# LOS ALAMOS SCIENTIFIC LABORATORY OF THE UNIVERSITY OF CALIFORNIA O LOS ALAMOS NEW MEXICO 

## METHODS OF OPACITY CALCULATIONS

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# LOS ALAMOS SCIENTIFIC LABORATORY OF THE UNIVERSITY OF CALIFORNIA LOS ALAMOS NEW MEXICO 

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## METHODS OF OPACITY CALCULATIONS

by
Harris Mayer

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ABSTRACT

Methods for caloulating the opacity of materials at high temperatures are discussed in this report. Minor improvements are outlined for the treatment of continuous absorption processes, and a small error usually made in treating the scattering process is corrected. In oontrast to all previous caloulations of opacity, the effect of line absorption is carefully examimed, for it may well be the dominant process under certain conditions of temperature and density. Detailed methods for caloulating the line absorption contribution are, therefore, developed. To illustrate the prinoiples involved, the opacity of pure iron at a temperature of 1000 volts and normal density is worked out in detail. For this case, the opecity
 The matio of the opacity including line effects to the opacity without lines is 5.2 .

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FORUULAE FOR THE NON-RELATIVISTIC ONE ELECTRON GAUNT FACTORS
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IV THE FREQUENCY FACTOR $g(0, y)$ ..... 151

A Reciprooal length, defined by (6.21).
A Helmholz free energy, (AI. A . $^{\text {. }}$
$2_{0}$ Bohr radius $=.5291 \times 10^{-8} \mathrm{~cm}$ 。
$a_{n i}$ defined by (5.72), (6.46).
$a_{Z}$, radius of ionic sphere, (4.24)
$B=\frac{M c^{2}}{2 k T u{ }_{0}^{2}}$ in (5.33)
$B=\frac{2 h \nu^{3}}{o^{2}} \frac{e^{-u}}{2-e^{-u}}$ as defined by (3.8)
$b_{b a}(\nu)$ dispersion factor for line absorptions (2.6)
$b^{\prime}$ (u) dispersion factor for line absorption (5.1la)
c velooity of light
$o\left(\varepsilon_{f}\right)$ density of states of the free electron per unit energy interval (2.29)
$c_{i Z}$ degeneracy of the ith ionic level.
$c_{f}$ density of initial state per df (2.26)
$D=\frac{u^{3} \mu_{c}}{A}$ reduced absorption coefficient (6.20)
$\mathrm{E}_{\mathrm{a}}$ onergy of quantum state a of an ion (2.4)
$E_{J}$ energy of quantum state $J$ of entire system (4.11)
e positive eleotronic charge
$f_{b a, f} f_{i}$ eleotron number defined by (2.3)
$F(\gamma, w)$ function defined by (2.34)
$g_{b \rightarrow f}(\nu)$ Gaunt factor for bound-free transitions (2.22)
$g_{f f}(\nu)$ Gaunt factor for free-free transitions
Eff Averacse caunt factor defined by (6.27)
$E_{n k}$ (u) Frequency factor (5.82)
f. ${ }^{\prime}$ nk( $u$ ) Frequency factor (5.71)

## Glossary of Notation (continued)

h Planoks constant
$\hbar=\frac{h}{2 \pi}$
$I(2)$ Speoific intensity of radiation see (2.2)
$I_{n}$ Ionization energy of an electron bound in the $n$ shell, (2.21)
K Mean opacity coefficient per unit mass om?/gm.
k Boltemann's constant
$\overrightarrow{\mathbf{r}}$ Wave number vector in direction of propagation, (2.1)
$l_{\nu}$ Mean free path for radiation of frequenoy $\nu /(3.15)$
$M$ Mass of absorbing atoms, (2.16)
$M_{Z} \quad$ Gram atomic weight, (6.1)
$M_{K} \quad$ Number lines in $K$ th class, $(6.44)$
$m$ Mass of eleotron, (2.3)
$N_{0}=6.023 \times 10^{23}$ Avogadro's Number
IV b Nuber of atoms in initial state b (2.9)
ab Number of bound electrons
$n_{f}$ Number of free electrons
$n=n_{b}+n_{f}$ Total number of electrons in system (4.8)
N Total number of atoms in system (4.6)
$\mathrm{N}_{\mathrm{Z}}$ Number of nucleii of atomic number Z in system (2.29)
$n_{0}$ Initial state of the scattering particle (2.32)
$\hat{n}$ Unit vector (3.17)
$\overline{\bar{n}}_{\nu}$ Average number of quanta in each radiation oscillator (2.1)
$n\left(\varepsilon_{f}\right) d\left(\varepsilon_{f}\right)$ Number of states of free electron in energy interval $d\left(\varepsilon_{f}\right)$
$n(\vec{v}) d \vec{N}$ Number of eleotron with velocity vector betweenv and $v+d v(3.3$
$P=\frac{1}{1+Y}(5.5)$
P Pressure, (6.17)

Glossary of Notation (continued)
$P_{J} \quad$ Probability given by (4.13)
Q Electronic partition function of the system (AI.1)
q Availability of final state (2.29)
$r=\frac{\mu_{l}}{\mu_{c}}=$ ratio of Iine absorption to continuous background $(5,5)$
$r^{2 l}$ Value of $r^{2}$ averaped over the bound wave
$r_{0}=\frac{e^{2}}{m c^{2}}$ Classical electrons radius (2.34)
Rhc Rydberg energy $=13.607$ volts (2.25)
$S(u)$ Stromgrem function Tabulated in Table VI (6.30)
S Eq. (5.11a)
S Entropy (AI.11)
T Absolute temperature (2.16)
$u=\frac{h \nu}{k I}$ as defined by (3.8a)
V Volume of system (2.9)
$V_{i j}$ Sum of Couloumb and exchange eleotron interactions (4.12)
W Transition probability as defined by (2.1)
$W=\operatorname{Cos} \theta(2.34)$
${ }^{\mathbf{w}} \mathbf{a b}^{\mathbf{s}}(\nu)$ Transition probability for spontaneous emission (2.10)
$\operatorname{wn}_{a b}^{i}$ (v) Probability for induced emission (2.11)
$Z^{\prime}$ Effective nuclear change (2.25)
$Z^{*}$ Effective nuclear charge (6.7)
Z Effective nuclear charge (AI.15)
$\alpha=e^{2} / \mathrm{hc}=1 / 137.03$ fine structure constant
$\alpha^{\prime}$ See (4.13)
$\alpha^{\prime \prime}$ See $(4.17)$
$\alpha^{*}:-\mathrm{kT} \alpha^{*}$ is free energy of electrons (4.4la)
$\vec{\alpha}_{j}$ Dirac velocity matrix for jth electron (2.1)

Glossary of Notation (cont inued)
$\beta=\frac{1}{k T}$
$\Gamma=\frac{\gamma}{4 \pi k T}$ Dimensionless half breadth, defined in (5.21)
$\frac{\gamma}{2 \pi}$ Breadth of spectral line (2.15)
$\gamma=\frac{h \nu}{m c^{2}}(2.34)$
$\vec{\varepsilon}$ Direction of polarization (2.1)
$\mathcal{E}(r)$ Potential energy (4.49)
$E_{f}$ Energy of free electron
$\bar{\varepsilon}_{1}$ Ionization energy of the fth electron (4.22)
$\bar{\varepsilon}_{i}$ Average energy of electron in fth state (AI.10)
$\theta$ Scattering angle (2.32)
$\Lambda$ Rosseland mean-free path (3.18)
$\Lambda_{c}$ Mean free path ignoring lines (5.7)
$\Lambda_{\boldsymbol{l}}$ Contribution of lines to Resseland mean free path (5.8)
$\mu$ Absorption coefficient, as defined in (5.3)
$\mu_{c}$ Absorption due to continuous processes alone (5.4)
$\mu_{l}$ Absorption due to lines (2.9), (5.4)
$\mu_{s}$ Scattering coefficient, i.e. inverse mean free path for scattering. (2.33)
$\nu$ Frequency
$\rho$ Density (3.25)
$\sigma_{i, j}$ Screening constants, defined by (4.23) Tabulated in Table II
$d \sigma$ Surface element (3.1)
$\phi_{0}=\frac{8 \pi}{3}\left\{\frac{e^{2}}{\mathrm{mc}^{2}}\right\}^{2}=.6654 \times 10^{-24} \mathrm{~cm}$. Thomson cross-section (6.24)
$\phi_{b a}(\nu)$ Absorption cross-section. Eq. (2.8)

## Glossary of Notation (continued)

$\varnothing_{f}$ Electrostatic potential, defined by (4.26)
$\varnothing$ Differential scattering oross-section (2.32)
$x_{k}$ Electron wave function (4.2)
$\Omega J$ Density of states, defined by (4.14)

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## I. INTRODUCTION

The opacity of materials at extremely high temperatures is important in the study of stellar interiors. As such, it has engaged the interest of astrophysioists for the last thiris years. The most recent study of this problem is by Morse ${ }^{(1)}$. Following the tradition in the field, Morse considers only three fundamental processes as contributing to the opacity - photocelectric absorption (boundfree transitions), collision absorption (free-free transitions) and Conpton scattering. It was pointed out by Edward Teller that a fourth process, line absorption, might often prove to be the determining one in the opacity. At his suggestion, therefore, the problem was investigated anew.

Previous workers ${ }^{(2)}$ had reoognized that sufficient absorption strength was present in the line speotrum to inorease the opacity by factors of 2 to 4 . It was argued, however, that individual lines are extremely sharp, and although they are very strong compared to the continuous absorption processes, they would simply make a very small region of the spectrum opeque to radiation. Since the mean free path for radiation (freciprocal of the opacity) is a weighted frequency average of the reciprocal absorption coefficient, the blackness of these small regions would not appreciably alter the mean free path. Ieller pointed out two reasons for suspeoting these conolusions: 1) Individual lines were apt to be very much broader under the extreme temperature and density conditions of stellar interiors than under normal terrestrial conditions 2) There would be an enormous number of lines arising from the oomplioated electrostatio interactions at these temperatures.

The reasoning behind these suspicions is qualitatively very simple. At high temperatures, an element will be found in many states of ionization. Furthermore each ion type will exist in a number of excited statese For example, caloulations show that iron at temperatures of 1000 volts and densities 5 times normal, i.e. 39.3 (1) Astrophysical Journals 92 pp. 27-49, 1940.
(2) Menzel and Perkeris: Monthly Notices of the Royal Astronomical Society, V. 96 , p. 77. (1935); S. Rosseland, Handbuoh der Astrophysik, V. 3 ' 3 teil.
grams/om ${ }^{3}$, (conditions available in stellar interiors) will nave on the average 2 K electruns, 1.24 Lelectrons, 1.10 if electrons, etc. - in all a total aterage of 4.3 bound electisons. We may expeot appreciable numbers of ions with fror 2 to 7 bound elactron. Liorfover of the ions with say 6 bound electrons there will be some with $2 \pi$, 4L, others with 2K, 3L, 1H, still otiners with 2K, 2L, 2M, and so on. Each configuration of these ions gives a ratiner complex line spectrum - the total number of lines from the assemblage will be enormous. Mixtures of elements will give even a richer line spectrun.

Amons the causes for line broadening are 1) Natural breadth; 2) Collision breadth; 3) Stark breadth; 4) Doppler breadth. Each of these is much more important at high tempratures and densities than at ordinary oonditions. Because many electrons are in excited orbitals a large number of emission transitions can destroy the initial state of an ion. In addition the intense temperature radiation present can induce absorption transitions. Both effects enhance the natural breadth. At high temperatures collisions with free elsotrons will frequently occur - a process almost entirely excluded under ordinary conditions. Moreover the high velocity of the ions creates an appreciable Doppler broadening. It also anables relatively high oharged ions to approach so close that enormous fields are available to cause Stark broadening. In many cases the lines will be so broadened as to form a virtually continuous absorption band, very effective in producing high upacity.

It will be raalized that it is the high exoitation of the material, combined with the pervading electrostatic interactions that makes line absorption so important a feature in opacity calculations. Previous treatments of the problem essentially negleoted all these interactions except those between a nucleus and its own bound electron. It was felt worthwhile to include these interactions in some moderately satisfactory form in order to insure the accuracy of the calculations of the line absorption. This problen is treated at some length in Chapter IV of this paper.

An immediate byproduct of the statistical mechanical calculation which is there performed, is the thermodynamic properties and the equation of state of material. It is generally assumed in work on stellar interiors that the material obeys a perfect gas equation of state with an effective number of particles equal to the sum of all the electrons and nuclei in the system. In Appendix $I$, the appropriase modification of the equation taking into account the presence of bound electrons, electrostatio interactions, and a small correction for relativity effects is given.

The oomplexity of the line spectrum precludes any etraightforward attack on the problem. Tèler, however, when he proposed considering the line absorption, also suggested the use of a statistical method of finding the contribution to the opacity. The method is developed in detail in Chapter $V$ which constitutes the significant nev contribution to the study of opacity.

During our reexamination of the opacity problem, Dr. Maria Mayer noticed that the scattering contribution has always been treated by analogy with the absorptionemission contribution. This proves to be incorrect, although the mumerical error is small. The difference between the two types of process lies in the effects of induced emission and induced scattering. The presence of induced emission strengthens a beam in its passage through matter, thus tending to reduce the opacity. In the scattering process, on the other hand, the induced scattering out of a beam compensates for the induced scattering into the beam. The correct calculation of opacity due to scatiering therefore gives a result greater (actually $5 \%$ greater) than the conventional one.

Although there are many features in common, the treatment of the opacity of heavy elements, is different in many details from the treatment for light elements. This results because the relativistic effects completely change the natiure of the line spectrum of heavy compared to light elements. A companion paper to this by (3) Boris Jacobsohn, gives speoific attention to the calculation of opacities for the (3)

Dissertation, University of Chicago, Dept. of Fhysics, 1947.

## heary elements.

The author wishes to thank Dr. Marda Mayer and Dr. Edward Teller for suggesting the proble:n and for much valuable assistance in attacking it. It is a sinoere pleasurf, moreover, to acknowledge my indebtedness to Dre Boris Jacobsohn for many discussions on every phase of this proolem, and to kr. Rudy Sternheimer for his valuable wark on line broadening without which this paper would have little practical significance.
II. THE PROCESSES CONTRIBUTING TO THE OPACITY

Any process capable of removing quanta from an incident beam, or emitting quanta into that beam must be oonsidered in the opacity problem. The most important of these prooesses is simpla absorption and its inverse. In this oase a quantum is absorbed by an atomic system, its energy being transferred to excitation of the electrons. We may distinguish with profit three different types of absorption. In the first an electron in a bound state of the atom is excited to another bound state. This is the familiar line absorption. In the second the bound state electron is renoved from the atom, going into one of the continum of free states available, the usual photoeleotric effect. In oontrast to line absorption, this process is possible for any energy of the incident quantum greater than the ionization energy of the electron. Lastly, an electron in a free state may absorb energy by a transition to another free state. Any amount of energy whatever may absorbed in this proceiss. The inverse processes to the three mentionad are, ill order: line emission, electron capture, and Bremsstrahlunge

A second absorption process is pair produation with its inverse process annihilation.

The last group of proossses is scattering. Here an incident quantum is deviated from its original path by an atom or eleotron. If the atom remains in its initial state after the scattering, we have ordinary ooherent soattering; if it does not; the phenomenon is known as Raman scattering. A special case ocours if the frequency of incident radiation is equal to the frequenoy of an absorption line of the atom. Soattering by free electrons is termed Compton scattering.

The remainder of this chapter giviss the transition probabilities for these processes.

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1. Line Absorption

The three types of simple absorption - line absorption, photoelectric absorption, and absorption due to freenfree transitions differ only in the nature of the initial and final states of the atom. It is therefore possible to develop the treatment of all three at once up to an advanced point and there introduce the differences.

The transition probability for the absorption of a quantum of energy hy from an incident beam is

In this formula $\vec{\alpha}_{j}$ is the Dirac velocity matrix for the jth eleotron. $\overrightarrow{\mathcal{E}}$ is the direction of polarization of the inoident light beam which has the wave number reotor $\vec{k}$ in the direction of propagation. The matrin element connects the inie tial state of the atom $b$ with the final state a and is averaged over all directions of polarization and orientations of the atom - which latter is the same as the average over directions of propagation of the light. The incident beam has on the average $\bar{n}_{y}$ quanta in each radiation osoilletor, that is the intensity of the beam direoted within a solid angle $d \Omega$ is
(2.2) $I(\nu) d \nu d \Omega=h \nu 0 \bar{n}_{\nu} \frac{2 \nu^{2} d \nu d \Omega}{0^{3}}$.

It is convenient to introduce the olectron number defined by
(2.3) $f_{b a}=\frac{2 m e^{2}}{h z_{a b}^{\prime}} \sqrt{\sum_{j}\left(\left(\vec{\alpha}_{j} \cdot \vec{\varepsilon}\right) e^{i \vec{k} \cdot \vec{r}_{j}}\right)_{a i j} \prod^{2} \vec{\varepsilon}, \vec{k}}$
where
(2.4) $\quad h \nu_{a b}=\left(E_{a}-E_{b}\right)$.

This definition reduces to the usual one for one electron atoms in non-relativitic approximation, when $e^{i \vec{F} \cdot \vec{r}}$ is replaced by unity - the conventional dipole approximation. Combining (1), (2), and (3) we get the transition probability as


In any actual physical system the absorption is not confined to a single frequancy, but it is possible for a range of frequencies in the neighborhood of $\nu_{a b}$. Wo shall therefore introduce the dispersion factor for line absorptions bba $(\nu)$ by the definition that the probability of absorting light of frequency between $\nu$ and $\nu+d \nu$ by the transition $b \rightarrow a$ is
(2.6) $\quad \omega_{b a}(\nu) d \nu d \Omega=\frac{\pi e^{2}}{m c} \frac{\nu / a b}{\nu} P_{b a} \frac{I(\nu)}{h \nu} d \Omega_{b_{b a}}(\nu) d \nu$.

The dispersion factor will have a sharp maximum at $\nu=\mathcal{L}_{a b}$ and furthermore
(2.7) $\quad \int_{0}^{\infty} b(\nu) d \nu=1$.

The absorption oross-section of this tiansition per atom for light of frequency $\nu$ is then
(2.8) $\quad \phi_{b a}(\nu)=\frac{\pi \rho^{2}}{m 0} f_{b a} b_{b a}(\nu)$,
where we have put $\nu=\nu_{a b}$ in the slowly varying function of frequency multiplying $b(\nu)$. From the oross-section we get ivmediately the absorption coefficient or inverse mean fres path for line absorption
(2.9) $\quad \mu_{e}(\nu)=\sum_{b a} \frac{H^{b}}{V} \varphi_{b a}(\nu)=\frac{\pi e^{2}}{m c i} \sum_{b a} \frac{x^{b}}{V} f_{b a} b_{b a}(\nu)$. where $\mathbb{N}^{b}$ is the number of atoms in initial state $b$, and the summation extends over all tramitions whioh may absorb the frequenoy $\nu$.

The inverise process, line emission, is the sum of two terms, one for spontaneous and one for induced emission. The traxsition probability for the spontaneous emission of a quantum $h \nu$ within a solid angle $a \Omega d u e$ to an atomic transition from state $b$ to a is

while the probability for induced omisuion is

$$
\begin{equation*}
\frac{\pi_{a b}^{i}}{}(\nu) \mathrm{d} \nu \mathrm{~d} \Omega=\frac{\pi \theta^{2}}{m c} \left\lvert\, r_{a \mathrm{~d}} \int \frac{I(\nu)}{h \nu}{ }^{1}{ }_{a b}(\nu) \mathrm{d} \nu \mathrm{~d} \Omega\right. \tag{2.11}
\end{equation*}
$$

The quantur theory moreover gives the general result required by thermodymamios that
(2.12) $\left|f_{a b}\right|=f_{b a} ; b_{a b}(\nu)=b_{b a}(21) ;$
so that
(2.13) $\quad \frac{\sigma_{a b}(\nu)}{w_{b a}(\nu)}=1+\frac{2 h \partial^{3}}{c^{2}} \frac{1}{I(\nu)}$.

An adequate discussion of the eleatron numbers is given by Jacobsohn ${ }^{(1)}$. He oonsiders an ion with only one electron and treats different order terms arising from the expansion of the factor $e^{i \vec{k} \cdot \vec{r}}$ in (\%.3). The leading term gives the major Praotion of the electric dipole strength. Nuserical caloulations for this term in the non-relativistic case which extend and correct similar caloulations by Bethe ${ }^{(2)}$ are presented in Table 1 of Jacobsohn's paper and are reproduced here in Table 1. It should be noted that the $f$ number in these tables is the average from a level ( $n($ ) to a level ( $n \cdot \ell$ ) found by suming over all states in the final level. In order to divide the $f$ number properly between the transitions $n \ell j \rightarrow n \cdot l^{\prime} j{ }^{\prime}$ we use the relations

$$
f_{n} \ell_{x \rightarrow n^{\prime}} \ell^{\prime} k+1=f_{n} l_{n \rightarrow n^{\prime} \ell^{\prime}}
$$

$$
\begin{align*}
& f_{n} l_{k \rightarrow n^{\prime}} l^{\prime} k-1=\frac{(k-1)(2 k+1)}{k(2 k-1)} f_{n} l \rightarrow n^{\prime} l^{\prime} .  \tag{2.14}\\
& f_{n} \ell_{x \rightarrow n^{\prime}} l^{\prime}(-k)=\frac{1}{k(2 k-1)} f_{n, b \rightarrow n^{\prime} l^{\prime}} .
\end{align*}
$$

where $|k|=j \nmid \frac{1}{2}$ and $k$ the Dirac theory. Jacobsohn also presents some formula and tables for the relativistic electric dipole and nonarelativistio electric quadripole oscillator strength.
(I) Op. Cit.
(2) H. Bethe, Hiandbuch der Physik (2nd Ed.) 2 V, Part 1, pp. 443.
-. 10
For approximate calculations, and to preserve the analogy between treatments of line absorption and continuous abscrption, we may use an asymptotic expansina for the dipole $f$ number (enorgy of transition $h \nu$ ) of a hydrogenic atom averaged over all init!lal states in a shell of the principal quantum number $n$. This is ${ }^{(3)}$ (2.14a) $\quad \bar{f}_{n \rightarrow m}=\frac{2^{6}}{3 \pi \sqrt{3}}\left\{\frac{1}{n^{2}}-\frac{1}{m^{2}}\right\}^{-3} \frac{1}{n^{3} m^{3}} \frac{5_{b}}{2 n^{2}}$
where $m$ is the principal quantum number of the fimal, $n$ of the initial state, and (2.14b) $\quad \delta_{b} \sim 1=0.1728\left(\frac{h \nu \nu}{R h o z^{2}}\right)^{1 / 3} \cdot\left[\frac{2}{n^{2}} \frac{R h o z^{2}}{h \nu}-1\right]+$ This expansion becomes better as both $n$ and $m$ increase.

The dispersion is due to the finite breadth of the atomio-staties in the system caused by the interactions with light and with other partioles. This problem has been treated in a thorough going fashion in an unpublished paper by R. Sternheimer. For dispersion oaused by the interactions with the radiation field, the line has a natural breadth $\frac{\gamma_{b a}}{2 \pi}$ and a dispersion
$\gamma_{b a}$

$$
\begin{equation*}
b_{b a}(\nu)=4 \pi^{2}\left(\nu-\nu_{0}\right)^{2}+\frac{\gamma^{2} b_{a}}{4} \tag{2.15}
\end{equation*}
$$

We note that when $2 \pi\left(\nu-\nu_{0}\right)=\frac{\gamma_{b a}}{2}$, the cross-section has fallen off to half its maximum value. Interactions with fast electrons will oause collision brcadening of the lines with the same dispersion form as (2.15). However, interactions with slow moving partioles like neighboring nuclei are best treated by an adiabatic approximation and the dispersion ourve then falls off exponentially with distance from the line senters. Another cause of exponential type dispersion is the DopplerEffeot which gives

$$
\begin{equation*}
b(\nu)=\left\{\frac{m_{0}^{2}}{2 \pi k T}\left(\frac{1}{\nu_{a b}^{2}}\right)\right\}^{\frac{1}{2}} \exp \left\{\frac{\mu_{0}^{2}}{2 k T} \frac{\left(\nu-\nu_{a b}\right)^{2}}{\nu_{a b}^{2}}\right\} . \tag{2.16}
\end{equation*}
$$

$M$ being the mass of the absorbing atoms and $T$ the absolute temperature.
(3) Menzel and Perkeris. Op. Cit.

## 2. Photo-leotric absorption.

In the case of line absorption we found a non-zero probability for absorption of frequencies differing slightly from that of the atomic transitions due to the splitting of the atomic states into a continuum by interactions with, for example, the radiation field. For boundfreo absorption (photoeleotric absorption) the final state is already a member of a continuum, and we will therefore have a finite absorption probability for a continuum of incident frequencies. The probability of absorbing light with frequenoy between $\nu$ and $\nu+\Delta \nu$ is from (2.5)

$$
\text { (2.17) } \quad w_{b i}(\nu 1) \Delta \nu=\frac{\pi e^{2}}{m c} \frac{I(\nu)}{h \nu} \sum_{\substack{\text { all } i \\ \text { in } \Delta \nu}} f_{b f i} .
$$

The oross section may therefore be writtien as
(2.18) $\quad \phi_{b f}\left(\lambda^{\prime}\right)=\frac{\pi e^{2}}{m c} \quad \frac{d}{d^{\nu}} f_{b f}$.
where by $\frac{\mathrm{df}_{\mathrm{bf}}}{\mathrm{d} \mathrm{\nu}}$ we mean
(2.19) $\frac{d f_{b r}}{d \nu}=\sum_{\substack{a i l i \\ \text { in } \Delta \nu}} \frac{f_{b r i}}{\Delta \nu} \quad$ for $\Delta \nu$ small.

The abaorption sooffioient is, then
$(2.20) \mu_{b f}(\nu)=\sum_{b} \frac{y^{b}}{\nabla} \varphi_{b f}(\nu)$,
where the sumaliion is extended over all staites $b$ which can absorb the frequency $2 /$. Naturally precisely the same expression (2.13) as in line absorption relates the probability of the inverse process of eleotron oapture with photoelectron absorption.

We shall follow historical precedent in transforming (2.18) and (2.20). When Kramers first attacked the problem of the photoelectric effeot using the machinery of the old Bohr theory, he found the oross-section for a complete shell of prinoipel quantum number at to be

$$
\begin{equation*}
\varphi_{n f}(: \nu)=\frac{2^{5}}{3 \sqrt{3}} \frac{h \rho^{2}}{m o} \frac{n}{h \nu}\left(\frac{I_{n}}{h \nu}\right)^{2} \tag{2.21}
\end{equation*}
$$

where $I_{n}$ is the ionization energy of axi electron bound in the $n$ shell. Later workers, Gaunt ${ }^{(4)}$ and Stobbe ${ }^{(5)}$, computing the cross section with the new wave mechanics, expressed their result in terms of Kremer's formula (2.21), correoted by a factor, the so-called Gaunt factor $E$. It is userul to retain this notation, since Menzel \& Pekeris ${ }^{(6)}$ have shown that the Gaunt factor $g$ per electron is close to unity when averaged over a complete shell for transitions to free states near the ionization limit (the region of interest in opacity calculations). To rewrite our oross-section (2.18) in terms of this notation, we define the Gaunt factor by

$$
(2,22) \quad g_{b f}(\nu)=\frac{3 \sqrt{3} \pi}{2^{4}} n\left(\frac{h \nu}{I_{n}}\right)^{2} \nu \frac{d f_{b f}}{d \nu}
$$

Thus (2.20) becomes

$$
\text { (2.23) } \mu_{b i}(\nu)=\sum_{b} \frac{\mathrm{~N}^{b}}{\nabla} \frac{2^{4}}{3 \sqrt{3}} \frac{h e^{2}}{m c} \frac{1}{n} \frac{1}{h}-\left(\frac{I_{n}}{h 2 j}\right)^{2} g_{b f}(\nu)
$$

As defined here, the $f$ number, and hencu the Gaunt factor refer to atomic transitions and $N^{b}$ is the atomic ocoupation number. It is frequently more convenient to use $f$ numbers and hence Gaunt factors per elestron. The atcmic occupation $\mathbb{N}^{\mathrm{b}}$ must then be replaced by $n_{1}$, the number of electrons in the initial state capable of absorbing the frequency $\mathcal{H}$. The transition probabilities must then be multiplied by a factor $q$, the probability that the final state will be available. For non-degenerate free electrons this is practically unity.

A disoussion of the Gaunt factors with numerical tables is given by Jacobsohn. He shows that for absorption by $L$ electrons, the dipole Gaunt factors for individual transitions variss considerably from transition to transition and moreover vary with frequency, espersially in the region ncar the ionization edge. The average Gaunt

[^1]factor per electron is, however, close to unity. Moreover using relativistic wave functions, while it drastically effeotis the results for individual transitions, has little effect on the average, even for the very heavy elements. The Gaunt factors for quadrupole transitions are smaller than those for dipole transitions by the factor $.13(z \alpha)^{2}$ for the $L$ shell at the ionization limit ( $\alpha=\frac{1}{137}=$ fime struoture constant), and even smaller lior the higher sholls. For many oases, therefore, it will be a sufficient app:oximation to replace g per electron by unity.

## 3. Froe-Froe Absorption.

The ability of an eleotron to absorb light depends on its binding to a mucleus, for a free electron oannot conserve momentum in absorption. Hence we expect a tightly bound sleotron to have a much larger photoelectrio orossesection than a loosely bound one, and therefore the free-free transitions should be a small effeot. There are two factors which nevertheless make it worthwhile to oonsider the process. In our highly ionized medium a great pertion of the electrons are freemin extreme cases there may be no bound electrons at all. Furthermore, absorption by a boundfree transitions can only take place for frequencies higher than the ionization potential for remoral of the eleotron in question. For low frequencies the freefree transition may be the only possible one, or at least can oompete favorably with a bound-free transition of a very loosely held bound electron.

The transition probability from one definite initial free state $f$ to a range of final free states with the absorption of light of frequoney betweon $\nu$ and $\nu+\Delta \nu$ is of course the same as (2.16) for the bound-free transitions. Hoverer, the same frequency can be absorbed by starting from any initial free state. Hence, to find the absorption oseffioient, we must maltiply the oross seotion (2.17) by the number of electrons per unit volume $\frac{n\left(\varepsilon_{\rho}\right)}{\bar{V}} d \mathcal{E}_{\mathrm{F}}$ cocupying a range of free states between $\mathcal{E}_{f}$ and $\mathcal{E}_{f} \notin \mathrm{~d} \mathcal{E}_{\mathrm{E}}$ and integrating over all enorgies, keeping $\nu$ constant.
(2.24)

$$
\mu_{f f^{\prime}}(\nu)=\int_{\varepsilon_{f}} \frac{n\left(\varepsilon_{f}\right)}{\nabla} \frac{\pi e^{2}}{m o} \frac{d f_{f f^{\prime \prime}}}{d \nu} d \varepsilon_{f}
$$

Mancel and Pekeris ${ }^{(7)}$ derive an asymptotio expression for $\frac{d f_{f f i}}{d \nu}$ whose leading term is that given by the Kramer's formula. We can see heuristically what the result must be by analogy with their asymptotic expansion (2.14a,b) for bound-bound transitions. If the initial free state is specified by a quantum number $f$, and the final free state after absorption of light energy $h z$ is specified by $k$, then
(2.25)
(a) $h z^{\prime}=\varepsilon_{k}-\varepsilon_{f}$;
(b) $\varepsilon_{f:}: \frac{\mathrm{RhcZ}^{2}}{\mathrm{f}^{2}}$;
(0) $\varepsilon_{k}=\frac{\mathrm{Rhoz}^{2}}{k^{2}}$
where $Z^{\prime}$ is the effective nuclear charge. The $f$ number for this transition is obtainod from (2.14a) by replacing $n$ by $i f, m$ by ik, the degeneracy $2 n^{2}$ by $c_{f}$, the density of initial states per df, and the bound Gaunt factor $g_{b}$ by a free-free factor $\mathrm{g}_{\mathrm{ff}}$ - Using the relations (2.25) wo obtiain
(2.26) $\quad f_{f l}=\frac{2^{6}}{3 \sqrt{3} \pi} \frac{\left(\mathrm{Rh}_{\mathrm{C}}\right)^{3 / 2} z^{3}}{(\mathrm{~h} \nu)^{3}} \frac{\varepsilon_{f^{\prime}}^{3 / 2}}{\mathrm{~h}^{3}} \frac{\mathrm{~g}_{\mathrm{ff}}}{0_{f}}$
where
(2.27) $\quad \mathrm{E}_{\mathrm{ff}} \ddot{\vdots} \mathrm{幺} 1+0.1728\left(\frac{\mathrm{~h} \nu}{\mathrm{Rhoz}^{2}}\right)^{1 / 3}\left[1 \neq \frac{2 \varepsilon_{\mathrm{f}}}{h \nu}\right]+\ldots$.

Then

$$
\begin{equation*}
\frac{d f}{d \nu}=f_{f k} \frac{d k}{d \nu}=\frac{2^{5}}{3 \sqrt{3 \pi}} \frac{(\text { Rhc })^{\frac{2}{2}} z^{2}}{(h \nu)^{3}} \quad \frac{\left(\varepsilon_{f}\right)^{3 \beta}}{o_{f}} \varepsilon_{f f} \tag{2.28}
\end{equation*}
$$

It might be thought at first that a factor of 2 is lacking from the above equation, since the number of states within a frequency $d \nu$ is $2 \frac{d k}{d \nu} d \nu$, since each level $k$ may have an electron with either of two spias. But for a dipole one electron transition. the $f_{f l}$ would be zero for a change of spin direction, so we need count only the states of one spin.

We now substitute (2.28) into (2.24). We shall later show (see Chapter IV) that $n\left(\varepsilon_{p}\right)=ब\left(\varepsilon_{q}\right)\left\{0^{\alpha *}+\beta \varepsilon_{f}+1\right\}^{-1}$, where $a\left(\varepsilon_{f}\right)$ is the density of states of the free
(7)
ibid.
eleotron per unjit energy interval and $B=1 / K T$. We obtain froin ( 3.25 b ) $o\left(\varepsilon_{f}\right) / o_{f}=\frac{N_{0}}{2} d \varepsilon_{f} / d_{f}=\frac{N_{Z}}{N_{Z}}$ (Rho $)^{\frac{2}{2}} z^{\prime} / \varepsilon_{f}^{3 / 2}$ where $N_{Z}$ is the number of nuolei of atomic number $Z$ in our system and the factor 2 takes oare of eleotron spin. Gathering together the terms gives

$$
\begin{equation*}
\left.\mu_{f:^{\prime}}(\nu)=\sum_{z} \frac{2^{4}}{3 \sqrt{3}} \frac{h \theta^{2}}{m 0} \frac{(R h o) z^{\prime 2}}{(h z)^{3}} \frac{k_{Z}}{\nabla} \int_{0}^{\infty} \frac{g_{3} q d \varepsilon_{f}}{\left(\alpha^{+}+\beta \varepsilon_{i}\right.}\right) \tag{2.29}
\end{equation*}
$$

where a faotor $q$ for the availability of tithe fimal state has been added. For nondegenerate free electrons we may safely neglect the term 1 in the denominator and put $q=1$. Then the integral gives $\frac{1}{\beta} e^{-d^{*}}$ so that

where
(2.31) $\bar{g}_{\mathrm{ff}}: 1+.1728\left(\frac{\mathrm{kP}}{\mathrm{Rhoz}^{\prime 2}}\right)^{1 / 3}\left(\frac{h \nu}{\mathrm{KT}}\right)^{\frac{7}{2}}\left[1+\frac{2 \mathrm{KP}}{h \nu}\right]$. and $\alpha^{*}$ is given by (4.4la) below.

Just as with photoelectric absorption we have the relation (2.13) for the probability of the inverse process.
4. Pair Produotion.

To produce a free electron-positron pair requires inoident quanta of energy at least $2 \mathrm{mc}^{2}$, while even the production of a bound electron and a positron requires energiss greater than mc ${ }^{2}$. In all our discussions the temperature will be so low that the amount of radiation of such high energies is negligible. We may, therefori, safely neglect pair production. Moreover there will be practically no positrons in equilibrium at these temperatures and so we may negleot the inverse process of anninilation.
5. Soattering.

The oross-seotions for scattering - coherent, Raman, and Compton - may be found, for exemple, in Heitler ${ }^{(8)}$. There, however, the case in whioh initially only one quantum is in the radiation osoiliator of the incident beam and none at all in the soattered beam is considered. In order to find the transition probability we must multifily these oross-sections by the incident intensity and by a factor $\left(I+\bar{x}_{b}\right)$, $\bar{n}_{b}$ heing the average number of quanta in the radiation osoillator of the soattered beami. The term in $\bar{n}_{\mathrm{b}}$ is the indured soattering.

Consider then an inoident beam in the direction speoified by the vector $\vec{l}_{\text {, }}$ having specific intensity $I_{\nu}(\vec{X})$. The probability that a transition will take place, scattering a quantum of energy $\bar{J} \prime^{\prime}$ in a direction specified by $\overrightarrow{\ell^{\prime}}$ within $\alpha \Omega^{\prime}$ is
(2.32) $\nu_{\nu \nu \eta_{n_{0}}} d \nu d \Omega d \Omega^{\prime}=d \varphi\left(\nu, n_{0}, \theta\right) \frac{I(\nu, \vec{B}}{h \nu} d \nu d \Omega\left\{+\frac{o^{2}}{2 h \nu^{i s}} I_{\nu},\left(\overrightarrow{\ell^{\prime}}\right)\right\}$,
where $n_{0}$ specifies the initial state of the soattering particle, $\theta_{\text {is }}$ scattering angle and $d \phi$ is the usual differential scattering oross section. The scattering ocofficient, i.e., the inverse mean free path for soattering is then (2.33) $\mu_{B}\left(\nu^{\prime}\right)=\sum_{n_{0}} \frac{\mathbb{X}}{\left(n_{0}\right)} \int_{\Omega^{\prime}}^{\pi_{\nu} \nu} n_{0} d \Omega^{\prime} \frac{h_{2}}{I Z X}$,
where the sum extends over all states vich can scatter this frequency.
For Compton scattering from free electrons initially at rest, the differential oross seotion is
(2.34) $d \varphi(: \nu, 0, \theta)=\frac{r_{0}^{2}}{2} d \Omega^{\prime} F(\gamma, w)$,

$$
P(\gamma, w)=\frac{1+w^{2}}{[1+\gamma(1-w)]^{2}}\left\{1+\frac{\gamma^{2}(1-w)^{2}}{\left(1+\alpha^{2}\right)(1+\gamma[1-w])}\right\} .
$$

$$
\tau=\operatorname{sos} \theta .
$$

$$
\gamma=l 2 / / \mathrm{mo}^{2} \text {, and } r_{0}=0^{2} / \mathrm{mo}^{2}=\text { olassical electron radius. }
$$

(8)
W. Heltler." Quantum Theory of Radiation, pp. 129-157.

