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Radiative Transfer with Scattering

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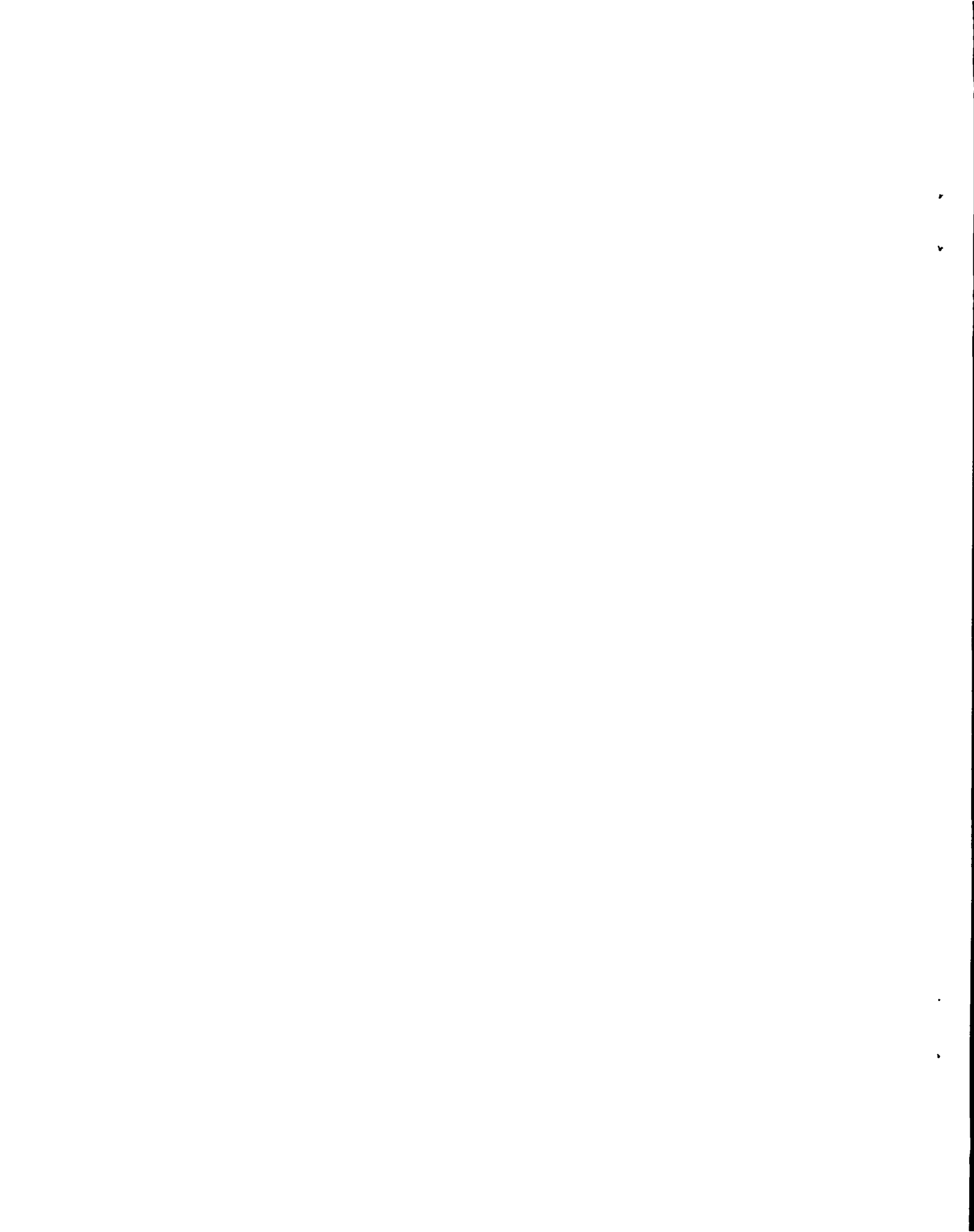
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**Radiative Transfer with Scattering**

by

G. C. Pomraning





# RADIATIVE TRANSFER WITH SCATTERING

by

G. C. Pomraning\*

## ABSTRACT

The equation of radiative transfer including the effects of photon scattering is discussed. The case of scattering from a Maxwellian gas of free electrons is considered in detail. The mathematical description of this process in the equation of transfer is formulated in two different ways, one differential and the other integral in character. Both descriptions have the potential of being incorporated into a general-purpose radiative transfer code.

