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SOME REMARKS ON THE HYDRODYNAMICAL THEORY OF WAVE PROPAGATION WITH AN APPLICATION TO A PROBLEM IN THE FLOW OF METALS INCOMPRESSIBLE AT LOW PRESSURES

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ABSTRACT

The behavior of a fluid at a surface S of discontinuity of higher order (i.e., not a shock) are discussed without reference to the equation of state. In particular, the formulae

$$\rho\, v_x\, \Delta\left(\frac{\partial^n v_x}{\partial x^n}\right) = -\Delta\left(\frac{\partial^n p}{\partial x^n}\right),$$

$$\Delta\left(\frac{\partial^n(\rho v_x)}{\partial x^n}\right) = 0,$$

$$\Delta\left(\frac{\partial^n p}{\partial x^n}\right) = c^2\, \Delta\left(\frac{\partial^n \rho}{\partial x^n}\right)$$

for a discontinuity of order n, $n \geqq 1$ are established, where x is the direction normal to the surface, p is pressure, ρ density, v_x velocity in the x-direction, c the velocity of propagation of S, and Δ the discontinuity of the indicated function. If $dp/d\rho$ exists the third formula yields the theorem of Hadamard that such surfaces propagate with sound-velocity (if there is material flow across them). The third formula is also applied to show that a surface forming the boundary between a metal in an incompressible phase where $p < p_o$, $\rho = \rho_o$ and in a phase where $p = p_o$, $\rho > \rho_o$, must be a discontinuity of order greater than 1, and at the same time gives the velocity of propagation of such boundary.

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SOME REMARKS ON THE HYDRODYNAMICAL THEORY OF WAVE PROPAGATION WITH AN APPLICATION TO A PROBLEM IN THE FLOW OF METALS INCOMPRESSIBLE AT LOW PRESSURES

I. The familiar treatments of wave propagation in gases assume ordinarily a differential relationship between pressure p and density ρ. Recently, however, applications have been found for the concept of a gas whose Watt diagram is a broken line, in which case, of course, the derivative $dp/d\rho$ does not exist everywhere. In the light of this fact, it is worthwhile to consider de novo the behavior of gases at a wave-front, without reference to the existence of the derivative $dp/d\rho$. In the case of shock-waves, of course, this involves nothing new, since these are discontinuities of the quantities p, ρ, and the velocity vector $v = (v_x, v_y, v_z)$ and the behavior of derivatives of those quantities does not enter into the discussion. We shall, therefore, be concerned with waves of higher order, that is to say, with waves whose presence is manifested by discontinuities in the derivatives of p, ρ, v. However, certain precise and interesting analogies between the theory of higher order discontinuities developed here and the theory of shocks are pointed out, and will be discussed further at a later date. In addition, a result concerning the flow of ideal metals incompressible at pressures less than some critical value but infinitely compressible at that value, which has been previously established by Teller and von Neumann is derived as an immediate consequence of one of the general laws here set forth.

II. We consider then a body of gas which occupies, at the time t, a volume V_t of space. The state of motion of such a gas during a time period T is completely characterized by the definition of the functions p, ρ, v over the re-

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gion $V = V_t \times T$ in space-time, these functions being constricted to satisfy the equation of motion

$$\text{grad } p = -\rho \mathring{v} \tag{1}$$

and the equation of continuity

$$\mathring{\rho} = -\rho \text{ div } v \tag{2}$$

Since we are not concerned with shocks we assume that p, ρ, v are continuous[1] throughout V.

III. Now let us consider a three-dimensional differentiable sub-manifold S of V and denote by S_t its cross section in V_t, i.e., S_t is a surface within the region V_t whose position and shape are changing with time. We assume that the first derivatives of p, ρ, v in directions parallel to S are continuous across S, but admit the possibility that the first derivatives of these quantities in directions transverse to S may be discontinuous. In order to study the nature of these discontinuities, we choose coordinates so that at the point x, y, z, and time t, the (y,z)-plane is tangent to S_t, and so that velocities are measured relative to material velocities in the (y,z)-plane, relative to the velocity of S_t normal to the (y,z)-plane. Thus, in effect, at the point and instant in question, the surface is at rest and the flow of matter is normal to it. Then, if F is any one of the functions p, ρ, v_x, v_y, v_z, we have, by virtue of these conventions

$$dF/dt = \partial F/\partial t + v_x \, \partial F/\partial x \tag{3}$$

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1) This assumption also rules out vortex sheets, that is to say surfaces across which ρ, v are discontinuous, but across which no matter flows. But the remark concerning shocks in paragraph I applies also to these.