To first order in $\gamma$, this reduces to

$$
\begin{equation*}
\mathrm{d} \phi(\nu, 0, \theta)=\frac{\Sigma_{0}^{2}}{2} \mathrm{~d} \Omega^{\prime}\left(18 w^{2}\right) \quad[1-2 \gamma(1-w)] \tag{2.35}
\end{equation*}
$$

which is sufficiently acourate for our purposes. To get the oross section for scattering frism an eleotron moving with velooity $\overrightarrow{\mathcal{L}}$, we simply apply a Lorente transformatiou. The result, to the same order as (2.35) is

$$
\mathrm{d} \phi(\gamma, \vec{\nabla}, \theta)=\frac{r_{0}^{2}}{2} \mathrm{~d} \Omega^{\prime}\left(1-\mathbb{W}^{2}\right)\left\{\left[\begin{array}{ll}
1 & -\operatorname{zr}(1-w)] \tag{2.36}
\end{array}\right]\right.
$$

$$
\left.+\frac{v}{0} \cos \alpha+\left(v^{2} / c^{2}\right)\left[\left(00 \varepsilon^{2} \alpha+\frac{1}{2}\right)+\left(\cos \alpha-\cos \alpha^{\prime}\right)^{2}\left(1-\mathbb{w}^{2}\right)-1\right]\right\}
$$

Another relation we shall need is the conneotion between incilient and scattered frequencies
(2.37) $\frac{h \dot{\nu}}{h \nu}=\frac{1}{1+\gamma(1-w)} ; \frac{h \nu}{h \nu}=\frac{1}{1-\gamma^{3}(1-w)} ; \gamma^{\prime}=h \nu j / m o^{2}$.

The Kramer's-Feisenberg formula for coherent or Raman soattering gives a orosssection which, except for frequencies near a resonance line of the atom, is of the same order of magnitude as (2.35). This formula gives the scattering oontribution of bound electrons. Now there are usually many more free electrons than bound so that the major part of the soattering will be of the form (2.36). Moreover, the scattering crosseseotion for bound electrons is generally small compared to the photo-eleotric cross-seotion. We will, therefore, never comnit a serious orror if we use the orassesection (2.36) as if it held for all the electrons, bound and free.

The question of resonance scattering must now be disposed of, for the orosssections then become exceedingly large and it would be improper to use the simple formula (2.36). Heitler discusses just, this question and he concluded that if the atom is irradiated by a continuous spectrum "the resonance fluorescence behaves with regard to the shape of the line alssorbed and reemitted exactly as if two independent processes, an absorption and a subsequent emission took place." We must not regard absorption and resonanoe sosttering as two independent fates which may overtake an atom, but rather consider that an atom may be excited to an intermediary state, and either remain there, in which case we have true absorption, or else returas to a lower state in which case we have resomance fluorescence. In calculat-
ing the total oross-section for all pricesses, we see that resonance fluoresoence has been included in the term for line absorption.
III. THE OPACITT COEFFICIENI

The opaci.ty coefficient is a partioular weighted average of the absorption coefficient disoussed in the previous chapter. This relation results from a consideration of the equation for energy transport by radiation. A very oomplete treatment of this equation of radiative transfer and the solution appropriate to stellar interiors is given in Chandraselchar ${ }^{(1)}$. Chandraseichar's disoussion, howover, is restricted to simple absorption and emission processes. other workers (2) In the field of stellar opacities have, by analogy, used the same relations for the soattering prociess, or combinations of soattering and absorption. It was pointed out by Dr. Maria G. Mayer that this is only approximately true: for pure soatterinc, the error made overestimates the mean free path for radiation by $5 \frac{1}{2} \%$ Although the difference is small, it is necessary to see how it arises. We shall, therefore, the repeat, conventional derivation of the opacity coefficient, including the soattering terms.

We shall first write the equation of radiative transfer in terms of the quantun mechanical oross-sections or transition probabilities discussed in Chapter II. Wei consider a beam of photons having on $\nu \mathrm{d} \nu \mathrm{d} \Omega \mathrm{d} \sigma$ quanta of frequenoy between $\mathcal{\nu}$ and $\nu / d \nu$ travelling within the solid angle $d \Omega$ of a specific direction, and normally incident per unit time upon a surface element do. The specific intensity of the beam is then $I(\nu)=h \nu c n_{\nu}$. In traversing a length ds in the direction of propagation some photons will be absorbed or knocked out of the beam, while others will be added to the beam by emission or scattering. The number absorbed from the beam in unit time is
(3.1) Absorbed $=\sum_{b a} \omega_{b a}(\nu) d \nu d \Omega \frac{\frac{1}{b}^{b}}{\nabla^{b}} d s d \sigma_{0}$

The sum is over all transitions, bound-bound, bound-free, or free-free, which can absorb the frequency $工$. The number enitted into the beam per unit time is
(1) S. Chandraselhar, Stellar Structure, pp. 198-213
(2) Morse Astrophysioal journal, 92, p. 30, 1940. APPROVED FOR PUBLIC RELEASE
-20-
(3.2) Emitted $=\sum_{a b} \pi_{a b}(\nu) d v d \Omega \frac{1^{a}}{V}$ de d $\sigma$.

The number per unit time scattered out of the beam is
(3.3) Scattigred out $=\int_{\vec{V}} \int_{\Omega^{\prime}} \frac{n(\vec{v})}{\nabla} d \vec{v} \pi_{\nu} \nu^{\prime} \vec{v} d \nu d \Omega d \Omega^{\prime}$ de do.
where $n(\vec{v}) d \vec{v}$ is the number of electrons with velocity vector between $\vec{v}$ and $\overrightarrow{\mathrm{F}}$ \& $\mathrm{d} \overrightarrow{\boldsymbol{v}}$. The number scattered into the beam is
(3.4) Scattered in $=\int_{\vec{\nabla}} \int_{\Omega^{\prime}} \frac{n}{\nabla}(\vec{\nabla}) \mathrm{d} \overrightarrow{\mathrm{V}} w \nu^{\prime \prime} \nu \vec{\nabla} \mathrm{d} \nu^{\prime \prime} \mathrm{d} \Omega \mathrm{d} \Omega^{\prime} \mathrm{ds} \mathrm{d} \sigma$.

Adding up the 4 contributions (3.1) to (3.4) with the proper signs gives the net gain in number of photons $d\left\{0_{\nu} d \nu d \Omega d \sigma\right\}$ in the beam as it traverses is in the steady state. Then


$$
+\int_{\vec{v}} \frac{x\left(\vec{v}^{\prime}\right)}{\nabla} d \vec{v} \int_{\Omega^{\prime}}\left(w_{\nu^{\prime \prime}} \nu \vec{v} \frac{d \nu^{\prime \prime}}{d \nu}=w_{\nu \nu^{\prime} \vec{v}}\right) d \Omega^{\prime} .
$$

We now introduce the assumption that, at each point in our medium local thermodynamic equilibrium exists. Then $(3.6) \frac{N^{a}}{N^{b}}=e^{-\frac{E_{a} E_{b}}{L T}}=e^{-\frac{h \nu_{a b}}{K^{T}}}=e^{-u_{a b}}$.

Furthermore introducing the relation (2.13), the terms involving absorption procusses give

$$
\begin{aligned}
& 8 \text { give } \sum_{b a} w_{b a}(\nu) \frac{N^{b}}{\nabla}\left\{e^{-u_{a b}}-1+\frac{2 l 2 \nu^{3}}{s^{2}} \frac{e^{-u_{a b}}}{I(\nu)}\right\} \\
& =\quad \sum_{b a} \frac{\phi_{b a}}{h_{\nu}}(\nu) \frac{N^{b}}{\nabla}\left(1 e^{-u_{a b}}\right)\left\{\frac{2 h \nu^{3}}{0^{2}} \frac{e^{-u_{a b}}}{1 \omega_{0}^{-u_{a b}}}-I(\nu)\right\}
\end{aligned}
$$

Since $\varphi_{b a}(\nu)$ has a sharp maximum at $\nu_{a:} \nu_{a b}$, it is permissible to put $u_{a b}=u$ in all places in the sum except in $\phi_{b a}(\nu)$.

With the use of definitions like (2.9), (2.20), we then get

$$
\begin{aligned}
& \text { (3.7) } \frac{d I_{\nu}}{d s} \text { (absorption) }=\mu_{a b s}(\nu)\left(1-e^{-u}\right)\left(B_{\nu}-I_{\nu}\right) \text {, } \\
& (3.8) \quad B_{\nu}=\frac{2 h \nu^{3}}{c^{2}} \frac{e^{-u}}{1-e^{-u}} \text {; } \\
& \text { (3.8a) } u=\frac{\ln 2}{x 2} .
\end{aligned}
$$

The terms involving scattering may now be simplified by replacing the transtion probabilities by (2.32). This gives

$$
\begin{aligned}
& \text { (3.9) } \frac{d I_{\nu}}{d s} \text { (scattering) }=\int_{\vec{v}} \frac{n(\vec{v})}{V} d \cdot \vec{v} \int_{\Omega^{\prime}} \int^{0} \frac{d \phi\left(\nu, v, \theta^{\prime}\right) I_{\nu}^{\prime}\left(\vec{l}^{\prime}\right)}{d \Omega} \\
& \frac{d \nu}{d \nu} \frac{\nu}{\nu}{ }^{\prime}\left(1+c^{2} \frac{I_{\nu}(\vec{l})}{2 h \nu^{3}}\right. \\
& \left.-\frac{d \phi\left(I_{j}, \overrightarrow{\nabla_{2}}, \theta\right)}{d \Omega!} I_{\nu}(\vec{l})\left(1+\frac{0^{2} I_{2}{ }^{\prime}\left(\ell^{\prime}\right)}{2 h \nu^{i \xi}}\right)\right\} d \Omega^{\prime} .
\end{aligned}
$$

It is worthwhile to note that since the oross-sections involve $\theta$ only in the form $W=\cos \theta$, and since $\theta^{\prime}=-\theta$, it is permissible to replace $\theta^{\prime}$ by $\theta$. For the simple case of Thoms on scattering from electrons at rest - the most important case in practise, we get

$$
\begin{gathered}
(3.10) \frac{d I_{\nu}}{d s} \text { (scattering) }=\frac{n}{\nabla} \frac{r_{0}^{2}}{2} \int_{\Omega:}\left\{\left(1+w^{2}\right) I_{\nu}\left(\overrightarrow{l^{\prime}}\right)\left(1+\frac{I_{\nu}(\vec{l})_{a^{2}}^{2}}{2 h_{2}{ }^{3}}\right)\right. \\
\left.-\left(1+w^{2}\right) I_{\nu}(\vec{\ell})\left(18 \frac{I_{\nu}\left(\vec{l}^{0}\right) o^{2}}{2 h \nu^{3}}\right)\right\} d \Omega^{\prime} .
\end{gathered}
$$

It can be seen immediately that the terms in induced scattering cancel exactly, leaving
(3.11) $\frac{d I_{\nu}}{d s}($ scat. $)=\frac{n}{V}\left\{-\frac{8 \pi}{3} r_{0}^{2} I_{\nu}(\vec{\ell})+\frac{r_{0}^{2}}{2} \int_{\Omega}\left(1+r_{r}^{2}\right) I_{\nu}\left(\vec{\ell}^{\prime}\right) \mathrm{d} \Omega^{\prime}\right\}$.

The exact solution to the equation of transfer for absorption alone may be Worked out. The result can then be expanded, the zero order term giving isotropic
radiation, and the first order term giving the diffusion theory type of approximatron. It is olaly in this type of approximation that the Rosseland mean opacity appears - however, the approximation is an excellent one in systems we shall consider. We, therefore, employ this expansion to solve the combined equation (3.7)t (3.11), by putting
(3.12) $I_{\nu}=I_{\nu}^{0}+l_{\nu} \frac{\partial}{\partial_{8}} I_{\nu}^{0}+h_{1} \frac{\partial}{\partial s} L_{\nu} \frac{\partial I_{\nu}^{0}}{\partial_{2}}+\cdots \cdot$

Substituting this expansion in (3.7) acid (3.11), and collecting terms not containing $\frac{\partial}{\partial s} I_{\nu}^{0}$ gives

$$
0=\mu_{a b s}\left(1-e^{-u}\right)\left(B_{\nu}-I_{\nu}^{0}\right)_{0}
$$

since the scattering terms cancel to this order. Hence we must have
(3.13) $\quad I_{j \prime}^{O}=B_{2}$.

Tho terms in $\frac{\partial}{\partial u} I_{\nu}^{0}$ give

$$
\begin{gathered}
\left(3_{0} 14\right) \frac{\partial I_{\nu}^{0}}{\partial_{s}}\left(1-\frac{\partial l_{\nu}}{\partial_{s}}\right)=\mu_{a b s}(\nu)\left(1-\theta^{-u}\right) l_{\nu} \frac{\partial I_{\nu}^{0}}{\partial_{s}}\left(1-\frac{\partial}{\partial_{s}} l_{\nu}\right) \\
+l_{\nu} \frac{\partial I_{\nu}^{0}}{\partial s}\left(1-\frac{\partial}{\partial s} h_{\nu}\right) \frac{n}{V} \frac{8 \pi}{3} r_{0}^{2}
\end{gathered}
$$

since the contribution of the integral in (3.11) is negligible to this order. In order to satisfy $(3.14) \mathscr{l}_{\nu}$ must be

$$
\begin{equation*}
\boldsymbol{l}_{\nu}=\frac{1}{\mu_{a b s}(v)\left(1-e^{-u}\right)+\mu_{8}(v)} \tag{3.15}
\end{equation*}
$$

where

$$
\begin{equation*}
\mu_{8}(2)=\frac{n}{v} \quad \varphi_{0}: \varphi_{0}=\frac{8 \pi}{3} r_{0}^{2} \tag{3.16}
\end{equation*}
$$

Thus we see that in this approximation scattering and absorption must be treated in different fashion. The physical reason for this is that the stimulated scattering tending to weaken the beam is exactly compensated by stimulated scattering, tending
to strengthen the beam. On the other hand the stimulated emission strengthens the beam and ncthing can compensate for this. The effeotive absorption coefficient must, therefore, be reduced by the factior ( $1, e^{-u}$ ).

Using the zero and first order texms in (3.12) we can compate the flux of energy of frequencies between $\nu$ and $\nu$ id $d \nu$

$$
\begin{aligned}
& \text { (3.17) } \hat{n} \bullet \vec{F}(\nu) d \nu=\int_{\Omega} \hat{n} \bullet I(\nu) d \nu d \Omega=-\frac{4 \pi}{3}, \ell_{\nu} \operatorname{n}_{\mathrm{n}} \operatorname{grad} \quad \mathrm{~B} \nu \mathrm{~d} \nu \\
& =-\frac{4 \pi}{3} l_{2} \frac{d}{d T} B_{\nu} d y \hat{n} \circ g \mathrm{grad} T \text {. }
\end{aligned}
$$

where $\hat{n}$ is a unit veotor normal to the aurface over which the flux is desired, and we have assumed that the spatial variations in the conditions of the system are solely due to a temperature gradient. the integrated flux is then $\vec{F}=\int_{0}^{\infty} F(\nu) d \nu$. By introducing mean free path properly averaged over all frequencies-othe somcalled Rosseland mean
(3.18) $\Lambda=\frac{\int_{0}^{\infty} l_{\nu} \frac{d}{d T} B_{\nu} d \nu}{\int_{0}^{\infty} \frac{d}{d T} B_{\nu} d \nu}$
the integrated :Plux can be written
(3.10) $\quad \vec{F}=-\frac{4 \pi}{3} \quad \Lambda_{\operatorname{grad}} T \frac{\mathrm{~dB}}{\mathrm{dT}}$.
where
(3.20) $\quad B=\int_{0}^{\infty} B y^{d} y^{\prime}=\frac{2 \pi^{4} k^{4}}{150^{2} h^{3}} \mathrm{~T}^{4}$.

The energy density may be found from (3.12) since

$$
(3.21) \quad u_{\nu}=\int_{\Omega} \frac{I_{2}}{0} d \Omega=\frac{B_{1}}{0} d \Omega \ldots \frac{4 \pi}{0} B_{\nu} .
$$

whence
$(3.22) u=\int_{0}^{\infty} u_{\nu} d \nu=\frac{4 \pi}{0} B_{0}$

The first ordor term in $h_{\nu i n}$ both these equations vanishes identically, so that (3.21) and (3.22) are correot to the same order as (3.19). Combining (3.22) with (3.19), we have
(3.23) $\vec{F}=-\frac{0}{3}$ Agrad u.

By introducing the expression (2.8) for $B_{\nu}$ and (3.20) for $B$, the expression (3.18) simplifies to

$$
\begin{equation*}
\Lambda: \frac{15}{4 \pi^{4}} \int_{0}^{\infty} l_{\nu} u^{4} e^{u}\left(e^{u}-1\right)^{-2} d u . \tag{3.24}
\end{equation*}
$$

The effective mean free path for radiation may be alternatively expressed in terms of the mean opacity coefficient per unit mass $\mathbb{K}$ by the relation
(3.25) $\quad K=\frac{1}{e \Lambda}$.
$\rho$ being the density. It is this quantity which is usually used in astronomical dise cussions.

The effect of the present treatment of scattering, compared to the usual practice in opacity disoussions is now olear. If there is no absorption, our treatment gives

$$
\begin{equation*}
\Lambda=\frac{1}{\Gamma^{2}}=\frac{15}{4 \pi^{4}} \int_{0}^{\infty} \frac{\nabla}{n \varphi_{0}} u^{4} e^{u}\left(e^{u}-1\right)^{-2} d u=\frac{\nabla}{n \varphi_{0}}=\frac{1}{\mu_{s c a t t}} \tag{3.26}
\end{equation*}
$$

while the conventional treatment gives

$$
\begin{equation*}
\Lambda=\frac{15}{4 \pi^{4}} \int_{0}^{\infty} \frac{\nabla}{n \phi_{0}} u^{4} e^{2 u}\left(e^{u}-1\right)^{-5} d u=\frac{1,055}{\mu_{s c a t t}} \tag{5.26a}
\end{equation*}
$$

In most cases of astrophysical interest, scattering is not the most important process involved and the error is correspondimyly much less.

We can now see in outline the stigps meeded to oarry out the calculation of the opacity ocefficient. We must first determine the absorption and scattering ooeffioients. This requires a knowledge of the cross-sections listed in Chapter II and the ocoupation number - the subject of the next chapter. The averaging process indicated by (3.24) must then be oarriod out.

TV. STATISTXCAL MECHANICS OF IONIZED ATMOSPRERES

1. Introductiony.

The methods of statistical mechanies will furnish the oocupation numbers nesded in the calculation of tho absorption coeffioient. The succeeding seotions develop a corrvenient method for perfarming this calculation to good approximation. In addition, statistical meahanios gives a desoription of phenomena related to the breadth end dispersion of spectral lines. This angle is disoussed in sections 6 and 7 . Lastly, it is a simple matiter to calculate the thermodymamic functions of our material once the ocoupation numbers have been treated. While this is not sotually noeded in a calculation of the opacity coefficient, it is an extremely useful by-product. We orrry out this treatment in the Appendix I。
2. Quantum Vechanioal Desoription of the Systeme

We assume that the system we deal with is in thermodymamic equilibrium at a temperature $I$ and occupies a volume $V$. Although our entire system is not in suoh an equilibrium, the gradients of the thermodynamic variables are so amall, that we may consider that at each point such a looal thermodymmic equilibrium does exist. Finthermore, the temperatures we disciass will be so low that we may oompletely ignore nuolear reactions and pair production. Then we may desoribe our system as composed of $N$ auclei of whioh $N_{Z}$ have atomic number $Z$, assooiated with $n$ eloctrons just suffioient in number to make $N$ neutral atons. That is

$$
(4.1) \quad N=\sum_{Z} N_{Z}, \quad n=\sum_{Z} Z N_{Z}
$$

Clearly we have a system of many parti.oles with strong interactions. Following the usual method of separating out the effeots of the nuclear motions, we then express the eleotronic wave function of the system as a properiy antisymatrized produot of olze electron funotions obeying the Hartree-Fook Equations ${ }^{(1)}$. (1)

[^2]The essential features of the one eleotron wave functions are apparent from physical considerations, and may also be derived by inspection of the Fock Hamiltonian. For large energies, the kinetic energy term in the Hamiltonian dominates, and the wave functions spproach those of a free electron. They are, therefore, independent of the positions of the nuolei. For low energies, on the other hand, the interaction term with the molear potential becomes of equal importance to the kinetic energy terme Because of the singuls.rity in the potential at the position of each nucleus, the behavior of the wave function at any position is largely conditioned by the field of the nearest; nucleus, secondarily by the nearest neighbors, and is hardly affected by more distant nuolei. We, therefore, expect that near a nuoleus, the one election function will approximate the shape of the atomic wave function of the isolated ion. In this extrome the wave functions depend only on the distance from the nearest nuoleus - and are independent of the relative position of the nuolei, just as for the free eleotron extreme.

A model which embodies these essential foatures is the orystalline solid. At first sight this appears to be a violent distortion of the actual state of the system, since we should not expecti any long range orystalline order at the high temperatures with which we deal. The model will, however, provide the proper qualitative features of the wave functions for a system of many nuclei throughout which the electrons are free to roame Naturally, any features charaoteristic of the strict periodici.ty of the lattioe are simply introduced artificially by our model. Those featiures of the orystal model, however, which depend on near neighbors only should apply to our system, for there will be a local ordering effect corresponding to that present in ordinary liquids.

Precisely, as in the usual theory of metals, we can use the Block approximation of periodio wave functions for the electrons. In the low energy case the energy levels will correspond closely to those in the isolated ion, except that a single ionic state is $N_{Z}$ fold degenerate. This results because we can
construot $\mathrm{N}_{\mathrm{Z}}$ independent periodic Bloch funotions from the $\mathrm{N}_{\mathrm{Z}}$ zero order function，each of which corresponis to the electron being on any one of $N_{Z}$ different ionsa This degeneracy is ramoved by interactions with neighboring nuclei．so that finally we shall have in our orystal a narrom band of $\mathrm{N}_{\mathrm{Z}}$ states in the neighborhood of each state of the isolated ion．The wave functions are of the form
（4．2）$\quad x_{k}=\frac{1}{\sqrt{N_{Z}}} \sum_{n=1}^{N_{Z}} e^{2 \pi i \vec{i} \cdot \vec{r}_{n} u\left(\vec{r}_{n} \vec{r}_{n}\right)}$
where $i\left(\vec{r}_{-\vec{F}_{n}}\right)$ is an atonic wave function with origin at the nucleus located at $\vec{r}_{n}$ ．As the energy is inoreased，the atonic wave functions of neighboring ions overlap more and more，thus widening the band．Eventually，the band widths will exceed the distance between atomic levels，and we shall have a quasi－continuum of states．At about this energy the approximation of losalized atomic type wave funotions breaks down，for the functions overlap several nuolei．Moreover，atomic functions fron several levels must be considered in building up a good approxima－ tion from（4．2）．The transition stage of the onset of the continuum leads naturally into the stage when the atomio wave functions beoome constant throughout the orystal． For high enough energies the funotions（4．i）are of the free－eleotron type．

The natune of the eigenfunctions in the transition region is complicated．We are fortunate，therefore，that in our system（in contrast to the usual metallic state）only a very small fraction of the electrons will populate states in this region．This results because the Boltzmand factor in the probability of ocoupa－ tion of these states is rather small compared to that of the clasely bound low energy states ${ }_{\wedge}$ while the a prigri probability factor is not yet so large as in the high energy free states．The contribution of these transition states to the partition function of the system is，therefore，small，and for the thermodynamic properties of the system we may treat them roughly．The approximation we shall use in our stetistical mechanios is to ignore the details of these transition states completely．For the low energy states，we shall use atomic wave funotions
and hence term them bound states. For energies greater than a oertain limit whioh we term the out-off onergy, we shall use the eigenfunctions for free eleotrons. Since the transition states are statistically unimportant, the exact position of the out off energy is not oritical. We shall return later to the question of fixing the cut of:i energy.

The ciroumstance that the trans ition states do not affeot the thermodynamic properties of our system is no guarantee that they will not seriously affect the optical properties. We shall sea later that the most critical element in the Rosseland mean opacity is the presence or absence of absorption in certain frequency regions. We must, therefore, examine whether the onset of a continum of one electron energy states will lead to a continuous absorption spectrum. In the Bloch scheme an examination of the transition probabilities proves this to be the case. This is not necessarily true in other approximation sohemes. The irregularities in our lattice, however, will undoubtedly provide the continuous absorption we assune.

There is still another and more serious short-coming of our onemleotron approximation. This is the neglect of oorrelations between electrons positions, except for that diotated by the Paull. prinoiple. These correlation energies are so small that they do not affect the ocoupation numbers of the one eleotron states. They are, however, decisive in determining the line absorption contribution to the opacity. This is so because the mumber of bound energy levels in our complicated crystal, is, in the one eleotron function approximation, exactly the same as in an isolated one electron atom. The speotrum would then appear to consist of just a very fow very strong lines. Taking corrslations into account would split these into very many lines of the same total absorption strength. The next chayter shows that heving the absorption strength distributed anong many lines very much enhances thejr effect on the opacity. The treatment of these correlations by
the "ionio" method is the topio of seotion 5 .
Another fisature of our approximatlon whioh bears watohing is the "missing" bound states wich have been excluded ty the out off. Of course, these are not missing at all but have been merged witih the continum. The absorption strength of the out of bound states should be distributed at the beginning of the free states.

With this quantum mechanical apprcximation, we now prooeed to examine the statistical mershanios of our idealized model.
3. Statisticai Mochanics - Independenti Eleotron Approximation-

The object of this section is to vise the methods of statistical meohanics to find the mumber of electrons in each eleotronic state in equilibriwn at the temperature T. We shall here assume for simplicity that the eleotrons are completely independent of each other. The next section will treat the eleotronic interactions, but the final results can be thrown into essentially the same form as for independent eleotrons.

The nature of the energy levels and the density of states has been disoussed In the previous sections. If $c_{i Z}$ is the degeneracy of the $i^{\text {th }}$ ionic level, energy $\mathcal{E}_{1 Z}$, of a muojeus of oharge $Z$, there will be a band of $\mathbb{N}_{Z} C_{i Z}$ states at this enorgy in the system. Such bands will persist for all energies $\mathcal{E}_{\mathrm{iZ}}<\mathcal{E}_{c}$ less than the out off. Since eleotrons obey the Fermi-Dirac statistios, the mmber of electrons in the $i_{Z}$ th level will then be

$$
\begin{equation*}
n_{1 Z}=\frac{N_{Z}^{o_{1 Z}}}{e^{\alpha+\beta c_{1 Z}+1}} ; \quad \varepsilon_{1 Z}<\varepsilon_{0} \tag{4.3}
\end{equation*}
$$

where $\beta=\frac{1}{k^{I}}$ and $\alpha$ is the normalization constant with the physical interpretation that $\mu=-2$, $\alpha$ is the free energy of thine electrons.

For energies greater than $\mathcal{E}_{c}$, the density of state will be that of a free electron with kinetic energy $\varepsilon-\varepsilon_{0}$. This is, in the relativistic case (4.4) $\quad o_{\rho}(\varepsilon)=\frac{8 \pi \nabla}{(a h)^{3}}\left\{2\left(\varepsilon-\varepsilon_{c}\right) m x^{2}+\left(\varepsilon-\varepsilon_{0}\right)^{2}\right\}^{\frac{3}{2}}\left\{\left(\varepsilon-\varepsilon_{c}\right)+m 0^{2}\right\}$. or expanding for kinetic energies sma. 11 compared to $\mathrm{mo}^{2}$
(4.4a) $\quad c_{f}(\varepsilon)=\frac{8 \pi \nabla m}{h^{3}}\left\{2\left(\varepsilon-\varepsilon_{0}\right) m\right\}^{2}\left\{2+\frac{5}{4} \frac{\varepsilon_{-} \varepsilon_{c}}{m 0^{2}}+\ldots\right\}$.

Then the number of free electrons with enargies between $\mathcal{E}$ and $\mathcal{E}+\mathrm{d} \mathcal{E}$ is (4.5) $\quad n_{f}(\varepsilon) \mathrm{d} \varepsilon=\frac{c_{f}(\varepsilon) \mathrm{d} \varepsilon}{\alpha \neq \beta \varepsilon+1}$.

The total number of bound electrons is found from (4.3)

$$
\begin{equation*}
n_{b}=\sum_{z} \sum_{i} n_{12} \tag{4.6}
\end{equation*}
$$

while the total number of free electrons is from (4.5)

$$
\begin{equation*}
n_{f}=\int_{\varepsilon=\varepsilon_{0}}^{\infty} n_{f}(\varepsilon) \mathrm{d} \varepsilon \tag{4.7}
\end{equation*}
$$

Of course, the total number of eleotrons in the system is the sum of bound and free

$$
\begin{equation*}
n=n_{b}+x_{p} \tag{4.8}
\end{equation*}
$$

and it is this condition which determines the normalization oonstant $\alpha$ of (4.3) and (4.5) 。

To use these occupation aumbers (4.3) and (4.5) we must determine $\alpha$. We note that (4.7) is the equation for a free electron gas, except for the one fact that $n_{f}$ is not a constant given by the physical nature of the system, but instead varies with the temperature and rolums. $x_{f}$ may, however, be a rather insensitive function, and we may then employ the follewing soheme of successive approximations to determine $\alpha$. Assume a trial $n_{f} ;$ usually we may start by taking $n_{f}=n$ the
total number of electrons. Then use (4.7) to oaloulate $\alpha$. For the oase in which relativistic and degeneracy effects ane small corrections - the region of greatest interest - a suitable expansion of ( 4,7 ) gives
(4.9) $\alpha^{*}=\ln \frac{2 V}{n_{f}}\left(\frac{2 \pi m k T}{h^{2}}\right)^{3 / 2}+\frac{15 j}{8} \frac{k T}{m c^{2}}-\frac{e^{-\alpha^{*}}}{2^{3 / 2}}$.
where
$(4.10) \quad \alpha^{*}=\alpha+\beta \varepsilon_{c}$.

Knowing $\alpha$, the sum in (4.6) must be carried out explioitly, giving $n_{b}$ and by ( 4,8 ) a seoomi approximation to $n_{f}$. The ofcle is very rapidly comvergent.

If a long series of computations must be made, it is more convenient to fix $\alpha$ to start. Then using (4.6) we find $n / \mathbb{N}$ while (4.7) gives $n_{f} / V_{0}$. (Here N is the total number of atous in the system). Then we may find out to what value of $n / T_{3}$ i.e.., to what density; the value of $\alpha$ eorresponds.
4. Statistioal Mochanios - Dependent Eleotions.

We now treat our system including the elettronic interactions by the method of the canonical ensemble. A state of the entire system, symbolized by $J$, will be determined if we know the number of electrons in each onemelectron orbital of the Hartree-Fook set of equations. Although each of these nondegenerate orbitals may have either one eleotron or none at all, we find it more oonvenient to group degenerate or nearly degenerate orbitals together and suoh groups we will designate by small subscript $i$ or $j$ or $\ell$. The raumber of suoh orbitals in the $i \frac{\text { th }}{}$ energy group; 1.e. the degeneracy; we denote by $c_{1}$. The energy of the state $J$ is (4.11) $\quad E_{J}=\sum_{i} n_{J i}\left\{\varepsilon_{J i}+\frac{1}{2} \sum_{j \neq i} n_{J j} \gamma_{J i j}+\frac{1}{2}\left(n_{J i-1}\right) \nabla_{J i i}\right\}$. $\|_{J i}$ is the number of eleotrons occupying orbitals in the energy interval when the system is in the state J. Now the parliition function will involve sums over all
states $J$ of the system, but only states near the equilibrium value will contribute heavily. Since the dependence on $J$ of the interactions $\mathcal{E}_{J i}$ and $\nabla_{J i j}$ is not pronounoect, we may insert some average value $\mathcal{E}_{i}$ and $\nabla_{i j}$ independent of $J$ instead. Moreover, we introduce the set of numbers $\bar{n}_{j}$, at present wholely arbitrary but later destined to represent some average occupation of the region. Then (4.11) bercomes

$$
\begin{aligned}
& \text { (4.12) } E_{J}=\vec{p}_{\frac{1}{2}} n_{J i}\left\{\left[\varepsilon_{i}+\frac{2}{2} \sum_{j \neq 1} \vec{n}_{j} V_{i j}+\frac{2}{2}\left(\vec{n}_{i}-1\right) \vec{V}_{i i}\right]\right. \\
& \left.+\left[\frac{1}{2} \sum_{j \neq 1}\left(n_{J j}-\bar{n}_{j}\right) \nabla_{i, j}+\frac{1}{2}\left(n_{j i}-\vec{n}_{1}\right) \nabla_{i i}\right]\right\} .
\end{aligned}
$$

The first term in square brackets in (4.12) is independent of the cocupations and may be regarded as the zero order approximation to the effective energy of the electron. The second term in square brackets gives the difference between the detailed interaction between electrons and the average interaction. We may expect this to be small and hence treat it as a perturbation.

Now accorifing to the canonical onemble treatment, the probability of finding our entire system in an energy level between $E_{J}$ and $E_{J} \& \Delta E_{J}$ is

$$
\begin{equation*}
P_{J}=\Omega_{J} e^{-\alpha^{\prime}-\beta E_{J}} \tag{4.13}
\end{equation*}
$$

where
(4.14)

$$
\Omega_{J}=\prod_{i} \frac{c_{i} t}{n_{J i} t\left(o_{i}-n_{j i}\right) s}
$$

is the number of states in the energy interval $\Delta E_{J}$ and $\alpha^{\prime}$ is the normalication constant deternined so that

$$
\begin{equation*}
\sum_{J} P_{J}=1 \tag{4.15}
\end{equation*}
$$

Hence in equiljbrium in our system the number of electrons in the $l$ th region will be
(4.16)

$$
n \mathscr{L}=\sum_{J} n_{J L^{p}}^{p}
$$

Substitute in (4.16) the appropriate expressions from (4.12),(4.13), (4.14), introduce the quantity $\alpha^{\prime \prime}$ by the relation
(4.17) $\quad \alpha^{\prime}=n \alpha^{\prime \prime}=\alpha^{\prime \prime} \sum_{i} n_{J i}$
and carry out the indicated operations to first order terms in the $\nabla_{i j}$. The treatmont is characteristic of the grand ensemble method. The manipulations are tedious and somewhat tricky but the result is comparativively simple, namely

$$
\begin{align*}
n_{l} & =c_{l} p \ell\left\{1-\beta_{q l} \quad\left\{\left[\left(\overline{n_{l}}-1\right)-2\left(\mathcal{C}_{\ell}-1\right) p_{l}\right] \frac{\nabla l l}{2}\right.\right.  \tag{4.18}\\
& \left.\left.+\frac{1}{2} \sum_{i \neq l} n_{i} \nabla_{l i}-\sum_{i \neq l} o_{i} p_{i} \nabla_{i l}\right\}\right\} .
\end{align*}
$$

where

$$
\begin{align*}
& q_{8}=1-p_{8} .  \tag{4.19}\\
& p \ell=\left\{1+\exp \left\{\alpha^{\prime \prime}+\beta\left[\varepsilon_{!}!+\frac{1}{2} \sum_{j \neq \ell} \bar{n}_{j} \bar{\nabla}_{\ell j}+\frac{3}{8}\left(\bar{n}_{\ell}-1\right) \bar{\nabla} \ell \ell\right]\right\}\right\}^{-1} \quad . \tag{4.20}
\end{align*}
$$

We now choose the arbitrary parameters so that the first order terms in (4.18) vanish identically. This gives

$$
\begin{array}{rl}
(4.21) & \bar{n}_{i} \\
=2 c_{i} p_{i} & i \neq \ell \\
\bar{n}_{\ell} & =1+2\left(\mathscr{Q}^{-1}\right) \mathrm{p}_{\ell}
\end{array}
$$

Although the two equations in (4.21) seem contradictory, this really is not so, for there is absolutely nothing which forbids us to use a different set $\bar{n}_{j}$ for each $n \ell$ in (4.016). Substituting (4.21) into (4.18) and (4.20) gives our answer
(4.18a) $\quad n_{\ell}=O^{\circ}{ }^{\mathrm{p}} \ell$
(4.20a) $\quad \mathrm{p}_{\ell}=\left\{1+\exp \left(\alpha^{n}+\beta \vec{\varepsilon}_{\ell}\right)\right\}^{-1}$

$$
\bar{\varepsilon}_{\ell}=\varepsilon_{\ell}+\sum_{j \neq \ell} s_{j} p_{j} \nabla \ell_{j}+\left(o_{\ell}-1\right) p_{l} \nabla \ell \ell
$$

Comparing our results of the last throe equations with those of the independent electron treatment, ( 4.3 ) for example, shows them to be of the same form, except the energy $\varepsilon_{i Z}$ of the independent electron case is replaced by an effective energy $\mathcal{E}_{\ell}$ of (4.22). We have, therefore, justified using an independent electron approximation with each electron subject to some averaged potential of its neighbors, and we have found that potential. correct to first order.

We can considerably simplify the result (4.22). Suppose $l$ represents a bound level. Then the interaction $\nabla \ell_{j}$ between the two bound levels can be shown to be exactly the same as that calculated using atomic wave functions which localize both the $l$ and $j$ orbitals on one particular ion. The terms in (4.22) due to the interaction of a bound electron with the other bounds is, therefore,
$\sum_{j} \frac{n_{j Z}}{N_{Z}} \nabla_{\ell_{j}} \& \frac{Y_{Z}}{N_{Z}}\left(\rho_{Z}-1\right) \nabla_{\ell l}$, where now the $V^{\prime}$ e are the ordinary coulomb and exchange interactions of atomic theory. By far, the largest contribution comes from the spherically symmetric part of the coulomb integral, usually denoted by $F^{0}(i, j)$ in theoretical spectroscopy. It is more convenient, however, to use screening constants $O_{1, j}$ instead of the $\mathrm{F}^{\mathrm{O}_{\mathrm{s}}}$, defined by

$$
\begin{equation*}
\text { oi.j= } \frac{F^{0}(i, j)}{-\left.\frac{d \varepsilon_{i}^{0}}{d z}\right|_{z}} \tag{4.023}
\end{equation*}
$$

The interaction energy of a bound electron with the frees can likewise be approximated by the interaction of an atomic wave function with the charge density of the free electrons in the neighborhood of one particular ion. Suppose, therefore, we break up our crystal into polyhedra, each containing a nucleus and of suffiorient size to enclose a negative charge $Z^{\prime}$ 'e where $Z^{\prime}=Z-\frac{n_{Z b}}{N_{Z}}$ is the average charge of the nucleus and its bound electrons. Approximate the polyhedra by spheres of the same volume, with radius $a_{Z}$ '. We then have
(4.24)

$$
\sum_{z} \frac{4}{3} \pi \quad a_{z}^{3} \cdot N_{z}=V_{0}
$$

If the electrons are really completely free, the charge density will be uniform throughout the crystal, and therefore

$$
\begin{equation*}
\frac{4}{3} \pi a_{z^{\prime}}^{3}=\frac{z^{\prime} v}{n_{f}} \tag{4.25}
\end{equation*}
$$

Moreover, there will be an electrostatic potential

$$
\begin{equation*}
\varphi_{f}=\frac{z^{\prime} e}{2 a_{z}{ }^{\prime}}\left\{\frac{r^{2}}{a_{z}^{2}}-3\right\} \tag{4.26}
\end{equation*}
$$

within each sphere due to the free electrons, and we obtain for the bound $l$ to free interaction energy
(4.27) $\left.\quad \infty \int \psi_{l}\right|^{2} \phi_{f} \mathrm{~d} \tilde{\tau}=\frac{z^{\prime} e^{2}}{2 a_{2}}\left\{3-\left(\overline{\left.\frac{r}{a_{z}}\right)^{2}}{ }^{\ell}\right\}\right.$
where $\bar{r}^{2}$ is the value of $r^{2}$ averaged over the $l$ bound wave function. For a bound state $l$, then, the energy $\overline{\mathcal{E}}_{l}$ becomes


$$
+\frac{z_{0}^{\prime}{ }^{2}}{2{a_{z}}^{\prime}}\left\{3-\left(\frac{\left.\frac{r}{a_{2}}\right)^{2}}{}{ }^{\text {bound }}\right\}\right.
$$

To first order in $\frac{d \mathcal{L}}{\mathrm{~d} Z}$, this is the same as

$$
\begin{equation*}
\bar{\varepsilon}_{D}=\varepsilon_{l}^{0}\left(z_{l}^{*}\right)+\frac{z_{0}^{\prime}{ }_{0}^{2}}{2 a_{Z^{\prime}}}\left\{3-\left(\overline{\left.\frac{r}{a_{Z^{\prime}}}\right)^{2}}\right\}\right. \tag{4.?9}
\end{equation*}
$$

where

$$
\begin{equation*}
z=z-\sum_{j \neq \ell} \frac{n_{j z}}{N_{z}} \sigma_{\ell, j}-\frac{n_{\ell z}}{Z_{z}}\left(1-\frac{1}{\sigma_{\ell Z}}\right) \sigma_{l} l \tag{4.30}
\end{equation*}
$$

We now must rewrite (4.22) for the case that $\ell$ represents a free electron. For this purpose, wo assume the bound electrons are localized at the nucleus. This is generally an excellent approximation. The free electrons move in the potential
field, which is in each ionic sphere
(4.31) $\varphi=\frac{z^{\prime} a}{2 \varepsilon_{Z^{\prime}}}\left\{\frac{2 a_{Z}!}{r}-3+\frac{r^{2}}{a_{Z!}^{2}}\right\}$.

