \alpha\right\} + r(1-s)\,\alpha^3 + \left\{(1+\sqrt{s})^2 - 4r\right\}\alpha}{\alpha^4 + 2(1+s-2r)\,\alpha^2/r(1-s) + 1/r^2} \tag{6.5}$$

If then we proceed as we did in arriving at expressions for $\overline{K}(x)$, we secure:

for $s > r$,

$$g(\tau) = \sqrt{\frac{r(1-s)}{s}}\int_0^\infty\left[\frac{1}{\sqrt{1-r}}\sin\sqrt{\frac{1-r}{r(1-s)}}\,z\,\cos\sqrt{\frac{s-r}{r(1-r)}}\,z \right.$$

$$\left. - \frac{1}{\sqrt{s-r}}\cos\sqrt{\frac{1-r}{r(1-s)}}\,z\,\sin\sqrt{\frac{s-r}{r(1-s)}}\,z\right]\frac{J_1\left\{\sqrt{(1-s)/s}\ (\tau+z)\right\}}{\sqrt{(1-s)/s}\ (\tau+z)}\,dz \tag{6.6}$$

for $s = r$

$$g(\tau) = \int_0^\infty\left[\sin\frac{z}{\sqrt{r}} - \frac{z}{\sqrt{r}}\cos\frac{z}{\sqrt{r}}\right]\frac{J_1\left\{\sqrt{(1-r)/r}\ (\tau+z)\right\}}{\sqrt{(1-r)/r}\ (\tau+z)}\,dz \tag{6.7}$$

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and for $s < r$

$$g(\tau) = \tfrac{1}{2}\, e^{-\sqrt{(r-s)/r(1-s)}\;\tau} \left[\cos \sqrt{\frac{1-r}{r(1-s)}}\;\tau - \frac{\sqrt{s}-r}{\sqrt{(1-r)(r-s)}} \sin \sqrt{\frac{1-r}{r(1-s)}}\;\tau\right]$$

$$+ \tfrac{1}{2}\sqrt{\frac{r(1-s)}{s}} \int_0^{\tau} e^{-\sqrt{(r-s)/r(1-s)}\;(\tau-\sigma)} \left[\frac{1}{\sqrt{1-r}} \sin \sqrt{\frac{1-r}{r(1-s)}}\;(\tau-\sigma) + \frac{1}{\sqrt{r-s}} \cos \sqrt{\frac{1-r}{r(1-s)}}\;(\tau-\sigma)\right] \frac{J_1\left(\sqrt{(1-s)/s}\;\sigma\right)}{\sqrt{(1-s)/s}\;\sigma}\, d\sigma$$

$$+ \tfrac{1}{2}\sqrt{\frac{r(1-s)}{s}} \int_0^{\infty} e^{-\sqrt{(r-s)/r(1-s)}\;z} \left[\frac{1}{\sqrt{1-r}} \sin \sqrt{\frac{1-r}{r(1-s)}}\;z + \frac{1}{\sqrt{r-s}} \cos \sqrt{\frac{1-r}{r(1-s)}}\;z\right] \frac{J_1\left\{\sqrt{(1-s)/s}\;(\tau+z)\right\}}{\sqrt{(1-s)/s}\;(\tau+z)}\, dz \qquad (6.8)$$

Numerical results have been obtained in one special case, namely for $f(\tau)$ of this form, and for the particular choice of parameters:

$$r = 1/2 \qquad s = 1/3 \qquad (6.9)$$

This corresponds to $\gamma = 3$ in the equation of state, for:

$$s = r/\gamma(1-r) = 1/3 \quad \text{if} \quad \gamma = 3 \quad \text{and} \quad r = 1/2$$

The solution was not obtained from the above analytical expressions, but by the IBM machines which solved the integral equation (4.16) by finite differences. The graph of $g(\tau)$ for this case is shown in Fig. 1. The amplitude has fallen to about 7.5% of its initial value when the second maximum is reached, which is about by the time the shock has traveled the distance of two wave lengths into the undisturbed material.

