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LA REPORT 236

[redacted]

March 16, 1945

This document contains 19 pages

DYNAMICS OF A MASS OF HOT GAS RISING IN AIR

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ABSTRACT

The theory developed here may be used to treat the dynamics of a ball of fire produced in a large explosion. At first this ball is a sphere of hot gases which starts to rise through the air. Its dynamics is considered in Part II. Its behavior is very similar to the rise of an air bubble through water. It soon flattens out like the top of a mushroom, its top remaining spherical while its bottom becomes flat. As it rises the stream of cold air flowing over its surfaces eats away the hot gases in turbulent convection. Most of the material eaten away comes from the lower surface. As the ball rises it maintains its shape but becomes smaller due to the loss of material. Finally after a rise of 3.7 times its original radius, the ball is completely dissolved in turbulence. The figure 3.7 depends on two considerations. First, it is known experimentally for air bubbles in water that the drag coefficient, C_D, is approximately 0.7. Secondly, the amount of air which gets sucked into the turbulent stream can be estimated from experimental observations of the motion of air produced above a long straight hot wire (thus the constant a is taken as 0.2).

The fate of the turbulent column of air is treated in Part I. The original ball of fire is considered to be comparable to an instantaneous point source of heat. Knowing the total amount of heat input, and the rate of rise of potential temperature with height in the atmosphere (taken to be 5° per 1000 meters) the dynamics of the turbulent column can be estimated. This theory predicts in agreement with observations that the column should rise to a maximum height and then mushroom out. The height should vary as the energy of the explosion to the one-fourth power and inversely as the one-fourth power of the rate of rise of potential temperature in the air. In the case of the Port Chicago explosion involving 1600 tons of H.E., this theory predicts that the maximum height should be 11,000 feet. Observations would in-

dicate that this height was actually between 8,000 and 12,000 feet. The theory is admittedly rough. An attempt is made to justify it by using the same sort of development for the case of turbulence produced by a hot wire in air and comparing it with the results of Schmidt's more accurate theory. Schmidt's theory is presented in the Appendix.

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DYNAMICS OF A MASS OF HOT GAS RISING IN AIR

PART I. Approximate Theory of Convection Currents

I. LINE SOURCE IN ATMOSPHERE WHOSE POTENTIAL TEMPERATURE VARIES WITH HEIGHT

The effect of mixture between the hot air in the convection current rising with velocity u and the surrounding atmosphere will be taken as being due to a current equal to αu flowing into the convection current from each side and mixing with it. A volume of air which at a certain height x_o has height $\delta\ell_o$ and width $2y_o$ may after mixing have a height $\delta\ell$ and a width $2y$. It is assumed that u is constant across the convection current. The equation for increase in volume of the element is

$$\frac{D}{Dt}(2y\,\delta\ell) = 2\alpha u\,\delta\ell \tag{1}$$

Since the motion is steady,

$$\delta\ell/\delta\ell_o = u/u_o \tag{2}$$

so that (1) may be written

$$u\frac{d}{dx}(yu) = \alpha u^2 \tag{3}$$

If θ_o is the potential temperature, or rather the rise in potential temperature at height x above that at the ground level the equation of motion is

$$\rho\frac{D}{Dt}(2yu'\,\delta\ell) = \frac{\rho(\theta-\theta_o)}{T}\,g(2y\,\delta\ell) \tag{4a}$$

Or in view of Eq. (2):

$$u\frac{d}{dx}(yu^2) = \frac{g}{T}(\theta-\theta_o)\,yu \tag{4b}$$

The equation for conservation of heat is

$$\frac{D}{Dt}(2y\rho\sigma\theta\delta\ell) = 2\alpha\rho\sigma u\,\theta_o\,\delta\ell$$

Or making use of Eq. (2):

$$u\frac{d}{dx}(yu\theta) = \alpha\,\theta_o\,u^2 \tag{5}$$

These equations may be reduced to

$$\left.\begin{aligned} \frac{du}{dx} &= \frac{g}{T}\left(\frac{\theta-\theta_0}{u}\right) - \frac{\alpha u^2}{yu} \\ \frac{d}{dx}(yu) &= \alpha u \\ \frac{d\theta}{dx} &= \frac{\alpha u(\theta_0-\theta)}{yu} \end{aligned}\right\} \qquad (6)$$