This is due te the nucleus and bound electrons, and the frees. The energy $\mathcal{E}_{\boldsymbol{l}}$ of (4.22) in this case includes the kinetic energy $\mathcal{E}_{f}$ and the interaction with the nuclei, while the other terms in (4.22) give the interactions with other bound and free electrons. All these interactions are just $\omega \bar{\phi}, \bar{\varphi}$ being averaged over the volume of the ion. Hence, for a free electron
(4.32) $\bar{\varepsilon}_{f}=\varepsilon_{f}-\frac{3}{5} \sum_{z} \frac{N_{Z}}{n_{f}} \frac{z^{\prime 2}{ }_{e} z}{2 a_{Z}^{\prime}}$.

We note that the interactions have raised the energies of the bound electrons from the zero order approximation of interaction with the nucleus alone, while for free electrons the energy has been lowered from the different zero order approximation of no interactions. We now shift the zero of energy, so that a free electron with zero kinetic energy has zero total energy, by adding the constant $\frac{3}{5} \sum_{Z} \frac{N_{Z}}{x_{f}} \frac{Z^{\prime} e^{2}}{8 a_{Z}}$ - This will have the aclvantage that the density of states for the free electrons takes the simple form $o\left(\overline{\mathcal{E}}_{f}\right) \mathrm{d} \overline{\mathcal{E}}_{f}=o\left(\mathcal{E}_{f}\right) \mathrm{d} \mathcal{E}_{f}$ and hence (4.23) becomes
(4.33) $n\left(\bar{\varepsilon}_{f}\right) \mathrm{d} \bar{\varepsilon}_{f}=\frac{c\left(\varepsilon_{f}\right) d \varepsilon_{f}}{\alpha^{*}+\beta \varepsilon_{f}+1}$.

Of course, changing the zero of energy has no effect on the occupation numbers, since it merely replaces $\alpha^{\prime \prime}$ by another normalization constant
$\alpha^{*}=\alpha^{\prime \prime}-\frac{3}{5} \beta \sum_{Z} \frac{\mathbb{N}_{Z}}{n_{f}} \frac{z^{2} e^{2}}{2 a_{Z}}$. With this change in zero of energy, (4.29)
becomes


Now we shall somewhat arbitrarily place our division into bound and free states at the zero in our new energy scale. This means that whenever $\overline{\mathcal{E}}_{\ell}^{\prime} \mathrm{Z}$ of (4.34) is positive, that state is not bound. In most cases the higher states of an ion have their electrons rather uniformly distributed in the ions sphere and, therefore, $\left(\frac{r}{a_{2}}\right)^{2} \sim 3 / 5$. We can, therefore, generally "out off" the bound states at about

$$
(4.35) \quad=\varepsilon_{l}^{0}\left(z_{l}^{*}\right)=3 \frac{z_{0}^{1} e^{2}}{2 a_{z^{1}}}
$$

The relevant equations for calculating the occupation numbers will now be summarized in final form.

$$
\begin{equation*}
\frac{n_{\ell z}}{N_{z}}=\frac{{ }^{0} z}{e^{\alpha^{*}+\beta \hat{B}^{\prime}} \ell_{z}^{\prime}} \tag{4.36}
\end{equation*}
$$

$$
\$ 1
$$


(4.38) $\quad z_{l}^{*}=z-\sum_{j \neq \ell} \frac{n_{j z}}{N_{z}} \sigma_{\ell j}-\frac{n \rho_{z}}{N_{z}}\left(1-\frac{1}{\delta_{\ell}}\right) \sigma_{l_{0}, l}$.
(4.39) $n\left(\bar{\varepsilon}_{f}^{\prime}\right) d a \bar{\varepsilon}_{f}^{\prime}=\frac{\frac{8 \pi \nabla_{m}}{h^{3}}\left(2 m \bar{\varepsilon}_{f}^{\prime}\right)^{\frac{2}{2}}\left\{1+\frac{5}{4} \frac{\bar{\varepsilon}_{f}^{\prime}}{m_{0}{ }^{2}}+\ldots 0\right\}}{e^{\alpha^{+}+\beta \varepsilon_{p}^{\prime}}+1}$,
(4.40) $n_{b}=\sum_{Z} N_{Z} \sum_{\ell}^{n}{ }_{\ell Z} \quad$ sum for all $\bar{\varepsilon}_{l Z}^{\prime}<0$,
(4.41) $n_{f}=\int_{\bar{\varepsilon}_{f=0}^{\prime}}^{\infty} n\left(\bar{\varepsilon}_{f}^{\prime}\right) d \bar{\varepsilon}_{f}^{\prime}$.
(4.42) $n=n_{b}+n_{p}$ 。
(4.43) $\frac{4 \pi}{3} a_{Z}^{3}=\frac{Z^{\prime} v}{n_{f}} ; \quad$ (4.43a) $v=\frac{4 \pi}{3} \sum_{Z} a_{Z}^{3} N_{Z}$
(4.44) $\quad z^{\prime}=2-\frac{\sum}{\ell} \frac{n_{l z}}{\Sigma_{Z}}$
and as an explioitly equivalent of ( 4,41 )
$(4.41 a) \alpha^{*}=l_{n} \frac{2 V}{n_{f}}\left(\frac{2 \pi m k T}{h^{2}}\right)^{3 / 2}+\frac{15}{8} \frac{k T}{110^{2}}-\frac{15}{16}\left(\frac{k T}{m c^{2}}\right)^{2}$

$$
\frac{e^{-\alpha^{*}}}{2^{3 / 2}}\left(1-\frac{15}{16} \frac{\mathrm{kI}}{\mathrm{mc}^{2}}\right)+e^{-3 \alpha^{*}}\left(\frac{1}{3^{3 / 2}}-\frac{1}{16}\right)+\ldots
$$

This set of equations is somewhat decosptively siraple in form, for it must entually be solved by a somewhat lengthy series of sucoessive approximations.
 an $\alpha^{*}$, and a set of $a_{Z}$, satisfying (4.43). (The latter is, of course, trivial if our system has only one element.) We then caioulate $z\left(\begin{array}{l}\text { b }\end{array}\right.$ by (4.38) and by interpolation in a table of energy levels find $\mathcal{E}_{\ell}^{0}\left(Z_{l}^{*}\right)$. Meanwille, by (4.44) oalculate $z^{\circ}$. We can then immediately get $\varepsilon_{l}^{\prime} z_{\text {a }}$ and by (4.36) a ner set of $n l Z$. By (4.40) wo get $n_{p}$ which with (4.42) gives $n_{f}$. Employing $n_{\rho}$ in (4.4la) we arrive at a new $\alpha^{*}$. Moreover, using (4.43a) gives a new set of az'. Continuing this oyole, we can establish our final ocoupation numbers. The tables of soreening oonstants $\sigma_{i, j}$ energy levels, and $\bar{r}^{2}$ needed for the caloulation are presented here as Table 2, 3, and 4 respectively.

One approximation made in the foregoing set of equation is the assumption of the undform oharge distribution of tine free electrons. An improvement on this approximation, which also demonstrates the range of its validity, is given in Appondix II.
5. Ionic Oooupation Numbers.

A somewhat different model for our syatem was mentioned briefly in the last seotion - the lonic model. We shall now desoribe the basis for this model somevinat more fully, show its relevancy to the opacity problem, and indioate how our previous results may be applied in this case.

We know that if an olectron is in a staje of sufficiently low energy, it will be temporarily bound to one nucleus. Beoause of the high coulomb barrier between neighboring nuclei, it will remain bound for a considerable time - indeed it would be f.mproper to use atomic wave functions for the eleotron unless it remained bound for times long compared to the time of revolution of its Bohr orbit. Naturally, several electrons may be bound to the same nuoleus at the samo time. The interactions between the electrons will not be expressible in terms of the ireatment we have hitherto used, for the correlations wioh were neglected are now of deaisive importance. For example, a nucleus with two bound $K$ eleotrons will behave much differently, partioularly with regard to its speotrum, than one with $2 K$ and four bound $L$ electronis. We can take these correlations into aocount by abandoning our simple "prociuct of one eleotron funotions" approximation and using instesd functions whjoh depend upon all the coordinater of the bound eleotrons of each nucleus. Ihis is equivalent to describing our system as composed of many different ions in a dynamic equilibrium in a sea of free electrons. Applying the statistiosl mechanics appropriate for systems undergoing "ohemioal" reactions, we can get, for example, the number of ions of each type in our system, and the distribution of the ions among ionic quantum states. Essentially the same result is obtained by tho use of the oanonioal ensemble treatment for dependent particle systems if we appropriately express the energy of the system as the sum of ionic onergies, free eleotron energies, and interaotion energies between these components. The latter method has the advantage that we are able to take into account, to first order at any rate, the interactions of the free electrons and ions. This model, which is certainly to be preferred to that of the previous seotion, gives the spectrum of our system as the superposition of the many differem; ionic speotra. It is preoisely this very rioh line spectrum which causes the lines to be so important in the opacity problem.

Despite the apparent dissimilarity oi the ionic and "one-eleotron funotion" models, we an show an intimate relationship between the two. In our ionic model we may use for each ion the oonventional treatment of many electron ions. This consists in expressing the wave function of the ion as antisymmetrized products of onemeleotron functions and then carrying out perturbation oqloulations - usually only to first order. Suppose we then neglect all but the spherically symmetrio coulomb interaction. This makes many of the ionic levels degenerate, of course, but the energy changes are so slight as not to ohange the ionic ocoupations. If we then take the average number of eleotrons in a particular orbital throughout all the ions in our system, we gel;, to first order, the results of section 4. The useful point about this relationship is that we can use the occupation numbers of sootion 4, giving the average ocoupation, to find the ionic cocupations to good accuracy without the neod of starting off afresh in a laborious calculation from the ionia model. Thus the work of the provious section gave the number of eleotrons $\frac{n_{l}}{H_{2}}$ in the $l^{\text {th }}$ level of an ion of nuelear change $Z$, or alternatively $p_{l Z}=\frac{n \ell Z}{Q_{Z}}$ the probability of ocoupation of the states of that level. From this we caloulate the probability of finding an ion in our system with several bound eleotrons arranged to give some particular quantum state of the ion. For example, the probability of having an ion with gleotron oonfiguration (1g) ${ }^{2}$ ( $2 \pi \frac{1}{2}$ ) ( $3 p_{3 / 2}$ ) in the $K$, $L$ and $M$ shells, whatever the configuration of the higher shells may be, is (4.45) $p_{1 s}^{2} \times 2 p_{2 s} q_{2 s} \times q_{2 p}^{2} p_{2}^{7} \times q_{2 p 3 / 2}^{4} \times q_{3 s}^{2} \times q_{3 p_{2}^{1}}^{2} \times 4 p_{3 p 3 / 2} q_{3 p 3 / 2}^{3} \times q_{3 d 3 / 2}^{4} \times q_{3 d 5 / 2}^{6}$ where
(4.46) $q_{\ell}=\left(1-p_{\ell}\right)$ 。

In general the probability of having ani ion with $\nu /$ eleotrons in the $l$ th level regardless of the occupation of the other levels is given by the binomial distribution

As is well known, for large ${ }^{\circ} \ell$ and small Pl . (which is the case for the higher bound levels of an ion) this reduces to the Poisson distribution

The ionic model gives a simple physical interpretation to the rather surpriseing looking formulae of section 4. Consider, for example, the term $-\frac{\mathrm{d} \varepsilon_{l}}{\mathrm{dZ}} \frac{n_{j} Z}{N_{Z}} \sigma_{\dot{b}, \mathrm{j}}^{\prime}$ in (4.28) the energy $\bar{\varepsilon}_{\ell}$. The factor $-\frac{\mathrm{d} \varepsilon_{l}}{\mathrm{~d} Z} \sigma_{\ell_{0 j}}$ is the interaction between the $j^{\text {th }}$ and the $l^{\text {th }}$ bound electron levels in an ion. Averaging over all the ions of the system, some having no electrons in the fth level, others having one, others having two, etc. gives precisely the term re are considering. Again consider the ions which definitely have electron in the $l$ th level. The average occupation among these ions of the other $0 \mathrm{lz}^{-1}$ states in the level is ${ }^{\mathrm{p}} \ell_{2}\left({ }^{0} l_{Z}-1\right)$. The average interaction energy between one particular $\ell$ electron and the others then is just $\mathrm{p}_{\ell Z}\left(\sigma_{\ell Z^{-1}}\right) O_{l_{0} \ell}\left(\frac{-\mathrm{d} \varepsilon_{l}}{\mathrm{dZ}}\right)$. This is precisely the third term in (4.28). We see that the energies involved in the dependent electron treatment are averages over the ions of the system. Going a little further, we can show that (4.28) is actually the average ionization energy of anlth level electron in our ionic system. To prove this, wo note that if an ion has $x_{j}$ electrons in the $j^{\text {th }}$ level, its energy to first order is approximately
$E\left(x_{1}, x_{2} \bullet \cdots x_{i} \bullet\right) E=\sum_{j} x_{j} E_{j Z}^{0}+\frac{1}{2} \sum_{i} \sum_{j \neq i} x_{i} x_{j} F^{0}(i, j)+\sum_{i} x_{i}\left(x_{i}-1\right) F^{0}(1, i)$
so that the ionization energy of the $l^{t h}$ election is

$$
\begin{aligned}
&\left.E\left(x_{1}, x_{2} \cdots\right)_{\ell} \bullet \cdots\right)-E\left(x_{1}, x_{2} \cdots \cdots l_{-1} \ldots\right) \\
&=\varepsilon_{l Z}^{0}+\sum_{j \neq l} x_{j} F^{0}\left(l_{0} j\right)+\left(x_{\ell}-1\right) F^{0}\left(l_{0}, \ell\right)
\end{aligned}
$$

Avaraging this ionization energy over all the ions having at least one bound $l$ electron gives (4.28) except for the terms involving the interacticns with the free eleotrons.

This completes our discussion of the ocoupation numbers in our system, which are needed to get the absorption cooffioient.
6. The Influence of Nuolear Motion.

Thus far we have considered the nuclei as fixed in a lattice position. This is justified, since neither the bound nor free wave functions depends appreciably on the relative position ol' nucle\{. All our ocoupation numbers are, therefore, correot. There are two phenomena, however, whioh depend on the nuclear motions 1) the total enerey of the system has a contribution from the kinetic ani potential energies of the nuclei; 2 ) in their motion, nuclei will exert varying eleotric fields upon the bound electrons of neighbors, thus causing Stark affect $s$ hifts and splittings of the spectral lines. The first effect is of sisme small importance in the thennodymamic properties of our system, While the secord may be very important in iafluenoint the effect of lines on the opacity.

The result of separating the wave equation for our entire system so as to give the eleotronic energ separately, describes the motion of the nuclei as if proceeding in a potential determined by the elsotronio energy. This potential is in our case approximately the classical potential of an assembly of positive ions moving In a uniform charge density due to the iree electrons. We treat this potential in two limiting cases. Assume first that the muclei are at lattioe positions, whioh configuration represents the zero of poisential energy. For amall deviations from this position the potential inoreases. For example, the change in potential energy. if a single ion of effeotive charge $Z^{\prime}$ fis at a small distance $r$ from its equilibrium posf.tion is
(4.49) $z^{\prime \prime} e\left[\varphi_{f}(r)-\phi_{f}(0)\right]=\frac{z^{\prime 2} e^{2}}{2 a_{Z^{\prime}}} \frac{r^{2}}{a_{z^{\prime}}}=\varepsilon(r)$.
where $\varnothing_{f}$, given by (4.26), is the electrostatic potential of the free electrons. So long as (4.49) holds, the nuclei vill perform simple harmonic vibrations about their position of equilibrium. The heavy mass of the nuclei will make the frequency so low that the contribution to the energy of the system is olassical
(4.50) $\quad E_{\text {nuelei }}=3 \mathrm{NKT}$ 。

Continuing the treatment of this approximation, we consider the Stark offect due to this motion. The number of ions with displacement $r$ to ridr is
(4.51) $N(r) d r=\mathbb{N}_{Z} \frac{4 \pi r^{2} \exp (-\varepsilon(r) / \operatorname{rr}) d r}{\int_{0}^{Z^{r}} 4 \pi r^{2} \exp -(\varepsilon(r) / \pi r) d r}$.
where $\mathcal{E}(r)$ is given by (4.49). Carrying out the integration gives
(4.52)

$$
\frac{N(r) d r}{N_{Z^{\prime}}}=\frac{\left(\frac{\mathrm{sr}}{\mathrm{a}^{\prime}}\right)^{2} \exp -\left(\frac{\mathrm{sr}}{\left.\mathrm{a}^{\prime}\right)^{1}}\right)^{2} \mathrm{~d}\left(\frac{r}{\mathrm{a}^{\prime} Z^{\prime}}\right)}{\sqrt{\frac{\pi}{4}} \text { erf } \mathrm{s}-(\mathrm{s} / 2) \exp \cdots \mathrm{s}^{2}} .
$$

where
(4.52a) erf $x=\frac{2}{\sqrt{\pi}} \int_{0}^{x} 0^{-y^{2}} d y$,
and

One important result of this formula is obvious: the Stark effeot dispersion will fall off exponentially with distance from the line oenter.

To examine the range of applicability of this approximation, we calculate the average displacement
-4400
(4.54)

$$
\frac{\bar{r}}{a_{Z}}=\int_{0}^{L_{Z}} \frac{p}{a_{Z}} \frac{N(x) d x}{N_{Z}}=\frac{\frac{1}{28}\left\{1-e^{-s^{2}}\left(s^{2}+1\right)\right\}}{\frac{\sqrt{\pi}}{4} \text { eris }-(8 / 2) e^{-8^{2}}}
$$

For s $\gg 1$ this becomes
(4.54a) $\frac{\overline{5}}{a_{z}} \sim \frac{2}{\sqrt{\pi 8}}$.
while for $\ll 1$
$(4.54 b) \quad \frac{\bar{r}}{a_{Z}}=\frac{3}{4}\left(1-\frac{8^{2}}{18}\right)$ 。
The expansion shows that f or $\mathrm{s} \ll 1$, that is high temperature, low density (large $a_{Z}$ ) or small nuclear charge, the average displacement of the mallei will be $3 / 8$ of the average internuclear distance. This is the result if the nucleus could be with equal probability at any point in the sphere. In this ouse we could scarcely speak of harmonic vibrations and the method of treatment is not applicable. For large $s$ how aver, that is low temperature, high density, and/or high nuclear charge, (4.54a) shows our approximation to be adequate. Figure 2 shows the behavior of the $\vec{r} / a_{Z}$, as a function of $s$.

For the cases where the approximation is valid, the electric field on a nucleus situated at $r$ is directed toward the lattice point and has magnitude (4.55) $|\vec{\varepsilon}|=\frac{d}{d r} \varphi_{f}=\frac{z^{\prime} \theta}{a_{z}^{\prime}{ }^{\prime}} \frac{p}{a_{Z^{\prime}}}$.

The distribution of nuclear positions will lead to a distribution of electric field magnitudigs, and hence a continuous dispersion of the observed spectral lines of the assembly of ions. The fraction af ions of effective charge $z^{\prime}$ which will be subject to a f:leld of between $|\vec{E}|$ and $\left|\vec{E} \prod_{i}\right| \vec{E} \mid$ is


Formula (4.55), while correct in order of magnitude, does not toll the whole story, position for neighboring ions will be displaced from their equilibrium and give rise to a dipole field. The resultant of all dipoles will give a field of the same order as (4.55). Moreover, in applications to Stark effect, it should be remembered that the field (4.55) is radial, not linear as in the usual considerations.

We now turn to the case where $S \ll 1$ and harmonic vibrations do not describe the motion. Here the nuclei may wander rather freely about, except when one nucleus makes a very close approach to another. The energy contribution will be essentially that of a perfect gas

## (4.57) $E_{\text {nuol.ei }}=3 / 2$ NET.

The spatial distribution of the ions will be determined by their manual
potential energy for close approaches. To good approximation this is simply


$$
\mathcal{E}\left(r_{12}\right)=0 \quad \text { elsewhere }
$$

Then
(4.59) $N\left(r_{12}\right) d r_{12}=\frac{N_{Z}}{\nabla} e^{\infty \varepsilon\left(r_{12}\right) / R e l t} 4 \pi r_{12}^{2} d r_{12}$.

The electric field felt by the ion $Z^{\prime \prime}$ as it approaches $Z^{\prime}$ is
(4.60) $|\vec{\varepsilon}|=\frac{z^{n} e^{r_{12}}}{r^{2}}$.
so that the number of ions in fields between $|\overrightarrow{\mathcal{E}}|$ and $|\overrightarrow{\mathcal{E}}|+\mathrm{d}|\overrightarrow{\mathcal{E}}|$ is


Here again we can see that the Stark effect dispersion will fall off exponentially.
7. Fluctuations.

The quantum mechanical treatment we have used did not actually find the stationary energy levels of our systom. The approximations whioh we were forced to introduce had the result that we treated the ions as if they were independent syatems, and then we introduced interactions between other ions and the free electrons as perturbations. The truls stationary states, of course, will resemble the zero order approximation except tihat interactions will have removed 8 ans of the degeneracy. This splitting is Fary important as a sourde of line breadth. To inolude this splitting we can conaider the interactions as time dependent perturbations or fluotuations.

Ons of these fluatuations is caused by the nuclear motions just disoussed in seotion 6. The nuclei being so messive onn be thought of classically. Since they have in equilibrim the same epergy as the electrons, their velocity will be a. Pactor $\sqrt{\frac{m}{m}}$ smaller. ( $M$ is the anoleer mass, $m$ the eleotronic mass). The motion will be so slow compared to that of the electrons that we may use the adiabatic approximation for the interaotions. The result is that eleotronio levels are shifted by a Stark effect vinen two naclei approach, but no eleotronic transitions are induced.

If we thought of the free eleotrons olassioally, they would be randomily distributed fin space. We Fould then obtain sonsiderable density fluotuations in the neighborthood of each ion and it might be imagined these effeots must be considered. Our quantum mechaniaal treatments of the free electrons, however, is much cleser to the truth. Fif must really colusider the wave function of a free eleotron to extend throughout the solid. This eliminates the density fluotuationse But, because we have negleoted eorrelations, there is another effeot we have missed. This is the collision of free electrons with the ions. Since this is a fast process, it will induce transitions from one ionic state to another, giving the states a collision breadth. It is well known that such breadth gives the same form of dispersion as the maturalAbreadtro FOR PUBLIC RELEASE
V. EFFECT OF LINES ON OPACITY. METHODS FOR TREATING LINE CONTRIBUTION.

1. Separation of Lire Contribution

The Rosseland mean opacity $K$ as is shown in Chapter III (3.24) and (3.15)
is a weighted average absorption coefficient given by
(5.1) $\quad \Lambda:: \frac{1}{\rho \mathbb{X}}=\int_{0}^{\infty} \frac{W(u)}{\mu u^{3}} d u$.
where
(5.2) $\quad W(u)=\frac{15}{4 \pi^{4}} u^{7} e^{2 u}\left(e^{u}-1\right)^{-3}$.
and
(5.3) $\quad \mu=\mu_{a b s}+\mu_{\text {sat }}\left(1-e^{-u}\right)^{-1}$.

It is usually convenient to consider the absorption coefficient resolved into two terms
(5.4) $\quad \mu_{0}=\mu_{0}+\mu_{l}$.
$\mu_{0}$ being the absorption due to continuous processes alone, and $\mu_{\ell}$ the absorption due to the lines. The reason for this division is that $\mu_{0}$ is moderately smooth function of frequency except at the location of an absorption edge, while $\mu \boldsymbol{l}$ is a very ragged function with sharp maxima at the frequency of each absorption line. Substituting (4) in (1) and introducing; the notation (5.5) $\quad r=\frac{\mu_{l}}{\mu_{c}} ; \quad P=\frac{1}{1+2} \quad$.
we get for the mean free path
$(5.6)$

$$
\Lambda=\frac{1}{\rho^{K}}=\int_{0}^{\infty} P \frac{W(u) d u}{\mu_{0} u^{3}}=\Lambda_{0}-\Lambda_{l} .
$$

where

$$
\begin{equation*}
\Lambda_{c}=\int_{0}^{\infty} \frac{w(u) d u}{\mu_{c} u^{3}} \tag{5.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\Lambda_{l}=\int_{0}^{\infty} \frac{r}{1+N} \frac{w(u) d u}{\mu_{c} u^{3}} . \tag{5.8}
\end{equation*}
$$

The latter form of ( 5.5 ) shows how the lines roduce the mean free path from the value $\Lambda_{c}$ obtained by oonsidering continuous processes alone. The oontribution $\Lambda_{c}$ of the continuous processes has been treated by all workers in the field of ste:llar opacities with varying degrees of completeness, but the contribution of lines has hitherto been ignored.
2. Effect of a single line

To understand the effect lines make on the opacity, and to help in developing methods of treating lines, we start by considering the simplest case of a spectrum with only one line. The lins absorption coefficient in that oase is (o.f. equation (2.9))
(5.9) $\mu_{i}(v)=\frac{\pi e^{2}}{m c} \frac{N^{b}}{v} f_{b a} b_{b a}(v)$.
where $b(2)$ the dispersion factor shows the frequency dependence of the absorption. $b(\nu)$ has a sharp maximum at $\nu=\nu_{0}$, the frequency of the center of the absorption line, and is so normalized that

$$
\begin{equation*}
\int_{0}^{\infty} b(z) d v=1 . \tag{5.10}
\end{equation*}
$$

Without considering further details, ve can see qualitatively what is the contribution of this single line to the opacity. Dropping the subsoripts for the moment, we have
(5.11)

$$
\frac{r}{1+r}=\frac{1}{1+\frac{S}{N f b^{\prime}}} .
$$

where
(5.11a)

$$
S=\frac{k T}{h} \frac{m e}{\pi e^{2}} \nabla \mu_{0}
$$

$b^{\prime}(u)=\frac{k T}{h} \quad b(v)$

The frequency variation of this function for a typioal onse is show in the ace companying figure. It is seen that this furiction

approaches unity when $r$ is a maximu at the frequency of the center of the line, decreases lowly with displacement from the center until it becoues $\frac{7}{2}$ when the Iine absorption is equal to the continucus absorption background, The variation of the function in the neighborhood of $\frac{1}{\mathrm{k}}$ is rapid, the transition from values near unity to values near zero occurring within a small frequency range. For greater displacements the function falls off rafidly, spon behaving simply as $x$. The contrast between the function $r$ which one might naively expect to determine the line effect and $r / 1 \neq \vdash^{\circ}$ is marked. Whereas $r$ drops to $\frac{1}{2}$ its value at (u-u $u_{0} \Gamma$, we find that $r / 1 \nmid r$ drops to half its value only at ( $u-u_{0}$ ) $\sim 3 \Gamma$.

The integral ( 5.8 ) giving the contribution of the line is approximately

$$
\left.\Delta_{l}=\int_{0}^{\infty} \frac{r}{1+w^{x}} d u \frac{w(u)}{\mu_{0} u^{5}}\right]_{u=u_{0}}
$$

since the integrand is negligible except for frequeneies near $\mathcal{V}=\mathcal{V}_{0}$ What is
$-50$
important then is $\int_{0}^{\infty} \frac{r}{1 \ngtr r}$ du. The greatest part of the contribution to this integral comes in the range where $r /(1+r) \cong 1 ;$ practically nothing is contributed by regions where $r /(1+r)<\frac{1}{2}$. Thus if $w$ is the distanoe between $u_{0}=h \nu_{0} / k T$ and the frequency $u_{\text {a }} \boldsymbol{z} / \mathrm{kT}$ at which $\mathrm{r}_{\mathbf{I}} 1$, the integrol is approxinately
(5.13) $\quad \Lambda_{2}=\left.\frac{W(u)}{\mu_{c} u^{3}}\right|_{v=u_{0}} \times 2 W$.

The quantity $2 w$ we shall term the wingspread of the line upon its continuous background. It is this quantity, as is shown by (5.13), rather than the dispersion breadth of the line which determines the contribution of the line to the opacity. We may actually think of the line, acsording to ( 5.13 ), as leaving the transmission of light unaffected throughout the spactrum except in the region of its wingspread, where it completely blocks the transmission.

Thewingspread of the line is devermined by the condition

$$
\begin{equation*}
\frac{N f}{S} \quad b^{\prime}\left(\sigma-u_{0}\right)=1 ; \tag{5.14}
\end{equation*}
$$

thus it depenis on the ratio of line atrength $N f$ to continuous background, and the dispersion. Even a line which has very small dispersive breadth may have a considerable wingspread if it is strong enough. On the other hand, a broad line may have very small or zero wingspread. if it is weak compared to its background. Arguments for neglecting line contributions bscause of the small dispersive breadths alone are, therefore, incorrect. Another important conclusion wry mara is the following. Sjace the wingspread does not depend on the position of the line, the line contribution in the case of an isolated line is not sensitive to position.

Suppose we consider first an artifioial example of a line with a rectangular shaped dispersion

$$
\begin{cases}b(\nu)=\frac{1}{2 \Delta \nu} & \text { for } \nu_{0}-\Delta \nu \leqslant \nu \leqslant \nu_{0}+\Delta \nu  \tag{5.15}\\ b(\nu)=0 & \text { otherwise }\end{cases}
$$

Then
(.5.15)

$$
\int_{1 D}^{\infty} \frac{r}{1 \& r} d u=\int_{u_{0}-\Delta u}^{u_{0}+\Delta u} \frac{d u}{1 \neq \frac{S}{N L}} 2 \Delta u \quad=\frac{1}{\frac{1}{2 \Delta u}+\frac{S}{N I^{\prime}}}
$$

In the line absorption is very strong, compared to the continuous absorption, we o.btaln tine oijvinus rosult that

$$
(5.17) \quad \Lambda_{\mathbb{E}}=\left.\frac{W(u)}{\mu_{c} u^{3}}\right|_{u=u_{0}} \times 2 \Delta u_{0}
$$

that is, tine line eliminates the entire transinission of the frequency intervals
 on Nf, the lime strength, for this implies that a line 10 times as strong as another will have just the same effect on opacity. But a little reflection shows that if a line completely absorbs the radiation in an interval, it already has a maximun effect in reducing the transmission. A stronger line can do no more. On the other hand, if the line absorption coeffioient is weak compared to the continuous background, that is
(5.18) $\frac{N \mathrm{~N}}{2 \Delta \mathrm{u}} \mathrm{S} \quad \ll 1$
we get
(5.19) $\Lambda_{l}=\left.\frac{W(u)}{\mu_{c} u^{3}}\right|_{u_{=u_{0}}} \frac{N f}{S} \quad$.

Here we get the important result (in contrast to (5.17) above) that the effect of the line is direocily proportional tio its integrated strength $N f$, but is indepondent of the dispersion interval $2 \Delta \nu$. As an jmediate consequence of this, we have that $\Lambda_{l}$ is independent of the dispersion shape whatever that may be, so long as the analogie to (5.18) is fulfilled. This is a very important result, for weak lines; that is; lines for which Nf $b_{\text {nax }}^{\prime}(u) / s \ll 1$ always give the contribution (5.19) and it is unncessary to inquire into the details of the dispersion. Sinoe most of the lines in a speotrum are of this natire, (5.19) solves a great deal of the problem.

Proceeding now to the actual types of dispersion we shall encounter, the specific casi of natural and/or collision broadth has the disparsion formula (0.f. (2.15).,

where $\gamma / 4 \pi$ is the half breadth, and

$$
\text { (5.21) } \quad \Gamma^{\prime}=\frac{h \gamma}{4 \pi R T} .
$$

The wingspread wis obtained from (5.14) giving
( 5.22 ) w $:\left\{\frac{M r \Gamma}{\pi S}-r^{2}\right\}^{\frac{1}{2}}=\left\{\frac{\pi \Gamma}{\pi S}\right\}^{\frac{1}{2}}$,
the latter approximation being zalid if

$$
(5.23) \quad \frac{\pi \Gamma^{8} \Gamma}{N T} \ll 1
$$

i.e. if $\Gamma \ll \omega$, a condition which is Proquentily the case. The wingspread is then proportional to $\Gamma^{\frac{1}{2}}$ and to the square root of the line strength (Nf) ${ }^{\frac{1}{2}}$. Carrying out the integration of (5.12), we get
(5.24) $\quad \Lambda_{\ell}=\left.\frac{W(u)}{\mu_{0} u^{3}}\right|_{u_{=u_{0}}} \pi\left(\frac{(N \Gamma \Gamma}{\pi S}\right)^{\frac{1}{2}}\left\{1 \% \frac{m-\Gamma \Gamma}{N I}\right\}^{-\frac{1}{2}}$.

In the case where $\pi S \Gamma /(N f) \ll 1$ we may put the last factor equal to 1 . In just this case the wingspread is given by the simplified form of (5.?2) and the result is (5.25) $\Lambda_{l}=\frac{\pi}{2} \times\left. 2 w \frac{\pi(u)}{\mu_{c} u^{3}}\right|_{u_{=u_{0}}}$ •

The fact that the line blacks out a frequency interval $\pi / 2$ times the wingspread is a confirmation of the general qualitative result in (5.33). The reason the numerical facior is so different ( 1.57 instead of 1 ) is that the natural breadth dispersion gives approciabla absorption even rather far from the line center. Wo shall generally speak of the extra contribution to the opacity of regions
beyond the wing spread as a "tail" el'feot, because it is due to the tail ends of the dispersion curve.

In the other extreme case of weak lines (5.24) reduces to

$$
\begin{equation*}
\Lambda_{p}=\left.\frac{W(u)}{\mu_{0}^{u} u^{3}}\right|_{u=u_{0}} \frac{N P}{S} \tag{5.26}
\end{equation*}
$$

This result is precisely the same as (5.19) for the case of step-like dispersion, and is a special illustration of the general validity of the weak line formula. regardless of type of dispersion or breadth.

Turning next to the case of Doppler broadening, the dispersion is

$$
\begin{equation*}
b\left(u^{\prime}\right)=\frac{h}{k T}\left\{\frac{M_{0}^{2}}{2 \pi k u_{0}^{2}}\right\}^{\frac{3}{2}} \exp -\left\{\frac{M_{0}^{2}}{2 k T^{2}} \frac{\left(u-u_{0}\right)^{2}}{u_{0}^{2}}\right\} \text {. } \tag{5.27}
\end{equation*}
$$

The half breadth is given by the value $\gamma$ which makes
(5.28) $b\left(;-\nu_{0}\right)=\frac{1}{2} b\left(\nu_{0}\right)$ 。

We see from (5.28) and (5.27) that
(5.29) $\Gamma=\frac{h \gamma}{K T}=u_{0}\left\{\frac{2 K T}{M 0^{2}} \ln 2\right\}^{\frac{7}{2}}$.

The wingspread, however, is given by the condition that

$$
\frac{\operatorname{Nf}}{S}\left\{\frac{M_{0}^{2}}{2 \pi \mathrm{KT} u_{0}^{2}}\right\}^{\frac{2}{2}} \quad \exp =\left\{\frac{\mathrm{Mo}_{0}^{2} w^{2}}{2 k u_{0}^{2}}\right\}=1 .
$$

whence

$$
(5.30) \quad=\left\{\frac{i k T u_{0}^{2}}{\mathrm{Kc}^{2}} \quad \ln \left[\frac{N P}{S}\left(\frac{\mathrm{Ma}_{c^{2}}^{2}}{2 \pi k T u_{0}^{2}}\right)^{\frac{2}{2}}\right]\right\}^{\frac{1}{2}} .
$$

Thus
(5.31) $W=\Gamma\left\{1.443 \ln \left[.4696 \frac{\mathrm{Nf}}{\mathrm{S} \Gamma}\right]\right\}^{\frac{1}{2}}$.

Aside from the logarithmic factor, the expressions for the wingspread (5.30) and the dispersion breadth agree. This is, of course, due to the exponentially falling off of the dispersion curve with distance from the line center; as a conses-
quence, the line has effective absorption only in the region of its dispersion breadth.

The actual contribution of the line with Doppler breadth to the opacity is from (5.11), (5.12), and (5.27)
(5.32)

$$
\Lambda_{l}=\left.\frac{W(u)}{\mu_{0} u^{3}}\right|_{u=u_{0}} \int_{0}^{\infty} \frac{d u}{1+\frac{\sqrt{\pi}-s B^{-\frac{\lambda}{2}}}{N F}} e^{+3\left(u-x_{0}\right)^{2}},
$$

where

$$
(5.33) \quad B=\frac{M_{0}^{2}}{2 k u_{0}^{2}} .
$$

We can express the integral as

$$
(5.34) \Lambda_{l}=\left.\frac{M(n)}{\mu_{0} u^{3}}\right|_{u=u_{0}} \frac{\sqrt{\pi}}{a} \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{B^{\frac{1}{2}}}{a}\right)^{n-1} \frac{1}{n^{\frac{1}{2}}} ; \quad a=\frac{\sqrt{\pi}}{\pi r} .
$$

which for small values of $B^{\frac{2}{2}} / a$ develops as
(5.35)

$$
\Lambda_{l}=\left.\frac{W(u)}{\mu_{0} u^{3}}\right|_{u=u_{0}} \frac{M I}{S}\left\{1-\frac{N f}{S} \frac{1}{\sqrt{2 \pi}}\left(\frac{M a_{0}^{2}}{2 k T u_{0}^{2}}\right)^{\frac{2}{2}}+\cdots\right\}
$$

The leading terms of this expansion is the weak line result (5.19) as should indeed be expected, for the condition

$$
(5.36) \frac{B^{\frac{1}{2}}}{2} \equiv \frac{\mathrm{HI}_{\Psi}}{S}\left(\frac{M_{0}^{2}}{2 \pi K u_{0}^{2}}\right)^{\frac{1}{2}} \ll 1
$$

moans that the lines are weak.
For large values of $\mathrm{B}^{\frac{1}{2}} / a$, the analytic form (5.34) is inconvenient for caloulatimon. We then develop the integral from (5.32) as

$$
\int_{0}^{\infty} \frac{\frac{-}{}_{\frac{1}{2}}^{d x}}{\frac{B^{\frac{1}{2}}}{2}+e^{x}}=\int_{0}^{\ln ^{n} B^{\frac{1}{2}}} \frac{x^{-\frac{1}{2}} d x}{\frac{B^{\frac{1}{2}}}{2}\left(1+\frac{B^{\frac{1}{2}}}{2} e^{x}\right)}+\int_{l_{x} \frac{B^{\frac{1}{2}}}{2}}^{\infty} \frac{x^{-\frac{1}{2}} \frac{d x}{x^{x}\left(1+\frac{B^{\frac{1}{2}}}{a} e^{-x}\right)} . .}{}
$$

The leading term in the development is $2\left(a / B^{\frac{1}{2}}\right) \sqrt{h_{a} B^{\frac{1}{2}} / a}$
whence
$(5.37) \quad \Lambda_{l}=\left.\frac{W(u)}{\mu_{c} u^{3}}\right|_{u_{m} u_{0}} \times 2 w$.
The higher order terms constitute a tiail effeot.
It is instructive to compare the effect of two lines having the same total strength and the same half breadth, elthough the dispersion in one is oaused by natural and/br collision breadth, while in tine other it is caused by Doppler breadth. For weal lines (5.19) tells us the result is identical. For strong lines, we have
(5.38) $\frac{\Lambda_{l} \text { natural }}{\Lambda_{l} \text { Doppler }}=1.57\left\{\frac{N I}{\pi S \Gamma}\right\}^{\frac{1}{8}}$.