The analytical expression for this case is:

$$g(\tau) = \tfrac{1}{2}\, e^{-\tau/\sqrt{2}} \left[\cos \sqrt{3/2}\;\tau - (2-\sqrt{3}) \sin \sqrt{3/2}\;\tau\right]$$

$$+ \tfrac{1}{2} \int_0^{\tau} e^{-(\tau-\sigma)/\sqrt{2}} \left[\sin \sqrt{3/2}\;(\tau-\sigma) + \sqrt{3} \cos \sqrt{3/2}\;(\tau-\sigma)\right] \frac{J_1(\sqrt{2}\;\sigma)}{\sigma}\, d\sigma \qquad (6.10)$$

$$+ \tfrac{1}{2} \int_0^{\infty} e^{-\sigma/\sqrt{2}} \left[\sin \sqrt{3/2}\;\sigma + \sqrt{3} \cos \sqrt{3/2}\;\sigma\right] \frac{J_1\left\{\sqrt{2}(\sigma+\tau)\right\}}{\sigma+\tau}\, d\sigma$$

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The integrals contribute terms which for large $\hat{t}$ behave as $\hat{t}^{-3/2}$ times an oscillating function.

VII. Solution for Infinite Wave Length

Since, in the above discussion, we have employed as independent variables: $\hat{t} = krUt$ and $\rho = kz$, we have precluded the possibility of getting from the results the solution for $k = 0$. However, this is easily derived.

For $k = 0$, the original differential equations are:

$$\frac{\partial u_x^1}{\partial t} = 0$$

$$\frac{\partial u_z^1}{\partial t} - c^2 \frac{\partial}{\partial z}\left(\frac{v^1}{v}\right) = 0 \qquad (7.1)$$

and

$$\frac{\partial u_z^1}{\partial z} - \frac{\partial}{\partial t}\left(\frac{v^1}{v}\right) = 0$$

It is clear that u_x^1 is zero. Adding $-c$ times the third equation to the second:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial z}\right)\left(u_z^1 + c\,\frac{v^1}{v}\right) = 0 \qquad (7.2)$$

which states that the quantity $u_x^1 + c(v^1/v)$ is propagated as a constant along the characteristics $z + ct = \text{const}$. The characteristic of this family through the point $(-rUt_1, t_1)$ is:

$$z + rUt_1 = c(t_1 - t) \qquad (7.3)$$

which intersects the $t = 0$ line at $\left\{(c-rU)t_1,\ 0\right\}$ or $\left(rU\left[(1-\sqrt{s})/s\right]\ t_1, 0\right)$, using the definition of s. Hence we have, as $u_z^1 + c(v^1/v)$ has the same value at all points of (7.3):

$$\left(u_z^1 + c\,\frac{v^1}{v}\right)_{\substack{t=t_1 \\ z=-rUt_1}} = \left(u_z^1 + c\,\frac{v^1}{v}\right)_{\substack{t=0 \\ z=rU\left[(1-\sqrt{s})/s\right]t_1}} \qquad (7.4)$$

[redacted]

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Applying the boundary and initial conditions, then:

$$\frac{1+\sqrt{s}}{1-\sqrt{s}}(1-r)\frac{dh(t_1)}{dt} = u_z^1\left(rU\,\frac{1-\sqrt{s}}{\sqrt{s}}\,t_1, 0\right) + \frac{rU}{\sqrt{s}}\,\frac{v^1\left(rU\left[(1-\sqrt{s})/\sqrt{s}\right]t_1, 0\right)}{v} \tag{7.5}$$

Integrating:

$$h(t) - h(0) = \frac{1-\sqrt{s}}{(1+\sqrt{s})(1-r)} \int_0^t \left[u_z^1 + \frac{rU}{\sqrt{s}}\,\frac{v^1}{v}\right]_{\substack{t=0 \\ z=rU\left[(1-\sqrt{s})/\sqrt{s}\right]t_1}} dt_1$$

or

$$h(t) = h(0) + \frac{\sqrt{s}}{(1+\sqrt{s})(1-r)} \int_0^{rU\left[(1-\sqrt{s})/\sqrt{s}\right]t} \left\{\frac{u_z^1(z,0)}{rU} + \frac{1}{\sqrt{s}}\,\frac{v^1(z,0)}{v}\right\} dz \tag{7.6}$$

For an initial disturbance which is finite in extent, then u_z^1 and v^1 vanish for all $z > z_o$ for some fixed z_o. For all values of $t > \left[\sqrt{s}/(1-\sqrt{s})\right] z_o/rU$ the above integral and hence $g(\tau)$ remain constant.

As a result, for an initial finite disturbance, the shock front advances a fixed amount ahead of its undisturbed position, due to the zero-order Fourier component (in the variable x, the coordinate along the shock) of the disturbance; meanwhile it is undergoing the damped oscillations in the higher-order components as studied in the preceding sections.