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Moreover, if Δ denotes a discontinuity across S, it follows from the fact that the surface S_t is instantaneously at rest that

$$\Delta\,(\partial F/\partial t) = 0 \tag{4}$$

which combined with (3) yields

$$\Delta\,(dF/dt) = v_x\,\Delta\,(\partial F/\partial x) \tag{5}$$

Finally, since S_t is tangent to the (y,z)-plane, we have

$$\Delta\,(\partial F/\partial y) = 0, \quad \Delta\,(\partial F/\partial z) = 0. \tag{6}$$

IV. Now let us apply Δ to the equations (1), (2). We obtain then, by virtue of the results of the preceding paragraph,

$$\left.\begin{aligned} \rho\,\Delta\,(dv_x/dt) &= -\,\Delta\,(\partial\rho/\partial x),\\ \Delta\,(dv_y/dt) &= 0,\\ \Delta\,(dv_z/dt) &= 0, \end{aligned}\right\} \tag{7}$$

$$\rho\,\Delta\,(\partial v_x/\partial x) = -\,\Delta\,(d\rho/dt). \tag{8}$$

From (5), we can write the first of the equations (7), and equation (8) in the form

$$\left.\begin{aligned} \rho\,v_x\,\Delta\,(\partial v_x/\partial x) &= -\,\Delta\,(\partial p/\partial x),\\ \rho\,\Delta\,(\partial v_x/\partial x) &= -\,v_x\,\Delta\,(\partial\rho/\partial x), \end{aligned}\right\} \tag{9}$$

which yield

$$\Delta\,(\partial p/\partial x) = v_x^2\,\Delta\,(\partial\rho/\partial x), \tag{10}$$

or alternatively,

$$\Delta\,(dp/dt) = v_x^2\,\Delta\,(d\rho/dt), \tag{11}$$

if $v_x \neq 0$.

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Now, recalling that coordinates were so chosen that the surface S_t is instantaneously at rest, we see that, in effect, $-v_x$ is the velocity of S_t at the point x, y, z and time t, relative to the material at that point. If we denote that velocity by c, we have

$$c^2 = \Delta\,(dp/dt)/\Delta\,(d\rho/dt) \tag{12}$$

if $d\rho/dt \neq 0$. If, at the point in question, $dp/d\rho$ exists, this becomes

$$c^2 = dp/d\rho \tag{13}$$

and yields, therefore, the usual result that a surface of discontinuity of $d\rho/dt$ is a sound-wave front. On the other hand, even if $dp/d\rho$ does not exist, equation (12) provides a means of calculating wave-velocity, where $d\rho/dt$ is discontinuous.

V. Suppose now that the first derivatives are continuous, or more generally that all derivatives of order n-1 or less are continuous, but that derivatives of order n are discontinuous. Let us assume, however, that derivatives of order n in directions parallel to S are continuous. Then if we differentiate (3) n-1 times with respect to t, and then apply Δ, we obtain the precise analogons of formulae (4), (5), (6), that is to say

$$\Delta\,(\partial^n F/\partial t^n) = 0, \tag{4'}$$

$$\Delta\,(d^n F/dt^n) = v_x^n\,\Delta\,(\partial^n F/\partial x^n), \tag{5'}$$

$$\Delta\,(\partial^n F/\partial y^n) = \Delta\,(\partial^n F/\partial z^n) = 0. \tag{6'}$$

Moreover, if we now differentiate (1), (2), (n-1) times with respect to t, and then apply Δ, we obtain

$$\left.\begin{aligned} \rho\,\Delta\,(d^n v_x/dt^n) &= -\,v_x^{n-1}\,\Delta\,(\partial^n p/\partial x^n), \\ \Delta\,(d^n v_y/dt^n) &= 0, \\ \Delta\,(d^n v_z/dt^n) &= 0, \end{aligned}\right\} \tag{7'}$$

$$\rho\, v_x^{n-1}\,(\partial^n v_x/\partial x^n) = -\,\Delta\,(d^n\rho/dt^n) \tag{8'}$$

