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TRANSPORT PHENOMENA IN A MIXTURE OF ELECTRONS AND NUCLEI

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ABSTRACT

At extremely high temperatures atoms are stripped of all or most of their electrons. The mean free path of the electrons moreover is proportional to the square of their kinetic energy. The electrons will therefore cause the ionized gas to conduct both electricity and heat quite easily. By solving the Boltzmann equation for assumed gradients of density, electric potential and temperature, we find the velocity distribution of the electrons as an expansion in Laguerre polynomials. From the first two coefficients of this expansion we find the electric and thermal conductivities. The long range of Coulomb forces leads to difficulties (divergent integrals) if one restricts the discussion to binary collisions. This is avoided by considering a shielding effect due to the rearrangement of the electrons in the neighborhood of a colliding pair.

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TRANSPORT PHENOMENA IN A MIXTURE OF ELECTRONS AND NUCLEI

At extremely high temperatures atoms are stripped of all or most of their electrons. The free electrons cause the ionized gas to conduct both electricity and heat quite easily. This investigation uses the kinetic theory of gases to find expressions for the electric and thermal currents carried by the free electrons due to gradients of density, potential and temperature[1). To do this one has to determine the velocity distribution function of the electrons from the Boltzmann equation. We shall treat the problem in the approximation wherein the heavy particles show no deviation from the Maxwell distribution. It is not necessary to assume the electrons and the nuclei to be at the same temperature. We shall, however, not consider the resulting heat exchange and assume time-independent distribution functions. We confine ourselves to the linear problem with all gradients and currents in the x direction. For the electron distribution, we try the form:

$$\phi(x, \vec{v}) = f(x,v) \left[1 + v_x h(v)\right] \tag{1}$$

with

$$f(x,v) = n \beta^3 \pi^{-3/2} e^{-\beta^2 v^2} \tag{1a}$$

$$n = n(x), \quad \beta = \beta(x) = \sqrt{m/2kT_e}$$

n is the number of electrons per cm^3.

For the nuclei of type i, we assume a distribution

$$F_i(xv_i) = N_i B_i^3 \pi^{-3/2} e^{-B_i^2 v_i^2}$$

$$N_i = N_i(x) \qquad B_i = B_i(x) = \sqrt{M_i/2kT_i}$$

1) The method used here is essentially the one described by Chapman and Cowley in their book "The Mathematical Theory of Non-Uniform Gases" (Cambridge, 1939).

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The Boltzmann equation can be written as

$$D(\phi) = - J_{ee}(\phi,\phi) - \sum_i J_{ei}(\phi,F_i) \tag{2}$$

where

$$D(\phi) = v_x \frac{\partial\phi}{\partial x} + \frac{eE}{m}\frac{\partial\phi}{\partial v_x}, \quad (e = -4.8 \times 10^{-10}\,\text{esu}) \tag{3a}$$

$$J_{ee}(\phi\phi) = \iint w\sigma_{ee}(w\theta)\left\{\phi(\vec{v})\phi(\vec{v}_1) - \phi(\vec{v}')\phi(\vec{v}_1')\right\} d\vec{v}_1 \, d\Omega \tag{4a}$$

$\vec{w} = \vec{v} - \vec{v}_1$, θ is the angle between $\vec{w}$ and $\vec{w}'$, and $d\Omega$ is the element of solid angle in the direction of $\vec{w}'$. $\sigma_{ee}(w\theta)$ is the cross section for electron-electron scattering. Similarly for the collisions with nuclei we have

$$J_{ei}(\phi F_i) = \iint w_i\sigma_{ei}(w_i\theta)\left\{\phi(\vec{v})F_i(v_i) - \phi(\vec{v}')F_i(v_i')\right\} d\vec{v}_i d\Omega \tag{5a}$$

The standard procedure to obtain an approximate solution of the Boltzmann equation is to replace $D(\phi)$ by $D(f)$ but leave ϕ unchanged on the right-hand side. If we do this the left-hand side can be expressed in the following manner. We note first that:

$$\frac{1}{f}\frac{\partial f}{\partial x} = \frac{1}{n}\frac{dn}{dx} - \frac{1}{T_e}\frac{dT_e}{dx}\left(3/2 - \beta^2 v^2\right)$$

and

$$\frac{1}{f}\frac{\partial f}{\partial v_x} = -2\beta^2 v_x$$

which by combination leads to

$$D(f) = \left[\frac{1}{n}\frac{dn}{dx} - \frac{eE}{KT_e} - \frac{1}{T_e}\frac{dT_e}{dx}\left(3/2 - \beta^2 v^2\right)\right] v_x f \tag{3b}$$

On the right-hand side of (4a) we substitute ϕ from (1), and obtain:

$$J_{ee} = \iint w\sigma_{ee}(w\theta)f(v)f(v_1)\left[v_x h(v) + v_{1x}h(v_1) - v_x' h(v') - v_{1x}' h(v_1')\right] d\vec{v}_1 d\Omega \tag{4b}$$

