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SOLUTION OF THE TRANSPORT EQUATION BY S_n APPROXIMATIONS

Report written by:
Bengt G. Carlson

PHYSICS & MATHEMATICS

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PHYSICS & MATHEMATICS

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Los Alamos Report Library	1-20
AF Plant Representative, Burbank	21
AF Plant Representative, Seattle	22
AF Plant Representative, Wood-Ridge	23
ANP Project Office, Fort Worth	24
Argonne National Laboratory	25-32
Armed Forces Special Weapons Project (Sandia)	33
Army Chemical Center	34
Atomic Energy Commission, Washington	35-37
Battelle Memorial Institute	38
Brookhaven National Laboratory	39-41
Bureau of Ships	42
California Research and Development Company	43-44
Carbide and Carbon Chemicals Company (C-31 Plant)	45
Carbide and Carbon Chemicals Company (K-25 Plant)	46-47
Carbide and Carbon Chemicals Company (ORNL)	48-53
Carbide and Carbon Chemicals Company (Y-12 Plant)	54-57
Chicago Patent Group	58
Chief of Naval Research	59
Columbia University (Havens)	60
Commonwealth Edison Company	61
Department of the Navy - Op-362	62
Detroit Edison Company	63
Directorate of Research (WADC)	64
duPont Company, Augusta	65-67
Foster Wheeler Company	68
General Electric Company (ANPP)	69-71
General Electric Company, Richland	72-75
Goodyear Atomic Corporation	76-77
Hanford Operations Office	78
Iowa State College	79
Kirtland Air Force Base	80
Knolls Atomic Power Laboratory	81-84
Massachusetts Institute of Technology (Kaufmann)	85
Monsanto Chemical Company	86
Mound Laboratory	87-89
National Advisory Committee for Aeronautics, Cleveland	90
National Bureau of Standards	91
Naval Medical Research Institute	92
Naval Research Laboratory	93-94
New Brunswick Laboratory	95
New York Operations Office	96-97
North American Aviation, Inc.	98-100
Nuclear Development Associates, Inc.	101
Patent Branch, Washington	102
Phillips Petroleum Company	103-106
Pratt & Whitney Aircraft Division (Fox Project)	107
RAND Corporation	108
Sandia Corporation	109
USAF-Headquarters	110
U.S. Naval Radiological Defense Laboratory	111
UCLA Medical Research Laboratory (Warren)	112
University of California Radiation Laboratory, Berkeley	113-117
University of California Radiation Laboratory, Livermore	118-120
University of Rochester	121-122
Vitro Corporation of America	123-124
Walter Kidde Nuclear Laboratories, Inc.	125
Westinghouse Electric Corporation	126-129
Yale University	130
Technical Information Service, Oak Ridge	131-145

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ABSTRACT

A method for reducing the Transport Equation, an integro-differential equation, to a set of ordinary differential equations is introduced. The reduction is applied to the case of spherical symmetry, and numerical methods for solving the resulting set of equations are developed. The stationary, as well as the time-dependent, equations are considered in detail.

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1. <u>Introduction.</u> The present report describes, in brief outline, a new approach to the solution of a large class of diffusion problems. This approach, referred to here as the S_n Method, is, in generality of application, comparable to the well known Spherical Harmonic Method.

The equation which represents the mathematical formulation of diffusion problems involves, in general, both an integral operator and a first-order partial differential operator. In neutron diffusion work, it is usually referred to as the Transport Equation. Exact analytical solutions are obtainable only in the very simplest cases. Many approximate methods have, therefore, been proposed and explored. Some of these, including the S_n Method, can be used to obtain approximate analytical solutions if sufficiently simple problems are considered. In general, regardless of method, one has to resort to numerical methods, usually difference techniques. Here the S_n Method seems to offer many advantages over earlier methods.

2. <u>The Transport Equation.</u> For the case of spherical geometry, one neutron velocity group, and isotropic scattering the Transport Equation has the following form:

$$(1) \qquad \left[\mu D_r + \frac{1-\mu^2}{r} D_\mu + \sigma_k\right] N(r,\mu) = \sigma_k c_k N(r).$$

$N(r)$ is given by (2), and the bracket denotes a partial differential operator. $N(r,\mu)$ represents the neutron flux (neuts/cm^{2}sec) at the radial distance r (cm) in the direction θ ($\mu = \cos\theta$) with respect to the r-direction. The parameters σ_k (colls/neut·cm) and c_k (neuts/coll) describe the media which make up the sphere, and are regarded as step functions of r. For the central sphere k equals one, for the first shell k equals two, etc. The average flux $N(r)$ is defined by:

$$(2) \qquad N(r) = \frac{1}{2}\int_{-1}^{1} N(r,\mu)d\mu.$$

Equation (1) also implies a time-independent situation and the absence of separate neutron sources. Generalizations of this case are discussed in Sections 10 and 11.