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Applying (5') in (7'), (8'), we have

$$\left.\begin{aligned} \rho\, v_x\, \Delta\,(\partial^n v_x/\partial x^n) &= -\,\Delta\,(\partial^n p/\partial x^n) \\ \rho\, \Delta\,(\partial^n v_x/\partial x^n) &= -\,v_x\, \Delta\,(\partial^n \rho/\partial x^n) \end{aligned}\right\} \qquad (9')$$

and so, as before

$$\Delta\,(\partial^n p/\partial x^n) = v_x^2\, \Delta\,(\partial^n \rho/\partial x^n) \qquad (10')$$

or, alternatively

$$\Delta\,(d^n p/dt^n) = v_x^2\, \Delta\,(d^n \rho/dt^n) \qquad (11')$$

Thus, again, if c is the velocity of propagation of the wave in question relative to the moving gas, we have

$$c^2 = \Delta\,(d^n p/dt^n) \big/ \Delta\,(d^n \rho/dt^n), \qquad (12')$$

where n is the smallest integer for which $d^n\rho/dt^n$ is discontinuous. Since the right member of (12') reduces to $dp/d\rho$ when that derivative exists, we have incidentally established the theorem of Hadamard that all discontinuities of order greater than zero in p, ρ, v propagate with sound velocity. At the same time, formula (12') gives the velocity of wave propagation irrespective of any assumption regarding the relationship between p, ρ. While we have not proved it, it is interesting to observe that (12') is true for $n = 0$ also if we insert a factor $1 + \Delta\rho/\rho$ on the right, and becomes in that case

$$c^2 = (\rho_1/\rho_0)(p_1 - p_0)/(\rho_1 - \rho_0)$$

the usual formula for shock-wave velocity relative to the material in front of it. In this connection, indeed, one may observe that the equations (9') may be written

$$\left.\begin{aligned} \rho\, v_x\, \Delta\,(\partial^n v_x/\partial x^n) &= -\,\Delta\,(\partial^n p/\partial x^n) \\ \Delta\,(\partial^n (\rho v_x)/\partial x^n) &= 0 \end{aligned}\right\} \qquad (14)$$

which are also true for $n = 0$, becoming in that case simply the laws of conservation of momentum and mass across a shock. The possible significance of these analogies will be studied later.

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VI. We have, throughout the preceding discussion, implicitly assumed $v_x \neq 0$, and it is worthwhile to consider briefly the alternative case assuming that there is a discontinuity of order n. From formula (5') it follows in this case that all total time derivatives are continuous across S, and this leads with the equations of motion to the conclusion that $\partial^n p/\partial x^n$, $\partial^n v_x/\partial x^n$ are continuous. Nothing, however, can be said about the quantities $\partial^n \rho/\partial x^n$, $\partial^n v_y/\partial x^n$, $\partial^n v_z/\partial x^n$, and thus discontinuities of order n which do not propagate but along which one or more of these three derivatives are discontinuous may be expected. These are the precise n^{th} order analogons of vortex sheets, or of density discontinuities if $\partial^n v_y/\partial x^n$, $\partial^n v_z/\partial x^n$ are continuous.

VII. Equations (10), (11') have the following important application. Consider a substance whose equation of state has the form

$$\begin{aligned} &\rho = \rho_o = \text{const.}, \ p < p_o \\ &p = p_o = \text{const.}, \ \rho_o < \rho < \rho_1 \end{aligned} \qquad (15)$$

where ρ_1 may be ∞, and we are in any case interested only in the behavior of the gas at densities $\rho_o \leqq \rho < \rho_1$. Consider a surface S_t which at the time t is a boundary between the substance in the incompressible phase $\rho = \rho_o$, and the substance in the phase $p = p_o$. Then clearly, from (15), at any point of S_t,

$$\Delta(\partial p/\partial x) = - a \, \Delta(\partial \rho/\partial x), \ a \geqq 0$$

Combined with (10) in the case that there is material flow across the surface, this yields

$$\Delta(\partial p/\partial x) = \Delta(\partial \rho/\partial x) = 0, \qquad (16)$$

and formula (11') provides the means for determining the velocity of propagation of S_t in terms of the behavior of p, ρ on the two sides. If there is no material flow, then of course, $\partial^n \rho/\partial x^n$ may be discontinuous for some n, but $\partial^n p/\partial x^n$ is

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continuous. Equation (16) has been established before both by Teller and von Neumann. Special importance attaches to it, since problems in the flow of materials with such an equation of state as (15) may be expected to be underdetermined if no account is taken of it.

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