A. Case where $\theta_0 = 0$, Atmosphere has uniform potential temperature

In this special case the solution to the Eqs. (6) is:

$$\begin{aligned} y &= x\,\alpha \\ u &= \text{constant} \\ \theta &= u^2T/(gx) \end{aligned} \qquad (7)$$

The total rate of heat transfer H is given by the equation

$$H = 2u\rho y\theta\sigma$$

And from Eq. (7) this is

$$H = 2\alpha\, u^3(\rho\sigma T/g) \qquad (8)$$

Comparison with Schmidt's Calculations[1]

It will be seen that so far as the dependence of u, y, and θ on x is concerned, Eq. (7) is in agreement with Schmidt's calculations. In considering how the absolute magnitude may be compared, it must be remembered that Schmidt's distribution of θ and u with $\eta (= y/x)$ are rather like that of an error curve. If a given amount of heat were uniformly distributed it could be imagined to spread out over the same total volume as in Schmidt's calculations. In this case u would be smaller than Schmidt's u_0 and α would be the same as Schmidt's limiting value of η. Or alterna-

1) Schmidt's theory is presented in the appendix.

[redacted]

tively the value of u could be taken as identical with Schmidt's u_o in which case α would be less than the limiting value of Schmidt's η. If u is taken as identical with Schmidt's u_o, Eq. (8) may be compared with Eq. (14) of the appendix and they will be found to be in agreement if

$$A\ K^{-2/9}\ I^{-1/3} = \alpha^{1/3} \tag{9}$$

When the calculated values of A and I are inserted and the value of K which makes Schmidt's calculation agree with observation are inserted in (9), according to my memory, $\alpha = 0.2$. This corresponds with a limiting value of η in Schmidt's case of about $\tan^{-1} 19^{\circ} = 0.32$. It seems that the present theory in which the effect of turbulent mixing is represented by an inflow of surrounding air through the outer surface of the heated zone at a velocity α times the velocity of the heated gas is in agreement with Schmidt's more accurate theory for the case where the surrounding air has a uniform potential temperature. It is proposed therefore to apply the approximate theory to cases where the analytical difficulties prevent the more accurate theory from being applied.

B. <u>Case where $\Theta_o = \beta x$, Atmosphere has uniform increase in potential temperature with height</u>

In this case the Eq. (6) may be reduced to non-dimensional form by taking:

$$\begin{aligned} u &= u'\,U \quad \text{where} \quad U = (Hg/2T\alpha\rho\sigma)^{1/3} \\ x &= x'\,X \quad \text{where} \quad X = U(T/\beta g)^{1/2} \\ \Theta &= \Theta'\,\circledR \quad \text{where} \quad \circledR = \beta X = U(\beta T/g)^{1/2} \end{aligned} \tag{10}$$

The value of X was chosen so that $X = U^2T/(gX)$. The Eqs. (6) then become:

$$\begin{aligned} \frac{du'}{dx'} &= \frac{\Theta'-x'}{u'} - \frac{\alpha u'}{y'} \\ \frac{d}{dx'}(y'\,u') &= \alpha u' \\ \frac{d\Theta'}{dx'} &= \frac{\alpha u'(\Theta'-x')}{y'u'} \end{aligned} \tag{11}$$

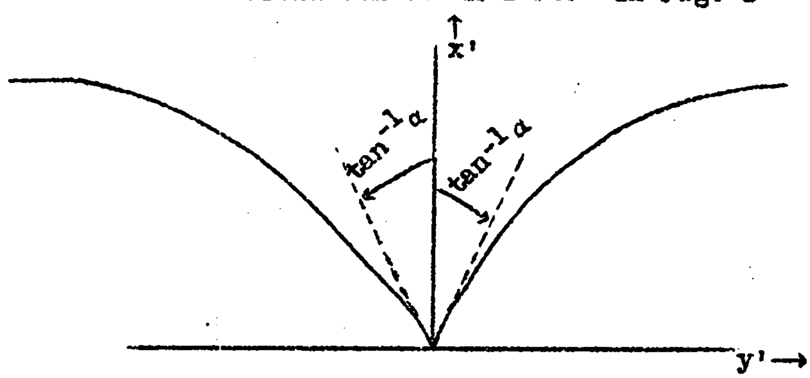
[redacted]