3. <u>Definition of S_n Approximations.</u> We divide the μ-interval (-1, 1) into n intervals (μ_{j-1}, μ_j), $j = 1,2,\ldots,n$, $\mu_0 = -1$, $\mu_n = 1$, and approximate $N(r,\mu)$ by n connected straight line segments as follows:

$$(3) \qquad N(r,\mu) = N(r,\mu_{j-1}) + \frac{\mu-\mu_{j-1}}{\mu_j-\mu_{j-1}}\left[N(r,\mu_j) - N(r,\mu_{j-1})\right],$$

where $\mu_{j-1} \le \mu \le \mu_j$, $j = 1,2,\ldots,n$. The integro-differential equation (1) can now be reduced to a system of n + 1 ordinary differential equations in the functions $N(r,\mu_j)$, $j = 0,1,\ldots,n$. The reduction is accomplished by:

(A) substituting $\mu = -1$ directly in (1), obtaining one equation, and by:

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(B) substituting (3) in (1) and then integrating both sides of (1) over μ from $\mu = \mu_{j-1}$ to $\mu = \mu_j$, $j = 1, 2, \ldots, n$, thus obtaining n additional equations.

4. The S_n Equations. Performing the above operations we obtain the following general S_n equations:

(4) $$\left[-D_r + \sigma_k\right] N(r,-1) = \sigma_k c_k N(r), \text{ and}$$

(5) $$\left[a_2 D_r + \frac{b}{r} + 3\sigma_k\right] N(r,\mu_j) + \left[a_1 D_r - \frac{b}{r} + 3\sigma_k\right] N(r,\mu_{j-1}) = 6\sigma_k c_k N(r),$$

where $a_2 = 2\mu_j + \mu_{j-1}$, $a_1 = \mu_j + 2\mu_{j-1}$, and $b = 2(3 - \mu_j^2 - \mu_j\mu_{j-1} - \mu_{j-1}^2)/(\mu_j - \mu_{j-1})$. For reasons which we shall not go into here we prefer to divide the μ-interval (-1,1) into an even number of intervals of equal length.

5. The S_2 Equations: We have two intervals: (-1,0) and (0, -1), and denote N(r, -1), N(r,0) and N(r,1) by $\overleftarrow{N}(r)$, $\overline{N}(r)$, and $\overrightarrow{N}(r)$, respectively. From (4) and (5) we obtain the following three differential equations:

(6) $$\begin{cases} (-D_r + \sigma_k)\overleftarrow{N}(r) = \sigma_k c_k N(r), \\ (-D_r + \frac{4}{r} + 3\sigma_k)\overline{N}(r) = (2D_r + \frac{4}{r} - 3\sigma_k)\overleftarrow{N}(r) + 6\sigma_k c_k N(r), \\ (2D_r + \frac{4}{r} + 3\sigma_k)\overrightarrow{N}(r) = (-D_r + \frac{4}{r} - 3\sigma_k)\overline{N}(r) + 6\sigma_k c_k N(r). \end{cases}$$

6. The S_4 Equations: We have four intervals: (-1, -1/2), (-1/2,0), (0, 1/2), and (1/2, 1), and denote N(r, -1), N(r, -1/2), N(r,0), N(r, 1/2), and N(r,1) by $\overleftarrow{N}(r)$, $\overleftarrow{M}(r)$, $\overline{N}(r)$, $\overrightarrow{M}(r)$, and $\overrightarrow{N}(r)$, respectively. From (4) and (5) we obtain the following five differential equations:

(7) $$\begin{cases} (-D_r + \sigma_k)\overleftarrow{N}(r) = \sigma_k c_k N(r), \\ (-2D_r + \frac{5}{r} + 3\sigma_k)\overleftarrow{M}(r) = (\frac{5}{2}D_r + \frac{5}{r} - 3\sigma_k)\overleftarrow{N}(r) + 6\sigma_k c_k N(r), \\ (-\frac{1}{2}D_r + \frac{11}{r} + 3\sigma_k)\overline{N}(r) = (D_r + \frac{11}{r} - 3\sigma_k)\overleftarrow{M}(r) + 6\sigma_k c_k N(r), \\ (D_r + \frac{11}{r} + 3\sigma_k)\overrightarrow{M}(r) = (-\frac{1}{2}D_r + \frac{11}{r} - 3\sigma_k)\overline{N}(r) + 6\sigma_k c_k N(r), \\ (\frac{5}{2}D_r + \frac{5}{r} + 3\sigma_k)\overrightarrow{N}(r) = (-2D_r + \frac{5}{r} - 3\sigma_k)\overrightarrow{M}(r) + 6\sigma_k c_k N(r). \end{cases}$$

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7. The Numerical Procedure. We let N(r) in equations (4) and (5) be given initially as identically equal to unity or some other suitable trial function, and go through an iterative procedure to improve N(r), each time solving (4) and (5) for the n + 1 flux functions $N(r,\mu_j)$, obtaining the next iterate or improved version of N(r) from equation (8) below:

$$N(r) = \frac{1}{n} \sum_{j=0}^{n} w_j \, N(r,\mu_j), \tag{8}$$

where $w_0 = w_n = 1/2$, and $w_j = 1$, $j = 1,2,\ldots, n-1$.
A convergence test of some kind is applied to N(r) and its successor after each iteration. If it leads to the termination of the iterative procedure, we say that (4) and (5) have been solved numerically relative to this particular test.

The necessary stability conditions on the integration of equations (4) and (5) are: (a) That the first $\frac{n}{2} + 1$ equations be integrated in the negative r-direction, i.e., from the outer boundary (r = a) inward to the center, and (b) that the remaining $\frac{n}{2}$ equations be integrated in the opposite or positive direction. If these conditions are not satisfied, the significant figures in the calculation will rapidly be obscured by an accumulation of errors.

For the first group of equations ($\mu_j \leq 0$) the initial conditions are applied at r = a and given by: $N(a,\mu_j) = 0$. For the second group ($\mu_j > 0$) the initial conditions are applied at the origin and given by:

$$D_r\left[N(0,\mu_j)\right] = -D_r\left[N(0,-\mu_j)\right] = \mu_j\sigma_1\left[c_1 N(0) - \overleftarrow{N}(0,-1)\right] \tag{9}$$

At interfaces one simply changes parameters and applies continuity conditions on the flux functions.

Summed up, the S_n-Method has the following important features: (A) the n + 1 equations (4) and (5) need not be solved simultaneously, (B) further complications of the left hand sides of (4) and (5) do not require any changes in the basic procedure, and (C) the stability conditions as well as (D) the boundary conditions are simple in formulation and simple to apply.

To perform the numerical integrations a radial mesh must be introduced (if possible defining equal intervals in each medium) in the core and in each of the spherical shells. This gives rise to a set of r_i's, $i = 0,1,\ldots,I$, $r_o = 0$, $r_I = a$, where we let r_{i_k} denote the interfaces, $k = 1,2,\ldots,K$, $r_{i_K} = r_I = a$. To assure accuracy in the final result, no interval should be longer than 1/n mean free path units.

8. The S_2 Difference Equations: From (6) we derive the following difference equations if we center the functions at $i + \frac{1}{2}$ and use two-point formulae for derivatives and averages:

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$$
(10)\begin{cases}
\overleftarrow{N}_i = \dfrac{(2 - \sigma_k\Delta_k)\,\overleftarrow{N}_{i+1} + \sigma_k\Delta_k c_k(N_i + N_{i+1})}{2 + \sigma_k\Delta_k}, \\[2ex]
\bar{N}_i = \dfrac{(\frac{2}{3} - \sigma_k\Delta_k - 4s_{i+1})\bar{N}_{i+1} + (\frac{4}{3} - \sigma_k\Delta_k + 4s_{i+1})\overleftarrow{N}_{i+1} - (\frac{4}{3} + \sigma_k\Delta_k - 4s_i)\overleftarrow{N}_i + 2\sigma_k\Delta_k c_k(N_i + N_{i+1})}{\frac{2}{3} + \sigma_k\Delta_k + 4s_i}, \\[2ex]
\overrightarrow{N}_{i+1} = \dfrac{(\frac{4}{3} - \sigma_k\Delta_k - 4s_i)\overrightarrow{N}_i + (\frac{2}{3} - \sigma_k\Delta_k + 4s_i)\bar{N}_i - (\frac{2}{3} + \sigma_k\Delta_k - 4s_{i+1})\bar{N}_{i+1} + 2\sigma_k\Delta_k c_k(N_i + N_{i+1})}{\frac{4}{3} + \sigma_k\Delta_k + 4s_{i+1}},
\end{cases}
$$