### 1. INTRODUCTION

In the calculation of energy transport by radiative processes, three basic interactions of photons with matter must in principle be taken into account. These are the processes of emission, absorption, and scattering. In many applications, scattering plays a small role and it has been traditional in much of radiative transfer to simply neglect its contribution to the equation of transfer. Since the scattering terms are by far the most complex in the equation of transfer, this neglect leads to a great simplification in the mathematical description of radiative transfer. However, in certain physical situations, scattering is an important interaction and its neglect can introduce serious errors. Further, the speed of current computers makes it feasible, from a practical computational point of view, to include the effects of scattering in the description of radiative transfer. For these reasons, a present-day, general-purpose radiative transfer

code would seem incomplete without the ability to account for the effects of scattering.

This report discusses some of the physical and mathematical aspects of the scattering of photons in the radiative transfer context. In particular, we consider in some detail the description of scattering from a Maxwellian gas of free electrons. In the next section we discuss the equation of transfer as well as the scattering kernel which is the basic description of the scattering interaction. The last two sections describe two different simplifications of the scattering terms in the equation of transfer. Each has its characteristic advantages and disadvantages insofar as its ability to retain the essential physics of the situation and its applicability to a numerical solution of the problem on a digital computer are concerned. Much of this report is a review of work; an attempt has been made to give the pertinent references to the original papers and reports.

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## 2. THE EQUATION OF TRANSFER WITH SCATTERING

In the polarization independent approximation, the equation of transfer, including the effects of scattering, for the specific intensity  $I(\vec{r}, \nu, \vec{\Omega}, t)$  of photons can be written quite generally as<sup>1,2</sup>

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\vec{r}, \nu, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I(\vec{r}, \nu, \vec{\Omega}, t) &= \mu'_a [B(\nu) - I(\nu, \vec{\Omega})] \\ - \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \sigma_s(\nu \rightarrow \nu', \vec{\Omega} \cdot \vec{\Omega}') I(\nu, \vec{\Omega}) \\ &\times \left[ 1 + \frac{c^2}{2h\nu'^3} I(\nu', \vec{\Omega}') \right] \\ + \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \sigma_s(\nu' \rightarrow \nu, \vec{\Omega}' \cdot \vec{\Omega}) I(\nu', \vec{\Omega}') \\ &\times \frac{\nu}{\nu'} \left[ 1 + \frac{c^2}{2h\nu^3} I(\nu, \vec{\Omega}) \right], \end{aligned} \quad (1)$$

where, for notational simplicity, we have dropped the arguments  $\vec{r}$  and  $t$  in the specific intensity  $I(\vec{r}, \nu, \vec{\Omega}, t)$  on the right-hand side of this equation. Here  $\vec{r}$ ,  $\nu$ ,  $\vec{\Omega}$ , and  $t$  are the spatial, frequency, angular, and temporal coordinates, respectively,  $c$  is the speed of light,  $B(\nu)$  is the source term due to spontaneous emission,  $\mu'_a(\nu)$  is the absorption coefficient suitably modified for induced emission, and  $\sigma_s(\nu' \rightarrow \nu, \vec{\Omega}' \cdot \vec{\Omega})$  is the double differential scattering coefficient (cross section). Although not explicitly shown,  $B$ ,  $\mu'_a$ , and  $\sigma_s$  are in general also functions of both  $\vec{r}$  and  $t$ . We assume, however, that  $B(\nu)$  and  $\mu'_a(\nu)$  are independent of the direction  $\vec{\Omega}$  of travel of the photon, and, as shown, that  $\sigma_s(\nu' \rightarrow \nu, \vec{\Omega}' \cdot \vec{\Omega})$  is a function only of the scattering angle rather than  $\vec{\Omega}$  and  $\vec{\Omega}'$  separately. For an observer at rest with respect to the medium, this implies isotropic matter, i.e., matter with no preferential direction. For an observer moving with speed  $v$  with respect to the medium, this also implies the neglect of terms of order  $v/c$ .<sup>3</sup> If local thermodynamic equilibrium can be assumed, then  $B(\nu)$  is the Planck function

$$B(\nu) = \frac{2h\nu^3}{c^2} (e^{h\nu/kT} - 1)^{-1}, \quad (2)$$

and

$$\mu'_a = \mu_a (1 - e^{-h\nu/kT}), \quad (3)$$

where  $\mu_a$  is the absorption coefficient appropriate to thermodynamic equilibrium and the exponential factor is the effective decrease in absorption due to stimulated emission. Here  $T = T(\vec{r}, t)$  is the local temperature of the medium. Stimulated scattering is described by the quadratic terms in the intensity in Eq. (1). The factor  $\nu/\nu'$  in the in-scattering term accounts for the fact that the scattering coefficient is defined relative to a photon density in phase space whereas the intensity  $I$  is an energy density. One significant approximation we have made in writing Eq. (1), in addition to the inherent approximation that photon transport can be described by a classical equation of transfer, is that polarization effects need not be taken into account. If the scattering interaction is between photons and free electrons, analytical evidence suggests that, as far as energy transfer is concerned, this approximation introduces a very small error.<sup>4</sup> Of course, if one is specifically interested in polarization effects, Eq. (1) is not applicable. We have also assumed in writing Eq. (1) that the index of refraction of the medium for frequencies of interest is unity. Although dispersive effects and the like are easily included in a transport description of photon interactions with material,<sup>5</sup> such a description severely complicates the streaming terms, i.e., the left-hand side of Eq. (1). Throughout this report we shall assume that such effects are unimportant.