For the case of strong lines this ratio is always much greater than unity. We oan, therefore, conclude that the natural breadth dispersion is always as effeotive as or moro effective than Doppler dispersion in increasing the opacity.

## 3. Effeot of Two Lines.

Now that we understand the contribution of a single line to the opacity, we can investigate the effect of a line speotrum. It is characteristic of this problem that the superposition principle does not hold in general, i.e. the effeot of lines is not simply additive. Instead, it depends upon the relative positions of the Iines. To illustrate this most olearly, we shall oonsider the ocntribution of two lines to the opacity. The line absorption coefficient will then be

$$
\begin{equation*}
\mu_{l}=\sum_{j=1}^{2} \frac{\pi_{e}^{2}}{m c} \frac{\pi_{f_{f}}}{\nabla} b_{j}(\nu) \tag{5.39}
\end{equation*}
$$

Where $f$ denotiss the number of the line. Suppose first that the lines are very far amray from each other compared to their wingspread (not their breadth). Then the funotion $\frac{r}{f r}$ in the integrand of (5.8) will have two widely separated humps, as illustrated in the accompanying figure.


Since the individual line absorption coefficients in the regions outside the wingspread of a. line are very small compared to the continuous absorption, we will have, as suggested by (5.19) and demonstrated later on, that the contributions to the integral. of this region are almost precisely additive. Moreover, they are small, so it is not neoessary to worry about; the very slight deviations from additivity. Within the wingspread of each line, the contribution to $r$ of the neighboring line is small; the ratio $r /(1 \nmid r)$ is almost unitye Increasing $r$ silghtily will have even less effect on $r /(1 \& r)$. Thus, within the wingspread each line makes the same contribution to the opacity mether its neighbor is present or not. Hence, the overall effect of the two lines is rery neerrly additive.

The additivity feature breaks down as soon as the wingspread of the lines overlap appreciably. Going to the extrime case of overlapping, we consider two identioal lines at the same frequency. The line absorption coefficient will then be (5.40)

$$
\mu_{l}=2 \mu_{l}
$$

where $\mu_{l}$, indieates the line absorption coefficient of the single line. The ratio $r /(1 \not \& r)=2 r_{1} /\left(1 \nmid 2 r_{1}\right)$. Now whthin the wingspread of the line (if the line is strong $r_{1} \gg 1$ ) wo have that $2 r_{1} /\left(1 \& 2 r_{1}\right) \sim 1 \sim r_{1}\left(1 \not r_{1}\right)$. Hence, the two lines together have no more efrect on the transmission than the single line. This result was again foreshadowed by (5.17). The tails of the two lines go as $2 r_{1}$ compared to $r_{1}$ for a single line, and additivity will oharaoterize their contribution. However, the tail effeot is usually small, so roughly we have the result that two strongly overlapping lines do not increase the opacity much beyond that resulting from the stronger of the two lines.

Naturally for cases of intermediary overlapping we shall have the situation betreen the extremes of strict additivity for no overlapping, and no adied effect for complete overlapping. Thus, we conclude that the relative position of the
two lines is critical, although their absolute position is not.
We shall. now reinforce these qualitative oonclusions with examples for the case of stepwise dispersion, natural breadth dispersion, and Doppler dispersion. For the case of stepwise dispersion wo take
(5.41)

$$
\left\{\begin{array}{l}
b_{1}(\nu)=\frac{1}{2 \Delta \nu_{1}} \text { for } \nu_{1}-\Delta \nu_{1} \leqslant \nu \leqslant \nu_{1}+\Delta \nu_{1}, \\
b_{1}(\nu)=0 \text { otherwise, }
\end{array}\right.
$$

and similarly with $b_{2}(\nu)$. Then
(5.42) $\lambda_{l}=\int_{0}^{\infty} \frac{\pi(\dot{u})}{\mu_{0} u^{j}} \frac{r}{1+r} d u=\int_{0}^{\infty} \frac{W(u)}{\mu_{0} u^{3}} \frac{d u}{1+\frac{s}{\sum_{j} N^{j} f_{j} b_{j}^{\prime}(u)}}$.

If the two stisps do not overlap, that is $\nu_{2}-\nu_{1}>\Delta \nu_{1}+\Delta \nu_{2}$, we can break up the integral $\int_{0}^{\infty}=\int_{0}^{\nu^{\prime}}+\int_{\nu 1}^{\infty}$ where $\nu^{\prime \prime}$ is any frequenoy $\nu_{1}+\Delta \nu_{1}<\nu^{\prime}<\nu_{2}-\Delta \nu_{2}$ between the steps. In each integral, the integrand is exactly the same as for the case of each line tiaken alone. Thus

where we have extended the upper limit from $u^{\prime}$ to $\infty$ beoause the integrand is zero In that region. We thus obtain exact anditivity for the no overlapping case.

Suppose now there is some over lapping. Then $\sum_{j=1}^{2} \ln ^{j} f_{j} b_{j}(2)$ will behave as
folloms:

$$
\begin{aligned}
& (5.44)\left\{\begin{array}{l}
\sum_{j=1}^{2} \\
\end{array}\right. \\
& \begin{aligned}
N^{j} f_{j} b_{j}(\nu) & =0 ; & \nu<\nu_{1}-\Delta \nu_{1} \\
& =\frac{N^{1} f_{1}}{2 \Delta \nu_{1}} ; & \nu_{1}-\Delta \nu_{1} \leqslant \nu \leqslant a .
\end{aligned} \\
& =\frac{N^{3} \mathrm{f}^{1}}{2 \Delta \nu \nu_{1}}+\frac{N^{2} \mathrm{f}_{2}}{2 \Delta \gamma_{2}} ; \quad a \leqslant \nu \leqslant b \text {. } \\
& =\frac{N^{2} f^{2}}{2 \Delta \nu_{2}} ; \quad b<\nu \leqslant \nu_{2}+\Delta \nu_{2} \text { 。 } \\
& =0 ; \quad \nu_{2}+\Delta \nu_{2} \leqslant \nu \leqslant \infty \text {. }
\end{aligned}
$$

The function is illustrated in the accompanying figure.

where $u^{*}$ is some mean frequency of the two lines whose value is not critical.
In the case where the lines are weak compared to the background, we can neglect the unity in each of the three denominators. We then get
(5.46)

$$
\left.\Lambda_{l} \cong w(u) u_{c} \frac{u^{3}}{}\right|_{u=u^{*}}\left\{\frac{N^{\prime} f_{1}}{S}+\frac{N^{2} f_{2}}{S}\right\} .
$$

that is the contributions are still gacaculy additive despite the overlaying. Incidentally this shows that the contributions of any set of weak absorptions upon a strong continuous background are additive, for any line absorption coefficient can be approximated by a series of step functions. In particular the contribution of overlapping line tails are usually additive.

If the lines are strong compared to the background we may neglect the second term in each denominator giving
(5.47)

$$
\left.\Lambda_{l} \cong \frac{w(u)}{u^{3} \frac{u}{u_{c}}}\right|_{u=u^{*}}\left\{u_{2}+u_{1} u_{2}-\left(u_{1}-\Delta u_{1}\right)\right\} .
$$

Here we can definitely see the non additivity of the line contribution but attis even more striking if the lines exactly overlap. Then we get

$$
\begin{equation*}
\left.\Lambda_{-1} \cong \underset{u_{c} u^{3}}{\cong}\right|_{u=u^{*}} \quad \alpha \Delta u_{1} \tag{5.18}
\end{equation*}
$$

which is precisely the same as the effect of either line alone.
The general conclusions about additivity hold for the case of natural and/or collision breadth dispersion, bur are somewhat influenced by the pronounced tail in this type of dispersion. For the case of two identical lines when the wings spreads do not overlap the contribution to the opacity is $\Lambda_{\ell}=2 \Lambda_{l}$, where $A_{l}$ is the contribution of each line individually. But suppose the lines are exactly superposed. Then from (5.24) we get

$$
\begin{equation*}
A_{l}=\left.\frac{w(u)}{\mu_{c}} \frac{u^{3}}{}\right|_{u=u^{*}} \pi\left\{\frac{2 N^{\prime} f_{1}}{\pi S}\right\}^{1 / 2}\left\{1+\frac{\pi S r}{2 N f}\right\}^{-1 / 2} \tag{5.49}
\end{equation*}
$$

For strong lines the last factor is unity and we can see that $\Lambda_{l}=\sqrt{2} \Lambda_{l}$. The factor $\sqrt{2}$ is easy to understand, since the contribution of the region inside the wingspread is the same in the case of two lines as with one, while the tail region is additive. Referring to (5.25) we see that the tail contributes ( $1 / 2-1$ ) times the contribution within the wingspread. Thus

$$
\begin{equation*}
n_{l}=\left\{2\left(\frac{\pi}{2}-1\right)+1\right\} 2 w \frac{w(u)}{u_{c} u^{3}} \int_{u=u_{0}^{*}}=1.364 \Lambda_{l,} \ldots \tag{5.50}
\end{equation*}
$$

The numerical factor 1.364 is quite close to $\sqrt{2}=1.414$.
For weak lines, on the other hand we may neglect 1 compared to $\pi s \Gamma /(2 \mathrm{Nf})$, and we get
(5.51) $\Lambda_{l}=\left.\frac{W(u)}{\mu_{c}} \frac{u^{3}}{}\right|_{u=u^{*}} \frac{2 N f}{s}=2 \Lambda_{l}$,
and thus the contributions are additive.

## 4. Effeot of many line speotrum

From the preceding disoussion of the contribution of two lines, the features and difficulties of the treatment of many lines appears. The most critical factor is whether or not the wingspread of the lines overlap. If there is no overlapping the contributions are additive. The case of weak lines which are always aadi ive is really included in the category or nou-overlapping wingspread, because the wingspread of a weak line is zero. When overlapping exists no simple treatment is readily available, but we may say the contribution of the ines is less than in the non-overlapping case.

In general, even if the contributions of the lines are additive, the result is still not simple enough to permit ready caloulation for a complicared line speotrum, bsoause the effect of thousands of individual lines has to be computed and then summed. This requires knowing the strengths, positions and dispersions of every line. Such a calculation is practical only for a very simple speotrum like that oharacterigtio of a one-leotron ion. However, suoh cases are of some practical importance for often we shall have an assemblage of ions having either no bound electrons at all, or only 1 bound $K$ electron. Even the case of 2 bound F electrons is simple enough, as is a 180 the case of a single bound eleotron outside a closed shell. The weak line case, however, is much more readily adapted to computation. We can gee by our consideration of the step type dispersion that the contribution of the line is independent of the dispersion intervale Generalizing, since any dispersion curve may be made up by superposing steps, we conclude that the effect of weak lines is independent or the dispersion shape and breadth as well. This is conf'irmed by the specific results for natural breadth dispersion and Doppler disyersion. Hence the $j t h$ weak line gives a contrioution (5.19) and if
we take a group of lines in the neighborhood of a partioular frequenoy $u^{*}$, we get
(5.52) $\Lambda_{E}=\sum_{j} \Lambda_{Q_{j}}=\left.\frac{W(u)}{\mu_{c} u^{3}}\right|_{u=u^{*}} \sum_{j} \frac{N_{j} f_{j}}{S}$.
since the contributions are additive and the continuous absorption $\mu_{0}$ and the weighting fastors $W(u) / u^{3}$ do not alter much from one member of the group to another. The important point about this formula is that only the total strength $\sum_{j} \mathbb{N}_{\mathbf{f}} \mathbf{j}_{\mathbf{j}}$ of the group of lines enters. Thus, we nsed not calculate strengths of individual Iines, for often the total strength is given directly by the theory of the speotrum. There is alsu no need to calculate the dispersion. The resulting simplification of calculation is enormous. Since most of the lines are weak, equation (5.52) solves a great deal of our problem in a simple manner.

Because of its importance, we shall present another derivation of (5.52) which emphasizes a different aspect of its physIoal intepretation. Consider a group of maxy lines with centers in the intervall $u^{*}-\Delta u$ to $u^{*} ; \Delta u$, none of which very much exceeds the average in strength. Assume also that the lines are distributed fairly uniformly and thiokiy over the region. Then the absorption coofficiont for all these lines will no longer be a very jagged function; for, although it still may have many maxima and minima, and even more inflections, the variations from a smoothed average ourve will be smal.1. This is illustrated in the aocompanying figure

Fer lines in region


Many weak lines in region.

Since the contribution to the opacity is dotermined by an integral, it is only some
sort of average which is important．The average absurption coefficient may be obtained by considering the total strength of the group of lines $\sum_{j} N^{3} f_{j}$ uniformly＂sraeared out＂over the interval iAu centered at $l^{*}$ ．Then （5．c3）,$l_{l}=\frac{\pi e^{\alpha}}{m c} \sum_{i} \frac{N^{j} f_{j}}{\alpha \Delta u}$.

The line opnoity is then given in amiogy with（5．16）as （5．54）

$$
\Lambda_{l}=\left.\frac{w(u)}{u^{3} \mu_{c}}\right|_{u^{*}} \quad 2 \Delta u \frac{\cdot r}{1+. r} .
$$

If the entide absorption due to the group or lines is weak this reduces to（5． 52 ）． If we compare the contribution of continuous processes in the same interval we get from $(5.7)$ simply $u^{3} \mu_{C} \|_{\nu} * 2 \Delta u$－Comparing with（5．54）we see that a fraction $f(1+r)$ of the continuous transmission of the region remainse Because of this derivation we shall call（5．54）the＂strearing out＂approximation． We should emphasize some of the limitations of this approximation．First，all the absorption atrength has been artificially confined to the region $\|^{7}-\Delta u \quad$ to $u^{*}+\Delta u$ Because of the dispersion，there is a tall effect of some absarption outsice this region．If the region within which the etrengths were smeared out has been made very black，the inclusion of some extra absorption which should properly go into the tails will not change the contribution of this region．The absence of gbsorption in the tails，may，however，considerably over estimate the transmission there．The tail effect has thus caused us to underestimate the opacity．Balancing this is the fact that smearing out overestimates the opacity due to the contribution of lines in the smeared out regions．Furthermore when the total absorption coefficient due to lines is small compared to the background，the contributions are strictly additive，and not including the tails is exactly compensated by putting the extra strength into the interval $2 \Delta U$ ．We conclude that it is generally better to for－
get ebout the tail effect, unless somethjag is also done to improve the smearing out arrroxination.

A second limitation of this approximation occurs if one or a few lines cafiy the bulk of the strength. While smearing out is valid for the greater number of lires, the few strong lines should not be smeared out. A possible procedure to follow in this case is to smear out the weak lines and calculate their contribution to the line absorption coefficient $\mu_{l} w$. Add this to the continuum $\mu_{c}$ to form a new background, and superimposed the strong lines upon thise The strong line contribution will be given by (5.8) except that $r$ now has the sinnificance $r=\frac{\mu_{l} T_{\text {Thong }}}{\mu_{c}+H_{l} \text { wrate }}$.

The oprosing extreme to the smearing out approximation occurs when many lines very clearly overlap. This case is also extremely frequent because practically every line in a spectrum is accompanied by many close shadows - its fine structure components for example. In general it will be gufficient to determine the wingepread of the group of closely spaced lines and assume that the transmisgion is zero within the wingspread. For natural breadth dispersion this estimate must be increased by the faotor $\pi / 2$ to account for the pronounced tail effect. We may contrist the results in the case of natural breadth for the two situations 1) the total strength $2 N f$ of the lines is equally distributed among $M$ non-overlapping lines giving a strength $N / f$ to each 2) the $M$ lines are coincident. In 1) the contributions are additive and $(5.55) \quad \Lambda_{\ell}=M \Lambda_{\ell,}$ while in 2) wo find (5.56) $\quad \Lambda_{l}=M^{1 / i} \Lambda_{\ell,}$.

Intermodiary ceses will 110 between the two ixtremes.

Whi.:s we cannot carry out the treatment for the intermediary cases in detail except by a device, soon to be discussed, which treats the lines statistically, we can make an approximate treament ooryect to first order terme. Suppose we have a group of several lines all with about the same dispersion and all at almost the same frequenoy. Then if we consider the line absorption contribution to the opacity and expand the result in povers of the deviations of the lines from some average position, we get the result that the zero order term is precisely the same as would result if had a single line with the total strength of the group looated at the average position. The first order term may be made to vanish by appropriately ohoosing this average position. Calculations show that the proper method of averaging is to weight each line position and breadth by $N \Gamma$, the product of strongth and breadth of the lins. Indeed the prinoiple of a strengthe breadth weightied average is general.

The foregoing considerations will enable us to make rough estimates of the contribution of lines to the opacity. In many cases this will suffice, since the line contribution is small, or else may be of the type given by the extremes considered here. But we should examine the more general problem of an arbitrary array of lines. The line absorption coefficient is then (5.57) $\quad \mu_{b}=\frac{\pi_{0}^{2}}{m c} \sum_{j=1} \frac{N^{j} f_{j}}{V} b_{j}(2)$ and we merely need carry out the operations indicated in (5.6). But let us note what this requires. We need the following date for each individual line. 1) position, 2) strength, 3) dispersion. Then wh have to perform a very complicated numerioal integration. In principle all this may be done, - in practice the oomm plexity of the caloulations makes the job prohibitively long unless we wish to treat a small speotral region with fev lines.

## 5. General Statistical Features of Lines,

Statietical Treatment of Line Speotra."

The very complexity of a line spectrum nay be the means of providing a simple method of caloulating its effect on opacities. For, if the enormous number of lines precludes an individual treatment, it makes possible a statistiaal approach. This approach will be developed in general in the suoceeding paragraphs and then applied later in the special cases of interest.

Now the line absorption coefficient $\mu /$ and consequently $r$ is the sum of contributions from every line in the spectirum. The ith line gives an absorption coefficient $\mu_{i}$ which has a sharp maximum at the frequency $u=u_{i}$ of the oonter of the line and approaches zero for frequencies far from the oenter. The sum $r=\sum_{i} r_{i}=\sum_{i} \frac{\mu_{l_{i}}}{\mu_{0}}$ therefore appears as a very irregular function of $u$ with many maxima, which it is practioally hopeless to calculate. We see, hovever, that in ordor to caloulate the mean opacity, it is not strictly necessary to know all the details of the line absorption coefficient itself as a function of frequency, but it would suffice to know the average value. At first sight, however, the calculation of the proper average would seem to involve evaluation of the very same integral (5.6) as needed to find the opacity. Here it is that the statistical approach proves useful. Suppose in (5.6) instead of the actual value of $P$. we insert a statistical average $\bar{P}$, averaged over certain distributions of line position, strength, and breadth. It may turn out that this average is rather easily susceptible to calculation compared to Pitself. While the two funotions may not have the same deteiled dependence on frequency, we may hope that, if a sensible statistical average is used, the integral (5.6) itself will not be very much altered by the substitution of $\overline{\mathrm{P}}$ for $P_{\text {. }}$

To understand the physical basis for the statistics we shall use, let us consider the very idealized case of a line spectrum having just two lines, separated sufficiently $s 0$ that there is very little overlap of the dispersion curves of each line. Of course, in this case we can calculato $P$ and, therefore, the opacity $K$. Ve can also caloulate the opacity if the two lines were a little further apart or a little oloser together, and we would get substantially the same result, since the integral is inseasitive to the position of each line, except for the overlap whioh is assumed small. We oan indeed pick a number of different distributions of the positions of the two lines, which will not give very different values of $1 /(\rho \mathbb{K})$, calculate these values, and average them. The average will maturally agree rather olosely with the true value, since every member of the group averaged agreed rather olosely by itself. It will not affeot the average very much even if we include a few distributions (for example, one in which the centers of both lines coincide), whose resulbant opacity is quite different from the true value. Now, it is imaterial whether we caloulate $P$ for each distribution, integrate each one, and then average, or invert the order of averaging and integration, thus finding the average $\vec{P}$ for all the distributions and then integrating to find the averaged opacity.

The question now arises as to what distribution should be included in our average. To answer this, we look at a line speotrum composed of several groups of two lines, each group in a sightly different frequency range so that lines in different groups do not overlap to any extent. Some of these groups undoubtedly will have the two lines far apart and others will have them oloser together. We can treat each group separately by the averaging procedure because of the noneoverlap between groups. If in the distributions we averaged, we never included any in which the two lines strongly overlap, we mould estimate the opacity due to the groups with overlap too high, while we would be substantially correct for all the groups without overlap. It is better to make compensating expors by including in our average some distributions with strong overlap. For
then, while we would estimate the opacity of a group having little overlap too low, we would on the other hand estimate too high for the groups with strong overlap. If the proportion of strong to teak overlap distributions included in our averages is the sauce as the proportion of strong to weak overlap groups in the actual spectrum, the errors will exactly compensate. This principle is of course applicable to the general case of a line spectrum, as well as to the specific example discussed here.

Now there are laws which tell, in any actual spectrum, exactly where each line must be. These are extremely complicated, and because of the fact the distribution of line positions in the groups of a omplicaced spectrum is very nearly random, thai 18 , considering all groups, a line has about equal probability of occuring anywhere within the frequency range covered by the group. It is just this distribution of line positions then which we shall use in our statistical average. Now if the th line or a spectrum can with equal probability lie anywhere in the region $u_{i}^{*}-\Delta_{i}$ to $u_{i}^{*}+\Delta_{i}$, the expected average of $P\left(u_{;} u_{1}, u_{2}, \cdots u_{i}, \cdots\right)=1 /(1+r) \quad$ over all distributions of line positions consistent with this probability is
(5.58) $\quad u_{i}^{\prime}+\Delta_{1} u_{i}^{\prime}+\Delta_{i}$

$$
\bar{p}\left(u_{j} u_{1}^{*} \ldots u_{i}^{*}, \cdots ; \Delta_{1}, \cdots \Delta_{i} \cdots\right)=\frac{1}{\prod_{i} 2 \Delta_{i}} \int_{u_{i}^{*} \cdot \Delta_{1}}^{u_{1}+\Delta_{1} u_{i}-\Delta_{i}+\Delta_{i}} \ldots P\left(u_{;} u_{1}, \cdots u_{i} \cdots\right) d u_{i}, \cdots d u_{i} \cdots
$$

This iterated integral is even more hopelessly complicated than (5.6) but it yields its value to any desired degree of approximation by the use of an extremely ingenious device suggested by Dr. Edward Teller. Develop the function $P=1 /(1+x)$ as an exponential series

$$
\begin{equation*}
P=\frac{1}{1+r}=\sum_{m} a_{\mu} e^{-b_{N} r}=\sum_{m} a_{\mu} e^{-b_{m} \sum_{i} h_{i}}=\sum_{\mu}^{\prime} a_{\mu} \pi c^{-b_{\mu} R_{i}} \tag{5.59}
\end{equation*}
$$

By properly choosing the $a_{\mu}$ and $b_{m}$, it is possible to got good numerical agreement ( 2 or $3 \%$ ) between the series $\not \mathbb{N}_{\mu} e^{-b_{\mu}}$ and the function $1 /(1+\mu)$ in the range $0 \leq r \leq 100$, taking only four terms. This range of $\pi$ will usually be sufficient; in any case the contributions to the opacity of regions where $k>100$ will be negligible. The series wo shall use is $(5.60)$

$$
\frac{1}{1+r}=.470 e^{-1.70 r}+.370 e^{-.35 \mu}+.120 e^{-.10 r}+.040 e^{-.015 r}
$$

Inserting (5.59) into (5.58) reduces the iterated integral to a product of single integrals each of the same type.
(5.61)

$$
\vec{p}=\sum_{\mu} a_{\mu} \prod_{i} \frac{1}{2 \Delta_{i}} \int_{u_{i}^{*}-\Delta_{i}}^{u_{i}^{*}+\Delta_{i}} e^{-b_{\mu} k_{i}} d u_{i}
$$

Although the essential simplification has now been made, (5.61) can be transformed into more convenient form, as follows:
(5.62)

$$
\vec{P}=\sum_{\mu} a_{\mu} \pi \prod_{i}\left\{1-\frac{j_{m i}}{2 \Delta_{i}}\right\}=\sum_{M} a_{\mu} \exp \left\{\sum_{i}^{i} \ln \left(1-\frac{j_{\mu i}}{2 \Delta_{i}}\right)\right\}
$$

where
(5.53) $\frac{j, x_{i}}{2 \Delta_{i}}=\frac{L_{1}}{2 \Delta_{i}} \int_{u_{i}^{*}-\Delta_{i}}^{u_{i}^{*}+\Delta_{i}}\left\{1-e^{-b_{m^{\prime}} u_{i}}\right\} d u_{i}$.

Further defining
(5.64) $\quad E_{M}=-\frac{1}{b_{M}} \sum_{i} \cdot \ln \left(i-\frac{v_{M i}}{2 \Delta_{i}}\right)$,
equation (5.62) becomes
(5.65)

$$
\bar{p}=\sum_{x} a_{\mu} e^{-b_{M} E_{\mu L}}
$$

Tho quantity $P_{\text {, }}$ a function of frequency $u_{0}$ may now be inserted into (5.6) in place of $P$, with the expectation that the integral itself will not be very much altered. A straightforward numerical integration will then give the opacity.

We now examine the quantity $E_{\mu}$, une essential factor in $P_{\text {, more ciosely. }}$ It involves a sum over all lines in the entire spectrum. It appears then that we are up against the same difficulty which prevented the caloulation of $r={\underset{c}{i}}^{i} r_{i}$ itself, before we introduced the statistical approach, ramely too many lines to calculate individually. But closer exaninavion shows we have made some progress. First, Ew does not require knuwledge of the exaot position of every line, but only the limiting frequencies of the region within which it may be found in the statistical treatment. Many lines have these same limits, aud we thus have eliminated very much of the data required for the opacity caloulation. Second, it is generally possibie to group lines into classes such that the sum over the lines in a class is simple. As one important example of such a case, suppose the $M_{k}$ lines $i_{i}, i_{2} \ldots i_{k} \ldots i_{M h}$ which both fall into the same frequency region and have the same value for the integrai $j_{M i}$ are treavea togeuher. Then (5.66) $-\frac{1}{b_{m}} \sum_{i_{k}=1}^{M_{k}} \ln \left(1-\frac{j_{m i}}{2 \Delta_{i}}\right)=-\frac{M_{k 2}}{b_{m}} \cdot C_{k}\left(1-\frac{j_{n k}}{2 \Sigma_{k}}\right) \equiv E_{n k}$, and part of the sum occuring in $E_{n}$ has been performed by reducing it to one term. Other ways of grouping lines into classes may also be used, the common feature of all such devices being the reduction of the sum over all lines $E_{n}$ to a sum over classes of lines ${\underset{k}{\prime}}_{k}^{c} E_{\mu N}$, the sum over the lines in each class being already performed. Thus we no longer treat individual lines, but classes with tens, hundreds or thousands of lines. Furthermore, it may be possible to use overail properties of a class, for example the total absorption.strength of all the lines in the olass, or again the average breadth, instead of requiring detailed calculation of this data for each line. Looking further ahead we may even find features among the classes whioh facilitate sumaing over them. For the moment wo pause to oonsider the special cases with which we shall be mainly concerned in our applications.

We first consider the case in which natural and/or collision breadth are the determining factors in dispersion. In that sse the absorption coefficient for the ith line is given by (2.15) and (2.0). Inserting the value of $\mu_{i}=\mu_{i} / \mu_{c}$ into (5.63) we obtain:
(5.67)

$$
\frac{j_{\mu i}}{2 \Delta_{i}}=\frac{1}{2 \Delta_{i}} \int_{a_{i}^{2}-\Delta_{i}}^{u_{i}^{\prime}+\Delta_{i}}\left\{1-\operatorname{esp}\left(-b_{m} \frac{N^{i} f_{i} \bar{r}_{i}}{\pi s_{s}}\left(u-u_{i}\right)^{2}+r_{i}^{2}\right)\right\} d u_{i}
$$

Changing variable of integration in (5.67), $x=\frac{u_{i}-4}{\sqrt{i}} \quad$, gives
(5.68) $\frac{\delta_{\mu_{i}}}{2 \Delta_{i}}=\frac{\Gamma_{i}}{2 \Delta_{i}} \int_{\substack{\mu_{i}^{*}-\mu-\Delta_{i} \\ \Gamma_{i}}}^{\frac{\mu_{i}^{*}-\mu+\Delta_{i}}{\Gamma_{i}}}\left\{1-\operatorname{cosp}\left(\frac{-\sigma_{\mu} \frac{N_{i} f_{i}}{\pi S \Gamma_{i}}}{x^{2}+1}\right) d x\right.$.

The result is a definite integral which depends on one parameter and the limits of integration. By defining (5.69)

$$
F(a, u)=\frac{2}{\pi a} \int_{0}^{u}\left\{1-\operatorname{sesp}\left(-\frac{a}{1+x^{2}}\right)\right\} d x
$$

we can Write (5.68) as
(5.70) $\frac{j_{m_{i}}}{2 \Delta_{i}}=\frac{b_{x}}{2 \Delta_{i}} \frac{N^{i} f_{i}}{S} F\left(a_{M_{i}}, \infty\right) g_{M i}^{\prime}(u)$,
where
(5.71)

$$
g_{m i}^{\prime}(u)=\frac{F\left(a_{m_{i}}, \frac{u_{i}^{*}-u+\Delta_{i}}{r_{i}}\right)-F\left(a_{m_{i}}, \frac{u_{i}^{*}-u-\Delta i}{r_{i}}\right)}{2 F\left(a_{m i}, \infty\right)}
$$

and
(5.72)

$$
a_{M i}=\frac{b_{\mu} N^{i} f_{i}}{\pi S r_{i}}
$$

Values of the integral $\quad F(a, u) \quad$ are presented in Table $V$; so that it is a comparatively simple matter to compute $j_{M i} \quad$ -

If, in a certain frequency interval $u_{k}^{*}-\Delta_{k}$ to $u_{k}^{x}+\Delta_{k}$, there are many lines $M_{k}$ having the same value of $J_{M_{i}}$, such a set of lines can be considered as a class and formula (5.66) applies. This will occur for example if all lines had the same strength, breadth and dispersion. As pointed out previously this means a great simplification. If in addition the $j_{k k} / 2 \Delta_{k} \ll /$, wo can expand the logarithm obtaining

$$
\text { (5.73) } \quad E_{m k}=-\frac{M_{k}}{b_{m}} \ln \left(1-\frac{j_{m k}}{2 \Delta_{k}}\right) \sim \frac{M_{k}}{b_{m}} \frac{j_{m k}}{2 \Delta_{k}}=\frac{\mu_{k}}{2 \Delta_{k}} \frac{N^{j} f_{j}}{S} F\left(a_{m k}, \infty\right) g_{m k}^{\prime}(u) \text {. }
$$

The first set of factors $M_{k} N^{j} f_{j}\left(2 \Delta_{k} S\right)$ is independent of $n$, the term number in the series development of $/(U+r)$, and depends only slightly on frequency through the factor $\$$;also $M_{k} N^{j} f_{j} \quad$ is the total absorption strength of all lines in the group. The second factor $\left.F\left(a_{k k}\right)^{\infty}\right)$ is less than unity, approaching unity as a limit as $a_{m k} \rightarrow 0$. In a great many cases this limiting value can be inserted, if not for all values of $n$, at least for the higher values. The strong freequency dependence of $E_{M h}$ is exhibited in the factor $g_{M K}^{\prime}(u)$ which is olose to unity within the region $u_{k}-\Delta_{k} \leq u \leq u_{k}^{*}+\Delta_{n}$ and is close to zero outside this region. Similar to the factor $F\left(a_{n k}, \infty\right)$, the factor $g_{m k}^{\prime}(u)$ depends upon $n$ only through the appearance of $a_{n} k$ and if $a_{n k} \lll$, the dependene on $n$ disappears entirely. Thus if $a_{\mu k} \ll /$, Este is independent of $n$ and we have the interesting result that

$$
\begin{equation*}
\bar{P}=\sum_{n} a_{n} e^{-b_{m} \sum_{k}^{\prime} E_{m} k}=\frac{1}{1+\sum_{k} E_{m k}} \tag{5.74}
\end{equation*}
$$

We shall later give an important physical interpretation to this result after we have seen it appear in other connections.

It may happen that in a region there are lines which have the same breadth, but
not the same strength. While the simplification (5.73) does not apply, an even simpler result can be obtained in one important case by use of a treatment due to (1) Boris Jacobson. Let the number of lines of the group we are treating as a class which have a strength times breadth $N^{i} f_{i} \Gamma_{i}$ between $Q_{j}$ and $Q_{j}+\Delta Q_{j}$ be $N_{k}\left(Q_{j}\right) \Delta Q_{j}$. Then

or, if one may expand the logarithm, (5.76)

$$
E_{n, k} \cong \frac{1}{b_{n}} \sum_{Q_{j}} N_{k}\left(Q_{j}\right) \Delta Q_{j} \frac{j_{n 1} Q_{j}}{2 \Delta k}
$$

If there are very many lines they will form a practically continuous distribution in $Q_{j}$, and the sum over all $Q_{j}$ in (5.76) may be replaced by an integral. Referring to (5.68), however, we see that $j M_{J_{j}}$ is also an integral, but the variable of integration is related to the frequency $u_{i}$ of the center of the line. The order of the two integrations may be reversed and we obtain

$$
\begin{equation*}
E_{n k} \approx \frac{1}{b_{n}} \frac{\Gamma_{k}}{2 \Delta_{k}} \int_{a_{k}^{2}-u-\Delta_{2}}^{\Gamma_{k 2}} \int_{Q}^{\frac{u_{k}-u+\Delta_{k}}{\Gamma_{k}}}\left\{N_{k}(Q)-N_{k}(Q) \exp -\left(\frac{b_{n}}{\pi} \frac{N^{h} f_{h}}{\Gamma_{k}} \frac{1}{x^{2}+1}\right)\right\} d Q d x \tag{5.77}
\end{equation*}
$$

Now the essential point of the method is to find a distribution of strengths Which occurs frequently in practice, and which enables both integrations in (5.77) to be performed analytically. Such a distribution is

$$
\begin{equation*}
N_{k}(Q)=\frac{M_{k}}{\bar{Q}_{k}} \exp -\left(Q / \bar{Q}_{k}\right) \tag{5.78}
\end{equation*}
$$

where $M_{k}$ is the total number of lines in the $k t h$ class and $\bar{Q}_{k} i s$ the average $N f f$ of the class. Although it is physically impossible to have any lines of infinite
(1)

Op. cit.
strengti, the upper limit of the integrel over $Q$ may well be extended to infinity, since beceuse of the exponential nature of the distribution (5.78) the contribution of the very large $Q$ is negligible. The integrating over $Q$ from 0 to $\infty$ gives (5.79) $E_{n k} \simeq \frac{1}{b_{n}} \frac{M_{k} \Gamma_{k}}{2 \Delta k} \int \begin{aligned} & \frac{u_{k}^{*}-u_{i} \Delta k}{\Gamma_{k}} \\ & \frac{u^{*} k-u_{n}-\Delta_{k}}{\Gamma_{k}}\end{aligned} \frac{b_{n} \overline{N f} /(\pi s \Gamma)}{1 \& x^{2}+b_{n} \overline{N f} /(\pi S \Gamma)} d x_{\bullet}$

Integrating now over $x_{\text {g }}$ we have
(5.80) $E_{n k} \cong \frac{M_{k} \overline{N i k}}{2 \Delta k} s \frac{g_{n k}(u)}{\sqrt{1+a_{n k}}}$
where

$$
(5.81) \quad a_{n k}=b_{n} \frac{\overline{N f^{k}}}{\pi S \Gamma_{k}}
$$

is exactly the same quantity previously defined in comeotion with a distribution of lines of equal strength and
(5.82) $g_{n k}(u)=\frac{1}{\pi}\left\{\tan ^{-1} \frac{u_{k}^{*}-u_{n}+\Delta_{k}}{\Gamma_{k} \sqrt{1+a_{n k}}}=\tan ^{-1} \frac{u_{k}^{*}-u-\Delta k}{\Gamma k \sqrt{1+a n_{k}}}\right\}$.

To faoilitate computations of this function, nonographs have been prepered, (Fig. 3a, Fig. 3b, Fig. 4). A comparison of (5.80) and (5.73) shows the extreme similarity of the results for these two different distributions. The first set of factors $\frac{M_{x} \overline{M^{k}}}{2 \Delta k}$ giving the essential magnitude of $E_{n k}$ is identioal. The second factor $F\left(a_{n k} p\right)$ or $\left(1+a_{n k}\right)^{-\frac{7}{2}}$ is less than anity and independent of $a_{n k}$ as $a_{n k} \rightarrow 0$ 。 The last factor in both cases contains the important frequency dependence, and has the same qualitative features. In the limit $a_{n k} \rightarrow 0$ the two formulae become identical as oonsiderations of the properties of $F(a, u)$ shows. In that oase equation (5.74) applies to this type of distribution also.

To explain thy the results are identical in the limit $a_{n k} \rightarrow 0$, and what the simple form ( 5.74 means, weturn to the smearing out approximation (5.57).
 (5.83) $\quad P=\frac{1}{1+\frac{\sum_{j} j_{f}}{2 \Delta u}} \quad$.

The only difference between (5.83) and (5.74) is the fact that in the latter case $E_{n_{k}}$ is not quite zero outside the interval $u_{k}^{*} \Delta_{k}$ to $u_{k}^{*} \& \Delta_{k_{k}}$ and it is not quite equal to $r_{k}$, differing by the factor $g_{n k}$ which may be .8 to . 98 In typioal cases, Fithin the interval. The first difference mentioned, the socalled tail effect, is the more important. Now it is just under the assumptions used in making this derivation that the quantity $a_{n k} \ll 1$ and $\frac{J_{n k}}{2 \Delta k} \ll 1$. The ape proximation (5.74) is thus essentially equivalent to continuously smearing out the absorption strength of the lines in the regions where they occure

## 6. Statistioal Treatmont of Absorption Edget.

The very same type of statistical treatiment appropriate for lines should also be used to treat the bound-free (photoelectric) absorptions, for there will be a very large number of absorption edges. In this section we shall develop this method, and also discuss some rather less acourate approximations. The boundufree absorption cross-section given by (2.23) may be rewritten as (5.84) $\mu_{b f}(\nu)=\frac{2^{4}}{3 \sqrt{3}} \frac{h e^{2}}{k I m} \frac{1}{u^{3}} \sum_{b} \frac{N^{b}}{\nabla} \frac{1}{n}\left(\frac{I_{n}}{K I}\right)^{2} \quad g_{b f}(\nu)$

In the noighborhood of the iomization potential of an eleotron shell, say the $L$ shell, the function $u^{3} \mu_{b f}$ will have a large number of amall steps due to the appearance of a new term in the sum at each absorption edge. It may well have somewhat the shape indicated by the a.000mpanying figure


Between such groups of edges, $\mu_{b f}$ and conssquently $\mu_{c}$, the oontinuous absorption, is a smoothly varying function.