where $\Delta_k = r_{i+1} - r_i$, $s_i = \Delta_k/3r_i$. The following special formula is used to start the outward integration:

$$
(11)\quad \overrightarrow{N}_1 = \frac{4\overleftarrow{N}_o + \frac{4}{3}\bar{N}_1 - (\frac{8}{3} - \sigma_1\Delta_1)\overleftarrow{N}_1}{\frac{8}{3} + \sigma_1\Delta_1}
$$

9. The S_4 Difference Equations: From (7) we obtain, using the same difference technique as for S_2:

$$
(12)\begin{cases}
\overleftarrow{N}_i = \dfrac{(2 - \sigma_k\Delta_k)\,\overleftarrow{N}_{i+1} + \sigma_k\Delta_k c_k(N_i + N_{i+1})}{2 + \sigma_k\Delta_k}, \\[2ex]
\overleftarrow{M}_i = \dfrac{(\frac{4}{3} - \sigma_k\Delta_k - 5s_{i+1})\bar{M}_{i+1} + (\frac{5}{3} - \sigma_k\Delta_k + 5s_{i+1})\overleftarrow{N}_{i+1} - (\frac{5}{3} + \sigma_k\Delta_k - 5s_i)\overleftarrow{N}_i + 2\sigma_k\Delta_k c_k(N_i + N_{i+1})}{\frac{4}{3} + \sigma_k\Delta_k + 5s_i}, \\[2ex]
\bar{N}_i = \dfrac{(\frac{1}{3} - \sigma_k\Delta_k - 11s_{i+1})\bar{N}_{i+1} + (\frac{2}{3} - \sigma_k\Delta_k + 11s_{i+1})\overleftarrow{M}_{i+1} - (\frac{2}{3} + \sigma_k\Delta_k - 11s_i)\overleftarrow{M}_i + 2\sigma_k\Delta_k c_k(N_i + N_{i+1})}{\frac{1}{3} + \sigma_k\Delta_k + 11s_i}, \\[2ex]
\overrightarrow{M}_{i+1} = \dfrac{(\frac{2}{3} - \sigma_k\Delta_k - 11s_i)\overrightarrow{M}_i + (\frac{1}{3} - \sigma_k\Delta_k + 11s_i)\bar{N}_i - (\frac{1}{3} + \sigma_k\Delta_k - 11s_{i+1})\bar{N}_{i+1} + 2\sigma_k\Delta_k c_k(N_i + N_{i+1})}{\frac{2}{3} + \sigma_k\Delta_k + 11s_{i+1}}, \\[2ex]
\overrightarrow{N}_{i+1} = \dfrac{(\frac{5}{3} - \sigma_k\Delta_k - 5s_i)\overrightarrow{N}_i + (\frac{4}{3} - \sigma_k\Delta_k + 5s_i)\overrightarrow{M}_i - (\frac{4}{3} + \sigma_k\Delta_k - 5s_{i+1})\overrightarrow{M}_{i+1} + 2\sigma_k\Delta_k c_k(N_i + N_{i+1})}{\frac{5}{3} + \sigma_k\Delta_k + 5s_{i+1}},
\end{cases}
$$

and the following special formulae for the outward integrations:

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$$
(13)\quad
\begin{cases}
\vec{M}_1 = \dfrac{\overleftarrow{2N}_0 + \frac{20}{3}\bar{N}_1 - (\frac{13}{3} - \sigma_1\Delta_1)\overleftarrow{M}_1}{\frac{13}{3} + \sigma_1\Delta_1}, \\[2ex]
\vec{N}_1 = \dfrac{\overleftarrow{6N}_0 - (\frac{10}{3} - \sigma_1\Delta_1)\overleftarrow{N}_1 + (\frac{1}{3} + \sigma_1\Delta_1)\overleftarrow{M}_1 + (\frac{1}{3} - \sigma_1\Delta_1)\vec{M}_1}{\frac{10}{3} + \sigma_1\Delta_1}
\end{cases}
$$

In deriving (10) through (13), Δ has been assumed to vary with k only.