J_{ei} can be greatly simplified by the observation that the heavy particles

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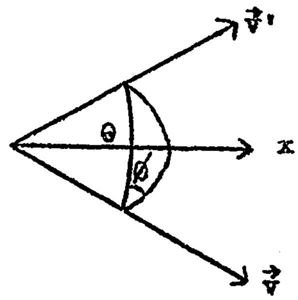
are much slower than the electrons so that w_i can be replaced by v. In addition, the only important collisions are those for which $1 - \cos\theta \ll 1$ so that $v_i' \approx v_i$ and $v' \approx v$. We may therefore write:

$$J_{ei} = v f(v) h(v) \iint \sigma_{ei}(v\theta) F_i(v_i)(v_x - v_x')\, d\vec{v}_i\, d\Omega$$

The integration with respect to v_i can be carried out at once and leads to:

$$J_{ei} = N_i v f(v) h(v) \int \sigma_{ei}(v\theta)(v_x - v_x')\, d\Omega$$



We note again that $v' \approx v$ and express v_x' thus:

$$v_x' = v(\cos\alpha \cos\theta + \sin\alpha \sin\theta \cos\phi)$$

Integration with respect to ϕ leads then to:

$$J_{ei} = 2\pi N_i v f(v) h(v) \int \sigma_{ei}(v\theta)(1 - \cos\theta)\, d(\cos\theta)$$

The cross section for collisions between electrons and nuclei with charge Z_i (Rutherford scattering) is:

$$\sigma_{ei} = \left(\frac{Z_i e^2}{m v^2}\right)^2 (1 - \cos\theta)^{-2}$$

We can thus reduce J_{ei} further to:

$$J_{ei} = 4\pi\lambda N_i \frac{v_x}{v^3} f(v)\; h(v) \qquad (5b)$$

where

$$\lambda = \frac{1}{2} \int (1 - \cos\theta)^{-1}\, d(\cos\theta) \qquad (6)$$

This last integration is carried out in Appendix I. After we enter (3b), (4b), and (5b) into the Boltzmann equation (2) we are left with the problem to find $h(v)$ from it. This can be done by expanding $h(v)$ in terms of Laguerre polynomials[2]. In particular we shall use the polynomials of order 3/2 and write for brevity:

$$L_r(\varepsilon) = L_r^{(3/2)}(\varepsilon)$$

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2) For a discussion of the properties of these polynomials the reader is referred to Chapter 5 of SZEGO, Orthogonal Polynomials.

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The L_r form a complete set and can be derived from their orthogonality relation:

$$\int_0^\infty \varepsilon^{3/2}\, e^{-\varepsilon} L_r(\varepsilon) L_s(\varepsilon)\, d\varepsilon = \frac{\Gamma(r + 5/2)}{\Gamma(r + 1)}\ \delta_{rs} \tag{7}$$

and $L_0(\varepsilon) = 1$. We note also that $L_1(\varepsilon) = 5/2 - \varepsilon$.

We express (3b) in the form:

$$D(f) = \left[\left(\frac{1}{n}\ \frac{dn}{dx} - \frac{eE}{KT_e} + \frac{1}{T_e}\ \frac{dT_e}{dx}\right) L_0(\beta^2 v^2) - \frac{1}{T_e}\frac{dT_e}{dx} L_1\ (\beta^2 v^2)\right]\ v_x f \tag{3c}$$

and substitute:

$$h(v) = \sum_{s=0}^{\infty} c_s L_s(\beta^2 v^2) \tag{8}$$

into (4b) and (5b).

We now multiply both sides of the Boltzmann equation by $v_x L_r(\beta^2 v^2)$ and integrate over $d\vec{v}$. On the left-hand side we obtain, using (7):

$$-A\delta_{0r} + B\delta_{1r} \tag{9}$$

where we have set:

$$A = \left(\frac{eE}{KT_e} - \frac{1}{n}\ \frac{dn}{dx} - \frac{1}{T_e}\ \frac{dT_e}{dx}\right)\ \frac{n}{2\beta^2} \tag{10a}$$

$$B = \frac{1}{T_e}\ \frac{dT_e}{dx}\quad \frac{5n}{4\beta^2} \tag{10b}$$

On the right-hand side we obtain: $-\Sigma\, C_s H_{rs}$

where $\quad H_{rs} = H_{rs}^{\ e} + \sum_i H_{rs}^{\ i}$

and where the $H_{rs}^{\ e}$ and $H_{rs}^{\ i}$ are defined as follows:

$$H_{rs}^{\ e} = \iiint w\sigma_{ee}(w\theta)\, f(v)\, f(v_1)\, v_x L_r(\beta^2 v^2)\, \Delta(v_x L_s)\, d\vec{v}\, d\vec{v}_1\, d\Omega \tag{11}$$

$$\Delta g = g(\vec{v}) + g(\vec{v}_1) - g(\vec{v}\,') - g(\vec{v}_1\,') \tag{12}$$

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and

$$H_{rs}^{\,i} = N_i Z_i^2 \; \frac{4\pi}{3} \lambda \left(\frac{e^2}{m}\right)^2 \int \frac{L_r L_s f}{v} \, d\vec{v} \tag{13}$$

The problem now consists in solving the set of equations

$$A\delta_{or} + B\delta_{1r} = \sum C_s H_{rs} \tag{14}$$

for the coefficients C_s. Actually we will need only the first two coefficients.