By means of a statistical treatment oompletely analogous to our treatment of lines, the function $1 / \mu$, jagged near each group of edges, is replaced by a smoothly varying average ohosen so that the final opacity is not falsified. To do this, wo artificially divided the absorption coefficient into three contributions:

$$
\text { (5.84a) } \quad \mu=\bar{\mu}_{c}+\left(\mu_{c}-\bar{\mu}_{c}\right)+\mu_{l} \text {. }
$$

where $\bar{\mu}_{0}$ is some average function approrimately representing the gross aspects of $\mu_{\mathrm{c}}$. Tnen
(5.85) $\mu=\bar{\mu}_{0}\left(1+r^{\prime}\right)(1+r)$,
where

$$
\text { (5.86) } \quad r^{\prime}=\frac{\mu_{c}-\bar{\mu}_{c}}{\bar{\mu}_{c}} ; \quad r=\frac{\mu_{l}}{\mu_{0}} ;
$$

and introducing
(5.87) $\quad P=\frac{1}{1+2^{2}}, \quad P^{\prime}=\frac{1}{2+r^{\prime}}$
the opacity formula bacomes

$$
\begin{equation*}
\Lambda=\frac{1}{\rho K}=\int_{0}^{\infty} \frac{P P^{\prime} N(u)}{u^{3} \bar{\mu}_{0}} d u \tag{5.88}
\end{equation*}
$$

We shall endeavor to replace $P^{\prime}$ by $\bar{P}^{\prime}$ its statistical average. Then since $1 / \mu_{0}=P^{\prime} / \bar{\mu}_{0}$ we have approximately
(5.89) $\frac{1}{\mu_{0}}=\frac{p^{\prime}}{\bar{\mu}_{0}}$;
(5.90) $\quad r=\frac{\mu_{l}}{\mu_{0}} \cong \frac{\mu \ell P^{\prime}}{\mu_{0}}$.

The entire statistical treatment of lines of the last section can be carried through unaltered but we now interpret $r$ by ( 5.90 ) which involves only the statistical average of $\mu_{c}$, not its small details.

Suppose now that the it absorption edge might with equal probability fall anywhere in the interval from $u_{i}^{*}-\Delta_{i}$ to $u_{i}^{*}+\Delta_{i}$. Then by following similar steps to those in the statistical treatment of lines which lead to (5.61), we get (5.91) $\vec{P}^{\prime}(u)=\sum_{n} a_{n} \prod_{i} \frac{1}{2 \Delta_{i}} \int_{u_{i}^{*}-\Delta_{1}}^{u_{i}^{*}+\Delta_{i}} e^{-b_{n} r_{i}^{\prime}} d u_{i}$,
where $r_{i}$ is the contribution to $r^{\prime}$ of the fth term in the sum (5.54). We thus see that

$$
\begin{array}{ll}
r_{i}^{\prime}=0 & \text { for } u<u_{i} \\
r_{i}^{\prime}=\rho_{i} & \text { for } u \geqslant u_{i}, \tag{5.92}
\end{array}
$$

and $P_{i}$ is a slowly varying function of frequency. Now consider the edges arranged in order of ascending ionization frequency. For the particular frequency $u$ at which $\bar{p}^{\prime}(u)$ is to be evaluated, assume that the edges $1,2 \ldots f(u)$ all certainly lie below $u$, that is
(5.93)

$$
u_{i}^{*}+\Delta_{1}<u \quad \text { for } 1 \leqslant 1 \leqslant j(u)
$$

Also assume that the regions in which edges $j(u)+1, \ldots \ldots(u)$ fall include $u_{0}$ that is
(5.94) $\quad u_{i}^{*}-\Delta_{i} \leqslant u \leqslant u_{i}^{*}+\Delta_{i} \quad$ for $j(u)+1 \leqslant i \leqslant k(u)$.

Lastly, the other edges will all certainly lie above $u_{\text {, }}$ that is
(5.95) $u<u_{i}^{*}-\Delta_{i}$ for $k(u)<i$.

For the edges of (5.95) the integral in (5.91) is just $2 \Delta_{i}$, for those of (5.94) it is $\left[u-\left(u_{i}^{*}-\Delta_{i}\right)\right] e^{-b_{n} P_{i}} \&\left(u_{i}^{*}+\Delta_{i}-u\right)$, while for those of (5.93) it is $2 \Delta_{i} 0^{-b_{n} P_{1}}$. We, therefore, have for (5.91)
(5.96) $\quad \vec{P}(u)=\sum_{n} a_{n} \prod_{i=1}^{j(u)} e^{-b} b_{i} \rho_{i=j} \prod_{i(u)+1}^{k(u)}\left\{1-\frac{u-\left(u_{i}^{*}-\Delta_{i}\right)}{2 \Delta_{i}}\left(1-e^{-b_{n} \rho_{i}}\right) \prod_{k(u)+1} 1\right.$

The absorption edges oocur in groups with long frequency intervals between groups. Let us follow the variation of $\vec{P}^{\prime}(u)$ with $u$ from a frequenoy $u=u_{1}$, whioh is below a particular group of edges--for ooncreteness say below the $L$ shell ionization edges, - to a frequenoy $u_{=} u_{2}$ above the group edges. At $u_{n} u_{1}$, there will be no terms in the second product, so that
whence by (5.59)

$$
\begin{equation*}
\bar{P}^{\prime}\left(u_{1}\right)=\frac{1}{1+\sum_{i=1}^{j\left(u_{1}\right)} \rho_{i}} \tag{5.97}
\end{equation*}
$$

Likewise for $u=u_{2}$, there will be no terms in seoond product of (5.96) but there will be additional terms in the first product givine

$$
\begin{equation*}
\bar{P}^{\prime}\left(u_{2}\right)=\frac{1}{1 \neq \sum_{i=1}^{j\left(u_{2}\right)} \rho_{i}} \tag{5.98}
\end{equation*}
$$

The form (5.97) will hola as $u$ inoreases fram $u_{1}$ until $u$ becomes equal to the limit of the region available to the lowest edge. Then some terms in the second product begin to appear and $\bar{P}^{\prime}(u)$ decrease until after $u$ has passed through the region of the edges, it takes the form (5.98). The variation through the region of the edges
takes a very simple form in the oase where there are very many, $M$, edges all having the same $\rho_{j} .=$ Pand the same region in winich they may fall. Then the second product becomes

$$
\left\{1-\frac{u-\left(u^{*}-\Delta\right)}{2 \Delta}\left(1-0^{-b_{n} \rho}\right\}^{M} \sim\left\{1-\frac{u-\left(u^{*}-\Delta\right)}{2 \Delta} \frac{b_{n} M \rho}{M}\right\}^{M}\right.
$$

Now assume that the total strength of the edges remains fixed but $M$ increases approaching $\infty$. Then the second product approaches exp $-\frac{u_{m}\left(u^{*}-\Delta\right)}{2 \Delta} b_{n}$ M(and (5.96) itself becomes

$$
(5.99) \quad \bar{P}(u)=\frac{1}{1+\sum_{i=1}^{j\left(u_{1}\right)} \rho_{i}+\frac{u-\left(u^{*}-\Delta\right)}{2 \Delta}} \text { MP}
$$

A reasonable choioe for $\bar{\mu}_{0}$ would be to malce $\bar{\mu}_{8} \mu_{0}$ between the groups of edges, but to have $\bar{\mu}_{0}$ jump in one step from its value before the group to its value after the group at the frequenoy of the highest edge in the group. The behavior of $u^{3} \mu_{0}, u^{3} \bar{\mu}_{0}$ and $\bar{\mu}_{c} u^{3} / \bar{p}^{\prime}$ in the noighborhood of a group of edges are illustrated schematioally in the acoompanying figure.


As a crude approximation, ono can simply use a single absorption edge to replace a whole group of edges. Whenever this is legitimate, the position of the edge is not oritical, and we may place it at the ionization energy of the ion With average ocoupation, that is at the ionization energies $\overline{\mathcal{E}} \ell z$ of (4.37). The reason the position of the edge is not oritical is that there are always a great number of lines near the edges. Their high absorption coefficient hides any small
alteration in edge absorptinn. Jacobsohn has worked out a refinoment of this treatment placing the effective edge of a group so as to make the average of $1 / \mu_{C}$ correct throughout the group but it is rarely necossary to use this treatment.
7. Simplified Practioal Treatment of Lines.

We have thus far discussed two detailed methods of treating lines. The straightforward approach is exact but impossibly oomplicated except for a small frequency region. The statistical approach brings the problem within the limits of human computation and should be nearly as acourate as the exaot treatment. However, even the statistical approaoh involves as much as 6 computer-months work to get a single value of the opacity. When flexible rapid eleotronic computing machines become available, the statistical method will oome into its ow as a good method of treating the lines. Until such time, we shall have to content ourselves with rougher approximations. It is these rougher treatments which we investigate here.

The clue to the problem is the smearing out approximation. Instead of using the full statistical treatment to give $\bar{P}$, we may use the approximate formula.

$$
\begin{equation*}
\bar{P}=\frac{1}{1+\sum_{k} E_{n k}} \cong \frac{1}{1+\sum_{k} r_{k}} \tag{5.94}
\end{equation*}
$$

where

$$
\begin{equation*}
r_{k}=\sum_{j k} \frac{N^{j k_{P}}}{S k k} \tag{5.,95}
\end{equation*}
$$

In the most detailed treatment of this type, ve may consider a class as composed of all the lines from a particular ion type arising from the same one electronic transition. We would also incorporate three features which will very much enhance the accuracy of our result; namely, 1) Treat etrongest lines individually by superimposing them upon the bacigeround absorption of the continuram plus the weak smeared out lines; 2) Incorporate an empirical correction factor to reduce the contribution of each group of linAs, since the smearing out treatment overemphasizes line effects.

This factor must be determinod by comparing the genuine statistical treatment with the smearing out treatment in several representative cases, 3) Take into acoount the tail effeot negleoted by smearing out treatment. Corrections $2 \& 3$ are of opposite sign and experience may show that it is sufficiently acourate to negleot bothe.

It may even prove possible to do a muoh less detailed smearing out treatment by cousidering a class of lines as composed of all lines from a single or small group of one electron transitions. This treatment should also include an empirioal factor designed to force its results to agree with the detailed statistical treato ment. By this method the work for computing the line effeot oould be reduced to 2 omputer weeks.

An ontirely different appronoh in modifying the detailed statistical treato ment is to use the soroalled pattern treatment. This treatment is based on the fact that relatire positions of classes of lines are the same for two groups of one elootron transitions differing only in the principal quantum number $n$ of the final state. Moreover, the ratio of strength of each line in one group to that In another is a constant for all lines. Then the value of $\sum_{k} E_{n k}$ at one frequency can be obtained from that at another, by the somcalled pattern transformation
(5.96) $\sum_{k} E_{n k}(u)=$ oonst. $\sum_{k} E_{n k}$ (upoonsto).
VI. SUMARY OF FORMULAE FOR COMPUTING OPACITIES.

In this Chapter all the formulae essential for opacity oalculations are oollected in form for computation. Energies are expressed in terms of the Rydberg energy Rhc $=13.61$ electron volts and lengths in terms of the first Bohr radius for Hydrogen $a_{0}=.5291 \times 10^{-8} \mathrm{~cm}$ 。

1. Formulae for Ocoupation Mumbers:

The volume $V$ is related to the density $\rho$ by
(6.1) $\frac{V}{N_{0}^{3}}=\frac{\sum_{Z} M_{Z} \frac{N_{Z}}{Z_{n}}}{\rho \mathbb{N}_{0}^{a} a_{0}^{3}}$.
$M_{Z}=\operatorname{gram}$ atomio weight
$N_{0}=6.023 \times 10^{23}=$ Avogadro's number.

The radius of the ion spheres is from (4.43),
(6.2) $\frac{a_{Z^{\prime}}}{a_{0}}=\left\{\frac{3}{4 \pi} \quad z^{\prime} \frac{V}{N a_{0}^{3}} \frac{N}{n_{f}}\right\}^{1 / 3}$.
where
(6.3) $z^{\prime}=\varepsilon-\sum_{\ell} \frac{n_{l z}}{\bar{F}_{Z}}$.

The number of bound electrons $n_{b}$ is from (4.40)
(6.4) $\frac{n_{b}}{N}=\sum_{Z} \frac{\mathbb{N}_{Z}}{N} \sum_{l} \frac{n_{g Z}}{N_{Z}}$.
and the number of frees $n_{f}$ from (4.42) is
(6.5) $\frac{n_{p}}{T}=\frac{n}{N}-\frac{n_{b}}{N}$.

The free energy -ici $\alpha^{*}$ of the electrons is given by (4.4ia)

$$
\begin{aligned}
(6.6) \alpha^{*}= & -3.1054+\ln \frac{V}{\mathrm{Na}_{0}^{3}}-\ln \frac{\mathrm{n}_{\mathrm{I}}}{\mathrm{~N}}+3 / 2 \ln (\mathrm{KT} / \mathrm{Rho}) \\
& +\frac{15}{16} \alpha^{2} \frac{\mathrm{KT}}{\mathrm{Rho}}-\frac{15}{64} \alpha^{4}\left(\frac{\mathrm{KT}}{\mathrm{Rho}}\right)^{2} \\
& -.35355 e^{-\alpha^{*}}\left(1-\frac{15}{64} \alpha^{2} \frac{\mathrm{KT}}{\mathrm{Kho}}\right)+.12995 e^{-2 d^{*}}
\end{aligned}
$$

where $\alpha=1 / 137.03$ is the fine structure onsstant．
From（4．38）we get
（6．7）$\quad z_{l}^{*}=z-\sum_{j \neq \ell} \frac{n_{j Z}}{N_{z}} \quad \sigma_{l_{0} j}-\frac{n \ell z}{N_{z}}\left(1-\frac{1}{\sigma_{L z}}\right) \sigma_{l_{0} l}$.
Table II gives the screening constants $\sigma_{\text {log }}$ ．Then from（4．37）

Table III gives the one electron energy levels $-\varepsilon_{l}^{0}\left(z_{l}^{\prime}\right) /$ Rho and table IV gives the average square radius $\left(\overline{r / a_{0}}\right)^{2}$－Finally，the occupation numbers are（4．36）
（6．9）$\frac{n / Z}{N_{Z}}=\frac{0 \ell z}{\alpha^{*}-\frac{R h c}{K T}\left(-\frac{\varepsilon_{l Z}^{\gamma}}{K h c}\right)}$ ．
The ionic occupations are found from（4．47）．The probability of having an ion with $\nu_{l}$ electrons in the $l$ level is

where
（6．11）$p_{l}=n_{l} / \sigma_{l}$ ．
（6．12）$q \ell=1-p \ell$ 。
These formulae must be modified if

$$
(6.13) \lambda=2 z^{\prime} \frac{a_{0}}{a_{2}} \frac{\text { Rho }}{k T}>1
$$

and the appropriate change is discussed in Appendix II．
2. Formulae for Thermodynamic Functions oboe Appendix I.

The total energy of the system $E$ is obtained from (AI 16, 17, 18) and follow-
ing discussion.

$$
\begin{aligned}
(6.14) & \frac{E}{N R h c}=\sum_{Z} \frac{N_{Z}}{N} \sum_{i} \frac{N_{i Z}}{N_{Z}} \frac{\varepsilon_{i}^{0}}{R h c}\left(\bar{z}_{i}\right)-\frac{9}{5} \sum_{Z} \frac{N_{Z}}{N} z^{N^{2}}\left(\frac{a_{0}}{\alpha_{Z}}\right) \\
& +(3 / 2) \frac{n_{f}}{N} \frac{k T}{R h c}\left\{1+\frac{5}{8} \alpha^{2} \frac{k T}{R h c}-\frac{5}{16} \alpha^{4}\left(\frac{k T}{R h c}\right)^{2} \cdot \cdots\right\}\left\{1+.17678 e^{-\alpha^{*}}\left(1-\frac{15}{32} \alpha^{2} \frac{k T}{R h c}\right) \cdot *\right\} \\
& +\left[\begin{array}{l}
3 \\
0 r \\
3 / 2
\end{array}\right] \frac{k T}{R h c},
\end{aligned}
$$

where oof. AI. 15

$$
\text { (6.15) } \bar{z}_{i}=z-\frac{1}{2} \sum_{j \neq i} \frac{n_{j Z}}{N_{Z}} \sigma_{i, j}-\frac{1}{2} \frac{n_{1 Z}}{N_{Z}}\left(1-\frac{1}{c_{i Z}}\right) \sigma_{i, i}=z-\frac{1}{2}\left(z-z_{i}^{*}\right)_{0}
$$

The zero of energy is taken as the state in which all the nuclei and electrons are infinitely separated and at rest. We should use the upper value 3 in the last term when
(6.16) $\mathrm{z}^{\mathrm{I}^{2}} \frac{\mathrm{a}_{0}}{\mathrm{a}_{\mathrm{z}}{ }^{4}} \frac{\mathrm{Rho}}{\mathrm{kT}}>11.6$
and the lower value $3 / 2$ in other oases.
The pressure $P$ is obtained from (AI.20). It is
(6.17) $\frac{P V}{N K T}=\frac{n_{f}}{N}\left\{1+.17678 e^{-\alpha^{*}}\left(1-\frac{15}{32} \alpha^{2} \frac{k T}{R h o}\right)\right\}-3 / 5 \sum_{Z} \frac{N_{Z}}{N} z^{2} \frac{R_{h c}}{k T} \frac{a_{0}}{a_{z}}$ $+\left[\begin{array}{c}3 / 2 \\ \text { or } \\ 1\end{array}\right]$.
where the condition ( 6.13 ) requires the use of upper value $3 / 2$ in square bracket when fulfilled, and the lower values when violated.

## 3. Continuous Opacity Formulae

The continuous transmission $\Lambda_{c}$ is defined by (5.7) as being the value of the mean free path found by neglecting line absorption. We write this in the form

$$
\begin{equation*}
A \Lambda_{0}=\int_{0}^{\infty} \frac{W(u)}{D(u)} d u \tag{6.18}
\end{equation*}
$$

where the weighting function
(6.19) $W(u)=\frac{15}{4 \pi^{4}} u^{7} e^{2 u}\left(0^{u}-1\right)^{-3}$
is recorded in Table $V I$, and the reduced absorption $D$ is
(6.20) $D=\frac{u^{3} \mu_{c}}{A}$.

The reciprocal length $A$ is introduced to give the quantities in the integral (6.15) simple coefficients and make them dimensionless. It has the value

$$
\begin{equation*}
A=\frac{2^{4}}{3 \sqrt{3}} \frac{h e^{2}}{m c} \frac{1}{k T} \quad \frac{N}{V}=4.762 \times 10^{6} \quad \frac{R h c}{k T} \frac{\rho}{\sum_{Z} N_{Z} \frac{N_{Z}}{N}} \tag{6.21}
\end{equation*}
$$

where c.g.e. units are used for $\rho$ and $\mathrm{N}_{\mathrm{Z}}$. The continuous absorption coefficient and consequently $D$ is the sum of 3 terms, $D_{s}$ from scattering, $D_{\text {ff }}$ from free-free transition and $D_{b f}$ from bound-free transitions
(6.22) $D=D_{s}+D_{f f}+D_{b f}$ -

The scattering term is
(6.23) $\quad D_{s}=\frac{n}{V} \frac{\Phi}{A} \frac{u^{3}}{1-\theta^{-u}}=\sqrt{\frac{3}{8}} \quad a^{3} \quad \frac{n}{N} \frac{k T}{R h c} \quad \frac{u^{3}}{1-e^{-u}} \frac{\Phi}{Q_{0}}$
where the scattering corns section is
(6.24) $\frac{Q_{0}}{Q_{0}}=\left(1-u a^{2} \frac{k T}{R h c}\right)\left[1+0\left(\frac{k T}{m c^{2}}\right)^{2}\right]$.

Here, of course, $a$ is the fine structure constant $=1 / 137.03$ and $\mathscr{Q}_{0}$ the Thompson cross section $\boldsymbol{Q}_{0}=\frac{8 \pi}{3}\left(e^{2} /\left(\mathrm{mc}^{2}\right)\right)^{2}=.6654 \times 10^{-24} \mathrm{~cm}^{2}$. The free-free absorption term follows from (2.30),
(6.25) $D_{f f}=\frac{R h c}{k T} \sum_{Z} \frac{N_{Z}}{N} Z^{2} e^{-a^{*}} E_{f f \cdot}$

If we neglect small correction terms in $a^{*}$, we obtain from (4.4la)
$(6.25 a)-a^{*}=\ln \frac{(4 \pi)^{3 / 2}}{2} \frac{\mathrm{n}}{\mathrm{N}} \frac{\mathrm{Na}}{\mathrm{V}}\left(\frac{\mathrm{Rhc}}{\mathrm{KT}}\right)^{3 / 2}=\ln ^{\mathrm{n}} \frac{1}{1.0075} \frac{2 \mathrm{f}}{\mathrm{NN}} \rho\left(\frac{\mathrm{Rhc}}{\mathrm{KT}}\right)^{3 / 2}$
where $\bar{M}=\sum_{Z} M_{Z} \frac{N_{Z}}{N}$, and putting this with (6.2) into (6.25) we get

$$
D_{f f}=3 \sqrt{\pi}\left(\frac{R h o}{k T}\right)^{5 / 2} \sum_{Z} \frac{N_{Z}}{N} z^{\circ 3}\left(\frac{a_{0}}{a_{Z^{0}}}\right)^{3} \bar{E}_{f f^{\circ}}
$$

where

$$
\begin{equation*}
\overline{\mathrm{E}}_{\mathrm{ff}}=1+.1728\left(\frac{\mathrm{kT}}{\mathrm{Rh}^{2} \mathrm{Z}^{2}}\right)^{1 / 3} u^{\frac{1}{2}}(1 \geqslant 2 / u) \tag{6.27}
\end{equation*}
$$

The bound free absorption tern (2.23) gives simply,

$$
\begin{equation*}
D_{b f}=\sum_{b} \frac{N^{b}}{N} \frac{1}{n}\left(\frac{I_{n}}{k T}\right)^{2} E_{b f}(u) \tag{6.28}
\end{equation*}
$$

the Gaunt factors being defined by (2.22). They are discussed further in Appendix III. Fig. I is a graph of these factors.

In most cases the function $D(u)$ varies rather slowly with frequency except in the neighborhood of an absorption edge, where $D_{b f}$ is discontinuous. It is the usual practice among astrophysioists to broak up the integration range into intervals within which $D(u)$ may be considered constant, and thus

$$
\begin{equation*}
A \Lambda_{c}=\sum_{k} \frac{1}{D\left(u_{k+\frac{1}{2}}\right)} \int_{u_{k}}^{u_{k+1}} w(u) d u=\sum_{k} \frac{s\left(u_{k+1}\right)-s\left(u_{k}\right)}{D\left(u_{k+\frac{2}{2}}\right)} \tag{6.29}
\end{equation*}
$$

where $S(u)$ is the Strömgren function
(6.30) $S(u)=\int_{0}^{u} w(x) d x$.

Values of this function are given in Table VI. Formula (6.29) is valuable for quickly obtaining approximate results, if we approximate all the absorption edges associated with a one electron ionization by a single edge.

For more accurate results, we should use the statistical treatment of absorption edges discussed in $V-6$. In this treatment $D(u)$ of (6.18) is replaced by $\bar{D}(u) / P^{\prime}(u)$ where from (5.99)
(6.31) $\frac{\vec{D}(u)}{\bar{P}^{\prime}(u)}=D(u)\left\{1+\sum_{i=1}^{j\left(u_{k}\right)} \rho_{i}+\frac{u-\left(u_{k}^{*}-\Delta_{k}\right)}{2 \Delta_{k}} u_{k} \rho_{k}\right\}$.

In this result, the $k$ th group of $M_{k}$ edges lies between $u_{k}^{*}-\Delta_{k}$ and $u_{k}^{*} \neq \Delta_{k}$. Each edge has a relative jump, fron (5.92), (5.86)
(6.32) $\rho_{i}=\frac{\mu_{c i}-\bar{\mu}_{0 i}}{\bar{\mu}_{c}}$,
and $\bar{\mu}_{c}$ is a function such that
(6.33) $\bar{\mu}_{0}(u)=\mu_{0}(u)$ between groups of edges

$$
\bar{\mu}_{0}(u)=\mu_{0}\left(u_{k}^{*}-\Delta_{k}\right) \text { for } u_{k}^{*}-\Delta_{k}<u<u_{k}^{*}+\Delta_{k}
$$

The reduced absorption $\overline{\mathrm{D}}(\mathrm{u})$ is
(6.34) $\bar{D}(u)=\frac{u^{3} \bar{\mu}_{0}}{A}$.
4. Line Absorption Contribution.

The line absorption coefficient is defined by (5.8) as the reduction in the continuous mean free path, due to the effect of the lines. It is (6.35) $A \Lambda_{\ell}=\int_{0}^{\infty} \frac{r}{1+r} \frac{\bar{P}^{\prime}(u) W(u)}{D(u)} d u$.
where
(6.36) $\quad r=\mu \ell / \overline{\mu_{0}}$
is the ratio of the line absorption coefficient to the continuous background statistically averaged in regions of absorption edges. For a line of natural and/or collision breadth dispersion this is c.f. (.5.24)

$$
\begin{equation*}
A \Lambda_{l}=\left.\frac{\bar{P}^{\prime}(u) W(u)}{\bar{D}(u)}\right|_{u=u_{i}} \pi\left(\frac{N_{i} f_{i} \Gamma_{i}}{\pi S}\right)^{\frac{3}{2}}\left(1+\frac{\pi^{S} \Gamma_{i}}{N^{i} i^{i}}\right)^{\frac{1}{2}} \tag{6.37}
\end{equation*}
$$

where
(6.38) $\quad \Gamma_{i}=\frac{h \gamma_{i}}{4 \pi k T}, \frac{h \gamma_{i}}{4 \pi}$ is the energy half breadth,

$$
\begin{equation*}
S=\frac{2^{4}}{3 \sqrt{3 \pi}} \frac{N \bar{D}(u)}{\bar{P}^{\prime}(u)} \quad=0.98015 \quad \frac{N \bar{D}(u)}{u^{3} \bar{P}^{\prime}(u)} \tag{6.39}
\end{equation*}
$$

$\mathbb{N}^{\mathbf{i}}$ is the number of ions in the initial state "and $f_{i}$ is the electron number for the transition.

If the absorption strength may be smeared out over the interval u*- $\Delta$ te $u^{*}+\Delta$, we get from (5.54)

When recourse is made to the statistical treatment of lines, the following formulae should be used:

$$
\begin{aligned}
(6.41) \quad A \Lambda & =\frac{A}{\rho R} \int_{0}^{\infty} \vec{P}(u) \frac{\vec{P}^{\prime}(u)}{\bar{D}(u)} W(u) d u \\
(6.42) \vec{P}(u) & =\sum_{n} a_{n} e^{-b_{n} E_{n}(u)} \\
& =.47 e^{-1.7 E_{2}}+.37 e^{-.35 E_{2}} \quad \& \quad .12 e^{-010 E_{3}}+.040 e^{\infty-015 E_{4}}
\end{aligned}
$$

(6.43) $\quad E_{n}=\sum_{i} E_{n i}=\sum_{k} \sum_{i_{k}} E_{n i k}=\sum_{i k} E_{n k}$. $a l l$ lines all all lines
olasses in a class
(6.44) $E_{n k}=-\frac{M_{k}}{b_{n}} \ln \left(1-\frac{j_{n k}}{2 \Delta_{k}}\right) \sim \frac{M_{k}}{b_{n}} \frac{j_{n k}}{2 \Delta_{k}} \quad$.
where $M_{k}$ is the number of lines in the kth olnsse
For natural breadth type dispersion, if a line may with equal probability fall anywhere in the region $u_{i}^{*}-\Delta_{i}$ to $u_{i}^{*}+\Delta_{i}$
$(6.45) \frac{j_{n i}}{2 \Delta_{i}}=\frac{b_{n}}{2 \Delta_{1}} \frac{W^{i} \underline{I}_{1}}{S} F\left(a_{n i}, \infty\right) g_{n i}^{\prime}(u)_{0}$
where
(6.46) $a_{n i}=\frac{b_{n}}{\pi_{i}} \frac{\pi^{i} f_{i}}{S}$.
$(6.47) g_{n i}^{\prime}(u)=\frac{F\left(a_{n i}{ }^{\left.\left(u_{i}^{*}-u \neq \Delta_{i}\right) / \Gamma_{i}\right)-F\left(a_{n j}\left(u_{i}-u-\Delta_{i}\right) / \Gamma_{i}\right)}\right.}{2 F\left(a_{n i}, \infty\right)}$.
and the function $F(a, m)$ is recorded in Table $V$.

If it is valid to use an exponential distribution of strength within a group
(6.48) $E_{n k} \cong \frac{M_{k}}{2 \Delta_{k} S} \frac{\overline{N f^{k}}}{\sqrt{1+a_{n k}}} E_{n k}^{(u)}$.
where $\overrightarrow{\mathrm{Nf}}^{\mathrm{k}}$ is the average strength of each line and
(6.49) $\quad a_{n k}=\frac{b_{n}}{\pi} \frac{1}{S} \frac{\overline{N f^{k}}}{\Gamma}$.

A nomograph of the function
(6.50) $g_{n k}^{(u)}=\frac{1}{\pi}\left\{\tan ^{-1} \frac{\left(u_{k}^{*}-u+\Delta_{k}\right)}{\Gamma_{k} \sqrt{1+a_{n k}}}-\tan ^{-1} \frac{u_{k}^{*}-u-\Delta_{k}}{\Gamma_{k} \sqrt{1+a_{n k}}}\right\}$
is given in Fig. III to facilitate calculations.

## VII Similarity Propertias and Limiting Foras of the Opacity

1. Similarity transformations for opacity calculations.

Since the calculation of even a single opacity coefficient is laborious, it is desirable to have approximate similarity transformations, which, if given values of the opacity ooefficient for one element at one temperature and density, will predict values for other elements under related conditions. This oan be done approximately if line absorption is not too important a factor.

We refer to our summary of formulae, Chapter VII, specializod for the case of a single element. From (6.9) we oan ses that the occupation numbers will be the same for two cases (subsoripts (1) and (2) ) provided that $\alpha^{*}$ is the same and (7.1)

$$
\frac{\bar{\varepsilon}_{\ell Z(1)}^{\prime}}{k T_{1}}=\frac{\bar{\varepsilon}_{l z(2)}^{\prime}}{k T_{2}} .
$$

The major contribution to these energies is just the interaction with the mucles, so that (7.1) is essentially equivalent to

$$
\begin{equation*}
z_{1}^{2} \frac{R H C}{R T_{1}}=z_{2}^{2} \frac{R H C}{k T_{2}} \tag{7.2}
\end{equation*}
$$

(1) has auggested a refinement which partially takes into account the screoning of the nuclei by the bou, de electrons. . He requires that

$$
\begin{equation*}
\gamma=\left(Z_{1}-N\right)^{2} \frac{R h e}{R T_{1}}=\left(L_{2}-N\right)^{2} \frac{R h c}{R T_{2}} \tag{7.3}
\end{equation*}
$$

where A is the screening of the level with probability of occupation $1 / 2$. This forces the occupations of the half filled levels to be the same; the lower energy levels will be completely filled anyway, while the higher ones will have so small an occupation as not to affect the opacity.
(I) John kage: Sinilarity Law for Opacity of Light Elements. Unpublished.

We next examine the continuous opacity of two different elements under conditions such that $\alpha^{\prime \prime}$ and $\gamma$ are the same for each. From (6.25) we can see that the free-free absorptions $D_{f f}$ will have the same contribution to $A \wedge_{c}$ in each case since (7.4) $D_{f f} \sim \gamma e^{-\alpha^{7}}$.

The same result is also evident for the bound-free transitions, since from (6.28) (7.5) $\quad D_{b f} \sim \sum \frac{N^{b}}{N} \frac{1}{n^{5}} \gamma^{2}$
(The sum extending over all states with $I_{m} / k T \sim \gamma / M^{2}<u$ ) and $N^{b} / N_{0}$ the occupation numbers are functions of $\alpha^{*}$ and $\gamma$ alone. The scattering contribution ( 6.23 ), however, cannot be written as a universal function of $\alpha^{*}$ and $\gamma$, and hence spoils the similarity transformation. If there are any bound electrons at all, however, scattering will play only a minor roll, while in the absence of bound electrons, the calculation of the opacity coefficient is simple enough so that no resort need be had to similarity transformations. We an say therefore that in most cases $A A_{c}$ of (6.18) is a function of $\alpha^{*}$ and $\gamma$ alone

From ( $6.25 a$ ) we see that if $\alpha^{k}$ is the game in two cases then we must have approximately

$$
\begin{equation*}
\frac{z_{1}}{M_{1}} \frac{\rho_{1}}{T_{1}^{3 / 2}}=\frac{z_{2}}{M_{2}} \frac{\rho_{2}}{\bar{T}^{3 / 2}}, \tag{7.6}
\end{equation*}
$$

or since $Z_{1} / M_{1} \sim \psi_{2} / \mu_{2} \sim 2$ for light elements except hydrogen

$$
\begin{equation*}
\frac{\rho_{1}}{\rho_{2}}=\left(\frac{T_{1}}{T_{2}}\right)^{3 / 2} \tag{7.7}
\end{equation*}
$$

But if $\gamma$ is the same in the two cases
(7.8)

$$
\frac{T_{1}}{T_{2}}=\frac{z_{1}{ }^{2}}{T_{2}^{2}}
$$

so that

$$
\begin{equation*}
\frac{\rho_{1}}{\rho_{2}}=\frac{z_{1}^{3}}{z_{2}^{3}} . \tag{7.9}
\end{equation*}
$$

Using (7.8) and (7.9) we can find the temperature and densities for which two different elements will have the same values of $A A_{c}$. Moreover since from (6.21) $A \propto \beta \rho /\left(z^{2} M\right)$ wet that, under these oonditions,

$$
K_{c,} \propto\left(Z_{1}^{2} M, A \Lambda_{c}\right)^{-1}
$$

$$
\begin{equation*}
\frac{K_{c}}{K_{c_{2}}}=\frac{z_{2}^{2} M_{2}}{Z_{1}^{2} M_{1}} \sim\left(\frac{Z_{2}}{Z_{1}}\right)^{3} . \tag{7.10}
\end{equation*}
$$

The line absorption contribution does not lend itself to a similarity treatment. The most important single factor in the effect of lines is the spread of each group. This spread, being due to electrostatic and exchange interaction, is proportional to $之$. If we keep $z^{2} /(k T)$ constant, then, on the frequency scale $u=k \nu /(k T)$ the spread of the groups will be inversely proportional to $\&$. Hence the line contribution will be more important for light elements than for heavy elements under conditions of similarity for the continuous opacity.

Relativity effects are to first order proportional to $Z^{\mu}$. This again spoils the similarity transformation, even when lines are unimportant. It also spoile any attempt to soale the line contribution separately. This is just another example of the qualitative difference in opacity calculations between the light and heavy elements.

## 2. Liniting values for the opaity.

At very high temperatures, all the eleotrons will be ionized so that the only processes contributing to the opacity are the free-free transitions and the scattering. Since scattering is proportional to the number of electrons per unit volume, while the free-free transitions are proportional to the square of the eleotron density, the former will be dominant at low densities, the latter at high. We examine the results to be expected under these extreme conditions.

If there are to be no bound electrons, the quantity $\alpha^{*}+\beta \bar{\varepsilon}_{l z}^{\prime}$ must be much greater than unity for all states. Putting $\bar{\varepsilon}_{\ell / 2} \sim-\Sigma^{2}$ the and using (6.6) for $\alpha^{*}$, this condition gives approximately

$$
\begin{equation*}
\ln \frac{1}{\rho}\left(\frac{k T}{R h c}\right)^{3 / 2}-z^{2} \frac{R h c}{k T}>=1 \tag{7.11}
\end{equation*}
$$

This result clearly indicates that increasing temperature favors increased ionizetion. But it also predicts that increasing the density always decreases the ioniation. This is not altogether true; the reason our result is misleading is our neglect of the interaction terms in $\bar{\varepsilon}_{l=}^{\prime}$. These give a pressure ionization for high densities.

In the low density, high temperature limit, where the only process contributing to the opacity is scattering wo get

$$
\text { (7.12) } \Lambda_{\text {seat }}=\frac{1}{\rho K}=\frac{75}{4 \pi^{4}} \int_{0}^{\infty} \frac{u^{4} \epsilon^{u}\left(\epsilon^{4}-1\right)^{-2} d u}{\frac{M}{V} \varphi_{0}\left(1-\frac{3 k T}{m c^{2}}\right)\left(1-2 u \frac{k T}{m c^{2}}\right)} \text {. }
$$

The denominator is independent of frequency except for the correction factor $\left(1-2 u \frac{k T}{m c^{2}}\right)$. We may replace $u$ by $u_{\text {make }}{ }^{4}$, the value maximizing the integrand in (7.12) without fear of serious error. Then
(7.13) $\Lambda_{\text {scat }}=\left\{\frac{n}{v} \phi_{0}\left[1-1 / \frac{k T}{m c^{2}}\right]\right\}^{-1}$,
or

$$
\text { (7.14) } K_{\text {scat }}<\frac{1}{\rho n}=N_{0} \psi_{0}\left[1-\| \frac{k T}{\left.i^{2 u c^{2}}\right] \frac{\bar{\zeta}_{1}}{\bar{M}_{b}}}\right.
$$

Where $\sum_{Z}^{M} \frac{N Z}{N}=$ ere molecular weight $\bar{Z}=\sum_{Z} \frac{N_{2}}{N}=$ average atomic number

$$
\begin{aligned}
& f_{0}=\text { Thomson oross-ssotion }=.5654 \times 10^{-24} \mathrm{om}^{2} . \\
& N_{0}=\text { Avogadro's number } 6.023 \times 10^{23}
\end{aligned}
$$

In this limit the opacity is independent of density and virtually independent of temperature. Moreover for the light elements $Z / i k y=1 / 2$ so that except for hydrogenous material the opacity is independent of composition.

In the hi th density high temperature region free-free transitions beoome the dominant process. Then using $(6.18)$ and $(6.25)$
(7.15)

$$
A \Lambda_{c}=\int_{0}^{\infty} \frac{w(u) d u}{\frac{R h c}{R T} \sum_{z}^{1} \frac{N_{z} Z^{2}}{N} e^{-\alpha \theta} \bar{g}_{f f}}
$$

The only frequency dependent factor in the denominator is the slowly varying Eff We replace $u$ by $u_{\max }{ }^{2} 7$ in this factor. Then wo can use $\int_{0}^{\infty} w(u) d u=S(\infty)$ giving. (7.15) $A \wedge_{c}=$ 196.5

$$
\frac{R k c}{k T} c^{-\alpha^{2}} \sum_{2}^{\prime} \frac{N_{2}}{N} z^{2}\left(1+.588\left[\frac{k T}{\operatorname{khc} Z^{2}}\right]^{1 / 2}\right),
$$

or putting in the value of $\alpha^{*}$ from (6.25a) wo get

$$
\begin{equation*}
k=\frac{1}{\rho \Lambda_{c}}=\frac{A}{\rho} \frac{4 \pi^{3 / 2}}{196.5} \frac{\bar{Z}}{A} \rho N_{0} a_{0}^{3}\left(\frac{R A c}{R e T}\right)^{5 / 2} \sum \frac{N_{2}}{N} z^{2}\left[1+.588\left(\frac{k T}{R A_{c} 4^{2}}\right)^{1 / 3}\right] \text {, } \tag{7.17}
\end{equation*}
$$

or inserting the value of $A / p$ from (6.21), we have in cotes. units (7.18)

$$
K=4.815 \times 10^{4} \frac{\bar{Z}}{M^{2}} \rho\left(\frac{R h c}{R T}\right)^{V / 2} \sum_{-2}^{T} \frac{N_{2}}{N} z^{2}\left[1+.588\left(\frac{k T}{R h c z^{2}}\right)^{1 / 3}\right] .
$$

In contrast to the scattering, the free -free opacity is proportional to the density, varies almost as $T^{-7 / 2}$ and is also approximately proportional to $Z$. It is because of this limiting form that astronomer have used the temperature dependence of the opacity as $T^{-7 / 2}$. But the limitations of this law should be clearly understood.
3. Variation of Opacity with Temperature, Density and Atomic Number.