10. Generalizations.

(A) Time-dependence: Insert $\frac{1}{v} D_t$ inside the operator bracket of (4), and $\frac{3}{v} D_t$ inside the operator brackets of (5), where t (shakes) is the time variable and v (cm/shake) is the neutron velocity. The time variable t is added to all flux functions. Cf Section 11.

(B) Many velocity groups: Attach a subscript g (for group, g = 1,2,...,G) to the flux functions in (4) and (5), and replace $\sigma_k c_k N(r)$ on the right hand sides by:

$$
(14)\qquad \sum_{g'=1}^{G} \sigma_{kg'}\, c_{kgg'}\, N_{g'}(r),
$$

where $\sigma_{g'}$ is the inverse mean free path as a function of velocity group, and $c_{gg'}$ is the number of neutrons transferred to group g per collision of neutrons of velocity $v_{g'}$.

(C) Anisotropic scattering: In the case of linear scattering with the scattering function (1/2) $(1 + 3b_k\mu)$ attached to elastic scattering (other processes assumed to be isotropic), $\sigma_k c_k N(r)$ in (1) is replaced by:

$$
(15)\qquad \sigma_k c_k \left[N(r) + 3\beta_k \mu \mathcal{N}(r)\right] ,
$$

where $\mathcal{N}(r) = (\frac{1}{2}) \int_{-1}^{1} \mu N(r,\mu) d\mu$, and $\beta_k = e_k\, b_k/c_k$; e_k being the probability of elastic collision. In the Transport Theory the anisotropic case is approximated by an isotropic situation in which σ_k is replaced by $\sigma_k(1 - c_k\beta_k)$ and c_k by $c_k(1 - \beta_k)/(1 - c_k\beta_k)$.

(D) Source term present: Replace $\sigma_k c_k N(r)$ in (4) and (5) by $\sigma_k c_k N(r) + S(r)$, where S(r) is the source density (neuts/cm^3 sec). An anisotropic source term can, of course, also be handled by going back to equation (1).

(E) Plane geometry: Drop all terms involving 1/r in the above equations, and let r be the perpendicular distance from some origin plane.

11. The Time-Dependent Case. We consider again equation (1) but with the time variable t added. Equations (4) and (5) are then replaced by (17) and (18) below:

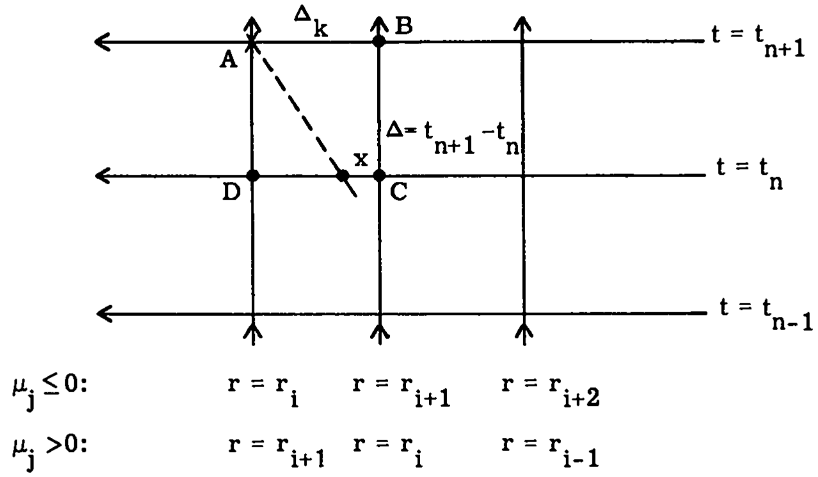
[redacted]

(17) $$\left[\frac{1}{v}D_t - D_r + \sigma_k\right] N(t,r,-1) = \sigma_k c_k N(t,r)$$

(18) $$\left[\frac{3}{v}D_t + a_2 D_r + \frac{b}{r} + 3\sigma_k\right] N(t,r,\mu_j) + \left[\frac{3}{v}D_t + a_1 D_r - \frac{b}{r} + 3\sigma_k\right] N(t,r,\mu_{j-1}) = 6\sigma_k c_k N(t,r).$$

If $N(t, r_i, \mu_j)$, $i = 0,1\ldots,I$, $j = 0,1\ldots,n$, and hence $N(t, r_i)$, are specified at time $t = t_0$ one can, by numerical integration of (17) and (18), obtain $N(t, r_i, \mu_j)$ at later times, say at $t = t_m$, $m = 1,2\ldots$. The following integration method may be used for the purpose. It has these main features: (A) The stability and boundary conditions of Section 7 are left unchanged, and (B) the resulting difference equations are similar to those of Sections 8 and 9.