We can write the current:

$$j = \int v_x \phi \, d\vec{v} = \int v_x^2 \, f \, \Sigma \, C_s L_s d\vec{v} = \frac{n}{2\beta^2} \, C_o \tag{15}$$

and the heat current

$$q = \int \frac{mv^2}{2} \, v_x \phi \, d\vec{v} = kT_e \int (5/2 \, L_o - L_1) v_x^2 \, f \, \Sigma C_s L_s d\vec{v} \tag{16}$$

$$q = \frac{5n}{4\beta^2} (C_0 - C_1) kT_e$$

It is convenient to take a factor:

$$\mu = \frac{8\sqrt{\pi}}{3} \left(\frac{ne^2}{m}\right)^2 \lambda\beta \tag{17}$$

out of the matrix elements and to write:

$$H_{rs} = \mu h_{rs} \tag{18}$$

C_o and C_1 can be written formally in terms of the dimensionless matrix elements h_{rs} as follows:

[redacted]

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$$
C_0 = \mu^{-1}\frac{\begin{vmatrix} A & h_{01} & h_{02}\cdots \\ B & h_{11} & h_{12}\cdots \\ 0 & h_{21} & h_{22}\cdots \\ - & - & - \\ - & - & - \\ - & - & - \end{vmatrix}}{\begin{vmatrix} h_{00} & h_{01} & h_{02}\cdots \\ h_{10} & h_{11} & h_{12}\cdots \\ h_{20} & h_{21} & h_{22} \\ - & - & - \\ - & - & - \\ - & - & - \end{vmatrix}} ; \qquad C_1 = \mu^{-1}\frac{\begin{vmatrix} h_{00} & A & h_{02}\cdots \\ h_{10} & B & h_{12}\cdots \\ h_{20} & 0 & h_{22}\cdots \\ - & - & - \\ - & - & - \\ - & - & - \end{vmatrix}}{\begin{vmatrix} h_{00} & h_{01} & h_{02}\cdots \\ h_{10} & h_{11} & h_{12}\cdots \\ h_{20} & h_{21} & h_{22}\cdots \\ - & - & - \\ - & - & - \\ - & - & - \end{vmatrix}} \tag{19}
$$

However, these determinants are infinite and the following limiting process is used to get convergent results. We cut both numerator and denominator determinants off beyond the row and column carrying the index n, take the ratio, and repeat with larger n. To carry this through we write:

$$
D^{(n)} = \begin{vmatrix} h_{00}h_{01}\cdots\cdots\cdots h_{0n} \\ h_{n0}h_{n1}\cdots\cdots\cdots h_{nn} \end{vmatrix} \tag{20}
$$

we also use the minors $D_{ik}^{(n)}$ which are obtained by deleting the ith row and the kth column.

Then we form

$$
R_{ik} = \frac{D_{ik}}{D} = \lim_{n\to\infty} \frac{D_{ik}^{(n)}}{D^{(n)}} \tag{21}
$$

[redacted]

By substitution into (19) we obtain:

$$C_o = (AR_{oo} - BR_{10})\mu^{-1}, \qquad C_1 = (-AR_{01} + BR_{11})\mu^{-1} \tag{22}$$

The limiting process (21) can be carried through by means of a theorem on determinants by Sylvester[3]. From this theorem we obtain the relation:

$$\begin{vmatrix} D^{(n)} & D^{(n+1)}_{n+1,k} \\ D^{(n+1)}_{i,n+1} & D^{(n+1)}_{ik} \end{vmatrix} = D^{(n+1)} \quad D^{(n)}_{ik}$$

so that

$$\frac{D_{ik}^{(n+1)}}{D^{(n+1)}} = \frac{D_{ik}^{(n)}}{D^{(n)}} + \frac{D^{(n+1)}_{i,n+1}\,D^{(n+1)}_{n+1,k}}{D^{(n)}\;D^{(n+1)}}$$

and further:

$$R_{ik} = \frac{D_{ik}^{(1)}}{D^{(1)}} + \frac{D_{i2}^{(2)}D_{2k}^{(2)}}{D^{(1)}\;D^{(2)}} + \frac{D_{i3}^{(3)}D_{3k}^{(3)}}{D^{(2)}\;D^{(3)}} + \ldots\ldots \tag{23}$$

It is interesting to note a simplification which is introduced if one imposes the condition $j = 0$ or its equivalent $C_o = 0$. In this case we can eliminate A from (22) and obtain:

$$C_1 = \mu^{-1}\;\frac{D_{11}D_{00} - D_{10}D_{01}}{D\;D_{00}}\;B = \mu^{-1}\;\frac{D_{11,01}}{D_{00}}\;B \tag{24}$$

by another application of Sylvester's theorem. We therefore do not need the zero row and column of our matrix if we only want the heat conductivity. If, however, we also want to know the value of A, that is the electric field in the case $j = 0$ or if we want the coefficients in the more general case when $j \neq 0$ we have to use the complete matrix.