The qualitative variation of the opacity with various factors is evident to a considerable extent by a consideration of the equations, although the quantitative ovaluation requires an immense amount of computation. Consicor first the varistion of opacity with density. At the lowest density the only importent process is acattering. So long as this is true the opectiy is independent of density. But as the density increases, the $K$ electrons become bound with apprecaible probability, unless the temperature is too high. For temperatures at which the $K$ electrons can become bound $\mathrm{kT}\left(\frac{1}{10}\right.$ to $\left.1 / 3\right) z^{2}$ Rho, the ionization limit of the Felectrons is low enough so that their photoelectric absorption will ocour in a region where there is considerable radiation. Their absorption will, therefore, be very importent in reducing tre transmission. For low densities the $K$ ocoupation will increase proportionally to the density; then $A \Lambda_{c} a^{-1}$ and Kap. Superimposed upon this linear variation of the contiruous opacity is the effect of lines. So long as only $K$ electrons are bound, this is usually a snall effect, ince the line spectrum is simple. As the density further increases, the $K$ shell becomes full. The increase ir density then can cause no further increase in $K$ occupation. For a short while, therefore, the opacity is af, in independent of tho density. But then the $L$ shell begins to fill. Initially the occupation of this shell again varies as and it may well be that considerable radiation is in the frequencies whioh they will absorb photolectrically. For awhile, therefore, we again have the continuous opacity $K_{c}$ aP. However, eventually tre $L$ Bhell will be filled and then $K_{c}$ will be independent of $\rho$. In this case, in contrast to the filling of the $K$ shell, the lines are very often of decisive importance. The line spectrum will be very simple for practically empty $L$ shell, inorease to maximum complexity as 4 or 5 electrons become bound, and then decrease in complexity as the shell fills up full. The interplay of the line and continuous process is shown schematically in the figure.


The ratio If, will have a broad maximum near the density at whioh the $L$ shell is half full. This pattern of behavior is repeated but much less distinctly as the higher shells fill in. Finally, however, inoreasing density "cuts off" more and more bound state. Eventually all the elections are pressure ionised and the determining process is free-free scattering. The opacity then increases proportional to the density. It is doubtful, however, that the methods outlined in this paper are adequate to cope in detail with the free-free absorptions at such high densities, since they are based upon hydrogenic wave functions for the electrons.

The variation with temperature is much more complicated than with density, because, not only do the ocoupation numbers change, but the frequency region in whioh the maximum radiation is present also changes. Av usua $\perp$ densities a shell will begin to fill up at a temperature from $1 / 10$ to $1 / 3$ times its ionization energy, and thus the absorption from the latest bound shell is always of decisive importance in determining the opacity. The occupations vary as exp-( $\left.T_{m} / k r\right)$, the bound free absorption coefficient as $D_{b f} \sim\left(I_{\mu} / k T\right)^{2}$ esp $-\left(I_{\mu} / k T\right)$ and hence $A \Lambda \sim\left(k T / I_{n}\right)^{2} \exp \left(I_{\mu} / k T\right) \quad$ - Fivally $\kappa_{c}=1 /\left(\rho I_{c}\right) \propto \frac{I_{m}^{2} R h c}{(k T)^{3}} \epsilon^{-I_{m} / k T}$ i.e. $K_{c} \propto T^{-3}$ exp $-($ conat $/ T)$. This variation is quite a bit more rapid than the $T^{-7 / 2}$ law for free-free processes. In addition the line contribution must be superposed upon tinis.

The variation with atomic numbers is related to that with temperature indeed we discussed a similarity transformation with the parameter $z^{\prime \prime}$ khc/k? Hence roughly $K_{c} \propto Z^{4}$ syp (cony $\frac{2}{2}^{2}$ ) - The line offects are important but too varied to discuss in brief. As was mentioned previously the lines ohange their qualitative behavior when relativity offects become important, Inat is for high $\#>60$ ot 70. A comprehensive discussion of the high \% phenomena is given by Jacobsohn.

## VIII Opacity of Iron

To illustrate the principles of the preceding chapters, the calculation of the opacity of pure iron, $z=26$, at normal density $e=7.85 \mathrm{grams} / \mathrm{cm}^{3}$ and at a temperature of $k T=1$ kilovolt will now be disoussed in detail.

## 1. Oocupation Numbers

The data for the calculation are

$$
\begin{array}{ll}
\mathrm{z}=26 \\
\mathrm{e}=7.1 \mathrm{grams} / \mathrm{cm}^{3} \\
& \mathrm{M}=55.85 \mathrm{grams}=\text { molar we ight } \\
& \frac{\mathrm{kT}}{\text { Rho }}=73.49
\end{array}
$$

Using the above data and (6.2) we determine the radius of the ion apheres - a particularly simple job in the case of a pure element. From (6.2)

$$
v=\frac{4 \pi}{3} N a_{Z}^{3}, ; \quad\left(\frac{a_{Z}^{1}}{a_{0}}\right)=\left(\frac{3}{4 \pi} \frac{v}{N a_{0}^{3}}\right)^{1 / 3}
$$

But
(c.2) $\frac{V}{N a_{0}^{3}}=\frac{M}{N_{0} \rho a_{0}^{3}}=79.75 \quad N_{0}=$ Avogadro's numbor
:Jenco
(8.z) $\frac{a_{z^{\prime}}}{a_{0}}=2.670$

The successive approximation procedure nust now be used to find the actual occupation nuriber. The last oycle of the approximation is sumnarized in the accompanying table ${ }_{0}$ Table A. Column (1) gives the orbital (specified by $l$ ), column (2) the degeneracy $C_{l Z}$ c and colume (3) the assumed values for the occupation. The sum $\sum_{l} \frac{n_{l} Z}{N}=\frac{n_{b}}{N}=2.225$ at the foot of the column gives the number of bound electrons, and hence $n_{f} / N=\frac{n-x_{b}}{s}=23.075$.
$\mathcal{E}_{l}^{0}$ (2*) can be determined by the use of (6.7) and irterpolation in Table III, the table of energy levels. This is recorded in column (4). With a knowledge of $a_{z}$ and $Z^{\prime}=z-\frac{K_{l}}{n_{l} Z} \frac{N}{N}=26-2.9251=23.075$ we can compute the correction terms in ( 6.8 ),
and the final value of $\overline{\mathcal{E}}_{2}$ ' is recorded in column (5) 。 Whenever this energy becomes positive we know that the bound state has been cut off, and merged with the free states. This happens for levels with $n>4 . \quad \alpha^{*}$, determined by ( 6.6 ), has the value 4.5826 so that by the use of ( 6.9 ) a final set of occupation numbers, listed in column (6), is obtained. The value of $n_{0} / \mathbb{N}=2.024$ is in olose agreement with $n_{b} / \mathrm{N}=2.925$ assumed in column (3) indicating that the sucoessive approximations have satisfactorily converged. It will be noted that only the $K$ shell is substantially occupied under these conditions.

The ionic occupation numbers may now be found. The fraction of ions having electrons in the lth shell is recorded in Table B following. More important for our later calculations is the number of $K$ electrons in ions of a given confjguration. This number is given in Table C.

Table A Occupation Numbers of Iron $2=26$
$e=7.85 \mathrm{grams} / \mathrm{cm}^{3} \mathrm{lT}=1$ xilovolt

| Level <br> l | Weight <br> ${ }^{c}{ }_{l}$ | Occupation $\frac{n}{n}$ | $-\frac{\mathcal{E}_{l}^{0}\left(Z^{*}\right)}{\text { Rhe }}$ | $\begin{gathered} \text { Ionization } \\ \text { energy } \\ -\frac{\overline{\mathcal{E}}^{\prime} \ell z}{\text { Phe }} \end{gathered}$ | Occupation $\frac{n_{l Z}}{N}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 18 | 2 | 1.9541 | 643.3 ? | 612.22 | 1.9540 |
| 28 | 2 | .09229 | 115.09 | 114.07 | .09214 |
| $2 \mathrm{p} \frac{1}{2}$ | 2 | . 03826 | 141.63 | 110.64 | .03813 - 8541 |
| 2 p 2 | 4 | . 17403 | 140.60 | 109.56 | . 2738 |
| 38 | 2 | . 03113 | 62.595 | 31.934 | . 03110 |
| $3 \mathrm{p} \frac{7}{2}$ | 2 | -03069 | 61.564 | 30.351 | . 03065 |
| $3 \mathrm{p} 3 / 2$ | 4 | . 06113 | 61.9 .60 | 30.547 | -6E126 62741 |
| 3d $3 / 2$ | 4 | . 06063 | 60.754 | 29.934 | .06055 ${ }^{-2441}$ |
| 3d5/2 | 6 | . 09082 | 60.666 | 29.336 | - $¢ \subseteq 0 \sim 1$ |
| 4 | 32 | . 34215 | 33.907 | 3.979 | - 5113 |
| Bound Plectrons $\sum_{E} \frac{M \ell Z}{N}=2.0252$ |  |  |  | ns $\sum_{Z} \frac{M \cdot l}{N}$ | $=2.024$ |

## Table $B$ Ionic Occupations of Iron $Z=26$

$\mu=7.85 \mathrm{grans} / \mathrm{cm}^{3} \quad \mathrm{kT}=1 \mathrm{kilovolt}$

| Shell | Fraction of ions with the following number of electrons in the shell |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 0 | 1 | 2 | 3 | 4 | 5 |
| K | . 200529 | . 04494 | . 95453 |  |  |  |
| L | - 59616 | . 25794 | . 241808 | . 203873 | . 000224 | neg. |
| M | . 75864 | . 21117 | . 027759 | 0022894 | .000133 | neg. |
| N | . 70922 | . 24.509 | . 041025 | . 004431 | .000347 | neg. |

Table $C$ Ionic Occupation of Iron $Z=26$
$e=7.35 \mathrm{grams} / \mathrm{cm}^{3}$
$\mathbf{k T}=1$ kilovolt

| Shell | Number of K electrons per <br> atom in ions having the follod- <br> ing number of electrons in the |  |  |
| :---: | :---: | :---: | :---: |
|  | 0 | 1 | 2 |
|  | 0 | .04494 | 1.9091 |

## 2. Thermodynamio Properties

Using the occupation numbers found in the preceding seotion and the formulae of Appendix I it is a simple matter to compute the energy and the pressure. The various terms are given in the following table.

$$
\rho=\frac{\text { Table D Energy and Pressure for Iron }}{7.85 \mathrm{~km} / \mathrm{cm}^{3}}
$$

| Binding energy of ions | $\frac{\mathrm{E}_{\mathfrak{b}}}{\mathrm{N} \text { Rho }}\left(A I_{.17}\right)$ | - 1381.6 |
| :---: | :---: | :---: |
| Potential energy | $\frac{P_{0} E_{0}}{\mathbb{R} \mathrm{RhC}_{C}}\left(A I_{.17}\right)$ | - 15.56 |
| Kinotio energy of freo electrons |  | 2550.1 |
| Energy of nuclei | $\frac{E \text { nuolei }}{\mathbb{N h o}}=\frac{3}{2} \frac{k T}{R h c}$ | 110.? |
|  | tal Energy $\underset{\text { F }}{\text { Fho }}$ | 1263.1 |

Free electrons
PV
NRW.
1695.0 $-5.19$

Muclei
$\frac{\text { PV }}{N \text { Rhe }}=\frac{\text { RT }}{\text { Rho }}$
73.49

Total pressure PV 1764.3

N Rho
PV
$(2+1)$ NET
,8897

For this oalculation the nuc lei were treated as free, because $s=\frac{Z r_{e}}{\sqrt{2 k T_{0} i^{\prime}}}=1.65 \ll 3.4$
(c.f. Appendix I). We note that the material does not behave like a perfeot gas of $(z+1) N$ particles, but instead behaves as if only $89 \%$ of the particles were completely free. Furthermore this percentage will vary rapidly with temperature and somewhat more slowly with density. With regard to the energy, we can see that the potential energy and nuclear energy terms are small corrections; it is not necessary to further refine our treatment of them. To the other terms we may attach a high degree of reliability. It is also worthy of note that the kinetic energy is almost twioo the total ionization energy of the remaining bound electrons.

## 3. The Continuous Opacity

The caloulation of the scattering and free-free absorption contributions is a straightforward application of ( 6.23 ) and ( 6.25 ). The bound-free contribution requires some care, however. We first make a preliminary calculation which assumes that all the ions have the same ionization potentials for a particular one electron transition. This is the average ionization potential $-\bar{\varepsilon}_{Z}$.ecorded in Table A oolumn 5. We notice that the subshells $2 \mathrm{~s}, 2 \mathrm{p} \frac{1}{2}, 2 \mathrm{p} 3 / 2$ have very nearly the same ionization potential and for simplicity we take an occupation weighted average value $-\overline{\mathcal{E}}_{2}^{\prime}=111.0$ Rhc. Similarly for the $n=3$ shell, we use $-\overline{\mathcal{E}}_{3}^{\prime}=30.36$ Rhc. Table $D$ summarizes the oontinuous opacity results. In column 1 the frequency $u=\frac{h V}{k T}$ for which the reduced absorption coefficient $D(u)$ is to be oalculated is given. Entries with an asterisk are the ionization frequencies. The contribution of the various absorption processes is given in on'גmns 2 to 7 and the total $D(u)$ in oolumn 8. Column 9 gives the difference $\left.\Delta S=S\left(u_{k} \dot{+}\right)_{2}\right) S\left(u_{\mathbf{k}}\right)$ of the Strongren functions for the interval between the two frequencies. This is a measure of how important the contribution of the region may be. In the final column is given $\Delta S / D(\bar{u})$, the contribution to the reduced mean free path $A \Lambda_{0}$ of the region.

For the very lowest frequencies, the only important process is free-free absorption. It is only above the ionization potential of the $L$ eleotrons that the bound-free absorptions become of dominant importanoe. Even here the free-free oontribution is $16 \%$ while APPROVED FOR PUBLIC RELEASE
the soattering contribution is $\sim 10 \%$ of $D(u)$. Sinoe it is the region between the $K$ and L ionization limits whioh, according to the last columa, contributes practically $100 \%$ to the mean Pree path, it is not possible to dismiss scattering and freo-free absorption lightly. Above the $K$ ionization limit, hoverers the bound-free absoiption of the $K$ eleotrons alone is sufficient to black out the region, so the other processes need hardly be considered.

In the $K$ to $L$ window the Gaunt factor for the $L$ eleotron absorption varies from .875 at the $L$ edge to 1.05 at the $K$ edge. The error made by putting the Gaunt factor exactly equal to unity, that is, by using the old Kramer's formula is not large.

The neighborhood of the $K$ edge is ons in whioh the weighting funotion $\mathbb{F}(u)$ is particularly large, and in which $D(u)$ take a very large jump. The details of the absorption in this region will therefore have a considerable effect on the final transmisgion. We therefore return to our ionic pioture for an accurate treatment of this region. The results are sumarized in Table E. The first column gives the ion type, that is the number of eleotrons in each of the $K, L, M$, and $N$ shells. The second column gives the number of $I$ eleotrons per atom of the system, which are in the various ions. The fourth column gives the contribution to the absorption coefficient of each ion type at the ionization frequency whioh is given by column 3. Sinoe the Gaunt factor varies slowly Fith frequency, we may assume these contributions do not ohange within the region oocupiea by the $K$ edges of the various ions. The next column (5) gives the absorption coefficient and the final column $\Delta S / D(\bar{u})$ the contribution of the region to $A A_{0}$, the reduced mean free path. Whereas this region gave a contribution 15.37 in the rough treatment of Table $D_{\text {, }}$ we now get .939, reducing the total transmission to $A \wedge_{0}=186.31$ a change of $7.7 \%$. Such a large change is only to be expeoted for even the few ions which have small ionization potential still have sufficient streagth in their $X$ absorption to wipe out most of the transmission left by the weak $L$ absorptions.

To avoid all the labor of this detailed calculation we could use a statistical treatment of the edges. This gives a transmission in the region of the edges as . 565 instead of the accurate result . 306 . The agreement within the region of the edges cannot be considered too good, but this is unimportant for the error in the total transmission is only $0.13 \%$.

TABLE E

CONTINUOUS OPACITY OF IRON Z』26

table $\mathrm{E}^{\text {s }}$
CONTRIBUTION TO THE CONTINOOUS OPACITY NEAR THE K EDGE: IONIC TREATMENT

| Ion Types <br> K L M N | Eleatron in Ion Ocoupation | $I_{n} / \mathrm{KI}$ or $u$ | $\begin{aligned} & \text { Contribution } \\ & \text { to } \mathrm{B}_{\mathrm{bf}} \end{aligned}$ | $D(u)$ | $\frac{\Delta_{\mathrm{s}}}{\mathrm{D}(u)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 9.0 |  | 109.3 | . 05078 |
| $1 \times x \times$ | . 04494 | 8.7630 | 2.7514 | 108.56 | . 08138 |
| 2000 | . 7151 | 8.4145 | 40.367 | 105.80 |  |
| 2001 | . 2471 | 8.3713 | 23.807 | 65.43 | . 01772 |
| 20010 | .1990 | 8.3358 | 13.318 | 51.63 | . 01691 |
| 20002 | . 0414 |  |  |  | . 02761 |
| 20011 | . 06878 | 8.2950 | 3.773 | 38.31 | . 02499 |
| 2020 | . 02617 | 8.2619 | 1.424 | 34.53 | . 00900 |
| 2100 | . 2649 | 8.2505 | 14.379 | 33.11 | . 05850 |
| 21001 | . 09156 | 8.2088 | 4.919 | 18.73 | . 06523 |
| 2110 | . 07375 | 8.1747 | 3.929 | 13.809 | . 2335 |
| 2200 | . 04295 | 8.0881 | 2.240 | 9.875 | . 3534 |
| Remainder | . 13836 | 7.9873 | 7.038 | 7.635 |  |
|  |  |  |  |  | $\Sigma=9390$ |

From our results on the continuous opacity, we see that the only important spectral region is that between the Lionization limit $u=1.510$ or $h=111.0$ Rho and the lowest lying edge of the $K$ ionization limit $u-7.987, h+587.0$ Rhe. The only lines in this region will be those in which a ls electron is raised to an exoited state, and wo may restrict our considerations to this rather simple portion of the speotrum.

## a. Position of the Lines

At first we examine the grose struoture of the line speotrum by considering only the spherically symetric part of the eleotrostatic interactionse Later we shall see that the splittings of the lines are unimportant. In order to get the positions of the lines in the correct relative positions to the absorption edges, it is easiest to compute the lines by the difference in ionization potentials of the eleotron involved in the transition. In an ion with $x_{j}$ eleotrons in the $j$ th level, the ionization potential of an electron in the kth level is approximately

$$
\begin{align*}
I_{k} & =-\mathcal{E}_{k}^{0}(Z)-\sum_{j \neq k} x_{j} F^{0}(j, k)-\left(x_{k}-1\right) F^{0}(k, k)  \tag{8.4}\\
& =\frac{z^{\prime} e^{2}}{2 a_{Z}^{\prime}}\left\{3-\left(\frac{r}{a_{Z}}\right)^{2} k\right\}-3 / 5 \sum_{Z} \frac{N_{z}}{n_{f}} \frac{z^{\prime 2} e^{2}}{2 a_{Z}^{\prime}}
\end{align*}
$$

The frequenoy of the line resulting from the one electron transition $k \rightarrow \ell$ is $(8.5)$

$$
\begin{aligned}
& I_{k}-I_{l}=-\varepsilon_{k}^{0}+\varepsilon_{l}^{0}+\cdots-z^{\prime} \frac{e^{2}}{2 \alpha_{0}}\left(\frac{a_{0}}{a_{z}}\right)^{3}\left\{\left(\frac{r}{a_{0}}\right)^{2^{l}}-\left(\frac{r}{a_{0}}\right)^{2}\right\} \\
& +\left\{\left(x_{k}-x_{l}-1\right) F^{0}(l, k)-\left(x_{k}-1\right) F^{0}(k, k)+x_{l} F^{0}(l, l)\right\} \\
& \quad-\sum_{j \neq k} \quad x_{j}\left[F^{0}(j, k)-F^{0}(j, l)\right] \\
& \quad j \notin l
\end{aligned}
$$

The firgt two terms are the frequency of the one eleotron line in the isolated ions the next tern is the lowering of the frequenoy caused by the eleotrostatic shielding of the
free electrons. This latter term is a oonstant for all ions with the same nuclear oharge. The next set of terms in (8.5) give the interaction with electrons in the same shell as the initial and final shells of the transition electron. The final set gives the eleotrostatic screening of the remaining bound electrons. This last contribution is additive for the electrons of any ion undergoing partioular transition. In Table $F$ are recorded the first two sets of terms of (8.5), that is the frequency of a one electron line in the free electron atmosphere.

Table F

| Transition | Frequenoy of Line in Isolated Ion $\mathrm{h} / / \mathrm{Rho}=\frac{-\mathcal{E}_{\mathrm{k}}^{0}+\mathcal{E}_{l}^{0}}{\text { Rhc }}$ | Screoning of Frees $z^{\prime}\left(\frac{a_{0}}{a_{z}},\right)^{3}\left\{\left(\frac{\bar{r}}{a_{0}}\right)^{2 l}\left(\frac{r}{a_{0}}\right)^{k}\right\}$ | Frequency of Line in free electron atmosphere $\mathrm{h} / \mathrm{Rhc}$ |
| :---: | :---: | :---: | :---: |
|  | 511.25 | . 048 | 511.20 |
| $1: \rightarrow 2 p^{3} / 2$ | 512.81 | . 048 | 512.76 |
| $1 \mathrm{~s} \rightarrow$ 3p $\frac{1}{\text { 咅 }}$ | 606.39 | . 317 | 606.07 |
| $1: \rightarrow 3{ }^{3} / 2$ | 606.85 | .317 | 606.53 |
| 1s $\rightarrow 4 \mathrm{p} \frac{1}{2}$ | 639.63 | . 941 | 638.69 |
| ls $\rightarrow 4 \mathrm{p}^{3} / 2$ | 639.82 | -941 | 638.88 |

If we neglect the small differences in soresning of ns, np, nd etc. electrons we can readily express the contributions of the last two sets of terms to the line frequency. The following Table $G$ gives the energy in units of $2 Z$ Rhe which must be subtracted from the values in Table $F$ for each bound electron in the ion in addition to the transition electron.

Table $G$ Contribution of Additional Bound Electron (Units of 2Z Rho)

| Shell of Additional Bound Eleotron |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Transition | $\mathrm{n}=1$ | $\mathrm{n}=2$ | $\mathrm{n}=3$ | $\mathrm{n}=4$ |  |
| $1 \mathrm{~s} \rightarrow 2 \mathrm{p}$ | .3822 | .0578 | .0090 | .0029 |  |
| $1 \mathrm{~s} \rightarrow 3 \mathrm{p}$ | .5162 | .1416 | .0347 | .0086 |  |
| $1 \mathrm{~s} \rightarrow 4 \mathrm{p}$ | .5633 | .1758 | .0559 | .0177 |  |

As mentioned above the contributions of the additional bound electrons in the ion are additive.

With the use of Tables $F$ and $G$, we prepare a list, Table $H$, of the stronger ls $\rightarrow n p \frac{1}{2}$ lines and their frequencies. The $1 s \rightarrow n p 3 / 2$ line will be split from their ls $\rightarrow n p / 2$ counterpart by just the splitting of Table $F$. We can see that for each transition the lines form into 2 groups, one formed from ions with a single $K$ electron, the other from ions with 2 K eleotrons. The $\mathrm{p} \frac{\mathrm{z}}{\mathrm{Z}}, \mathrm{p} / 2$ splitting, and also the electrostatic splitting will not change the group to which a line belongs, but will simply increase the number of lines in each group. The differences within each group are so small that we may well expect them to be smaller than the wingspread of the lines. In that event it is no longer necessary to consider all the details of each line. We therefore stop to consider the dispersion of the lines. b. The Doppler Breadth.

The energy half breadth for Doppler broadening is given by (5.29). The calculation for iron at $k T=1000$ volts gives to lines of frequenoy $\sim 7 \mathrm{KT}$ (the $\mathbb{K}$ electron lines) a half breadth of .0836 Rydberg units, or $\Gamma=\frac{h \gamma}{K 1}=1.138 \times 10^{-3}$. c. The Natural Breadth:

In opacity oalculations, the natural breadth phenomenon differs in two important respects from the ordinary case encountered in optical speotra. First the radiation density is so high that absorption and induced emission processes as well as spontaneous emission oontribute to the breadth of the levels. Second the atoms are so highly ionized that

TABIE H
Spectrum of $K$ Eleotron Lines (Fho unite)

many transitions contribute to the breadth.
In general the onergy half breadth at half maximum is $\pi \bar{\gamma} / 2$, and this breadth is the sum of the breadths of the initial and final states of the transition. The half breadth of a level is

$$
\begin{equation*}
\Gamma_{i}=\frac{\alpha \gamma_{i}}{2 \mathrm{KT}}=\frac{\alpha^{3}}{2} \frac{\mathrm{KT}}{\operatorname{Rhc}} \sum_{j} \frac{u_{i j}^{2} f_{i j}}{e^{u_{i j}}-1} \tag{8.6}
\end{equation*}
$$

where $u_{i j}=\frac{h_{2 h j}}{k T}$ and $h \gamma_{i j}$ is the energy of the transition $i$ to $j$. Wo note that $u_{i j}$ and $f_{i j}$ are both negative for dowmard transitions. The sum $\sum_{j}$ includes all possible transition which shift the frequency of the line in quastion by more than its breadth.

While (8.6) is very convenient for transitions between disorete states, we can put it into simpler form for transitions to and from the continurm. For transitions from a bound to a free state

$$
\begin{equation*}
\sum_{j} \frac{u_{i j}^{2} e_{i, j}}{e^{u} i_{j-1}} \quad \text { goes over into } \quad \int_{u_{l}}^{\infty} \frac{u^{2} \frac{d t}{d j} d u}{e^{u_{-1}}} \tag{8.7}
\end{equation*}
$$

where $u_{i j}$ is the ionisation potential in units of kT of the $i$ th level. Introduoing the result of (2.22) into (8.7) we get

$$
\sum_{j} \text { continuum }=\frac{2^{4}}{3 \sqrt{3} \pi} \frac{1}{n_{i}} u_{i}^{2} \int_{u_{i}}^{\infty} \frac{g_{i}(u) d u}{u\left(e^{u}-1\right)} .
$$

The maximum value of the integrand occurs at $u=u_{i}$ and we may replace the elowly varying funotion $g_{i}(u)$ by $g_{i}\left(u_{i}\right)$. Then the integral may be expressed in terms of the exponential integral - $E_{i}(-x)$ tabulated, for example, in Jahnke und $\mathbb{E m a}_{\text {m }}$, pp. 6 ff. The oontribution of the bound-free transitions to the breadth is thus


In addition to the bound free transitions, the process of free electron capture contributes to the breadth of a level. In thia oase wust multiply the contribution from a free to a bound state transition by the probability that the free state is occupied, $p \cong e^{-\alpha^{x}-\beta \varepsilon_{f}}=e^{-\alpha^{x}-u^{2}+u_{k}}$, and sum over all free states. Wo get exactly the same
integral as in (8.7) so the resulting contribution of these processes is

$$
\begin{aligned}
(8.9) \Gamma_{i \rightarrow b}= & \left.\sum \frac{\alpha^{3}}{2} \frac{k T}{R h o} e^{-\left(\alpha^{*}\right.}+u_{k}\right)\left\{\frac{2^{4}}{3 \sqrt{3}} \pi \frac{1}{n_{k}} u_{k}^{2} g_{k}\left(u_{k}\right) \sum_{\lambda=1}^{\infty}-F_{i}\left(-\lambda u_{k}\right)\right\} . \\
& \text { bound states } k \\
& \text { (not levels) }
\end{aligned}
$$

The contributions ( 8.8 ) and ( 8.9 ) are generally small compared to that of the bound-bound transitions. In Table I are recorded the natural breadth contributions of the various transitions for the case of iron $Z=26, \mathrm{kT}=1000$ volts and normal density. The natural breadths of the stronger $K$ electron lines is given in table $J$. The natural breadth is much smaller than the Doppler breadth and the collision breadth (see next section) and may therefore be neglected.

## d. The Collission Breadth:

The problem of collision broadening has been analyzed by R. Sternheimer. We quote his results without proof. The energy half breadth at half maximum is analogous to the natural breadth t $\delta / 20$ The breadth is the sum of the initial and final breadths of the states involved in the transition. In turn the breadth of a state is the sum of contributions from all the transitions which the ion can undergo which shift the line more than the breadth. A transition of an electron from an initial state to a final state $j$, both discrete states, gives the contribution to the energy half breadth on an ionic level

$$
\begin{equation*}
\left.\frac{2}{3} \sqrt{2 \pi} \frac{n_{f}}{V^{\prime}} \frac{1}{2^{2}} \sqrt{\frac{m_{0}^{2}}{k I}}\left(\frac{k}{m c}\right)_{m c^{2}}^{3} \frac{\ell_{\max }}{2 \ell_{i}+1} \right\rvert\, R_{n_{i} \ell_{i j}} n_{j} \ell_{j} I_{i j} \tag{8.20}
\end{equation*}
$$

where

$$
I_{i j}=e^{-u_{i j}} \ln \frac{\sin \beta \bar{\varepsilon}_{i z}^{\prime} \mid}{u_{i j}}-E_{i}\left(-u_{i j}\right) \quad \text { if } \quad\left|u_{i j}\right|>\frac{u_{i j}^{2}}{4\left|\beta \varepsilon_{i z}\right|} \frac{\pi a^{\prime}}{k I}
$$

$$
\text { (8.11) } \quad I_{i j}=-\mathbb{E}_{i}\left(-\frac{u_{i j}}{4\left|\beta \bar{\varepsilon}_{i 2}^{\prime}\right|}\right)
$$

$$
\text { if } \frac{u^{2} i_{j}}{4\left|\beta \overline{\mathcal{E}}_{i z}^{\prime}\right|}>\left.\right|^{u_{i j} \left\lvert\,>\frac{\hbar a^{\prime}}{\sum r^{\prime}}\right.}
$$

$$
I_{i j}=\ln \frac{4\left|\beta \xi_{i z}\right|}{\left(\frac{\hbar}{E} a^{1}\right)^{2}}-.577216
$$

$$
\text { if }\left|u_{i j}\right|<\frac{\delta_{2}}{\underline{⿺}}
$$

and, hal the plasma energy is given by (8.12) $\quad$ n $a^{\prime}=\sqrt{\frac{n_{f}}{\nabla} \frac{e^{2}}{\pi m}}=\sqrt{\frac{4}{\pi} \frac{n_{f}}{N^{2}} \frac{N a_{0}^{3 n}}{\nabla}}$ Rho while $\left|R_{n i} l_{i}, m_{j} l_{j}\right|^{2}$ is recorded in Tables by Bethe ${ }^{2}$. We must sum ( 8.10 ) over all final states $j$ available for the transition and all initial state $i$ occupied by electrons in the ion. We then get for the partial energy half breadth of the level due to discrete transitions

$$
\begin{equation*}
\frac{\hbar \gamma \ell}{2 R h c}=\frac{6}{3} \sqrt{\pi} \frac{n_{p}}{\pi} \frac{n_{0}^{3}}{V}\left(\frac{R h c}{2 T}\right)^{\frac{1}{2}} \frac{l_{2}}{2} \sum_{i} \sum_{j} \frac{\ell_{\max }}{2 \ell_{i}+1}\left|R_{i, j}\right|^{2} I_{i j} \tag{8.23}
\end{equation*}
$$

Transitions to and from the free states should also be included as contributing to the breadth. For the former we may use the results of Bethe ${ }^{3}$ quoted by Mott and Mosey. ${ }^{4}$ Fe gives the cross-section for ionization which leads to a collision breadth contribution of

$$
\begin{equation*}
x\left\{\operatorname{erfc} \left\lvert\, \frac{\bar{\varepsilon}_{m \rho} \mid}{k!} \ln \frac{80 \mathrm{kr}}{\left|\bar{\varepsilon}_{m p}^{\prime}\right|}+\chi\left(\frac{\bar{\varepsilon}_{m \Omega}^{\prime}}{k \mid}\right)\right.\right\} ; \tag{8.14}
\end{equation*}
$$

where
(8.15)

$$
\text { orfo } x=1 \text { - orf } x=\frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-y^{2}} d y
$$

$$
\begin{aligned}
& X(x)=\frac{1}{\sqrt{\pi}} \int_{x}^{\infty} y^{-\frac{1}{2} e-y} \ln y d y \\
& \text { is given in the following table. }
\end{aligned}
$$

Table of ${ }^{o_{n g}}$

| State | 18 | 2 s | 2 p | 3 s | 3 p | 3 d | 4 s | 4 p | 4 d | 4 f |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| ${ }^{0} \mathrm{n} \ell$ | 0.28 | 0.22 | 0.13 | 0.17 | 0.14 | 0.07 | 0.16 | 0.13 | 0.09 | 0.04 |

2. Eandbuch der Physios Second Edition, Vol. 24, 1; p. 442
S. Annalen der Physic, 5 (1930) pp. 325 H.
3. Theory of Atomic Collisions, Oxford 1933, p. 182.
-113-
Table Is Contributions to Enorgy Falf Breadth
of Eleatronio Trangition


The integral (8.15) has been evaluated through the good offices of Mr. Bengt Carlson, whose results are contained in Table IX.

Transitions involving the capture of an electron will be relatively unimportant, and transitions from one free state to another will not appreciably alter the frequency of the absorption line and hence must not be considered as contributing to the collision width.

In Table I, the contributions of the various transitions to the collision breadth is given. The largest contributions come from those transitions requiring very little energy change. The contribution of ionizing transitions also is appreciable except for the most tightly bound eleotrons. Values of the breadths for the strong $K$ lines of iron are reoorded in Table $J$ along with the natural breadths.

## O. Stark Broadening

To adequately discuss the Stark Effect broadening of the lines we shall have to refine slightly the treatment given in IV 6 to inolude effects of shielding by the free electrons. As before (0.f. 4.59) the number of ions with effective charge $z^{\prime}$ at a distance ri2 from a particular ion with effective charge $Z^{\prime \prime}$ is

$$
\begin{equation*}
N_{Z}^{\prime}\left(r_{12}\right) d r=\frac{N_{Z}}{V^{\prime}}\left\{e^{-\varepsilon\left(r_{12}\right) / k T}\right\} 4 \pi r_{12}^{2} d r_{12} \tag{8.15}
\end{equation*}
$$

The mutual potential energy $\mathcal{E}\left(x_{12}\right)$ is not given, however, by the eimple form (4.58), but instead by

$$
\begin{aligned}
& \text { (8.16) } E\left(r_{12}\right)=z^{i} z^{n} \frac{e^{2}}{a_{0}}\left\{\frac{a_{0}}{r_{12}}-\frac{3}{2} \frac{a_{0}}{a_{Z^{n}}}+\frac{r_{12}^{2}}{2 a_{0}^{2}}\left(\frac{a_{0}}{a_{z^{n}}}\right)^{3}\right\}^{2} r_{12} \leq a_{z^{n}} \\
& E\left(r_{12}\right) \sim 0, r_{12}>a_{Z^{n}}
\end{aligned}
$$

The elective field due to the invading ion is
(8.17)

$$
\vec{\varepsilon}=\frac{z^{1}}{a_{0}^{2}}\left\{\frac{a_{0}^{2}}{r_{12}^{2}}-\frac{r_{12}}{a_{0}}\left(\frac{a_{0}}{a_{2}^{n}}\right)^{3}\right\}
$$

direoted radially. This field is not uniform as is the oase in the laboratory Stark effect

However we may expand the field in spherical harmonics. Keeping only the leading term, we get the usual uniform field case. The terms we neglect in the development are of the sime order as effects not considered at all in this orude treatment, for example the resultant field due to all neighboring ions and free eleotrons acting as an assemblage of dipoles, and indeed these neglected terma are of about the order of the term considered.

The electric field $\overrightarrow{\mathcal{E}}$ will split and shift the levels of the ion affected, causing a displacement of the absorption line of magnitude.

$$
\begin{equation*}
\Delta=\left(c_{n}+o_{n}^{\prime}\right)|\vec{\varepsilon}|+\left.\left.\left(c^{\prime} n+c^{\prime} n^{\prime}\right)\right|_{\bar{\varepsilon}}\right|^{2}+\ldots \tag{8.18}
\end{equation*}
$$

where the first term gives the linear Stark effect shift, the seond term the quadratic, etc. The quantity $C_{n}|\overrightarrow{\mathcal{E}}|$ is the change in energy of the state $n$ caused by the linear Stark effeot of the field $|\vec{C}|$; for example, in a one electron ion

$$
\begin{equation*}
c_{n}|\vec{k}|=-\frac{3}{2} \frac{a_{0}}{2^{2}} n k e|\vec{k}| k=0, \pm 1, \pm 2 \ldots \tag{8.19}
\end{equation*}
$$

For the higher levels of an ion where the Stark effeot is most important, the linear Stark effect may be present. In this osse the number of ions whioh suffer a shift between $\Delta$ and $\Delta+d \operatorname{cinin}^{2}$ a particular line can be approximated by

$$
\begin{align*}
& \frac{N_{Z}{ }^{n}}{N_{2^{n}}}(D) d(h v)=\frac{N}{V} 2 \pi a_{0}^{3} \frac{\left(C_{n n \prime}\right)^{3 / 2}}{D 5 / 2}\left\{^{\prime}-3\left(\frac{C_{n n_{1}}}{D}\right)^{3 / 2}\left(\frac{a_{0}}{\left.a_{Z}\right)^{3}}+\ldots . \ldots d(h \nu)\right.\right.  \tag{8.20}\\
& x \exp \left\{-2 Z^{\prime} Z^{n} \frac{R_{h o}}{K T}\left[\left(\frac{D}{c_{n n^{\prime}}}\right)^{\frac{1}{2}}-3 / 2\left(\frac{a_{0}}{a_{Z}}\right)+\left(\frac{C_{n n^{\prime}}}{D}\right)\left(\frac{a_{0}}{a_{Z}}\right)^{3} \cdot \bullet \bullet\right\}\right.
\end{align*}
$$

where

$$
(8.21) \quad C_{n n^{\prime}}=\left(c_{n}+C_{n}^{\prime}\right) T^{\prime} \frac{e}{a_{c}^{2}}
$$

This asymptotio development is valid for large $A$, the region in which we are partioularly interested.

Fe cannot use this expression direotly to define a half intensity half breadth, for it diverges for no shift $\Delta=0_{4}$ However, we can compare the number of ions having a shift $\Delta_{2}$
with the number having a shift $\Delta_{2}$ provided both shifts are fairly large. This gives, with neglect of the onrrection terms

Now a great number of ions will be at a distance $a_{2}{ }^{n}$. To the same approximation as (8.22)
these will cause a shift
(8.23) $\quad \Delta_{1}=\left(c_{n}+c_{n^{\prime}}\right) \frac{\overline{z^{\prime}} e}{a_{z^{\prime \prime}}^{2}}=C n n^{\prime}\left(\frac{a_{0}}{a_{z^{n}}}\right)^{2}$

We con define somewhat arbitrarily a half breadth $\boldsymbol{\Delta}_{2}$, by requiring that the ratio (8.22) be 竞 when $\Delta_{L}$ is given by (8.23). This gives
(8.24) $\quad \frac{\Delta_{2}}{c_{n a I}}=\frac{(k T / R h c)^{2}}{4 Z^{2} Z^{n z}}\left\{\ln 2+\frac{2 \bar{Z}^{\prime} z^{n}}{(k T / R h c)} \frac{a_{0}}{a_{Z}}+5 / 2 \ln \frac{\Delta_{1}}{\Delta_{2}}\right\}^{2}$

For the case of iron at $\mathrm{kT}=1000$ volts, $\rho=7.85 \mathrm{gm} / \mathrm{om}^{3}$. This gives $\Delta_{2} / \mathrm{c} \mathrm{nnl}=.162$ or by means of (8.21) and (8.19)
(8.25) $\Delta_{2}=.486\left(n k+n^{\prime} k^{\prime}\right)$ Rho
which corresponds to a field of intensity
(8.26) $|\mathcal{E}|=.162 \frac{z^{\prime} e}{a_{0}^{Z}} \cdot$

We will get the maximum shift and split from the unperturbed line when $\mathbf{I}=(n-1)$. $k^{\prime}=(n-1)$. For $K$ electron lines this gives a half breadth in Rydberg units as follows

| Final state | $. \Delta_{2} /$ Rho |
| :---: | ---: |
| $n=2$ | .972 |
| $n=3$ | 2.92 |
| $n=4$ | 5.83 |

To test whether our approximation of the linear Stark shift is correct, we examine the splittings of the upper states of our ions. For example, we have the following deviations from the zero order energy.