[redacted]

Eqs. (11) can be solved numerically because the initial conditions near $x' = 0$ are

$$u' = 1, \; y' = \alpha x', \; \theta' x' = 1$$

The general nature of the solution is indicated in Fig. 1



II. <u>POINT SOURCE OF HEAT EMITTING H CALORIES PER SECOND</u>

Here the equations are:

$$u \frac{d}{dx} (\pi r^2 u) = 2\pi r \alpha u^2 \qquad \text{Increase in Volume}$$

$$u \frac{d}{dx} (\pi r^2 u^2) = \frac{g}{T} (\theta - \theta_0) \pi r^2 u \qquad \text{Equation of Motion}$$

$$u \frac{d}{dx} (\pi r^2 u\theta) = 2\pi r \alpha u \theta_0 \qquad \text{Conservation of Heat}$$

Here r is the distance from the source. Dividing the above equations by πu:

$$\frac{d}{dx} (r^2 u) = 2r\alpha u \tag{12}$$

$$\frac{d}{dx} (r^2 u^2) = \frac{g}{T} (\theta - \theta_0) r^2 \tag{13}$$

$$\frac{d}{dx} (r^2 u\theta) = 2\alpha \theta_0 ru \tag{14}$$

If $\theta_0 = 0$ corresponding to an atmosphere having a constant potential temperature, the solution of (14) is:

$$r^2 u\theta = \text{constant} = H/(\pi\rho\sigma) \tag{15}$$

Assume as solutions to Eqs. (12) and (13)

$$u = A/x^{1/3} \tag{16}$$

$$y = \beta x \tag{17}$$

[redacted]

Eq. (12) is satisfied if

$$\beta = 6\alpha/5 \tag{18}$$

Eq. (13) is satisfied if

$$A^3 = \frac{3gH}{4\pi\rho\sigma T\,\beta^2} = \frac{25gH}{48T\rho\sigma\,\alpha^2} \tag{19}$$

The value of θ is from Eq. (15)

$$\theta = \frac{H}{\pi\rho\sigma\,\beta^2\,A}\,x^{-5/3} \tag{20}$$

The solution contained in Eqs. (15) through (20) is analogous to Schmidt's more accurate solution in this case. The exponents of x in the Schmidt expression for u, y, and θ are the same as those in Eqs. (15) through (20).

III. INSTANTANEOUS POINT SOURCE OF HEAT

When the point source of heat is not continuous but is generated suddenly in a small volume, the principle embodied in Eq. (1) may still be applicable. But since the motion is not steady the relationship (2) between the variation in vertical depth of the heated layer and the vertical velocity no longer holds. Some assumption must be made to take its place. For this purpose it will be assumed that the depth of the heated volume is the same as the breadth. In fact it may be assumed that the heated volume is a sphere of radius r and that its volume increases because air enters all over its surface at the rate αu. Thus the equation representing the spread of the heated sphere due to turbulence is

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2\,\alpha u \quad \text{or} \quad \frac{d}{dt} r^3 = 3\alpha\, r^2\, u \tag{21}$$

The equation of motion is

$$\frac{d}{dt}\left(r^3\, u\right) = \frac{g}{T}\left(\theta - \theta_0\right) r^3 \tag{22}$$

The equation of heat conservation is

$$\frac{d}{dt}\left(r^3\, \theta\right) = 3\alpha\, r^2\, u\, \theta_0 \tag{23}$$

[redacted]

Since u = dx/dt, these equations may be written:

$$\frac{dr^3}{dx} = 3\alpha r^2 \text{ or } r = \alpha x \tag{24}$$

$$\frac{d}{dx}(r^3 u) = \frac{g}{T}(\theta - \theta_o)\frac{r^3}{u} \tag{25}$$

$$\frac{d}{dx}(r^3 \theta) = 3\alpha r^2 \theta_o \tag{26}$$

A. <u>Case where $\theta_o = 0$, Atmosphere has uniform potential temperature</u>

In this case we can integrate Eq. (26) to get:

$$r^3 \theta = \text{constant} = 3H/(4\pi\rho\sigma)$$

And Eq. (25) becomes:

$$r^3 u \frac{d}{dx}(r^3 u) = \frac{g}{T\alpha}\left(\frac{3H}{4\pi\rho\sigma}\right) r^3$$