3) See e.g. Kowalewski, THEORIE DER DETERMINANTEN, theorem 30.

[redacted]

By combining (10a), (10b), (17) and (22) we are led to the equations:

$$j = \frac{3\lambda^{-1}}{4\sqrt{2\pi}} \left(\frac{m}{e^2}\right)^2 \left(\frac{kT_e}{m}\right)^{5/2} \left[\left(\frac{eE}{kT_e} - \frac{1}{n}\frac{dn}{dx} - \frac{1}{T_e}\frac{dT_e}{dx}\right) R_{00} - \frac{5}{2}\frac{1}{T_e}\frac{dT_e}{dx} R_{01}\right] \tag{25}$$

$$q = \frac{75\lambda^{-1}}{16\sqrt{2\pi}} \left(\frac{m}{e^2}\right)^2 \left(\frac{kT_e}{m}\right)^{5/2} kT_e \left[-\frac{1}{T_e}\frac{dT_e}{dx}\left(R_{01} + R_{11}\right) + \frac{2}{5}\left(\frac{eE}{kT_e} - \frac{1}{n}\frac{dn}{dx} - \frac{1}{T_e}\frac{dT_e}{dx}\right)\left(R_{00} + R_{01}\right)\right] \tag{26}$$

we are particularly interested in the case $j = 0$ where we find:

$$\frac{eE}{kT_e} = \frac{1}{n}\frac{dn}{dx} + \frac{1}{T_e}\frac{dT_e}{dx}\left(1 + \frac{5}{2}\frac{R_{01}}{R_{00}}\right) \tag{27}$$

$$q = -\frac{75\lambda^{-1}}{16\sqrt{2\pi}} \left(\frac{m}{e^2}\right)^2 \left(\frac{kT_e}{m}\right)^{5/2} k \left(\frac{R_{00}R_{11} - R_{01}^{\,2}}{R_{00}}\right)\frac{dT_e}{dx} \tag{28}$$

That is we get for the heat conductivity:

$$k = \frac{75\lambda^{-1}}{16\sqrt{2\pi}} \left(\frac{R_{00}R_{11} - R_{01}^{\,2}}{R_{00}}\right) kc \left(\frac{e^2}{mc^2}\right)^{-2} \left(\frac{kT_e}{mc^2}\right)^{5/2} \tag{29}$$

In this formula c was introduced to put the dimensions of k into evidence. For the value of λ see equation (46).

The integrations (11) and (13) are carried out in Appendix II. The matrix h_{rs} is seen to depend on an effective nuclear charge:

$$Z = \frac{\Sigma N_i Z_i^{\,2}}{n} \tag{30}$$

The computations were carried through for Z value of 1, 2, 2.5, 3 and ∞. The case $Z = \infty$ means that the term $h_{rs}^{\,e}$ resulting from electron-electron scattering was neglected. This case is very important because the Boltzmann equation can

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also be solved in a closed form, so that we are able to check our theory. The table below shows the first 4 terms and their totals of the series for R_{00}, R_{01} and R_{11}. Obviously we have $R_{10} = R_{01}$ on account of the symmetry. It also shows the important combinations, $1 + \frac{5}{2}\frac{R_{01}}{R_{00}}$ and $\frac{75}{16\sqrt{2\pi}}\frac{R_{00}R_{11} - R_{01}^2}{R_{00}}$ which occur in (27) and (29).

$Z \rightarrow$	1	2	2.5	3	∞
Terms of R_{00}					
1st	1.9320	1.1590	.9748	.8430	$3.2500\ Z^{-1}$
2nd	.0179	.0015	.0002	.0000	$.1406\ Z^{-1}$
3rd	.0117	.0023	.0011	.0005	$.0039\ Z^{-1}$
4th	.0041	.0006	.0002	.0001	$.0005\ Z^{-1}$
total	1.9657	1.1634	.9763	.8436	$3.3950\ Z^{-1}$
Terms of R_{01}					
1st	.6213	.4393	.3832	.3398	$1.5000\ Z^{-1}$
2nd	-.0668	-.0192	-.0063	.0027	$.5625\ Z^{-1}$
3rd	.0053	.0011	.0006	.0004	$-.0234\ Z^{-1}$
4th	-.0015	+.0005	-.0003	-.0001	$-.0014\ Z^{-1}$
total	.5583	.4207	.3772	.3428	$2.0377\ Z^{-1}$
Terms of R_{11}					
1st	.4142	.2929	.2555	.2265	$1.0000\ Z^{-1}$
2nd	.2494	.2504	.2440	.2360	$2.2500\ Z^{-1}$
3rd	.0024	.0006	.0004	.0003	$.1406\ Z^{-1}$
4th	.0005	.0004	.0003	.0002	$.0039\ Z^{-1}$
total	.6665	.5443	.5002	.4630	$3.3945\ Z^{-1}$
$1 + \frac{5}{2}\frac{R_{01}}{R_{00}}$	1.710	1.904	1.966	2.016	2.5005
$\frac{75}{16\sqrt{2\pi}}\left(\frac{R_{00}R_{11}-R_{01}^2}{R_{00}}\right)$	~~[illegible]~~ .9498	~~1.192~~ .7334	~~1.041~~ .6629	~~.971~~ .6053	~~6.3786~~ Z^{-1} 4.0607