Table J
Energy Half Breadths of Strong K Eleotron Lines (Rho Units)

| Transition | Doppler <br> Breadth | Natural <br> Breadth | $2 \times \text { Collision }$ | Stark Breadth |
| :---: | :---: | :---: | :---: | :---: |
| $1 \mathrm{~s} \rightarrow 2 \mathrm{p}$ | . 084 | .00833 | .0599 | . 5 |
| $(1 \mathrm{~s})^{2} \rightarrow(1 \mathrm{~s})(2 \mathrm{p})$ | .084 | . 00466 | . 0599 | -5 |
| $(1 s)^{2}(2 s) \rightarrow(1 s)(2 s)(2 p)$ | . 084 | .0576 | . 1710 | - 5 |
| $(1 s)^{2}(2 \mathrm{p}) \rightarrow(1 \mathrm{~s})(2 \mathrm{p})^{2}$ | .084 | .00937 | .1796 | . 5 |
| $(1 s)^{2} \rightarrow(18)(3 p)$ | . 084 | . 0021 | . 818 | 2.9 |
| $(1 s)^{2} \rightarrow(1 s)(4 p)$ | . 084 | . 0044 | 2.10 | 5.8 |

Contribution of Additional Electrons to Collision Breadth

| Shell of <br> Additional Eleotron | $x$ Contribution <br> to Half Breadth Rho |
| :---: | :---: |
| 2 | 1.0941 |
| 3 | 2.176 |
| 4 | 2.49 |


| State | Energies in ${ }_{\text {R }}$ |  |  |
| :---: | :---: | :---: | :---: |
|  | $n \geq 2$ | Q 123 | n ${ }^{1} 4$ |
| (18) (ns) $1_{8}$ | .2318 | . 2053 | .060 |
| (18) (ns) $3_{8}$ | .1879 | .0937 | . 0552 |
| (1s) (np) $1_{P}$ | -2599 | . 1108 | . 0824 |
| (18) (np) $3_{p}$ | . 2257 | . 2.068 | .0607 |

For the $n=4$ state the largest splitting that affeots the oaloulation is 286 Rhe and for $n=3$ it is 0681 Rho so we are quite safe in using a lipear Stark effect for these lines. For $n=2$, however, the spilt is 1.97 Rhe compared to a linear Stark effect shift of .972 Rhc, so must use the formula for the quadratic Stark effect. This will give a result of approximatelyO. 5 Rho.

## 2. Treatment and offect of the Lines.

A glance at Tables $D$ and $\mathbb{F}$ show that only those lines with energies below 590 Rhe can affeot the tranemission appreciably. Referring to Table $H$ we find that three groups of lines $-1 s \rightarrow 2 p$ with $1 \mathbb{K}$ oleotron, $1 s \rightarrow 2 p$ and $1 s \rightarrow 3 p$ with $2 K$ eleotrons have appreciable strength below this limit.

From the table of line breadths, Table $J$, we see that the lines will be sufficiently broad so that different components of a line caused by electrostatic and spin interactions will overlap. Furthermore most of the lines from the different ions will overlap. Because of this extensive overlapping the smearing out formula will apply. In its most refined form, this approximation is given by (5.74) which correotly treats the oontribution of the lines outside of their groups. We have, therefore, for the line contribution (8.27)

$$
\Delta \Delta_{l}=\int_{0}^{\infty} \frac{\pi(u)}{D(u)} \frac{z^{\prime} E_{n k}}{1+\sum_{n k} E_{n k}} d u .
$$

where
(8.28) $\sum_{k} E_{n k}=\sum_{k} \frac{X^{k} f_{k} q_{k}}{2 \Delta_{k} S} g_{k}$ (u).

Here $\prod^{k}$ is the electron in ion occupation number. Table $C$, $f_{k}$ the one electron $f$ number, $q_{k}$ the probability that the final state of the transition is unocoupied and $2 \Delta_{k}$ the extent (in units of $h \nu / k T$ ) of the group of lines. The function $g_{k}$ can be obtained from the nomo graphs, Fig. IIIa, and $b$, with
(8.29) $\quad x=\frac{\left|u_{k}^{*}-u\right|}{\Gamma_{k}} \quad y=\frac{\Delta_{k}}{\Gamma_{k}}$.

For values of $x>y$, it is a valid approximation to use

$$
(8,30) \quad g(x, y)=\frac{2}{\pi} \frac{y}{x^{2}-y^{2}+1}
$$

The data for the three groups of lines is collected in the following table.
Table K:

| Group | $\begin{aligned} & \text { Group } 1 \\ & 18+2 p \\ & \text { IK Eleotron } \end{aligned}$ | $\begin{aligned} & \text { Goup } 2 \\ & l_{s} \neq 2 p \\ & 2 \mathbb{X} \text { oleotrons } \end{aligned}$ | $\begin{aligned} & \text { Group } \\ & \text { ls }+3 \mathrm{p} \\ & 2 \mathrm{~K} \text { Elootrons } \end{aligned}$ |
| :---: | :---: | :---: | :---: |
| $u_{k}^{*}$ in Pho | 508.2 | 486.8 | 568.2 |
| Nk | . 04494 | 1.9091 | 1.9091 |
| $\mathrm{f}_{\mathbf{k}}$ | . 4162 | . 4162 | . 07910 |
| $¢_{k}$ | . 9564 | . 9564 | . 9847 |
| $2 \Delta k$ in Rhe | 6.02 | 9.02 | 22.09 |
| $r_{k}$ in Rho | . 62 | . 68 | 1.00 |

For the breadths of the lines we have used a strength weighted average collision plus natural breadth for the ions present. Doppler breadth is small enough to neglect. The Stark broadening, though larger than the collision broadening has an exponential diepersion shape, and will not be important much outside the group limits. Within the group limits, it is the Stark breadth which effectively smears out the line strength. The formulae we have used are appropriate for a smeared out group of lines with the collision shape dispersion outside the group limits. Wie may factor out the slowly varying function $\frac{\text { W(u) }}{D(u)}$ - Then the integration (8.27) is done numerically in the neighborhood of the APPROVED FOR PUBLIC RELEASE
center of the group. The region far from the group center can be done analytically, for then

$$
\frac{\sum E_{n k}}{1+\sum E_{n k}} \sim \sum^{E_{n k}} \sim \frac{\sum^{n k_{f_{k}} q_{k}}}{2 \Delta k S} \frac{2}{\pi} \frac{\Gamma_{k} \Delta k}{\left|u_{k}^{*}-u\right|^{2}}
$$

and
(8.31)

$$
\int_{u_{1}}^{\infty} \frac{\sum_{n k} E_{n k}}{\Gamma+\sum_{n k}^{E_{n k}}} d u \sim \frac{\sum^{N^{k} f_{k} q_{k}}}{2 \Delta_{k} S} \quad \frac{2}{\pi} \quad \frac{\Gamma_{k} \Delta k}{\left|u_{k}-u_{1}\right|}
$$

The contribution (8.31), we term the long range tail effect.
The final result for the opacity of iron may now be given.

## Table L

$$
\begin{array}{rlrl}
\mathrm{A} \Lambda_{\mathrm{c}} & =186 & \mathrm{~K}_{\mathrm{c}} & =6.24 \mathrm{~cm}^{2} / \mathrm{gm} \\
\mathrm{~A} \Lambda & =129 & \mathrm{~K} & =20.2 \mathrm{~cm}^{2} / \mathrm{gm} \\
\mathrm{~A} \Lambda & =57.6 & \frac{\mathrm{~K}}{\bar{K}_{0}} & =3.24 \\
\Lambda & =6.31 \times 10^{-3} \mathrm{~cm} & &
\end{array}
$$

## Appendix Is Thermpdymanio Properties

This appendix contimee the statistical mechanical treatment of Chapter IV, 4. Whereas the latter section merely derived the occupation makers for the system, wo now proceed to osculate, the thermodynamic functions and the pressure. From the Last we get the equation of state.

The electronic partition function of the system is
(AID) $\quad Q=\sum_{J} \Omega_{J} e^{-\beta E_{J}}$
where $\mathbb{E}_{J}$ in given by ( 4.12 ) and $\Omega_{J}$ by ( 4.14 ). We can rewrite this by virtue of ( 4.13 ) ax

$$
\begin{equation*}
Q=e^{\alpha^{\prime}} \sum_{J} P_{J}=e^{n \alpha^{\prime \prime}} \sum_{J} P_{J} \tag{AI,2}
\end{equation*}
$$

Carrying out the equation to first order in $V_{i j}$, the interactions, wo get
(AID) $\ln Q=x \alpha^{\prime \prime}-\sum_{i} c_{i} \ln g_{i}$

$$
+\frac{\beta}{2} \sum_{i} V_{i i} c_{i} \phi_{i}\left(\bar{M}_{i}-c_{i} p_{i}-q_{i}\right)+\beta_{2} \sum_{i} \sum_{j \neq i} V_{i j} c_{i} \phi_{i}\left(\bar{M}_{j}-c_{j} p_{j}\right)
$$

We introduce the vale values for the arbitrary parameters $\bar{M}_{j}$ as wo did in the treatment of occupation numbers ( 4.21 ). There wore selected to make the first order term in the ocoupation numbers (4.18) vanish Such a choice then gave occupation number: identical in form to an independent electron treatment and agreeing with it to first order terms. Thin ohoioe has no particular advantage besides oonsisteuey for the present calculation. Then, since the Belmhols free energy is $A=-k T \ln Q$ (A IdA) $A=-\operatorname{LeT} \mu \alpha^{H}+\operatorname{kT} \sum_{i} C_{i} \ln g_{i}-1 / 2 \sum_{i} V_{i i} M_{i}^{2}\left(1-1 / c_{i}\right)$

$$
-1 / 2 \sum_{i}^{\prime} \sum_{j \neq i}^{\prime} V_{i j} M_{i} M_{j}
$$

From this equation we find the ohomical potential

$$
\left.\mu=\frac{\partial A}{\partial x}\right)_{V_{1} T}=-\phi T \alpha^{\prime \prime}
$$

By differentiation we can now get the other thermodynamic functions. It is easier in the case of the energy, however, to returns to our general treatment. The energy is

$$
(A I .4) \quad E=Z_{J} E_{J} P_{J}
$$

The energy If from ( 4.12 ) can be written as

$$
(A I .5) \quad E_{J}=\sum_{i} n_{J i}\left(E_{i}^{0}+E_{J i}^{\prime}\right)
$$

where
(AI.6) $\quad E_{i}^{0}=\varepsilon_{i}+1 / 2 \sum_{j \neq i}^{\prime} \bar{m}_{j} V_{i j}+1 / 2\left(\bar{m}_{i}-1\right) V_{i i}$
(AIT) $\quad E_{J i}^{\prime}=1 / 2 \sum_{j \neq i}\left(n_{J j}-\bar{n}_{j}\right) V_{i j}+1 / 2\left(n_{J i}-n_{i}\right) V_{i i}$

Then in ( $1 \mathrm{I} \cdot \mathrm{C}_{\text {) }}$ the summation order may be inverted, giving (A Ide)

$$
E=\sum_{i}^{\prime} E_{i}^{\cdot} \sum_{J} M_{J i} P_{J}+\sum_{i} \sum_{j} M_{J i} F_{J i}^{\prime} P_{J}
$$

We rooogaice in the first term the quantity $\quad M_{i}=\sum_{J} M_{S i} . P_{j}$ of ( $\mu, 16$ ). The second term contains parts all of mich scoured in the evaluation of the partition function. The result of the operations on (AI.8) is (AI.9) $E=\sum_{i} E_{i}^{0} r_{i} p_{i}\left\{_{0}^{0} 1+\theta / 2 g_{i} \cdot\left\{V_{i i}\left[\left(\bar{N}_{i}-1\right)-2\left(c_{i-1}\right) \phi_{i}\right]\right.\right.$

$$
\left.\left.+\sum_{j \neq i} v_{i j} \bar{M}_{j}-\sum_{j \neq i} V_{j i} \epsilon_{j} \phi_{j}\right\}\right\}
$$

$$
+\sum_{i}\left\{\frac{1}{2} v_{i i} r_{i} p_{i}\left(\kappa_{i} p_{i}-\bar{M}_{i}+q_{i}\right)+\sum_{j \neq i} 1 / 2 v_{i j} c_{i} p_{i}\left(c_{j} p_{j}-\bar{M}_{j}\right)\right\}
$$

Again introducing the values of $\bar{M}_{j}$, from (4.21) and $E_{i}^{0}$ from (A I.B) this reduces to (AI.10) $\quad E=\sum_{i} \cdot \mu_{i} \overline{\overline{\epsilon_{i}}}$

$$
\overline{\bar{\epsilon}}_{i}=\varepsilon_{i}+M_{i}\left(1-\frac{1}{c_{i}}\right) \frac{V_{i i}}{2}+\sum_{j=i}^{\prime} M_{j} \frac{V_{i j}}{2}
$$

Whee that for purposes of calculating the total energy of the system we can assume that sech electron has the energy $\overline{\hat{\varepsilon}}_{i}$ and that the energy of the system is just the sum of these anergies of the individual electrons. We should contrast this energy with $\vec{E}_{i}$ of (4.22). The latter gives the ionisation energy of the $i$ th electron, and the sum of the ionization energies is not the total energy of the system.

The entropy nay now be found from (A I.A) and (AI.10) by the equation

(AI.11) $\quad S=k \mu \alpha-k \underset{i}{Z} c_{i} \ln g_{i}+\frac{1}{T} \sum_{i} M_{i} \bar{\varepsilon}_{i}$, where
(AI.12) $\quad \hat{\varepsilon}_{i}=\varepsilon_{i}+M_{i}\left(1-\frac{1}{c_{i}}\right) V_{i i}+\sum_{j \neq i} M_{j} V_{i j}$
is the ionisation energy (4.22).
We now return to (AI.10) and introduce the same type of approximations which lead to (4.34). Fires we break up the energy
(AI.13)

$$
E=\sum_{i} M_{i} \overline{\bar{\varepsilon}}_{i}=\sum_{\sum} N_{z} \sum_{i} \frac{M_{i z}^{i}}{N_{i}} \overline{\bar{E}}_{i z}
$$

into gums of energy of average type lone To now compute $\overline{\bar{\varepsilon}}_{\text {ii }}$.
For a bound eleotron, we have the following contributions.

1) Kinetic energy plus full interaction with the moles $=\varepsilon_{i z}^{0}$,
2) $1 / 2$ interaction with all other bounds $=$

$$
-1 / 2 \frac{d \varepsilon_{i}^{0}}{d z}\left\{\sum_{j \neq i} \frac{M_{j z}}{N_{z}} \sigma_{i j j}+\frac{M_{i z}}{N_{z}}\left(1-\frac{1}{c_{i z}}\right) \sigma_{i, i}\right\},
$$

s) $1 / 2$ interaction with frees $: 1 / 2 \frac{z^{\prime} e^{2}}{2 a_{z^{\prime}}}\left\{3-\left(\frac{\bar{t}}{a_{z^{\prime}}}\right)^{2}\right\}$.

For a free electron, wo get

1) Kinetic energy $=\epsilon_{f}$
2) Full interaction with malleus $=-3 \sum_{*} \frac{N_{z}}{M_{f}} \frac{z^{\prime} \hbar e^{2}}{2 a_{z}}$
$\begin{aligned} & \text { 3) } 1 / 2 \text { interaction with bounds } \\ & \text { (considered at the moleus) }\end{aligned} \quad=\frac{3}{2} \sum_{z} \frac{N_{z}}{M_{f}} z^{\prime}\left(z-z^{\prime}\right) \frac{e^{2}}{2 a_{z}^{\prime}}$,
3) $1 / 2$ interaction with other frees $=\frac{6}{5} \sum_{2} \frac{N_{z}}{M_{f}} \frac{z^{\prime 2} e^{2}}{2 a_{z}^{\prime}}$.

Now the number of bound electrons attached to the average type ion of nucleus $Z$ is $M_{b Z} / N_{z}=\left(Z-Z^{\prime}\right) \quad$ - Using these relations, we gather all the terms contributing to (AI.18). It is a good approximation to consider the bound electrons at the nucleus in computing the interaction with the frees - hence $\bar{r}^{2}{ }^{i}$ is put equal to zero. We have finally

$$
\begin{aligned}
\text { (AI.14) } E= & \sum_{z} N_{z}\left\{\sum_{i}^{0} \frac{M_{i z}}{N_{z}} \varepsilon_{i}^{0}\left(\bar{z}_{i}\right)+\frac{e^{2}}{2 q_{z}}\left\{3 / 2 z^{\prime}\left(z-z^{\prime}\right)-3 z z^{\prime}+3 / 2 z^{\prime}\left(z-\xi^{\prime}\right)\right.\right. \\
& \left.\left.+6 / 5 z^{\prime}\right\}\right\}
\end{aligned}
$$

where
(AI.15)

$$
\vec{Z}_{i}=Z-\frac{1}{2} \sum_{j \neq i} \frac{M_{j z}}{N_{z}} \sigma_{i j}-1 / 2 \frac{M_{i z}}{N_{z}}\left(1-\frac{1}{C_{i z}}\right) \sigma_{i j i}
$$

We can rewrite (AI.14) in a way soon to prove significant.
(1I.18) $E=E_{b}+P . E .+(\dot{K} . E .)_{f}$, where
(AI.17) $\quad E_{b}=\sum_{Z} N_{2} \sum_{i} \frac{M_{i z}}{N_{z}} \quad \epsilon_{i}^{0}\left(\bar{z}_{i}\right)$
is the energy of the bound electrons excluding interactions with the frees
(AI.17) $\quad P_{1} E=-\frac{9}{5} \sum_{2} N_{2} \frac{z^{2} e^{2}}{2 a_{21}}$
is a potential energy term, and

$$
\begin{align*}
(K . E .)_{f} & =\int_{0}^{\infty} \varepsilon_{f} M\left(\varepsilon_{f}\right) d \varepsilon_{f}  \tag{AI.18}\\
& =3 / 2 M_{f} k T\left\{1+\frac{5}{4} \frac{k T}{M c^{2}}-\frac{5}{4}\left(\frac{k T}{M c^{2}}\right)^{2} \cdots\right\}\left\{1+\frac{e^{-\alpha^{*}}}{2^{5 / 2}}\left(1-\frac{15}{16} \frac{k T}{m c^{2}} \cdots\right) \cdots\right.
\end{align*}
$$

is the kinetic energy of the free electrons corrected for degeneracy and relativity. The pressure may be found from the Helmholtz free energy $\left.A_{\text {, since }} P=-\frac{\partial A}{\partial V}\right)_{T}$ This complicated computation can be avoided to the approximation we are working here, since all the forces are due to the coulomb interactions. (We have neglected exchange energies). Then we may use the virial theorem to find the pressure. In non relativistic theory this gives

$$
\begin{equation*}
P V=2 / 3 K . E . \rightarrow 1 / 3 \text { Potential Energy } \cdot \tag{AI.19}
\end{equation*}
$$

In relativistic theory as well as non-relativistic, the bound electrons include the proper balance of kinetic and potential energy to make the contribution to the pressure zero. The kinetic of the free electrons, however, does not contribute the full $2 / 3$ K.E. to the pressure because of the relativity correction. This is known to give exactly $P V=H_{+} k T \quad$ for the non-degenerate case, and we merely beep the additional degeneracy corrections. So, finally we get

$$
\begin{equation*}
P V=n_{f} k T\left\{1+\frac{e^{-\alpha^{*}}}{2^{5 / 2}}\left(1-\frac{15}{16} \frac{k T}{m c^{2}}\right) \cdot\right\}-\frac{3}{5} \sum_{2} N_{z} \frac{z^{2} e^{2}}{2 a_{z^{\prime}}} \tag{AI.20}
\end{equation*}
$$

Thus far we have not considered tie nuclear motion. This contribution has been worked out in Chapter IV, section 6. We get the following additive contributions to the energy and the pressure in the two limiting cases considered

## Free Fuclei

| Kinetic Energy | (3/2) MET | ( $3 / 2$ ) Nk T |
| :---: | :---: | :---: |
| Potential Pnerjy | 0 | (3/2) MLTT |
| Total Inergy | (3/2) MKT | 3 NKM |
| (PV) Muclei | NKP | (3/2) Mker |

The nuolear contribution to the energy and the pressure is so small (since $N \ll M_{f} \quad$ ) that me need not bother refining our treatment of them fiarther. We oan use a rough criterion that when $r / a_{z \prime}$ of (4.54) is less than $1 / 3$ we consider the nuciei to exert pure harmonic vibrations, while if $\bar{r} / a_{2^{\prime}}=1 / 3$ we shall oonsider the nuclei as free. Referriag to Fig. 2, we see that $r / a_{2}=\% / 3$ when $s=\frac{z^{\prime} e}{\sqrt{2 k T a_{2}}}=3.1$. For smaller values of $s$ we should use the free nuclei approsimation, winile for larger values of $S$ we should use the approximation of harmonic vibrations.

## Appendix II

The assumption that the free electrons are uniformly distributed was made thoughout Chapter IV. can correct this by the use of the olassicial statistical approach. Within each ionic polyhedron, assume we have an electrostatic potential $\phi$. Then the density of electrons in phase space will be (AII.1) $\quad M(\vec{g}, \vec{\phi})=\frac{2}{k^{3}} \exp \left\{-\alpha-\frac{\phi^{2}}{2 m k T}-\frac{e \phi}{k T}\right\} \sim \frac{2}{k^{3}}\left\{\exp \left(-\alpha-\frac{p^{2}}{2 m k T}\right)\right\}\left\{1+\frac{e \varphi}{k T}\right\}$. The density in configuration space $M(\vec{g})$ is found by integrating over the momentume This gives a charge density $\rho=-e M(\vec{g}) \quad$, and the potential $\varphi$ must satisfy Poisson's equation with this density. Since we have used the expansion exp- $\frac{e \phi}{k T}=1+\frac{e \phi}{k T}$ we keep terms in our result only to this order. The potential which replaces (4.31) is
(AII.2) $\varphi=\frac{z^{\prime} e}{r}\left[\frac{\sinh x\left(1-r / a_{z^{\prime}}\right)-x \cosh \times\left(1-n / a_{z^{\prime}}\right)}{\sinh x-x \cosh x}\right]-\frac{k T}{e}$,
where $x$ is a root of the transcendental equation
(AII ss) $\cosh x-\frac{\sinh x}{x}=\lambda$,
(AIT. 4 ) $\quad \lambda=z^{\prime} e^{2} /\left(a_{z}, k T\right)$.
Exapnding in powers of
(AII.5)

$$
\begin{aligned}
& x^{2}=3 \lambda-0.9 \lambda^{2}+.44358 \lambda^{3}+\cdots \\
& \frac{1}{x^{2}}=\frac{1}{3 \lambda}\left[1+0.3 \lambda-.05786 \lambda^{2}+\cdots\right]
\end{aligned}
$$

The quantity which replaces $\frac{3 / 5}{\frac{z^{\prime} e^{2}}{2 a_{z}^{\prime}}}$ in (4.32) is $\frac{3 \xi^{\prime} e^{2}}{a_{2 \prime} x^{2}}-k T$ and on putting in our expansion for $/ / x^{2}$ wet (AII.6)

$$
\bar{\varepsilon}_{f}=\varepsilon_{f}-\frac{1}{v} \sum_{2}^{\prime} N_{z} \frac{v \pi}{3} a_{z}^{3} \frac{z^{\prime} e^{2}}{2 a_{z^{\prime}}}(3 / 5-.1157 \lambda) .
$$

The radii $a_{Z^{\prime}}$ must be chosen so that (4.24) is satisfied. However, it is no longer appropriate to use (4.25), for the electron distribution is not uniform

Instead we must be sure that the free energy of the electrons is the same throughout the system. This is so if the quantity,
(AII.7) $Q=\frac{k T}{4 \pi e^{2} a_{Z^{\prime}}^{2}} x^{2} \sim \frac{3 z^{\prime} e^{2}}{4 \pi A_{Z^{\prime}}^{3}}\left[1-.3 \lambda+.44358 \lambda^{2}\right]$,
is independent of 2 . The $a_{z}$ then are chosen to satisfy (4.24) and (4.25).
In most cases the effect of nomuniform free eleotron distribution may be disregarded. The oriterion for this is
(AII.3) $\lambda=\frac{z^{\prime} e^{2}}{a_{z} \cdot k T}<1$.
Referring to Chapter IV, seotion 6, we find that the criterion for the nuclei to be considered as performing harmonic vibrations in a lattioe is $s=\frac{z^{\prime} e}{\sqrt{2 k T a_{2}}}>3.4$. Combining this with (AII. 8) we find the condition for uniformity in the distribution of free eleotrons, simultancously with a lattice structure for the nuclei is

AII.9)

$$
\frac{23}{z^{\prime}}<\frac{z^{\prime} e^{2}}{k T a_{2}}<1
$$

This can never be true for the very light nuclei, but is fullfilled by the heaviest mulei. This is another reason for the qualitative difference between the opacity of high $\geqslant$ and low $\rangle$ elements.

## Appendix III: Formulae and Tables of Gaunt Factors

This Appendix summarizes the formulae and numerical results or the boundfree Caunt factors applicatle to a non-relativistio electron in a Coulomb field. Only the leadine dipnle contribution is considered, so that these Gaunt faotors are the appropriatc analogue of the electron numbers recorded in Table I. Most of the resilits and computations are the worl of Dr. Boris Jacobsohn.

The Gaunt fretor is defined by (2.22).
(AIII.1) $\varepsilon_{b f}=\frac{3-\sqrt{3} \pi}{2^{4}} M\left(\frac{h v}{I_{n}}\right)^{2} \nu \frac{\mathrm{dfff}}{\mathrm{d} \nu}$.
The ionization potential $I_{n}$ is, however,
(AIII.2) $I_{n}=\frac{z^{2}}{n^{2}}$ Rhc.
We can also express the energy of the free electron after ionization by a quantum number $k$, defined so that
(AIII.3) $\varepsilon_{f}=\frac{z^{2} R h c}{k^{2}} \quad$
We then have for the frequenoy
(AIII.4) $h V^{\prime}=I_{n}+\varepsilon_{f}$ 。
whence
(AIII.5)

$$
\nu \frac{d f}{d v}=y \frac{d k}{d v} \frac{d f}{d k}=\frac{k^{3}}{2}\left[\frac{1}{n^{2}}+\frac{1}{k^{2}}\right] \frac{d f}{d k} .
$$

so that the Gaunt factor reduces to
(AIII.6)

$$
E_{n k}=\frac{3 \sqrt{3} \pi}{2^{5}} \frac{k^{3}}{n}\left[1+\frac{n^{2}}{k^{2}}\right]^{3}(-1)-\frac{d f}{d k}
$$

The $f$ numbers for bound-bound transitions have been computed by many previous workers. Since it is possible to find the bound-free $\frac{d f}{d k}$ rather simply from the appropriatie bound-bound $f$ nuriber, we have incladed a list of fommulas, Table I, for the latier. The procedure to be used in going from $f_{n^{\prime} \ell^{\prime} \rightarrow n_{l}}$ to $\frac{d f}{d k} n^{\prime} \ell^{\prime} \rightarrow k l$ is to substitute ik for $n$ in $f_{n^{\prime}} A^{\prime} \rightarrow n_{\ell}$ and multiply the result by $i /\left(1-e^{-7 . T k}\right)$. To
illustrate in the simplest case, if
(AIII.7)

$$
\left.f_{1 s \rightarrow n p}=\frac{2^{8}}{3} \frac{n^{5}}{\left(n^{2}-1\right.}\right)^{4}\left(\frac{n-1}{n+1}\right)^{2 n} .
$$

then we get

$$
\frac{d f_{1 s} \rightarrow k p}{d k}=\frac{i}{1-0^{-2} \pi k} \frac{2^{8}}{3} \frac{(i k)^{5}}{\left(k^{2}+1\right)^{4}}\left(\frac{i k-1}{i k+1}\right)^{2 i k}
$$

or
(AIII.8) $\frac{d f_{1 s}-k p}{d k}=-\frac{2^{8}}{3} \frac{k^{5}}{\left(k^{2}+1\right)^{4}} \frac{\exp \left(-1 k \operatorname{ctn}^{-1} k\right)}{1-\exp (-2 \pi k)}$.
Substitution in AIII. 6 gives the appropriate bound-free Gaunt factor. The resulting formulae for this and other cases are summarized in Table VII. Numerical values are recorded in Table VIII. For $n=1,2,3,4$, these values were calculated from the formulae of Table VII. For $n \in e 0$ it is possible to obtain the asymptotic expression presented in Table VII for the Gaunt factor, and the values are based upon this expression except for that at $k=1$ which was calculated exactly. The asymptotic formula is good to $.08 \%$ at $k=1$, and is even better for largo $k$. Values of the Gaunt factor for $n=5,6,7,8,9,10$ moro found by graphical interpolation. In most instances a plot of $g_{\pi} \rightarrow k$ VB. $1 / n^{2}$ for fixed $k$ gave a smooth curve which did not deviate markedly from a straight line, permitting accurate interpolation. Fig. I presents graphically the values for the Gaunt factor averaged over a complete shell.

1 Menzol \& Pekeris op. cit.

Table I

## Formulae for Dipole Oscillator Strengths

For Coulomb Field (Non-Relatioistic)
Transition
Electron Number
$18 \rightarrow n p$

$$
f=\frac{2^{8}}{3} \frac{n^{5}}{\left(n^{2}-1\right)^{4}} \frac{(n-1)^{2 n}}{(n+1)^{2 n}}
$$

ns $\rightarrow$ np

$$
f=\frac{2^{15}}{5} n^{5} \frac{\left(n^{2}-1\right)}{\left(n^{2}-4\right)^{5}} \quad\left(\frac{n-2}{n+2}\right)^{2 n}
$$

$2 p \rightarrow n s$
$2 p \rightarrow n d$

$$
P=\frac{2^{15}}{3^{3}} \frac{n^{7}}{\left(n^{2-4}\right)^{5}}\left(\frac{n-2}{n+2}\right)^{2 n}
$$

$$
f=\frac{2^{18}}{3^{3}} \frac{n^{7}\left(n^{2}-1\right)}{\left(n^{4}-4\right)^{6}}\left(\frac{n \infty 2}{n+2}\right)^{2 n}
$$

$3 s \Rightarrow n p$

$$
f=2^{8} 3^{4} n^{5}\left(n^{2}-1\right) \frac{\left(7 n^{2}-27\right)^{2}}{\left(n^{2}-9\right) 7} \frac{(n-3)^{2 n}}{(n+3)^{2 n}}
$$

$3 p \rightarrow n s$

$$
f=2^{11} 3^{3} n^{7} \frac{\left(n^{2}-3\right)^{3}}{\left(n^{2}-9\right)^{7}}\left(\frac{n-3}{n+3}\right)^{2 n}
$$

$3 p \rightarrow n d$

$$
f=2^{12} 3^{5} n^{7} \frac{\left(n^{2}-1\right)\left(n^{2}-4\right)}{\left(n^{2}-9\right)^{7}}\left(\frac{n-3}{n+3}\right)^{2 n}
$$

Bd $\rightarrow \mathrm{mp}$

$$
f=\frac{2^{12} 3^{4}}{5^{2}} \frac{n^{9}\left(n^{2}-1\right)}{\left(n^{2}-9\right)^{7}}\left(\frac{n-3}{n+3}\right)^{2 n}
$$

$3 d \rightarrow n f$

$$
f=\frac{2^{13} 3^{7}}{5^{2}} \frac{n^{9}\left(n^{2}-4\right)\left(n^{2}-1\right)}{\left(n^{2}-9\right)^{8}}\left(\frac{n-3}{n+3}\right)^{2 n}
$$

TABLE I a
DIPOLE OSCILLATOR STRENGTHS (NOM-RELATIVISTIC)
FOR COULOMB FIELD

| n | $\mathrm{f}^{\prime} \mathrm{ls} \rightarrow \mathrm{np}$ | $\mathrm{f}_{2 s \rightarrow n p}$ | $\mathrm{f}_{2 \mathrm{p} \rightarrow \mathrm{ns}}$ | $\mathrm{f}_{2 \mathrm{p} \rightarrow \mathrm{nd}}$ | $\mathrm{f}_{3 s \rightarrow \mathrm{np}}$ | $\mathrm{f}_{3 p \rightarrow \mathrm{~ns}}$ | $f_{3 p \rightarrow n d}$ | $\mathrm{f}_{3 \mathrm{~d} \rightarrow \mathrm{np}}$ | $\mathrm{f}_{3 \mathrm{~d} \rightarrow \mathrm{nf}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0 | -- | -. 13873 | - | $\cdots$ | -.026367 | -- | -- | --- |
| 2 | .41620 | 0 | 0 | --* | -. 040769 | -. 14495 | - | -41693 | -ma |
| 3 | .079102 | -43488 | . 013590 | . 69580 | 0 | 0 | 0 | 0 | $\cdots$ |
| 4 | . 028991 | . 10277 | .003044 | . 12181 | .48472 | . 032250 | . 618285 | . 010957 | 1.01432 |
| 5 | .012938 | . 04193 | .001213 | -04437 | .12102 | $7.428 \times 10^{-8}$ | .139235 | . 002210 | . 15649 |
| 6 | .007799 | .02163 | $6.180 \times 10^{-4}$ | .02163 | . 051394 | 3.032 | . 056136 | $8.420 \times 10^{-1}$ | . 05389 |
| 7 | .004814 | . 01274 | 3.613 | . 01233 | . 027369 | 1.579 | .029009 | 4.213 | . 02559 |
| 8 | .003183 | .00818 | 2.309 | .007757 | . 016549 | . 941 | . 017210 | 2.448 | .01442 |
| 9 | .002216 | .00658 | 1.570 | .005221 | . 010863 | . 612 | . 011153 | 1.564 | .00903 |
| 10 $n>10$ | .001605 $1.6 \mathrm{n}^{-8}$ | .00399 $3.7 \mathrm{n}^{-3}$ | $1.119 \times 10^{-4}$ $0.1 \mathrm{n}^{-8}$ | .003693 $3.3 \mathrm{n}^{-3}$ | .007554 $6.2 n^{-3}$ | $\begin{aligned} & .218 \times 10^{-6} \\ & .3 n^{-3} \end{aligned}$ | .003972 $6.1 n^{-3}$ | $\begin{aligned} & .552 \times 10^{-4} \\ & .07 n^{-3} \end{aligned}$ | .00314 $4.4 n^{-3}$ |
| Disorete  <br> Speotrum 08641 <br> Continuum 04359 <br> Potal  |  |  |  |  |  |  |  |  |  |
|  |  | . 638 | -. 119 | . 923 | - 707 | -. 121 | -904 | -. 402 | 1.302 |
|  |  | . 362 | .008 | . 293 | 293 | . 010 | . 207 | .002 | . 098 |
| Total | $\times 1.0000$ | 1.000 | -0111 | 1.111 | 1.000 | -0111 | 1.111 | $-0.400$ | 1.400 |

## PABIE II

NON-RELATIVISTIC SCRRENING CONSTANTS

$$
\sigma_{i j}=\frac{F_{0}(i, j)}{-\frac{d \varepsilon_{i}}{d \eta}}
$$



| $\bar{j}{ }^{\overline{1}}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | . 6250 | . 9383 | . 9811 | . 987 | . 994 | . 997 | . 999 | 1.000 | 1.000 | 1.000 |
| 2 | . 2346 | . 6895 | . 8932 | . 94 | .97 | . 984 | . 990 | . 993 | . 995 | 1.00 |
| 3 | -1090 | . 3970 | -7018 | .85 | .92 | .955 | . 97 | . 98 | -99 | 1.00 |
| 4 | . 06169 | . 2350 | . 4781 | . 705 | .83 | . 90 | -95 | -97 | -98 | -99 |
| 5 | .03976 | . 1552 | . 3312 | . 531 | . 72 | . 83 | . 90 | -95 | . 97 | . 98 |
| 6 | . 02769 | .1093 | . 2388 | .400 | - 854 | . 735 | -83 | -90 | . 95 | -97 |
| 7 | . 02039 | . 08082 | . 1782 | .3102 | -459 | . 610 | . 745 | . 83 | .90 | . 95 |
| 8 | .01563 | . 06250 | . 1378 | . 2425 | . 371 | . 506 | . 635 | . 750 | -83 | . 90 |
| 9 | .01234 | .04938 | .1106 | . 1936 | . 299 | . 431 | . 544 | . 656 | . 760 | . 88 |
| 10 | . 01000 | .04000 | .0900 | . 1584 | . 245 | . 353 | .466 | . 576 | . 67 | . 765 |