This integrates to give:

$$\frac{(r^3 u)^2}{2} = \frac{g}{T\alpha}\left(\frac{3H}{4\pi\rho\sigma}\right)\int r^3\, dr = \frac{3gH}{4\pi\rho\sigma T\,\alpha^2}\left(\frac{r^4}{4}\right)$$

Thus:

$$u = \left[\frac{3gH}{8\pi\rho\sigma T\alpha}\right]^{1/2}\frac{1}{r} = \left[\frac{3gH}{8\pi\rho\sigma T\,\alpha^3}\right]^{1/2}\frac{1}{x}$$

$$\theta = \left(\frac{3H}{4\pi\rho\sigma\,\alpha^3}\right)\frac{1}{x^3} \tag{27}$$

$$r = \alpha x$$

Eq. (27) is a solution to this problem.

B. <u>Case where $\theta_o = \beta x$, Atmosphere has uniform rate of increase of potential temperature with height</u>

In this case, Eq. (26) becomes

$$\frac{d}{dr}(r^3 \theta) = 3\, r^3\, \beta/\alpha$$

Which integrates to give:

$$r^3 \theta = \frac{3\beta}{4\alpha} r^4 + \frac{3H}{4\pi\rho\sigma}$$

[redacted]

or

$$\theta = \frac{3\beta}{4\alpha} r + \frac{3H}{4\pi\rho\sigma\, r^3} \qquad (28)$$

And Eq. (25) becomes

$$r^3 u \frac{d}{dr}(r^3 u) = \frac{g}{T\alpha} r^6 \left[\frac{3\beta}{4\alpha} r + \left(\frac{3H}{4\pi\rho\sigma}\right)\frac{1}{r^3} - \frac{\beta}{\alpha} r \right]$$

Which integrates to give

$$u^2 r^6 = \frac{2g}{T\alpha}\left[-\left(\frac{\beta}{4\alpha}\right)\frac{r^8}{8} + \left(\frac{3H}{4\pi\rho\sigma}\right)\frac{r^4}{4} \right] + \text{constant}$$

But this constant of integration must equal to zero so that:

$$u^2 = \frac{2g}{T\alpha}\left[\left(\frac{3H}{16\pi\rho\sigma}\right) r^4 - \frac{\beta}{32\alpha} r^8 \right] \qquad (29)$$

Thus there is a height at which u is equal to zero, i.e.,

$$u = 0 \quad \text{when} \quad r^4 = \left(\frac{6H}{\pi\rho\sigma}\right)(\alpha/\beta)$$

$$\text{or} \quad x = \left[\frac{6H}{\pi\rho\sigma\beta\,\alpha^3}\right]^{1/4} \qquad (30)$$

Suppose a mass, M, of explosive leaves a fraction γ of the total chemical energy which is released per gram, E, in the form of heat energy at the site of the explosion. Then

$$H = \gamma M E$$

And the smoke rises to a height of

$$x = \left[\frac{6\gamma ME}{\pi\rho\sigma\beta\,\alpha^3}\right]^{1/4} \qquad (31)$$

<u>EXAMPLE 1:</u> Explosion of ton of TNT

$M = 10^6$, $E = 1000$ calories per gram, $\rho = .0013$ grams/cm^3,

$\sigma = .25$, $\beta = 5^\circ$ per 1000 meters $= 5 \times 10^{-5}$ deg/cm

This value of β corresponds to a rise of potential temperature of 5° per 1000 meters. Since the adiabatic gradient gives rise to a fall of 10° per 1000 meters, a rise of 5°

[redacted]

per 1000 meters in potential temperature corresponds to a lapse rate in actual temperature of 15^{o} per 1000 meters. If we suppose that

$$\gamma = 0.5 \quad \text{and} \quad \alpha = 0.2$$

It follows from Eq. (31) that

$$x = 5.2 \times 10^{4} \text{ cms} = 1700 \text{ feet}$$

EXAMPLE 2: The Port Chicago Explosion

$$M = 1600 \text{ tons} = 1.6 \times 10^{9} \text{ grams}$$

Using the same values of the other quantities as in Example 1, it follows that

$$x = 3.3 \times 10^{5} \text{ cms} = 10{,}900 \text{ feet}$$

These results are likely to require modification when a more correct value of α is found using the result of Schmidt's paper and, in Example 2, when the change of density of air with height is allowed for.