For large Z we obtain by combining (3b) and (5b):

$$\frac{1}{n}\frac{dn}{dx} - \frac{eE}{kT_e} + \left(\beta^2 v^2 - \frac{3}{2}\right)\frac{1}{T_e}\frac{dT_e}{dx} = -4\pi\lambda n Z \left(\frac{e^2}{m}\right)^2 \frac{h(v)}{v^3} \tag{31}$$

[redacted]

[redacted]

For the electron and heat currents we have:

$$j = \int v_x \phi d\vec{v} = \int v_x^2 h f d\vec{v} = \frac{1}{3}\int v^2 h f d\vec{v} \tag{32}$$

$$q = \int \frac{m}{2} v^2 v_x \phi d\vec{v} = \frac{1}{3}\ \frac{m}{2}\int v^4 h f d\vec{v} \tag{33}$$

To carry these integrations through we need:

$$\int v^5 f\ d\vec{v} = 2n\pi^{-1/2}\beta^{-5} \int \varepsilon^3 e^{-\varepsilon} d\varepsilon = 2n\pi^{-1/2}\beta^{-5} \cdot 3! \tag{34}$$

and similarly

$$\int v^7 f d\vec{v} = 2n\pi^{-1/2}\beta^{-7}\ 4! \tag{35}$$

$$\int v^9 f d\vec{v} = 2n\pi^{-1/2}\beta^{-9}\ 5! \tag{36}$$

We substitute h(v) from (31) into (32) and (33) and use the integrals (34), (35), and (36) to obtain:

$$j = \frac{8}{\pi\sqrt{2\pi}}\lambda^{-1}Z^{-1}\left(\frac{e^2}{m}\right)^{-2}\left(\frac{kT_e}{m}\right)^{5/2}\left(\frac{eE}{kT_e} - \frac{1}{n}\frac{dn}{dx} - \frac{5}{2}\frac{1}{T_e}\frac{dT_e}{dx}\right) \tag{37}$$

$$q = \frac{32}{\pi\sqrt{2\pi}}\lambda^{-1}Z^{-1}\left(\frac{e^2}{m}\right)^{-2}\left(\frac{kT_e}{m}\right)^{5/2} kT_e\left(\frac{eE}{kT_e} - \frac{1}{n}\frac{dn}{dx} - \frac{7}{2}\frac{1}{T_e}\frac{dT_e}{dx}\right) \tag{38}$$

Comparing these equations with (25) and (26) we see that the following equalities should exist:

$$\frac{3}{4}R_{00} = \frac{8}{\pi}Z^{-1}; \qquad \frac{3}{4}\left(R_{00} + \frac{5}{2}R_{01}\right) = \frac{20}{\pi}Z^{-1}$$

$$\frac{15}{8}(R_{00} + R_{01}) = \frac{32}{\pi}Z^{-1}; \qquad \frac{75}{16}\left(R_{01} + R_{11} + \frac{2}{5}R_{00} + \frac{2}{5}R_{01}\right) = \frac{112}{\pi}Z^{-1}$$

That is we should have:

$$R_{00} = R_{11} = \frac{32}{3\pi}Z^{-1} = 3.395,305\ Z^{-1},\quad R_{01} = \frac{32}{5\pi}Z^{-1} = 2.03718\ Z^{-1}$$

[redacted]

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and for the derived quantities:

$$1 + \frac{5}{2}\,\frac{R_{01}}{R_{00}} = 2.5\ ; \qquad \frac{75}{16\sqrt{2\pi}}\,\frac{R_{00}R_{11} - R_{01}^{2}}{R_{00}} = \frac{32}{\pi\sqrt{2\pi}}\,Z^{-1} = \overset{4.0635}{\cancel{6.3830}}\,Z^{-1}$$

These values agree very well with those listed in the preceding table, so that we can be quite confident of the validity of our general method.