The integration requires a two-dimensional mesh, here defined by the perpendicular line families $r = r_i$ and $t = t_m$. To integrate (17) and (18) numerically is then equivalent to finding $N(t, r, \mu_j)$ at A from functions known at B, C, and D. See diagram below:



In the stationary case the difference equations were obtained by averaging derivatives and functions over the line AB. In the time-dependent case we choose the characteristic line or "particle path" Ax as the reference line. The slope of AE equals $-1/v$ for equation (17) and $3/va_2$ for equation (18), where a_2 is defined on p. 5. The magnitude of the slope (denoted by $1/dv$) may be greater (x on DC) or less (x on BC) than Δ/Δ_k. If a function F has to be evaluated at x, the following interpolation formula applies:

(19) $$F^x \equiv F(x) = \begin{cases} w_1 F(C) + (1-w_1)\, F(D), \quad w_1 = \dfrac{dv\Delta}{\Delta k}, & dv\Delta \le \Delta_k \\ w_1 F(C) + (1-w_1)\, F(B), \quad w_1 = \dfrac{\Delta k}{dv\Delta}, & dv\Delta > \Delta_k \end{cases}$$

12. <u>The S_4 Time-Dependent Difference Equations.</u> Averaging derivatives and functions in (17) and (18) over the appropriate characteristic lines we obtain (n = 4):

$$
(20)\quad \left\{
\begin{aligned}
\overleftarrow{N}_i^{m+1} &= \frac{(2-\sigma_k\Delta')N^x_{i+1} + \sigma_k\Delta' c_k(N_i^{m+1}+N^x_{i+1})}{2+\sigma_k\Delta'}, \\
\overleftarrow{M}_i^{m+1} &= \frac{(\frac{4}{3}-\sigma_k\Delta'-5s_{i+1})\overleftarrow{M}^x_{i+1} + (\frac{4}{3}-\sigma_k\Delta'+5s_{i+1})\overleftarrow{N}^x_{i+1} - (\frac{4}{3}+\sigma_k\Delta'-5s_i)\overleftarrow{N}_i^{m+1} + 2\sigma_k\Delta' c_k(N_i^{m+1}+N^x_{i+1}) + \overleftarrow{N}'_i}{\frac{4}{3}+\sigma_k\Delta'+5s_i}, \\
\bar{N}_i^{m+1} &= \frac{(\frac{1}{3}-\sigma_k\Delta'-11s_{i+1})\bar{N}^x_{i+1} + (\frac{1}{3}-\sigma_k\Delta'+11s_{i+1})\overleftarrow{M}^x_{i+1} - (\frac{1}{3}+\sigma_k\Delta'-11s_i)\overleftarrow{M}_i^{m+1} + 2\sigma_k\Delta' c_k(N_i^{m+1}+N^x_{i+1}) + \overleftarrow{M}'_i}{\frac{1}{3}+\sigma_k\Delta'+11s_i}, \\
\overrightarrow{M}_{i+1}^{m+1} &= \frac{(\frac{2}{3}-\sigma_k\Delta'-11t_i)\overrightarrow{M}^x_i + (\frac{2}{3}-\sigma_k\Delta'+11t_i)\bar{N}^x_i - (\frac{2}{3}+\sigma_k\Delta'-11t_{i+1})\bar{N}^{m+1}_{i+1} + 2\sigma_k\Delta' c_k(N^x_i+N^{m+1}_{i+1}) + \bar{N}'_i}{\frac{2}{3}+\sigma_k\Delta'+11t_{i+1}}, \\
\overrightarrow{N}_{i+1}^{m+1} &= \frac{(\frac{5}{3}-\sigma_k\Delta'-5t_i)\overrightarrow{N}^x_i + (\frac{5}{3}-\sigma_k\Delta'+5t_i)\overrightarrow{M}^x_i - (\frac{5}{3}+\sigma_k\Delta'-5t_{i+1})\overrightarrow{M}^{m+1}_{i+1} + 2\sigma_k\Delta' c_k(N^x_i+N^{m+1}_{i+1}) + \overrightarrow{M}'_i}{\frac{5}{3}+\sigma_k\Delta'+5t_{i+1}},
\end{aligned}
\right.
$$