[redacted]

PART II. Early Stages of the Rise of a Sphere of Hot Gas

When air is released in water the resulting bubbles are nearly spherical if their diameters are of the order of one millimeter. When they attain a radius of around one centimeter the variation in pressure distribution around their surface due to hydrodynamical flow causes them to flatten. They usually oscillate violently and frequently break up. If however a large amount of air is suddenly released in water, very large bubbles are formed. They are umbrella shaped and appear to be smooth on the upper surface and highly disturbed on the lower surface. Photographs of bubbles up to six inches in diameter show that the rim of the umbrella subtends at the center an angle of about 40° from the vertical axis. The pressure distributions have been measured over the surfaces of the segments of spheres set in a wind tunnel with the pole or axis of the segment in line with the wind. Such measurements show that the pressure over the greater part of the spherical segment differs little from that which would be calculated by classical hydrodynamics. There is reason to believe therefore that the pressure in the water near the spherical surface of a volume of gas which is rising in an umbrella or mushroom shaped form will be close to that which would be calculated by classical hydrodynamics for a sphere. This is

$$p = \tfrac{1}{2}\rho u^2(1 - 9\sin^2\theta/4) \qquad (1)$$

Here θ is the angle of the point of the sphere from the summit and u is the vertical velocity of rise. If θ is small, Eq. (1) may be written:

$$p = \tfrac{1}{2}\rho u^2(1 - 9\theta^2/4)$$

The above pressure is the excess due to hydrodynamical causes over the pressure in the fluid at infinity. Since there is a gravitational field, the actual pressure in the water at the surface is

$$p = g\rho h_\theta + \tfrac{1}{2}\rho u^2(1 - 9\theta^2/4) \qquad (2)$$

[redacted]

Here h_θ is the depth of any point on the surface of the bubble below the surface of the water. If θ is small

$$h_\theta = h_0 + \tfrac{1}{2}\, a\, \theta^2 \tag{3}$$

Here $\underline{a}$ is the radius of the bubble. The true pressure at the surface of the bubble is therefore

$$p = p_0 + g\rho\, h_0 + \tfrac{1}{2}\rho u^2 + \tfrac{1}{2} g\rho a\, \theta^2 - 9\rho\, u^2\, \theta^2/8 \tag{4}$$

Here p_0 is the atmospheric pressure.

It seems from Eq. (4) that the pressure in the water at the surface of the bubble can be constant and therefore in equilibrium with the rising gas if

$$u^2 = 4\, ga/9 \quad \text{or} \quad u = (2/3)(ga)^{\frac{1}{2}} \tag{5}$$

<u>Resistance of the Bubble</u>

Though the radius of curvature of the top of the bubble is related to the velocity of rise by Eq. (5), further information is required before the rate of rise can be determined. Experiments on the rate of rise of large bubbles of known volume give rather variable results but it was found by measuring the horizontal diameter of the bubble that the resistance coefficient, C_D, is of the order of[2)]

$$C_D = 0.5 \text{ to } 1.0$$

If the bubble were flat at the bottom, the volume of a segment subtending an angle $2\theta_0$ at the center is

$$\begin{aligned} V &= \frac{4}{3}\pi a^3 \left[\frac{2\pi(1-\cos\theta_0)}{4\pi}\right] - \frac{1}{3}\pi a^3 \sin^2\theta_0 \cos\theta_0 \\ &= \frac{1}{3}\pi a^3 \left[2(1-\cos\theta_0) - \sin^2\theta_0 \cos\theta_0\right] \end{aligned} \tag{6}$$

For such a bubble to rise with the velocity u, the drag coefficient, C_D, must be given by the relation:

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2) This figure is from memory and may need correcting.