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APPENDIX I. CALCULATION OF λ

The integral λ given by (6)

$$\lambda = \frac{1}{2} \int_{\theta_1}^{\theta_2} (1 - \cos\theta)^{-1} \, d\cos\theta = \frac{1}{2} \ln(1 - \cos\theta) \Big|_{\theta_1}^{\theta_2}$$

diverges if one uses $\theta_1 = 0$ for the lower limit. The reason is that the kinetic theory, as it is used, restricts itself to the consideration of encounters between only two particles at a time. The Coulomb force law is, however, of such a nature that the possibility of interactions between more than two particles must not be excluded. Another, less catastrophic, difficulty arises out of the uncertainty principle which excludes the possibility of head-on collisions, because one has to consider an electron as being spread out over a region of the order of magnitude of its de Broglie wave length. To remedy the situation we, first of all, express λ in terms of the collision parameter p.

$$\lambda = \frac{1}{2} \ln\left[1 + \left(\frac{mv^2}{Ze^2}\right)^2 p^2\right] \Big|_{p_1}^{p_2} \qquad (39)$$

The lower limit is, according to the uncertainty principle, the de Broglie wave length. That is we have:

$$p_1 \approx \frac{\hbar}{mv} \qquad (40)$$

The expression:

$$\left(\frac{mv^2}{Ze^2}\right) p_1 = \frac{137}{Z} \cdot \frac{v}{c}$$

is in our case considerably larger than one so that we can leave the one in front out. The same is obviously true at the upper limit so that we can write:

$$\lambda = \ln \frac{p_2}{p_1} \qquad (41)$$

 [redacted]

We chose the upper limit by excluding collisions which last longer than the time during which the electron gas would be able to rearrange its density distribution so as to give a shielding effect. The rate at which this will take place is determined by the frequency of the plasma vibrations[4].

$$\omega = \sqrt{\frac{4\pi n e^2}{m}} \tag{42}$$

The collision parameter will thus be given to the right order magnitude by the relation

$$p_2 \approx \frac{v}{\omega} \tag{43}$$

Thus we obtain:

$$\lambda = \ln \frac{mv^2}{\hbar\omega} = \ln \frac{3kT_e}{\hbar\omega} \tag{44}$$

or, after introducing (42) and rearranging to put dimensions into evidence:

$$\lambda = \frac{1}{2} \ln \left[\frac{9}{4\pi} \left(\frac{\hbar c}{e^2}\right) \left(\frac{kT_e}{mc^2}\right)^2 n^{-1} \left(\frac{\hbar}{mc}\right)^{-3} \right] \tag{45}$$

For convenience we introduce Avogadro's number N and obtain:

$$\lambda = 10.881 + \ln\left(\frac{kT_e}{mc^2}\right) - \ln\left(\frac{n}{N}\right) \tag{46}$$

Because of the slow variation with Te it will usually be sufficient to use an average Te in this expression.

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4) See Cobine, Gaseous Conductors (1941) p. 132.

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APPENDIX II. CALCULATION OF THE MATRIX ELEMENTS

In carrying through the integration (11) one can replace $v_x \Delta(v_x L_s)$ by $\frac{1}{3}\, \vec{v}\, \Delta(\vec{v}\, L_s)$ because the rest of the integrand is spherically symmetrical in the velocities. Let us furthermore express the velocities in terms of the velocity $\vec{u}$ of the center of mass and the relative velocities $\vec{w}$ and $\vec{w}'$ of the two particles before and after collision:

$$\vec{v} = \vec{u} + \frac{1}{2}\vec{w}, \quad \vec{v}_1 = \vec{u} - \frac{1}{2}\vec{w}, \quad \vec{v}' = \vec{u} + \frac{1}{2}\vec{w}' \qquad \vec{v}_1{}' = \vec{u} - \frac{1}{2}\vec{w}' \tag{47}$$

We shall collect these four equations symbolically in one and write:

$$\vec{v}_i = \vec{u} + \frac{1}{2}\vec{w}_i \quad (i = 1\ldots4) \tag{48}$$

We further introduce the angle Θ_i by

$$\vec{w}\cdot\vec{w}_i = w^2 \cos\Theta_i \tag{49}$$

That means that Θ_i assumes the values $0,\ \pi,\ \theta,\ \pi - \theta$.

The Jacobian of the transformation (47) has the absolute value one so that $d\vec{v}d\vec{v}_1 = d\vec{u}d\vec{w}$. Now let:

$$M_i = \int f(v) f(v_1) (\vec{v}\cdot\vec{v}_i)\, e^{-\beta^2(xv^2 + yv_i^2)}\, d\vec{u} \tag{50}$$

where

$$x = \frac{\xi}{1-\xi}, \quad y = \frac{\eta}{1-\eta} \tag{51}$$

Then, considering the generating function of the Laguerre polynomials:

$$(1-\xi)^{-5/2}\, e^{-x\epsilon} = \sum_r \xi^r L_r(\epsilon) \tag{52}$$

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we can write:

$$\sum_r \sum_s \xi^r \eta^s H_{rs}{}^e = \frac{1}{3}(1-\xi)^{-5/2}(1-\eta)^{-5/2} \iint w\sigma_{ee}(w\theta)(M_1 + M_2 - M_3 - M_4)\,d\vec{w}\,d\Omega \tag{53}$$