where $s_i = \frac{\Delta'}{3r_i}$, $s_{i+1} = \frac{\Delta'}{3(r_i+\Delta')}$, $t_{i+1} = \frac{\Delta'}{3r_{i+1}}$, and $t_i = \frac{\Delta'}{3(r_{i+1}-\Delta')}$. If $dv\Delta \leq \Delta_k$, then $\Delta' = dv\Delta$, $w_1 = dv\Delta/\Delta_k$, $F'_i = \frac{1}{3}w_1(F^m_{i+1} - F^m_i)$; and if $dv\Delta > \Delta_k$, then $\Delta' = \Delta_k$, $w_1 = \Delta_k/dv\Delta$, $F'_i = \frac{1}{3}\left[w_1(F^m_{i+1} - F^m_i) + (1-w_1)(F^{m+1}_{i+1} - F^{m+1}_i)\right]$, where d = 1 for $\overleftarrow{N}$, $\frac{2}{3}$ for $\overleftarrow{M}$, $\frac{1}{6}$ for $\bar{N}$, $\frac{1}{3}$ for $\overrightarrow{M}$, and $\frac{5}{6}$ for $\overrightarrow{N}$.

Since N_i^{m+1} is not known, we let $N_i^{m+1} = N_i^m$ for all i, solve the above equations to obtain a first approximation to N_i^{m+1}, and repeat the procedure with these values of N_i^{m+1} to obtain a final set. If the time intervals are large, the advance to $t = t_{m+1}$ may require more than one iteration.

In the case under consideration, $N(r,\mu)$ is linear in μ for small values of r. We may, therefore, let $\overrightarrow{M}_1^{m+1} = 2\bar{N}_1^{m+1} - \overleftarrow{M}_1^{m+1}$, and $\overrightarrow{N}_1^{m+1} = 2\bar{N}_1^{m+1} - \overleftarrow{N}_1^{m+1}$. The formulae corresponding to (13), although possibly more accurate, would be very complicated.

The methods described in this report, extended to the many-group case, have been coded for the IBM Type 701 electronic calculator and successfully applied to a variety of neutron diffusion problems. The tables at the end of this report compare S_2 and S_4 with other methods in a series of simple critical mass problems.

[redacted]

TABLE 1

UNTAMPED SPHERES

Critical radius (a) in mean free path units

c	EEM[1]	SWM[2]		S_4 Approx.		S_2
	a	a	% error[3]	a	% error[3]	a
1.1	4.873	4.722	-3.1	4.852	-.4	4.796
1.2	3.172	3.057	-3.6	3.156	-.5	3.095
1.3	2.425	2.331	-3.9	2.411	-.6	2.354
1.4	1.985	1.906	-4.0	1.971	-.7	1.920
1.5	1.690	1.621	-4.1	1.675	-.9	1.630
1.6	1.476	1.415	-4.1	1.463	-.9	1.420
1.7	1.312	1.258	-4.1	1.300	-.9	1.260
1.8	1.183	1.134	-4.1	1.171	-1.0	1.134
1.9	1.078	1.033	-4.2	1.066	-1.1	1.032
2.0	.990	.949	-4.1	.980	-1.0	.947
2.5	.707	.678	-4.1	.699	-1.1	.674
3.0	.551	.529	-4.0	.545	-1.1	.524

1. Extrapolated Endpoint Method, known to be in error by about -0.1% (LA-258).
2. Serber-Wilson Method (LA-234, 247, 756).
3. Compared to EEM.

[redacted]

TABLE 2

TAMPED SPHERES

Equal mean free path in core and tamper

Critical core radius (a) in m.f.p. units

c core	c tamper	Tamper thick-ness m.f.p.	EEM a	SWM a	SWM % error[2]	MDM[1] a	MDM[1] % error[2]	S_4 Approx. a	S_4 Approx. % error[2]	S_2 a
1.3	1.00	1.5	1.896	1.855	-2.2	1.868	-1.5	1.878	-.9	1.816
1.3	.95	1.5	1.973	1.931	-2.1	1.938	-1.8	1.959	-.7	1.894
1.3	.80	1.5	2.132	2.086	-2.2	2.076	-2.6	2.119	-.6	2.056
1.7	1.00	1.5	1.044	1.027	-1.6	1.001	-4.1	1.031	-1.2	.988
1.7	.95	1.5	1.075	1.058	-1.6	1.028	-4.4	1.063	-1.1	1.018
1.7	.80	1.5	1.148	1.127	-1.8	1.085	-5.5	1.136	-1.0	1.092
1.3	1.00	3.0	1.752	1.718	-1.9	1.722	-1.7	1.736	-.9	1.675
1.3	.95	3.0	1.886	1.849	-2.0	1.849	-2.0	1.869	-.9	1.808
1.3	.80	3.0	2.108	2.062	-2.2	2.049	-2.8	2.093	-.7	2.033
1.7	1.00	3.0	.992	.977	-1.5	.947	-4.5	.982	-1.0	.940
1.7	.95	3.0	1.043	1.026	-1.6	.993	-4.8	1.032	-1.1	.989
1.7	.80	3.0	1.137	1.117	-1.8	1.073	-5.6	1.125	-1.1	1.083