[redacted]

$$V \rho g = C_D(\tfrac{1}{2} \rho u^2)(a^2 \sin^2 \theta_o) \qquad (7)$$

Comparing Eqs. (5), (6), and (7), it seems that the relationship between the angle θ_o which defines the shape of a flat-bottomed bubble in the form of a spherical segment and C_D is

$$C_D = \frac{3}{2}\left[\frac{2(1 - \cos\theta_o)}{\sin^2\theta_o} - \cos\theta_o\right] \qquad (8)$$

The values of C_D for a few angles is shown below:

θ_o	C_D
40°	0.543
45	0.697
50	0.856
55	1.042

Thus the observed range of values of C_D for a solid segment would lead one to expect that if the bubble were flat-bottomed it would cover an angle θ from the vertical axis which lies with the range of from 40 to 55 degrees. This is in agreement with observation.

Mixture at the Spherical Surface of a Volume of Hot Gas

Suppose that a hot gas bubble is penetrating upwards into colder air. If the density of the hot gas is ρ', Eq. (5) must be modified to read:

$$u = (2/3)\left\{ga\left[1 - (\rho'/\rho)\right]\right\}^{1/2} \qquad (9)$$

The relationship between mixing and velocity of the cold air flowing radially outwards from above the vortex of the bubble is the same as that assumed in the simple theory given in Part I.

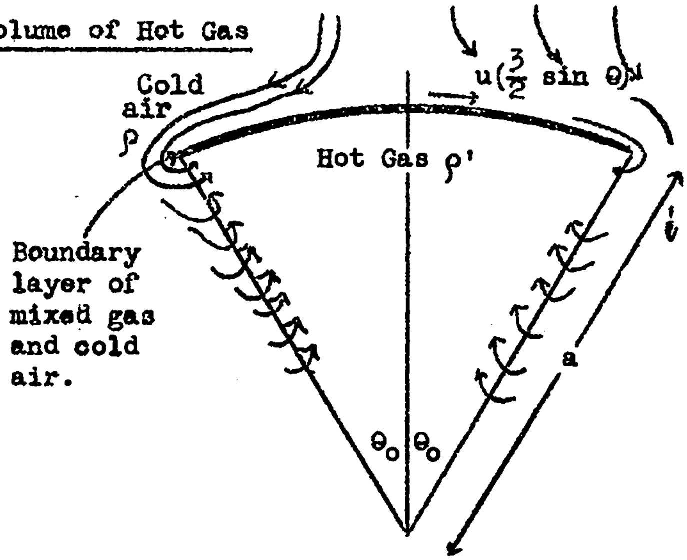

Fig. 1

[redacted]

Then the rate at which hot air is removed at the spherical surface of the hot gas bubble is α times the velocity of cold air over hot gas $= \alpha\left((3/2)\ u \sin\theta\right)$ cm^3 per cm^2. Thus the total rate of removal of volume from the whole surface is

$$\frac{dV}{dt} = \alpha \int_0^{\theta_0} \frac{3}{2} u \sin\theta (2\pi a \sin\theta)\ a d\theta$$

$$= \frac{3}{2} \alpha \pi u\ a^2 \left[\theta_0 - \tfrac{1}{2} \sin 2\theta_0\right]$$

If $\theta_0 = 45^\circ$, this equation becomes:

$$dV/dt = 1.35\ \alpha\ a^2\ u$$

And from Eq. (6), $V = 0.24\ a^3$. Therefore while the bubble rises a height x, the volume changes by

$$\delta V = 1.35\ \alpha\ a^2\ \delta x$$

So that

$$\frac{dV}{V} = \frac{1.35\ \alpha\ a^2\ \delta x}{0.24\ a^3}$$

Also since $dV/V = 3\ \delta a/a$ it appears that

$$dx/da = 3(0.24)/1.35\ \alpha = 0.53/\alpha$$

If $\alpha = 0.2$, this gives

$$x = 2.6\ a$$

Therefore in this case, the unmixed portion of the bubble might be expected to rise to a height of 2.6 a, i.e., $(2.6/\sin\theta_0)$ times the horizontal radius, which is

3.7 times the horizontal radius

before being completely mixed in the turbulent mass which might be expected to rise in the manner contemplated in the case analyzed in Part I.