$H_{rs}{}^e$ appears thus as a coefficient in expanding the expression (53) in powers of ξ and η. Our next step is therefore to determine the integrals M_i. Introducing (1a) we obtain:

$$M_i = n^2\left(\frac{\beta^2}{\pi}\right)^3 \int e^{-\beta^2\left[(1+x)v^2+v_1{}^2+yv_i{}^2\right]}(\vec{v}\cdot\vec{v}_i)\,d\vec{u} \tag{54}$$

Introducing the substitution (47) we rewrite:

$$(1+x)v^2 + v_1{}^2 + yv_i{}^2 = (u^2 + \frac{1}{4}w^2)(2+x+y) + \vec{u}\cdot(x\vec{w} + y\vec{w}_i)$$

$$= (2+x+y)g^2 + jw^2 \tag{55}$$

where:

$$\vec{g} = u + \frac{x\vec{w} + y\vec{w}_i}{2(2+x+y)} \tag{56}$$

$$j = \frac{2(1+x+y) + xy(1-\cos\Theta_i)}{2(2+x+y)} \tag{57}$$

(56) is simply a change of origin so that we have $d\vec{u} = d\vec{g}$. We express $\vec{v}_i$ in terms of $\vec{g}$ as

$$\vec{v}_i = \vec{g} + \vec{g}_i \tag{58}$$

with:

$$\vec{g}_i = \frac{(2+x)\vec{w}_i - x\vec{w}}{2(2+x+y)} \tag{59}$$

Similarly we have:

$$v = \vec{g} + \vec{g}_0 \tag{60}$$

with:

$$\vec{g}_0 = \frac{(2+y)\vec{w} - y\vec{w}_i}{2(2+x+y)} \tag{61}$$

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Entering (55), (58) and (60) into (54), we can carry out the integration and obtain:

$$M_i = \pi^{-3/2}\, n^2 \beta\, (2 + x + y)^{-5/2}\, e^{-j\beta^2 w^2}\left(\frac{3}{2} + \beta^2 (2 + x + y)\vec{g}_0 \cdot \vec{g}_i\right) \qquad (62)$$

where $\vec{g}_0 \cdot \vec{g}_i$ can be obtained from (59) and (61)

$$\vec{g}_0 \circ \vec{g}_i = \frac{-(x + y + xy) + (2 + x + y + xy) \cos \Theta_i}{2(2 + x + y)^2}\, w^2 \qquad (63)$$

In order to carry through the integration (53) we set:

$$\begin{aligned}
A &= \frac{3}{2}\, \pi^{-3/2}\, n^2\, \beta\, (2 + x + y)^{-5/2} \\
B &= \frac{x + y + xy}{3(2 + x + y)}\, \beta^2 \\
C &= \frac{2 + x + y + xy}{3(2 + x + y)}\, \beta^2 \\
D &= \frac{2 + 2x + 2y + xy}{2(2 + x + y)}\, \beta^2 \\
E &= \frac{xy}{2(2 + x + y)}\, \beta^2
\end{aligned} \qquad (64)$$

Thus we can write:

$$M_i = A(1 - Bw^2 + C \cos \Theta_i w^2)\, e^{-(D - E \cos \Theta_i) w^2} \qquad (65)$$

Now we form $\Delta M_i = M_1 + M_2 - M_3 - M_4$ and expand in powers of $\nu = \cos \theta - 1$. Actually, we will need only the linear term of the expansion because the scattering cross-section is proportional to ν^{-2} so that the quadratic and higher terms give small contribution to the integral as compared with the linear term.

Entering the proper Θ_i values we obtain:

$$\begin{aligned}
M_1 &= A(1 - Bw^2 + Cw^2)\, e^{-(D - E) w^2} \\
M_3 &= A(1 - Bw^2 + C \cos \theta\, w^2)\, e^{-(D - E \cos \theta) w^2}
\end{aligned} \qquad (66)$$

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and by subtracting

$$M_1 - M_3 = Ae^{-(D-E)w^2}\left[(1 - Bw^2)(1 - e^{Evw^2}) + Cw^2\left(1 - (\nu + 1)e^{E\nu w^2}\right)\right]$$

$$= -Ae^{-(D-E)w^2}\left[(1 - Bw^2)Ew^2 + Cw^2(1 + Ew^2)\right]\nu + O(\nu^2) \qquad (67)$$