1. Modified Diffusion Method, similar to EEM but with interface correction only at the outside boundary.
2. Compared to EEM, known to be in error by about -0.1%.

[redacted]

TABLE 3

TAMPED SPHERES

Core m.f.p. equal to 3/2 of tamper m.f.p.

Critical core radius (a) in m.f.p. units

c core	c tamper	Tamper thick-ness m.f.p.	"Exact"[1] a	SWM a	SWM % error[2]	MDM a	MDM % error[2]	S_4 Approx. a	S_4 Approx. % error[2]	S_2 a
1.3	1.00	1.5	1.999	1.960	-2.0	2.027	1.4	1.981	-.9	1.910
1.3	.95	1.5	2.057	2.017	-1.9	2.083	1.3	2.040	-.8	1.971
1.3	.80	1.5	2.184	2.135	-2.2	2.193	.4	2.167	-.8	2.099
1.7	1.00	1.5	1.103	1.088	-1.4	1.116	1.2	1.091	-1.1	1.044
1.7	.95	1.5	1.125	1.110	-1.3	1.137	1.1	1.114	-1.0	1.068
1.7	.80	1.5	1.180	1.160	-1.7	1.183	.3	1.169	-.9	1.124
1.3	1.00	3.0	1.893	1.860	-1.7	1.921	1.5	1.876	-.9	1.808
1.3	.95	3.0	1.996	1.957	-2.0	2.020	1.2	1.979	-.9	1.909
1.3	.80	3.0	2.165	2.117	-2.2	2.176	.5	2.149	-.7	2.084
1.7	1.00	3.0	1.068	1.053	-1.4	1.076	.7	1.057	-1.0	1.012
1.7	.95	3.0	1.104	1.088	-1.4	1.113	.8	1.093	-1.0	1.048
1.7	.80	3.0	1.174	1.153	-1.8	1.176	.2	1.163	-.9	1.119

1. Obtained by extrapolation from S_2 and S_4 using Table 2 as a guide.
2. Compared to "Exact."

[redacted]

TABLE 4

TAMPED SPHERES

Core m.f.p. equal to 3/4 of tamper m.f.p.

Critical core radius (a) in m.f.p. units

c core	c tamper	Tamper thickness m.f.p.	"Exact" a	SWM		MDM		S_4 Approx.		S_2
				a	% error[1]	a	% error[1]	a	% error[1]	a
1.3	1.00	1.5	1.825	1.780	-2.5	1.774	-2.8	1.811	-.8	1.754
1.3	.95	1.5	1.917	1.871	-2.4	1.851	-3.4	1.902	-.8	1.840
1.3	.80	1.5	2.103	2.052	-2.4	2.004	-4.7	2.088	-.7	2.026
1.7	1.00	1.5	.997	.981	-1.6	.928	-6.9	.987	-1.0	.946
1.7	.95	1.5	1.035	1.018	-1.6	.957	-7.5	1.025	-1.0	.983
1.7	.80	1.5	1.122	1.102	-1.8	1.021	-9.0	1.112	-.9	1.070
1.3	1.00	3.0	1.651	1.615	-2.2	1.593	-3.5	1.638	-.8	1.586
1.3	.95	3.0	1.811	1.772	-2.2	1.736	-4.1	1.796	-.8	1.737
1.3	.80	3.0	2.074	2.024	-2.4	1.968	-5.1	2.059	-.7	1.998
1.7	1.00	3.0	.934	.918	-1.7	.862	-7.7	.924	-1.1	.886
1.7	.95	3.0	.995	.979	-1.6	.914	-8.1	.985	-1.0	.945
1.7	.80	3.0	1.109	1.089	-1.8	1.005	-9.4	1.099	-.9	1.058

1. Compared to "Exact."

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