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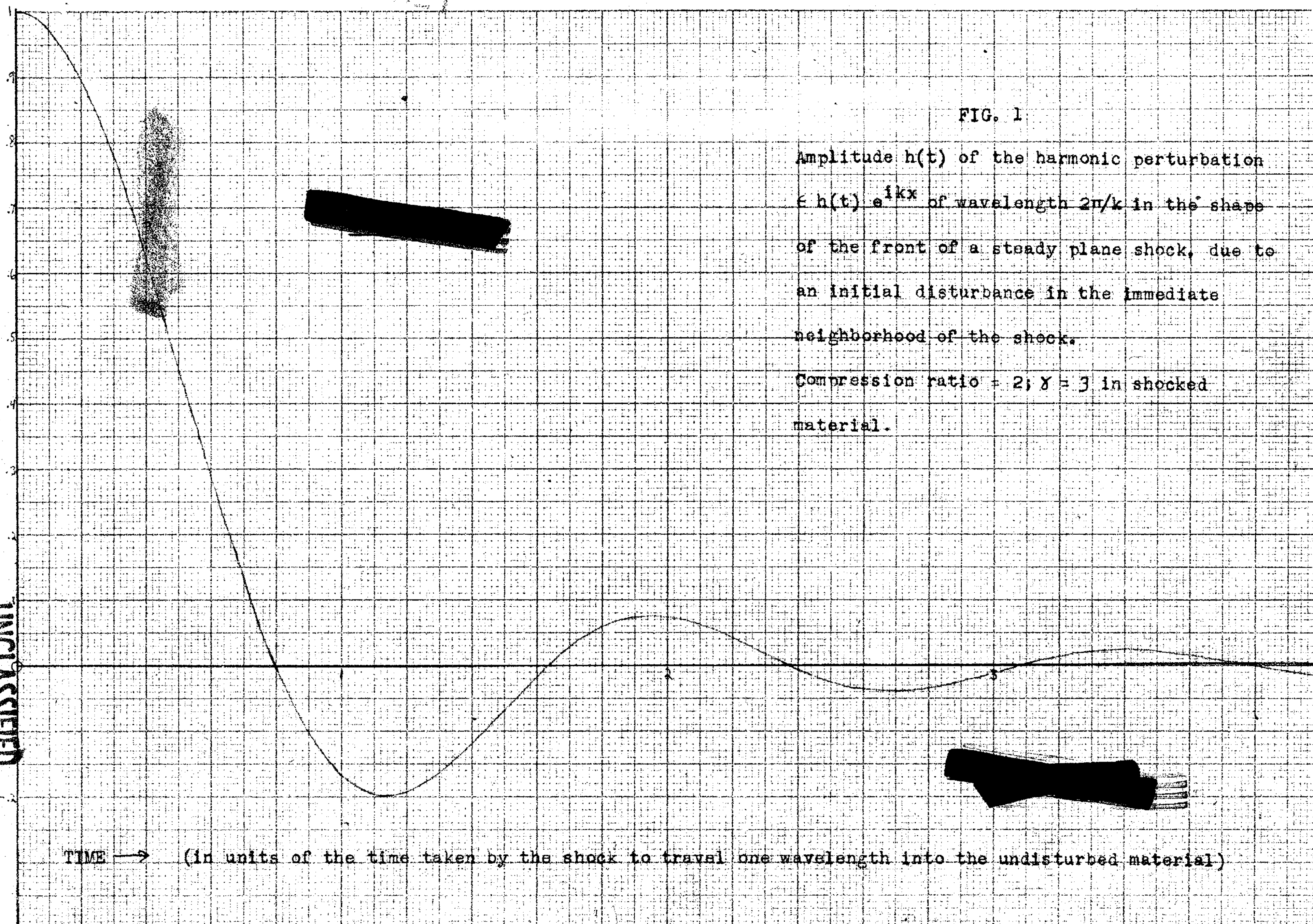

FIG. 1

Amplitude $h(t)$ of the harmonic perturbation $\epsilon\, h(t)\, e^{ikx}$ of wavelength $2\pi/k$ in the shape of the front of a steady plane shock, due to an initial disturbance in the immediate neighborhood of the shock.

Compression ratio = 2; $\gamma = 3$ in shocked material.

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