TABLE III
REIATIVISTIC ENERGY LEVELS FOR ELECTRON IN COULONB FIELD
Values tabulated are $-\frac{\varepsilon_{n_{n j}}^{0}(z)}{\text { Rho }_{0}}$

| 2 | $\begin{aligned} & x_{\text {m }}^{1} \\ & \mathrm{x}_{2}^{2} \end{aligned}$ | $\begin{aligned} & n=2 \\ & y=2 \end{aligned}$ | $\begin{aligned} & n=2 \\ & j=3 / 2 \end{aligned}$ | $\begin{aligned} & n=3 \\ & j=\frac{1}{x} \end{aligned}$ | $\begin{aligned} & n=3 \\ & f=3 / 2 \end{aligned}$ | $\begin{aligned} & n=3 \\ & f=5 / 2 \end{aligned}$ | nu4 | $n \mathrm{n}$ | п $=6$ | $8 \geq 7$ | $n \times 8$ | $n=9$ | $n=10$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 10 | 100.13 | 25.042 | 25.0083 | 11.1259 | 11.1160 | 11.1127 | 6.2521 | 4.0009 | 2.7782 | 2.0411 | 1.5626 | 1.2347 | 1.0001 |
| 11 | 121.20 | 30.3111 | 30.2622 | 13.4662 | 13.4516 | 13.4468 | 7.5655 | 4.8413 | 3.3617 | 2.4698 | 1.8908 | 1.4940 | 1.2101 |
| 12 | 144.28 | 36.087 | 36.0173 | 16.0308 | 16.0102 | 16.0034 | 9.0043 | 5.7619 | 4.0009 | 2.9393 | 2.2503 | 1.7780 | 1.4401 |
| 13 | 169.38 | 42.3694 | 42.2738 | 18.8202 | 18.7918 | 18.7825 | 10.5684 | 6.7626 | 4.6957 | 3.4497 | 2.6411 | 2.0867 | 1.6901 |
| 14 | 196.51 | 49.161 | 49.0320 | 21.8349 | 21.7967 | 21.7841 | 12.2580 | 7.8434 | 5.4462 | 4.0009 | 3.0631 | 2.4201 | 1.9602 |
| 15 | 225.68 | 56.4619 | 56.2922 | 25.0753 | 25.0250 | 25.0083 | 14.0730 | 9.0045 | 6.2523 | 4.5931 | 3.5164 | 2.7782 | 2.2503 |
| 16 | 256.88 | 64.274 | 64.0546 | 28.5420 | 28.4768 | 28.4552 | 16.0136 | 10.2459 | 7.1141 | 5.2261 | 4.0010 | 3.1611 | 2.5604 |
| 17 | 290.12 | 72.6002 | 72.3196 | 32.2355 | 32.1523 | 32.1248 | 18.0798 | 11.5675 | 8.0316 | 5.9000 | 4.5169 | 3.5687 | 2.8905 |
| 18 | 325.41 | 81.441 | 81.0875 | 36.1565 | 36.0518 | 36.0172 | 20.2718 | 12.9694 | 9.0047 | 6.6148 | 5.0640 | 4.2010 | 3.2406 |
| 19 | 362.75 | 90.7975 | 90.3589 | 40.3056 | 40.1755 | 40.1325 | 22.5896 | 14.4517 | 10.0336 | 7.3705 | 5.6425 | 4.4580 | 3.6108 |
| 20 | 402.15 | 100.67 | 100.134 | 44.6834 | 44.5235 | 44.4707 | 25.0333 | 16.0143 | 11,1183 | 8.1672 | 6.2523 | 4.9398 | 4.0010 |
| 21 | 443.62 | 111.069 | 110.413 | 49.2908 | 49.0961 | 49.0320 | 27.6030 | 17.6574 | 12.2587 | 9.0048 | 6.8934 | 5.4463 | 4.4112 |
| 22 | 487.16 | 121.99 | 121.196 | 54.1284 | 53.8936 | 53.8163 | 30.2987 | 19.5810 | 13.4549 | 9.8833 | 7.5659 | 5.9775 | 4.8414 |
| 23 | 532.78 | 133.431 | 132.484 | 59.1971 | 58.9162 | 58.8238 | 35.1207 | 21.1851 | 14.7069 | 10,8028 | 8.2697 | 6.5335 | 5.2917 |
| 24 | 580.48 | 145.40 | 144.277 | 64.4978 | 64.1642 | 64.0546 | 36.0690 | 23.0697 | 16.0148 | 11.7632 | 9.0048 | 7.1142 | 5.7620 |
| 25 | 630.28 | 157.903 | 156.576 | 70.0313 | 69.6378 | 69.5087 | 39.1437 | 25.0350 | 17.3786 | 12.7646 | 9.7713 | 7.7197 | 6.2524 |
| 26 | 682.19 | 170.94 | 169.382 | 75.7985 | 75.3374 | 75.1863 | 42.3450 | 27.0809 | 18.7982 | 13.8071 | 10.5692 | 8.3499 | 6.7628 |
| 27 | 736.21 | 184.506 | 182.694 | 81.8005 | 81.2633 | 81.0875 | 45.6730 | 29.2076 | 20.2738 | 14.8906 | 11.3984 | 9.0049 | 7.2933 |
| 28 | 792.36 | 198.61 | 196.514 | 88.0382 | 87.4158 | 87.2123 | 49.1278 | 31.4150 | 21.8052 | 16,0350 | 12.2590 | 9.6847 | 7.8438 |
| 29 | 850.64 | 213.261 | 210.842 | 94.5127 | 93.7953 | 93.5608 | 52.7096 | 33.7035 | 23.3927 | 17,1806 | 13.1510 | 10.3893 | 8.4144 |
| 30 | 911.05 | 228.46 | 225.678 | 101.225 | 100.402 | 100.133 | 56.4185 | 36.0725 | 25.0362 | 18.3872 | 14.0743 | 11.1186 | 9.0050 |

Values tabulated are $-\frac{\mathcal{E}_{\mathbf{n}_{n} f}^{0}(Z)}{\text { Rho }}$

| Z | $\begin{aligned} & n=1 \\ & j=1 / 2 \end{aligned}$ | $\begin{aligned} & n_{m} 2 \\ & j=1 / 2 \end{aligned}$ | $\begin{aligned} & n=2 \\ & j \pm 3 / 2 \end{aligned}$ | $\begin{aligned} & n=3 \\ & j=1 / 2 \end{aligned}$ | $\begin{aligned} & \mathrm{n}-3 \\ & j=3 / 2 \end{aligned}$ | $\begin{aligned} & n=3 \\ & j=5 / 2 \end{aligned}$ | $\begin{aligned} & n=1 \\ & f=1 / 2 \end{aligned}$ | $\begin{aligned} & n=A \\ & j=3 / 2 \end{aligned}$ | $\begin{aligned} & n=4 \\ & i=5 / 2 \end{aligned}$ | $\begin{aligned} & n=4 \\ & j \times \pi / 2 \end{aligned}$ | n¢5 | n=6 | $\mathrm{n}=7$ | n=8 | n-9 | nel |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 60 | 3791.1 | 960.08 | 911.07 | 421.15 | 406.56 | 402.15 | 234.57 | 228.46 | 226.59 | 225.69 | 145.2 | 100.6 | 73.8 | 56.4 | 44.6 | 36 |
| 61 | 3925.8 | 994.64 | 942.01 | 436.12 | 420.46 | 415.75 | 242.82 | 236.26 | 234.27 | 233.29 | 150.0 | 104.0 | 76.3 | 58.3 | 46.1 | 37 |
| 62 | 4063.4 | 1030.0 | 973.53 | 451.40 | 434.60 | 429.57 | 251.24 | 244.20 | 242.07 | 241.92 | 155.1 | 107.4 | 78.8 | 60.3 | 47.6 | 38 |
| 63 | 4203.9 | 1066.1 | 1005.6 | 466.99 | 448.99 | 443.62 | 259.82 | 252.28 | 250.00 | 248.88 | 160.2 | 111.0 | 81.4 | 62.2 | 49.1 | 3 C |
| 64 | 4347.5 | 1103.1 | 1038.3 | 482.89 | 463.63 | 457.91 | 268.57 | 260.50 | 258.06 | 256.86 | 165.3 | 114.5 | 84.0 | 64.3 | 50.7 | $4]$ |
| 65 | 4493.9 | 1140.8 | 1071.5 | 499.12 | 478.52 | 472.42 | 277.49 | 268.85 | 266.25 | 264.98 | 170.6 | 118.2 | 86.7 | 66.3 | 52.3 | 42 |
| 66 | 4643.0 | 1179.2 | 1105.2 | 515.56 | 493.66 | 487.16 | 286.57 | 277.34 | 274.57 | 273.23 | 175.9 | 121.8 | 89.4 | 68.3 | 54.0 | 47 |
| 67 | 4795.1 | 1218.5 | 1139.5 | 532.54 | 509.05 | 502.14 | 295.83 | 285.97 | 283.03 | 281.61 | 181.4 | 125.6 | 92.1 | 70.4 | 55.6 | 45 |
| 68 | 4950.3 | 1258.6 | 1174.3 | 549.76 | 524.59 | 517.34 | 305.26 | 294.74 | 291.62 | 290.11 | 186.9 | 129.4 | 94.9 | 72.6 | 57.3 | 46 |
| 69 | 5108.4 | 1299.6 | 1209.7 | 567.31 | 540.57 | 532.79 | 314.86 | 303.66 | 300.35 | 298.73 | 192.4 | 133.3 | 97.7 | 74.7 | 59.0 | 47 |
| 70 | 5269.6 | 1341.4 | 1245.6 | 585.21 | 556.71 | 548.45 | 324.65 | 312.71 | 309.21 | 307.49 | 198.2 | 137.2 | 100.6 | 76.9 | 60.7 | 4S |
| 71 | 5434.0 | 1384.0 | 1282.1 | 603.45 | 573.11 | 564.34 | 334.62 | 321.91 | 318.20 | 316.38 | 203.9 | 141.1 | 103.5 | 79.1 | 62.5 | 51 |
| 72 | 5601.5 | 1427.5 | 1319.2 | 622.03 | 589.76 | 580.44 | 344.77 | 331.24 | 327.31 | 325.40 | 209.8 | 145.2 | 106.5 | 81.4 | 64.2 | 52 |
| 73 | 5772.3 | 1471.9 | 1356.8 | 640.97 | 606.67 | 596.83 | 355.09 | 340.72 | 336.54 | 334.55 | 215.7 | 149.3 | 109.5 | 83.7 | 66.1 | 53 |
| 74 | 5946.5 | 1517.3 | 1394.9 | 660.29 | 623.33 | 613.45 | 365.61 | 350.35 | 345.91 | 343.32 | 221.7 | 153.4 | 112.5 | 86.0 | 67.9 | 54 |
| 75 | 6124.1 | 1563.6 | 1433.6 | 679.98 | 641.26 | 630.29 | 376.33 | 360.12 | 355.43 | 353.22 | 227.8 | 157.7 | 115.6 | 88.4 | 69.7 | $5 \epsilon$ |
| 76 | 6305.1 | 1610.8 | 1472.9 | 700.05 | 658.94 | 647.35 | 387.24. | 370.04 | 365.11 | 362.76 | 234.0 | 161.9 | 118.7 | 90.7 | 71.5 | 58 |
| 77 | 6489.6 | 1659.0 | 1512.7 | 720.49 | 676.88 | 664.65 | 398.34 | 380.10 | 374.91 | 372.42 | 240.3 | 166.3 | 121.9 | 93.2 | 73.5 | 59 |
| 78 | 6677.6 | 1708.2 | 1553.1 | 741.31 | 695.09 | 682.19 | 409.55 | 390.30 | 384.34 | 382.20 | 246.7 | 170.5 | 125.1 | 95.5 | 75.5 | 61 |
| 79 | 6869.0 | 1758.5 | 1594.1 | 762.51 | 713.57 | 699.97 | 421.15 | 400.54 | 394.89 | 392.11 | 253.1 | 175.1 | 128.3 | 98.1 | 77.4 | 68 |
| 80 | 7064.3 | 1809.7 | 1635.6 | 784.18 | 732.27 | 718.00 | 432.86 | 411.15 | 405.07 | 402.14 | 259.7 | 179.6 | 131.6 | 100.6 | 79.4 | 64 |
| 81 | 7263.1 | 1862.? | 1677.7 | 806.18 | 751.28 | 736.23 | 444.75 | 421.30 | 415.41 | 412.31 | 266.3 | 184.2 | 135.0 | 103.2 | 81.4 | 65 |
| 82 | 7465.3 | 1915.4 | 1720.4 | 828.62 | 770.55 | 754.70 | 456.87 | 432.60 | 425.38 | 422.51 | 273.0 | 188.9 | 138.3 | 105.7 | 83.4 | 67 |
| 83 | 7672.5 | 1969.9 | 1763.5 | 851.52 | 790.08 | 773.40 | 469.21 | 443.55 | 436.47 | 433.06 | 279.8 | 193.5 | 141.8 | 108.3 | 85.5 | 6! |
| 84 | 7883.1 | 2025.5 | 1807.4 | 874.86 | 809.87 | 792.33 | 481.77 | 454.65 | 447.18 | 443.64 | 286.7 | 198.2 | 145.2 | 111.0 | 87.6 | 7 C |
| 85 | 8097.8 | 2082.3 | 1851.8 | 898.61 | 829.94 | 811.53 | 494.55 | 465.90 | 458.05 | 454.33 | 293.7 | 203.0 | 148.8 | 113.7 | 89.7 | 72 |
| 86 | 8316.6 | 2140.2 | 1896.9 | 922.79 | 850.29 | 830.98 | 507.54 | 477.30 | 469.07 | 465.13 | 300.7 | 207.9 | 152.3 | 116.4 | 91.8 | 74 |
| 87 | 8539.7 | 2199.3 | 1942.4 | 947.46 | 870.87 | 850.63 | 520.76 | 488.79 | 480.19 | 476.06 | 307.9 | 212.8 | 155.9 | 119.1 | 94.0 | $7 \epsilon$ |
| 88 | 8767.0 | 2259.7 | 1988.6 | 972.53 | 891.76 | 870.54 | 534.21 | 500.44 | 491.46 | 487.18 | 315.2 | 217.9 | 159.5 | 121:9 | 96.2 | 77 |
| 89 | 8998.7 | 2321.5 | 2035.3 | 998.21 | 912.94 | 890.67 | 547.92 | 512.31 | 502.84 | 498.37 | 322.5 | 222.6 | 163.2 | 124.7 | 98.4 | 75 |
| 90 | 9235.0 | 2384.4 | 2082.7 | 1024.4 | 934.32 | 911.03 | 561.85 | 524.33 | 514.41 | 509.68 | 329.9 | 227.9 | 166.9 | 127.5 | 100.6 | 81 |
| 91 | 9476.1 | 2448.9 | 2130.6 | 1051.0 | 956.07 | 931.65 | 576.05 | 536.50 | 526.09 | 521.17 | 337.4 | 233.1 | 170.7 | 130.4 | 102.9 | 82 |
| 92 | 9722 .? | 2514.7 | 2179.2 | 1078.2 | 978.08 | 952.53 | 590.51 | 548.82 | 538.00 | 532.78 | 345.0 | 238.3 | 174.5 | 133.3 | 105.2 | 85 |

## TABLE IV

AVERAGE SQUARE RADIUS OF ELECTRON $\frac{\text { ORBITS }}{(r)^{2}}$
Values of $\left(\frac{r}{a_{0}}\right)^{2^{n}}$ and $\left(\frac{r_{0}}{a_{0}}\right)^{2}$
$\left(\frac{r_{0}}{a_{0}}\right)^{2^{n} \ell}=\frac{n^{4}}{z^{2}}\left\{1+3 / 2\left[1-\frac{l\left(\frac{L}{2}+1\right)-1 / 3}{n^{2}}\right]\right\}$.
$\left(\frac{r}{a_{0}}\right)^{2^{n}}=\frac{2 n^{4}}{z^{2}}\left\{1+\frac{1}{2 n^{2}}\right\}$


## THE FREQUENCY FACTOR $F(a, u) / F(a, \infty)$

FOR LINES OF EQUAL STBENGTH


| u | $t=\frac{1}{1+u^{2}}$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 1.0 | 1.5 | 2.0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 菏 | $\mathrm{F}(2, \infty)$ | 1.0000 | －9761 | ． 9528 | ．9307 | ．9096 | ．8896 | ． 80182 | ． 73138 | ． 67399 |
| ${ }_{\text {O }}^{0} \boldsymbol{\infty}$ | 0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| － 4.3589 | ． 05 | ． 8564 | ． 8530 | ． 8496 | ． 8461 | ．8427 | ．8393 | ． 8225 | ． 8061 | ． 7905 |
| O 3 | ． 10 | ． 7952 | ． 7902 | ． 7855 | ． 7808 | ． 7761 | ． 7714 | ． 7485 | ． 7265 | ． 7056 |
| 2.3805 | ． 15 | ． 7474 | ． 7411 | ．7354 | ． 7298 | ． 7235 | ． 7187 | ；6918 | ． 6662 | ． 6421 |
| $\stackrel{\rightharpoonup}{0}_{\substack{0}}$ | ． 20 | ． 7048 | －6984 | ．6921 | ． 6859 | ． 6796 | ． 6735 | ${ }^{6} 6438$ | .6157 | ． 5895 |
| 苗1．7321 | ． 25 | ． 6667 | ． 6597 | ．6529 | ． 6462 | ． 6395 | ． 6329 | ． 6011 | ． 5714 | ． 5439 |
| $\bigcirc .5275$ | ． 30 | ． 6310 | ． 6237 | ． 6165 | ． 6094 | ． 6024 | ． 5955 | ． 5622 | ． 5314 | ． 5032 |
| 罗1．3628 | ． 35 | ． 5970 | ． 5894 | －5819 | .5746 | ． 5673 | ． 5602 | ． 5260 | ． 4946 | －4661 |
| $\text { 何 } 1.2247$ | ． 40 | ． 5641 | ． 5563 | ． 5487 | ． 5412 | ． 5337 | ． 5265 | ．4918 | ．4602 | －4317 |
| ${ }_{\sim}^{1} 1.1055$ | ． 45 | ． 5319 | ． 5240 | ． 5163 | ． 5087 | ． 5012 | ． 4938 | ． 4591 | ． 4276 | ． 3995 |
| ${ }^{[\mathrm{T}} 1.0000$ | ． 50 | ． 5000 | －4921 | －4844 | .4767 | －4693 | －4619 | －4274 | ． 3965 | ． 3670 |
| ． 9045 | －55 | ．4681 | ． 4603 | －4526 | ． 4450 | ．4376 | ． 4304 | ．3965 | ．3663 | －3397 |
| ． 8165 | ． 60 | －4359 | －4282 | －4207 | ．4133 | ． 4060 | ． 3989 | ． 3659 | ． 3368 | ． 3113 |
| ． 7071 | ． 65 | ． 3918 | ． 3956 | ． 3882 | ． 3810 | ． 3740 | ． 3672 | ． 3354 | ． 3076 | ． 2834 |
| ． 6547 | ． 70 | ． 3690 | ． 3619 | ． 3548 | ． 3480 | ． 3413 | ． 3347 | ． 3046 | ． 2783 | ． 2558 |
| － 5774 | ． 75 | ． 3333 | ． 3266 | ． 3200 | ． 3135 | ． 3072 | ． 3011 | ． 2729 | ． 2486 | ． 2278 |
| ． 5 | ． 80 | ． 2952 | ． 2890 | ． 2828 | ． 2769 | ． 2711 | ． 2655 | ． 2398 | ． 2178 | ． 1990 |
| －4201 | ． 85 | ． 2532 | ． 2477 | ． 2422 | ． 2369 | ． 2318 | ． 2268 | ． 2041 | ． 1848 | ． 1685 |
| .3333 | ． 90 | ． 2048 | ． 2003 | ． 1957 | ． 1913 | ． 1870 | ． 1829 | ． 1640 | ． 1481 | ． 1347 |
| ． 2294 | ． 95 | ． 1436 | ． 1400 | ． 1368 | ． 1336 | ． 1305 | ． 1275 | ． 2148 | ． 1026 | ． 0932 |
| 0 | 1.00 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

## TABLE V (Continued)

| и |  | 4. | 5. | 6. | 8. | 10. | 15. | 20. | 25. | 30, |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\boldsymbol{P}(1, \infty)$ | . 58241 | .4770 | . 4402 | . 3860 | . 3478 | -2864 | . 2491 | . 2234 | . 2043 |
| $\infty$ | 0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 31.6070 | . 001 | -- | .9592 | --- | $\cdots$ | -** | .9311 | . 9206 | . 9114 | . 9032 |
| 9.9499 | . 01 | --> | . 8685 | --80 | --7 | --80 | . 7836 | . 7530 | . 7265 | . 7032 |
| 4.3589 | . 05 | . 7349 | -- | . 6893 | . 6511 | . 6187 | --- | --- | --- | --- |
| 3 | . 10 | . 6331 | . 6040 | . 5761 | . 5303 | -4929 | . 4232 | .3742 | . 3378 | . 3145 |
| 2.3805 | . 15 | . 5605 | -- | . 4987 | . 4509 | . 4132 | --- | --- | -- | $\cdots$ |
| 2 | . 20 | . 5027 | 4696 | .4395 | . 3921 | . 3560 | .2942 | . 2550 | . 2278 | . 2077 |
| 1.7321 | . 25 | .4547 | --- | .3918 | . 3462 | . 3122 | --- | $\rightarrow-$ | --- | --- |
| 1.5275 | .30 | .4133 | . 5804 | . 3521 | .5087 | -2772 | . 2260 | . 1952 | -1741 | . 2587 |
| 1.3628 | . 35 | . 8770 | --> | . 3181 | . 2773 | . 2482 | $\cdots$ | -- | --- |  |
| 1.2247 | .40 | . 3445 | . 8137 | . 2883 | . 2508 | . 2236 | . 1815 | . 1565 | . 1396 | . 1272 |
| 1.1055 | -45 | . 3149 | - | -2619 | -2288 | -2021 | --- | --- | --- | --- |
| 1 | . 50 | -2877 | -2608 | -2879 | .2053 | . 1829 | .1482 | . 1278 | . 1140 | . 1039 |
| .9045 | . 55 | . 2622 | -mm | . 2159 | . 1859 | . 1655 | - | --- | --- | -- |
| . 8185 | . 60 | . 2382 | .2146 | . 1958 | . 1679 | .1495 | . 1210 | . 1043 | . 0931 | . 0848 |
| . 7071 | . 65 | . 2151 | $\cdots$ | . 1759 | . 1610 | . 1348 | --- | --- | --- | --0 |
| . 6547 | .70 | . 1927 | . 1744 | . 1571 | . 1847 | . 1199 | .0970 | . 0839 | . 0746 | .0880 |
| . 5774 | . 75 | . 1705 | -- | . 2587 | . 1188 | .1057 | -0- | --- | --- | --- |
| . 5 | . 80 | sist | . 1885 | .2863 | . 1050 | . 0916 | .0741 | .0839 | .0570 | 046 |
| . 4201 | . 85 | . 1247 | --- | . 1010 | .0868 | .0769 | -- | -mer | -- | --. |
| . 3533 | . 90 | .0992 | .0890 | .0802 | .0888 | . 0811 | . 0494 | . 0428 | . 0580 | .0346 |
| . 2294 | .95 1.00 | .0685 | $\cdots$ | .0651 | $.0473$ | $\underset{0}{.0420}$ | $\cdots$ | $\cdots$ | $\cdots$ | 0 |

## TABLE VI

THE WEIGHTING FUNGTION

$$
W(u)=\frac{15}{4 \pi^{4}} \quad u^{7} e^{2 u}\left(e^{u}-1\right)^{-3}
$$

AND THE STRONGREN FUNCTION

$$
S(u)=\int_{0}^{u} W(x) d x
$$

| 4 | W(u) | S(u) | $u$ | W(u) | $s(u)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | 0 - 6 | 0 | 8.5 | 8.2005 | 6.2882 |
| . 1 | $4.0390 \times 10^{-6}$ | $8.015 \times 10^{-8}$ | 3.6 | 8.9572 | 7.1469 |
| . 2 | $6.7737 \times 10^{-5}$ | $2.669 \times 10^{-6}$ | 3.7 | 9.7404 | 8.0819 |
| . 3 | 3.5814x10-4 | $2.103 \times 10^{-5}$ | 3.8 | 10.546 | 9.0957 |
| . 4 | $1.1801 \times 10^{-8}$ | $9.185 \times 10^{-5}$ | 3.9 | 11.371 | 10.1902 |
| . 5 | $2.9948 \times 10^{-3}$ | $2.902 \times 10^{-4}$ | 4.0 | 12.211 | 11.3676 |
| . 6 | $6.4396 \times 10^{-3}$ | 7.454×10-4 | 4.1 | 13.064 | 12.6809 |
| . 7 | . 012339 | $1.660 \times 10^{-3}$ | 4.2 | 13.926 | 13.9808 |
| -8 | .021726 | 8.35Ix $10^{-3}$ | 4.3 | 14.793 | 15.4274 |
| $\bullet 9$ | .035822 | $6.170 \times 10^{-3}$ | 4.4 | 15.660 | 16.9409 |
| 1.0 | . 056077 | $1.071 \times 10^{-2}$ | 4:.5 | 16.525 | 18.5793 |
| 1.1 | . 084108 | $1.765 \times 10^{-2}$ | 4.6 | 17.384 | 20.2476 |
| 1.2 | . 12175 | $2.787 \times 10^{-2}$ | 4.7 | 18.232 | 22.0290 |
| 1.3 | - 17101 | $4.214 \times 10^{-2}$ | 4.8 | 19.067 | 23.8940 |
| 1.4 | . 23401 | $6.257 \times 10^{-2}$ | 4.9 | 19.884 | 25.8403 |
| 1.5 | . 31303 | $8.981 \times 10^{-2}$ | 5.0 | 20.681 | 27.8664 |
| 1.6 | -41042 | -1258 | 5.1 | 21.453 | 29.9725 |
| 1.7 | . 52859 | . 1724 | 5.2 | 22.199 | 32.1510 |
| 1.8 | . 67064 | . 2313 | 5.3 | 22.915 | 34.4014 |
| 1.9 | . 83681 | . 3066 | 5.4 | 23.599 | 36.7225 |
| 2.0 | 1.0316 | . 3994 | 5.5 | 24.249 | 39.1128 |
| 2.1 | 1.2564 | . 5129 | 5.6 | 24.861 | 41.5698 |
| 2.2 | 1.5134 | . 6502 | 5.7 | 25.436 | 44.0895 |
| 2.3 | 1.7970 | . 8144 | 5.8 | 25.970 | 46.6656 |
| 2.4 | 2.1307 | 1.009 | 5.9 | 26.463 | 49.2917 |
| 2.5 | 2.4940 | 1.238 | 6.0 | 26.918 | 51.9633 |
| 2.6 | 2.8948 | 1.505 | 6.1 | 27.319 | 54.6755 |
| 2.7 | 3.3314 | 1.815 | 6.2 | 27.682 | 57.4249 |
| 2.8 | 3.8128 | 2.173 | 6.3 | 28.000 | 60.2082 |
| 2.9 | 4.3302 | 2.582 | 6.4 | 28.278 | 63.0215 |
| 3.0 | 4.8857 | 3.048 | 6.5 | 28.502 | 65.8637 |
| 3.1 | 5.4791 | 3.568 | 6.6 | 28.886 | 68.7225 |
| 3.2 | 6.1087 | 4.1483 | 6.7 | 28.826 | 71.3978 |
| 3.3 | 6.7735 | 4.7923 | 6.8 | 28.923 | 74.4853 |
| 3.4 | 7.4715 | 5.5041 | 6.9 | 28.977 | 77.8810 |


| u | $w(u)$ | S(u) | u | W(u) | $s(u)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 7.0 | 28.991 | 80.2800 | 10.2 | 17.027 | 157.1759 |
| 7.1 | 28.961 | 83.1773 | 10.4 | 16.163 | 160.3624 |
| 7.2 | 28.895 | 86.0698 | 10.6 | 15.053 | 163.3490 |
| 7.3 | 28.789 | 88.9546 | 10.8 | 13.815 | 166.1404 |
| 7.4 | 28.546 | 91.8283 |  |  |  |
| 7.5 | 28.470 | 94.6856 | 11.0 | 12.530 | 168.7424 |
| 7.6 | 28.258 | 97.5219 | 11.5 | 10.787 | 174.4525 |
| 7.7 | 28.016 | 100.3346 | 12.0 | 8.4757 | 179.2535 |
| 7.8 | 27.742 | 103.1223 | 12.5 | 7.0717 | 182.9749 |
| 7.9 | 27.439 | 106.8825 | 13.0 | 5.4602 | 186.0382 |
| 8.0 | 27.110 | 108.6119 | 13.5 | 4.4377 | 188.4766 |
| 8.1 | 26.757 | 111.3053 | 14.0 | 3.3745 | 190.3846 |
| 8.2 | 26.379 | 113.9608 | 15.0 | 2.0121 | 193.0267 |
| 8.3 | 25.981 | 116.5797 | 16.0 | 1.1629 | 194.5800 |
| 8.4 | 25.562 | 119.1587 | 18.0 | -35896 | 195.9583 |
| 8.5 | 25.126 | 121.6939 | 20.0 | . 10157 | 196. 3685 |
| 8.6 | 24.673 | 124.1825 | 22.0 | .026787 | 196.4813 |
| 8.7 | 24.206 | 126.6251 | 24.0 | .0066659 | 196.5103 |
| 8.8 8.9 | 23.726 23.234 | 129.0210 131.3691 | 26.0 30.0 | .0015798 | 196.5174 196.5194 |
| 8.9 | 23.234 | 131.3691 | 30.0 | .000071419 | 196.5194 |
| 9.0 | 22.732 | 133.6682 | $\infty$ | 0 | 196.5194 |
| 9.1 | 22.222 | 135.9163 |  |  |  |
| 9.2 | $21.706$ | 138.1127 |  |  |  |
| 9.3 9.4 | $\begin{aligned} & 21.183 \\ & 20.658 \end{aligned}$ | 140.2571 142.3489 |  |  |  |
| 9.5 | 20.127 | 144.3881 |  |  |  |
| 9.6 | 19.596 | 146.3742 |  |  |  |
| 9.7 9.8 | 19.066 18.535 | 148.3066 150.1856 |  |  |  |
| 9.9 | 18.007 | 152.0119 |  |  |  |
| 10.0 | 17.480 | 153.7856 |  |  |  |

$$
\text { Define } F_{n}(k)=\frac{\exp \left(-4 k \tan ^{-1} n / k\right)}{1-\exp (-2 \pi k)}
$$

$$
\mathrm{g}_{1 s \rightarrow k p}=8 \pi \sqrt{3} \frac{k^{2}}{k^{2}+1} F_{1}(k)
$$

$$
\mathrm{g}_{2 \mathrm{~s} \rightarrow \mathrm{kp}}=2^{9} \pi \sqrt{3} \frac{\left(1+1 / x^{2}\right)}{\left(1+\frac{4}{k^{2}}\right)^{2}} F_{2}(k)
$$

$$
\mathrm{g}_{2 p \rightarrow k}=\frac{2^{7} \pi \sqrt{3}}{3^{2}} \frac{1}{\left(1+4 / k^{2}\right)^{2}} F_{2}(k)
$$

$$
g_{2 p \rightarrow k d}=\frac{2^{12} \pi \sqrt{3}}{3^{2}} \frac{\left(1+1 / k^{2}\right)}{\left(1+4 / k^{2}\right)^{3}} \quad F_{2}(k)
$$

$$
g_{2 p \rightarrow k}=\frac{2^{7} \pi \sqrt{3}}{3^{2}} \frac{\left(33+\frac{36}{k^{2}}\right)}{\left(1+4 / k^{2}\right)^{3}} \quad F_{2}(k)
$$

$$
E_{2 \rightarrow k}=\frac{2^{5} \pi \sqrt{3}\left(15+\frac{2}{k^{2}}+\frac{1}{k^{4}}\right)}{\left(1+\frac{4}{k^{2}}\right)^{3}} \quad F_{2}(k)
$$

$$
g_{3 p \rightarrow k}=2^{6} 3^{3} \sqrt{3} \pi \frac{\left(19+\frac{96}{k^{2}}+\frac{81}{k^{4}}\right)}{\left(1+9 / k^{2}\right)^{4}} \quad F_{3}(k)
$$

$$
g_{3 d \rightarrow k}=\frac{2^{7} 3^{4} \sqrt{3} \pi}{5} \frac{\left(1+1 / k^{2}\right)\left(11+\frac{45}{k^{2}}\right)}{\left(1+9 / k^{2}\right)^{5}} \quad F_{3}(k)
$$

$$
E_{z} \lambda_{k}=2^{3} 3^{2} \sqrt{3} \pi \frac{\left[13+\frac{7 e}{k^{2}}+\frac{81}{k^{4}}\right]\left[29+\frac{126}{k^{2}}+\frac{81}{k^{4}}\right]}{\left(1+9 / k^{2}\right)^{5}} F_{3}(k)
$$

$$
\begin{aligned}
k_{4} \rightarrow k= & \frac{2^{7} \sqrt{3} \pi}{3^{2}\left(1+\frac{16}{k^{2}}\right)^{7}}\left[539+\frac{6800}{k^{2}}+\frac{2073}{k^{4}}+\frac{12288}{k^{6}}\right] \\
& x\left[197+\frac{3152}{k^{2}}+\frac{13056}{k^{4}}+\frac{12288}{k^{6}}\right] F_{4}(k)
\end{aligned}
$$

$$
\operatorname{Lim}_{n \rightarrow \infty} E_{n} \rightarrow_{k}=\left\{1+\frac{.1728261}{k^{2 / 3}}-\frac{.0495957}{k^{4} / 3}\right.
$$

$$
\left.-\frac{.0171429}{k^{2}}+\frac{.0020450}{k^{8 / 3}}\right\}
$$

## Table VIII Values for Bound Free Gaunt Factors

| $1 / k^{2}$ | $k$ | $\mathrm{~g}_{1_{s} \rightarrow \mathrm{kp}}$ |
| :--- | :--- | :--- |
|  |  |  |
| 0 | 00 | .79730 |
| .001 | 31.623 | .79770 |
| .01 | 10 | .79989 |
| .05 | 4.4721 | .81001 |
| .1 | 3.1623 | .82167 |
| .2 | 2.2361 | .84331 |
| .5000 | 1.414 | .89240 |
| 1.000 | 1.000 | .94236 |
| 2.0 | .70711 | .98447 |
| 5.0 | .44721 | .98592 |
|  |  |  |

Table VIII Values for Bound Free Gaunt Factors (Continued)

| $1 / k^{2}$ | k | $\mathrm{E}_{2 s} \rightarrow \mathrm{kp}$ | $\mathrm{g}_{2 \mathrm{p} \rightarrow \mathrm{ks}}$ | $\mathrm{g}_{2 \mathrm{p} \rightarrow \mathrm{kd}}$ | $\mathrm{g}_{2 p \rightarrow k}$ | ${ }_{8}^{2} \rightarrow 2$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $\infty$ | . 93460 | . 025961 | . 83075 | . 85671 | . 87619 |
| . 0625 | 4. | 1.13689 | . 029723 | . 30845 | . 23818 | -91286 |
| . 12755 | 2.8 | 1.32462 | . 032632 | . 77965 | -81229 | . 94037 |
| . 25 | 2.00 | 1. 62532 | .036129 | . 72258 | . 75871 | .97549 |
| . 37180 | 1.64 | 1.87276 | . 037922 | . 67064 | . 70856 | -99860 |
| . 52510 | 1.38 | 2.12688 | . 038738 | -60977 | . 64852 | 1.01811 |
| . 74316 | 1.16 | 2.41104 | . 038420 | . 53946 | - 57790 | 1.03618 |
| 1.000 | 1.000 | 2.66420 | . 037003 | . 47363 | . 51064 | 1.04903 |
| 1.5625 | . 8 | 3.03132 | . 032860 | - 37165 | . 40452 | 1.06122 |
| 2.25 | . 66667 | 3.28819 | . 028104 | . 29228 | . 32039 | 1.06234 |


| 1/k ${ }^{2}$ | k | $\mathrm{E}_{3 \mathrm{~s}} \rightarrow \mathrm{kp}$ | $\mathrm{g}_{3 \mathrm{p} \rightarrow \mathrm{L}}$ | $g_{3 d \rightarrow k}$ | $g_{3} \rightarrow \mathrm{k}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $\infty$ | 1.06160 | 1.09768 | . 76259 | -90751 |
| . 02777 | 6. | 1.31070 | 1.23028 | . 68433 | -93591 |
| . 056689 | 4.2 | 1.55462 | 1.33175 | . 61340 | -95743 |
| . 11111 | 3.0 | 1.9759 | 1.4544 | . 5059 | . 9854 |
| . 23338 | 2.07 | 2.77863 | 1.55574 | . 34854 | 1.02095 |
| . 3333 | 1.732 | 3.32160 | 1.55697 | . 26988 | 1.03799 |
| . 4444 | 1.5 | 3.83427 | 1.52290 | . 21072 | 1.05073 |
| . 6944 | 1.2 | 4.73641 | 1.40219 | . 13321 | 1.06767 |
| 1.000 | 1.000 | 5.52416 | 1.24884 | .085635 | 1.07765 |

Table VIII Values for Bound Free Gaunt Factors (Conoluded)

| $1 / k^{2}$ | k | ${ }_{4}^{4} \rightarrow$ k | $\mathrm{E}_{5 \rightarrow \mathrm{k}}$ | ${ }_{6}{ }_{6}{ }^{\text {a }}$ | ${ }_{\text {g }}^{7 \rightarrow k}$ | $\mathrm{g}_{8 \rightarrow k}$ | $\mathrm{g}_{9 \rightarrow k}{ }^{*}$ | $\mathrm{g}_{\overline{10} \rightarrow \mathrm{k}}$ | $\stackrel{\mathrm{L}}{\substack{\text { a }}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $\infty$ | -92474 | . 9360 | . 9438 | . 9498 | -9545 | -9578 | .9605 |  |
| 0001111 | 30 | -92672* | . 9386 | . 9470 | . 9535 | . 9582 | . 9625 | . 9656 | 1.01735 |
| -01 | 10 | .941* | .9590 | . 9722 | . 982 | . 9895 | . 995 | 1.00 | 1.0349 |
| . 027778 | 6 | .9618* | .980 | -994 | 1.0035 | 1.0115 | 1.0175 | 1.022 | 1.04733 |
| . 06250 | 4 | -98992 | 1.0088 | 1.0210 | 1.0290 | 1.0353 | 1.0395 | 1.0427 | 1.05976 |
| . 1111 | 3 | 1.0132* | 1.0290 | 1.0401 | 1.0470 | 1.0518 | 1.0557 | 1.0579 |  |
| . 25000 | 2.00 | 1.04743 | 1.0597 | 1.0668 | 1.0712 | 1.0744 | 1.0767 | 1.0781 | 1.0853 |
| $\stackrel{5000}{ }$ | 1.414 | 1.0723* | 1.0807 | 1.0857 | 1.0888 | 1.0909 | 1.0923 | 1.0935 | 1.0982 |
| 1.0000 | 1.000 | 1.08874 | 1.0945 | 1.0983 | 1.1009 | 1.1026 | 1.1036 | 1.1044 | 1.1088 |

* Indicates interpolated valuse.

TABLE IX

The function $\chi(x)=\frac{1}{\sqrt{\pi}} \int_{x^{\prime}}^{\infty} y^{-\frac{1}{2}} e^{-y} \ln y d y$

| $x$ | $\chi(x)$ |
| :---: | :---: |
| 0 | -1.9636 |
| .01 | -1.2202 |
| .05 | .- .7183 |
| .1 | -.4627 |
| .2 | -.2149 |
| .3 | -.0907 |
| .4 | -.0190 |
| .5 | +.0241 |
| .6 | .0507 |
| .8 | .0754 |
|  |  |
| 1.0 | .0807 |
| 1.1 | .0799 |
| 1.2 | .0776 |
| 1.3 | .0744 |
| 1.4 | .0707 |
| 1.5 | .0666 |
| 1.6 | .0624 |
| 1.7 | .0582 |
| 1.8 | .0541 |
| 1.9 | .0501 |
| 2.0 | .0462 |
| 2.1 | .0426 |
| 2.2 | .0392 |
| 2.3 | .0359 |
| 2.4 | .0329 |
|  |  |


| $x$ | $\chi(x)$ |
| :---: | :---: |
| 2.5 | .0302 |
| 2.6 | .0276 |
| 2.7 | .0252 |
| 2.8 | .0230 |
| 2.9 | .0210 |
| 3.0 | .0191 |
| 3.2 | .0158 |
| 3.4 | .0131 |
| 3.6 | .0108 |
| 3.8 | .0089 |
| 4.0 | .0073 |
| 4.2 | .0060 |
| 4.4 | .0049 |
| 4.6 | .0040 |
| 4.8 | .0033 |
| 5.0 | .0027 |
| 5.5 | .0016 |
| 6.0 | .0009 |
| 6.5 | .0005 |
| 7.0 | .0002 |
| 7.5 | .0001 |
| 8.0 |  |
|  |  |
|  |  |


| 0.80.70.60.50.40.30.20.10.10.2 |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
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|  |  |  |  |  |  |  |  |  |

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FIG IIIG
NOMOGRAPH OF


$$
g(x, y)=\frac{1}{\pi}\left\{\tan ^{-1}(x+y)-\tan ^{-1}(x-y)\right\}
$$



$-150$.
FIG III b

$$
\begin{gathered}
\text { NOMOGRAPH OF } \\
g(x, y)=\frac{1}{\pi}\left\{\tan ^{-1}(x+y)-\tan ^{-1}(x-y)\right\}
\end{gathered}
$$





[^0]:    Contract W-7405-ENG. 36 with the U. S. Atomic Energy Commission

[^1]:    (4)
    (5) J. A. Gaunt, Phil Trans. A 229 pp. 163 (1930)
    ${ }^{(5)} \mathbf{u}_{0}$ Stobbe, Ann. d. Phys. 2 pp. 661 (1930)
    ${ }^{(6)}$ Op. Cit.

[^2]:    F. Seitz "Modern Theory of Solide, pp, 243 ff.