The physics of the scattering process is contained in the scattering kernel  $\sigma_s(\nu \rightarrow \nu', \mu)$ . We shall consider the case of photon scattering from a Maxwellian gas of electrons at temperature  $T$ . In the case of photon scattering from free electrons at rest, the scattering interaction is described by the well-known Klein-Nishina formula<sup>6</sup>

$$\begin{aligned} \sigma_s(\nu \rightarrow \nu', \mu) &= N_e r_0^2 \left( \frac{1+\mu^2}{2} \right) \left[ 1 + \gamma(1-\mu) \right]^{-2} \\ &\times \left\{ 1 + \frac{\gamma^2(1-\mu)^2}{(1+\mu^2)[1+\gamma(1-\mu)]} \right\} \delta \left( \nu' - \frac{\nu}{1+\gamma(1-\mu)} \right), \end{aligned} \quad (4)$$

where  $N_e$  is the electron density,  $\gamma = h\nu/mc^2$  ( $h$  = Planck's constant and  $mc^2$  is the rest energy of the electron),  $r_0 = e^2/mc^2$  is the classical electron

radius, and  $\delta(z)$  is the Dirac delta function. The case of scattering from a moving electron can be treated by using Eq. (4) in conjunction with a Lorentz transformation of Eq. (1). Finally, the case of scattering from a distribution of moving electrons can be treated by averaging the Lorentz transformed equation for a particular velocity over the velocity distribution of the electrons. For a Maxwellian distribution, the resulting scattering kernel will have three rather distinct characteristics, as pointed out by Dirac.<sup>7</sup> In the first place, a photon will, upon scattering, have its wavelength increased due to the usual Compton shift associated with scattering from an electron at rest. Secondly, it will undergo broadening due to the classical Doppler effect of scattering from a distribution of moving electrons. Finally, there will be a reduction in the wavelength upon scattering due to the relativistic effect that the photon density will appear more intense to an electron moving toward the photon than away from it. This last effect, the blue shift, is needed to "balance" the Compton red shift, for, as pointed out by Milne,<sup>8</sup> if blackbody radiation at a certain temperature scatters from a Maxwellian gas of free electrons at the same temperature, the scattered radiation must have the same distribution in wavelength as the incident radiation.

Dirac<sup>7</sup> was the first to consider the problem of determining the scattering kernel for photons scattering from a Maxwellian gas of electrons. He used the low frequency limit of Eq. (4), i.e.,

$$\sigma_s(\nu \rightarrow \nu', \mu) = N_e r_0^2 \left( \frac{1+\mu^2}{2} \right) \delta \left( \nu' - \frac{\nu}{1+\gamma(1-\mu)} \right), \quad (5)$$

to describe the scattering in the electron rest frame because, at that time (1925), the Klein-Nishina formula was not available. He obtained a result as a series in  $\sqrt{\alpha} \equiv (kT/mc^2)^{\frac{1}{2}}$  and found a relatively simple expression by explicitly carrying only the leading term. In this low-order approximation, the Compton red shift and the relativistic blue shift were lost, and Dirac's kernel showed only Doppler broadening. Munch<sup>9</sup> used this result in the equation of transfer to investigate the problem of line broadening due to scattering. Edmonds<sup>10</sup> re-considered the problem of computing the scattering

kernel in 1953 using the Klein-Nishina formula and carrying the calculation to second order in  $\sqrt{\alpha}$ . In this order he found a rather complex result exhibiting both the red and blue shifts as well as Doppler broadening. Subsequently, Edmonds used his result in various transport problems.<sup>11,12</sup> Recently, Cantwell<sup>13</sup> has undertaken a numerical calculation, using Monte Carlo methods, to obtain the scattering kernel. These results show a rather smooth dependence of  $\sigma_s(\nu' \rightarrow \nu, \mu)$  on the three variables involved.

We now consider two distinct possibilities for making use of these scattering kernel results in radiative transfer calculations. The first method extends the analytic approach of Dirac and Edmonds and leads to a relatively simple treatment of scattering in the equation of transfer. This description is well suited for analytic investigations of scattering in radiative transfer problems. The second approach makes direct use of the numerical results of Cantwell and in principle is a more accurate method, although it is also more cumbersome.