$M_2 - M_4$ is obtained by simply changing the signs of C and E. The cross-section of e - e scattering is:

$$\sigma_{ee} = \left(\frac{2e^2}{mw^2}\right)^2 \nu^{-2} \qquad (68)$$

We now determine the integral:

$$\int w\sigma_{ee}(w\theta)\,(M_1 - M_3)\,d\vec{w}\,d\Omega$$

$$= (4\pi)^2\,\lambda\left(\frac{2e^2}{m}\right)^2 A\int e^{-(D-E)w^2}\left[(E + C) - E(B - C)w^2\right] w\,dw \qquad (69)$$

$$= 32\pi^2\,\lambda\left(\frac{e}{m}\right)^{2\,2}\left[\frac{E + C}{D - E} - \frac{E(B - C)}{(D - E)^2}\right]$$

and

$$\int w\sigma_{ee}(M_2 - M_4)\,d\vec{w}\,d\Omega = 32\pi^2\lambda\left(\frac{e}{m}\right)^{2\,2} A\left[-\frac{E + C}{D + E} + \frac{E(B + C)}{(D + E)^2}\right] \qquad (70)$$

so that:

$$\int w\sigma_{ee}\,\Delta M_i\,d\vec{w}\,d\Omega = 64\pi^2\lambda\left(\frac{e}{m}\right)^{2\,2} AE\,\frac{D^2E + 2D^2C - E^3 - 2BDE}{(D^2 - E^2)^2} \qquad (71)$$

Now we have to express (71) as a function of ξ and η. If we set $a = (1 - \xi)^{-1}(1 - \eta)^{-1}$ we find:

$$\begin{aligned}
2 + x + y &= (2 - \xi - \eta)a \\
x + y + xy &= (\xi + \eta - \xi\eta)a \\
2 + x + y + xy &= (2 - \xi - \eta + \xi\eta)a \\
\ell + 2x + 2y + xy &= (2 - \xi\eta)a
\end{aligned} \qquad (72)$$

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and entering this into (64):

$$A = \frac{3}{2}\,\pi^{-3/2}\,n^2 \beta\,(2 - \xi - \eta)^{-5/2} a^{-5/2}$$

$$B = \frac{1}{3}\,\frac{\xi + \eta - \xi\eta}{2 - \xi - \eta}\,\beta^2$$

$$C = \frac{1}{3}\,\frac{2 - \xi - \eta + \xi\eta}{2 - \xi - \eta}\,\beta^2$$

$$D = \frac{1}{2}\,\frac{2 - \xi\eta}{2 - \xi - \eta}\,\beta^2$$

$$E = \frac{1}{2}\,\frac{\xi\eta}{2 - \xi - \eta}\,\beta^2$$

We enter these expressions into (71) and multiply according to (53) by $\frac{1}{3}(1 - \xi)^{-5/2}(1 - \eta)^{-5/2} = \frac{1}{3}\,a^{5/2}$ and get:

$$\sum_r \sum_s \xi^r \eta^s H_{rs}^{e} = \mu\sqrt{2}\;\frac{\xi\eta\left(1 - \frac{1}{2}(\xi+\eta) - \frac{1}{8}(\xi\eta) + \frac{1}{4}(\xi\eta)(\xi+\eta) - \frac{3}{8}(\xi\eta)^2\right)}{\left(1 - \frac{1}{2}(\xi+\eta)\right)^{5/2}(1 - \xi\eta)^2} \tag{74}$$

where μ is defined by (17). We can see immediately that all elements in the zero row and zero column are zero. By expanding the expression (74) in powers of ξ and η, we obtain the symmetrical matrix:

$$h_{rs}^{e} = \sqrt{2}\left\{\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 3/2^2 & 15/2^5 & 35/2^9 & --- \\ & & 45/2^4 & 309/2^7 & 885/2^9 & --- \\ & & & 5657/2^{10} & 20349/2^{12} & --- \\ & & & & 149749/2^{14} & --- \\ & & & & & \end{array}\right\} \tag{75}$$

The integration (13) requires considerably less labor. The angular integration gives a factor 4π and if we further set $\beta^2 v^2 = \varepsilon$ and use (1a) we have:

$$H_{rs}^{i} = \mu\,\frac{N_i Z_i^2}{n}\int_0^\infty L_r(\varepsilon) L_s(\varepsilon)\,e^{-\varepsilon}\,d\varepsilon \tag{76}$$

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as before, we make use of the generating function, and write:

$$\sum\sum \xi^r \eta^s h_{rs}^i = \frac{N_i Z_i^2}{n} (1-\xi)^{-5/2}(1-\eta)^{-5/2} \int_0^\infty e^{-(x+y+1)\varepsilon} d\varepsilon$$

$$= \frac{N_i Z_i^2}{n} (1-\xi)^{-3/2}(1-\eta)^{-3/2}(1-\xi\eta)^{-1} \tag{77}$$

By expanding this in powers of ξ and η, we obtain the symmetrical matrix :

$$h_{rs}^i = \frac{N_i Z_i^2}{n} \left\{ \begin{array}{lllll} 1 & 3/2 & 15/2^3 & 35/2^4 & 315/2^7 \\ & 13/2^2 & 69/2^4 & 165/2^5 & 1505/2^8 \\ & & 433/2^6 & 1077/2^7 & 10005/2^{10} \\ & & & 2957/2^8 & 28257/2^{11} \\ & & & & 288473/2^{14} \end{array} \right\} \tag{78}$$

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