Fig. 2 shows the successive positions of the residue of hot gas according to the above theory. It will be noticed that in this theory the mixed gases are supposed to be removed in the boundary layer which enters the turbulent air behind the spherical

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segment of unmixed hot gas and forms the turbulent sphere which is the subject of Part I. This process is indicated in Fig. 1.

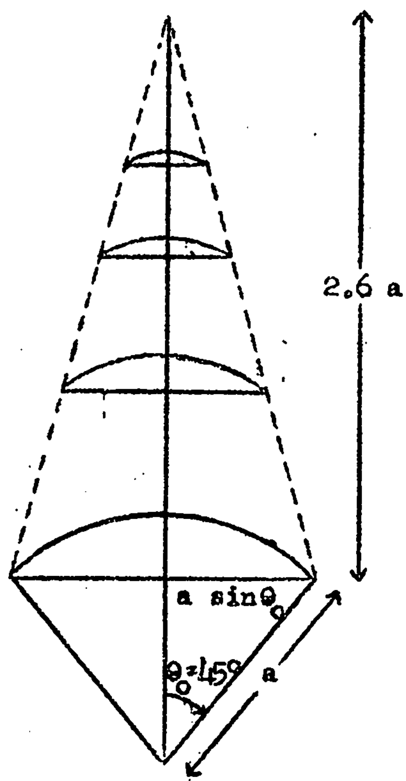

Fig. 2

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APPENDIX

SCHMIDT'S THEORY OF THE VERTICAL CONVECTION CURRENTS ABOVE A LINE SOURCE OF HEAT

Take x vertical, y horizontal,

$$\eta = y/x. \qquad (1)$$

Assume for stress $\widehat{xy} = \tau$,

$$\tau = \rho\left[\ell \frac{du}{dy}\right]^2, \qquad (2)$$

and ℓ, the mixture length, may be assumed as proportional to the scale of the eddies which may be taken as proportional to the width of the heated current. In the present case it turns out that all the equations are satisfied if vertical velocity u is a function of η only so that the width of the heated current is proportional to x. Thus we may take

$$\ell = Kx \quad \text{and} \quad \tau = \rho\, K^2\, x^2\left(\frac{du}{dy}\right)\left|\frac{\partial u}{\partial y}\right|,$$

i.e., virtual coefficient of viscosity $K^2\, x^2\left|\frac{\partial u}{\partial y}\right|$. Since u is a function of η only

$$\tau = \rho\, K^2\left(\frac{du}{d\eta}\right)\left|\frac{du}{d\eta}\right| \qquad (3)$$

Since the total heat flow per second, H, over any section is constant

$$H = \int u\rho\sigma\Theta dy = \rho\sigma\int u\Theta x dy \qquad (4)$$

when Θ is rise in temperature over the atmospheric temperature T; ρ density; σ specific heat. Thus Θx is a function of η only.

It is assumed that the variations in density are small, i.e., Θ is small compared with T and that u and v are to this order of approximation related by the incompressible relationship $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$. This enables u and v (which are both functions of η only) to be expressed in the form

$$u = f'(\eta), \; v = \eta f'(\eta) - f(\eta) \qquad (5)$$

$$\left(\text{since } \frac{\partial}{\partial x} = -\frac{\eta}{x}\frac{d}{d\eta}, \; \frac{\partial}{\partial y} = \frac{1}{x}\frac{d}{d\eta}\right)$$

[redacted]

for functions of η

$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right) = -\frac{\eta}{x}f'(\eta) + \frac{1}{x}\left[\eta f'(\eta) - f(\eta)\right]\frac{d}{d\eta} = -\frac{1}{x}f(\eta)\frac{d}{d\eta} \quad . \qquad (6)$$

Equation of Motion is:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = K^2 x^2 \frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)\left|\frac{du}{dy}\right|\right] + \frac{\theta g}{T} \qquad (7)$$

Consider only y positive. $\frac{\partial u}{\partial y}\left|\frac{du}{dy}\right|$ is negative throughout region where y is positive so that (7) is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -K^2 x^2 \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\theta g}{T}$$

or

$$-\frac{1}{x}f(\eta)\,f''(\eta) = -\frac{K^2}{x}(2f''\,f''') + \frac{\theta x g}{Tx}$$

and since θx is a function of η only this is an equation independent of x.

$$2K^2 f\,f'' - f\,f'' = (\theta x)\,g/T = (g/T)\,F(\eta) \qquad (8)$$

Equation of heat conductivity in horizontal direction (Schmidt neglects vertical conduction, I think) is

$$\frac{\partial}{\partial y}\left(c\,\frac{\partial\theta}{\partial y}\right) = \rho\sigma\left(u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y}\right) \qquad (9)$$

Where c is the heat conductivity and c is related to the initial coefficient of viscosity by the assumption that heat is transferred by exactly the same lateral eddy convection process as vertical momentum. Thus

$$C = K^2 x^2 \rho\sigma\left|\frac{\partial u}{\partial y}\right| = K^2 \rho\sigma x f''$$

writing $\theta = F(\eta)/x$, (9) becomes

$$K^2\frac{d}{d\eta}(f''\,F') = -fF' - f'F \qquad (10)$$

(8) and (10) can be put into a nondimensional form by using

$$\eta_1 = \eta^{-2/3} \qquad (11)$$

as the independent variable and substituting $F_1 = g\,K^{4/3}\,F/T$ thus (8) becomes

$$2f''\,f''' - f\,f'' = F_1 \qquad (12)$$

[redacted]

in future dashes refer to η_1. (10) becomes

$$f'' F_1'' + F_1' f''' + f' F_1'' + f F_1' = 0 \qquad (13)$$

The conditions at $\eta = 0$ are

$$f = 0, \; f'' = 0, \; F_1' = 0$$

We may take $F_1 = 1$ without loss of generality.

Near $\eta = 0$ the solution developed in the form

$$f = A\,\eta_1 - 4/15\,\eta_1^{5/2} + C\,\eta_1^{4} + D\,\eta_1^{5.5}$$

$$F_1 = 1 - 36\,C\,\eta_1^{3/2} + E\,\eta_1^{3}$$

There is only one arbitrary constant in this solution, namely A. i.e., C, D etc. are determined in terms of A. Schmidt determined the numerical value of A which makes F and f' vanish simultaneously, and he determined the (finite) value of η_1 at which this happened. (I think from memory that A was somewhere in the neighborhood of 0.9 and the limiting value of y was about 2. If k is known this gives the limiting value of η, and u was determined by comparing the observed distribution of Θ as η varies with that calculated by the theory. The limiting value of η was, I think, about 19° for the line source.)

From (12) and (13) it seems that F_1 can be multiplied by any number N provided f is multiplied by $\sqrt{N}$. To find the appropriate value of N from (4)

$$\frac{H}{\rho\sigma} = 2\int_0^{\eta_1 \text{limit}} u\Theta x\, d\eta = N^{3/2}\int \frac{df}{d\eta}\left(\frac{F_1\, T}{gK^{4/3}}\right) d\eta = \frac{2N^{3/2}T}{gK^{4/3}}\int f'\, F_1\, d\eta_1$$

here $\int f'\, F_1\, d\eta_1$ is a pure number and is obtained from Schmidt's solution. Writing this number I

$$\text{Thus } N = \left(\frac{Hg\;K^{4/3}}{2\rho\sigma T I}\right)^{2/3}$$

The maximum velocity at the center is equal to

$$U_0 = \left[\frac{df}{d\eta_1}\right]_{\eta_1 = 0}\left(\frac{d\eta_1}{d\eta}\right) = K^{-2/3}\,AN$$

where A is a pure number and is the value determined by Schmidt. Thus

$$U_0 = \frac{AK^{(-2/3+4/9)}}{2^{1/3}}\left(\frac{Hg}{\rho\sigma T I}\right)^{1/3} = \frac{AK^{-2/9}I^{-1/3}}{2^{1/3}}\left(\frac{Hg}{\rho\sigma T}\right)^{1/3} \qquad (14)$$

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