### 3. A DIFFERENTIAL MODEL OF SCATTERING

We assume that  $I(\vec{r}, \nu, \vec{\Omega}, t)$  is an analytic function of the frequency  $\nu$  so that  $I(\nu', \vec{\Omega}')$  in Eq. (1) can be expanded in a Taylor series about  $\nu' = \nu$ . For any given term in the Taylor series, the dependence of the integrand in Eq. (1) upon  $\nu'$  is then completely known, and the integrals over frequency can in principle be performed term by term. This procedure converts the integral operator in the frequency variable of Eq. (1) into an infinite order differential operator, with the coefficients of the derivatives involving certain frequency moments of the scattering kernel. While Dirac<sup>7</sup> and Edmonds<sup>10</sup> have shown that the analytic representation of the scattering kernel is very complex, Fraser<sup>3</sup> has shown that, correct to order  $\alpha = kT/mc^2$  and  $\gamma = h\nu/mc^2$ , these moments of the kernel are very simple. In most problems of interest,  $\alpha$  and  $\gamma$  are generally quite small since  $mc^2 \approx 0.51$  MEV, and it should introduce a small error if higher order terms in  $\alpha$  and  $\gamma$  are neglected. This has the effect of truncating the infinite-order differential operator to one of second order. It should be noted that an implicit assumption in this truncation is that  $\nu^n \partial^n I(\nu) / \partial \nu^n$  is of the

same order of all  $n$ . Accordingly, if the spectrum  $I(\nu)$  has very sharp peaks or dips, this truncation procedure becomes questionable.

The above analysis is due to Fraser<sup>3</sup> and leads to\*

$$\begin{aligned} & \frac{1}{c} \frac{\partial I(\vec{r}, \nu, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I(\vec{r}, \nu, \vec{\Omega}, t) = \mu_a' \\ & \times \left[ B(\nu) - I(\nu, \vec{\Omega}) \right] - \mu_s (1 - 2\gamma) I(\nu, \vec{\Omega}) \\ & + \mu_s \int_{4\pi} d\vec{\Omega}' \sum_{n=0}^3 \left( \frac{2n+1}{4\pi} \right) P_n(\vec{\Omega} \cdot \vec{\Omega}') S_n I(\nu, \vec{\Omega}') \\ & - \frac{3\mu_s}{16\pi} \frac{c^2}{h\nu^3} \gamma I(\nu, \vec{\Omega}) (1 - \nu \frac{\partial}{\partial \nu}) \int_{4\pi} d\vec{\Omega}' \\ & \times \left[ 1 - (\vec{\Omega} \cdot \vec{\Omega}') + (\vec{\Omega} \cdot \vec{\Omega}')^2 - (\vec{\Omega} \cdot \vec{\Omega}')^3 \right] I(\nu, \vec{\Omega}'), \end{aligned} \quad (6)$$

where  $\mu_s = 8\pi N_e r_e^2/3$  is the Thomson scattering coefficient,  $P_n(z)$  is the  $n$ th Legendre polynomial, and the operators  $S_n$  are defined as

$$S_0 = \left[ 1 - \gamma \left( 1 - \nu \frac{\partial}{\partial \nu} \right) - \alpha \left( 2\nu \frac{\partial}{\partial \nu} - \nu^2 \frac{\partial^2}{\partial \nu^2} \right) \right], \quad (7)$$

$$S_1 = \frac{2}{5} \left[ \gamma \left( 1 - \nu \frac{\partial}{\partial \nu} \right) - \alpha \left( 1 - 2\nu \frac{\partial}{\partial \nu} + \nu^2 \frac{\partial^2}{\partial \nu^2} \right) \right], \quad (8)$$

$$S_2 = \frac{1}{10} \left[ 1 - \gamma \left( 1 - \nu \frac{\partial}{\partial \nu} \right) - \alpha \left( 6 + 2\nu \frac{\partial}{\partial \nu} - \nu^2 \frac{\partial^2}{\partial \nu^2} \right) \right], \quad (9)$$

$$S_3 = \frac{3}{70} \left[ \gamma \left( 1 - \nu \frac{\partial}{\partial \nu} \right) + \alpha \left( 4 + 2\nu \frac{\partial}{\partial \nu} - \nu^2 \frac{\partial^2}{\partial \nu^2} \right) \right]. \quad (10)$$

Equation (16) is the photon analogue of the heavy gas model used in neutron thermalization work.<sup>19</sup>

For  $\alpha = 0$  (zero temperature), the second derivative terms in Eq. (6) vanish and it reduces to an equation similar to the Fermi age approximation in neutron transport theory.<sup>20</sup>

It has recently been shown<sup>21</sup> that this

\*Several classified reports (with unclassified titles) have also been written on this differential approach to scattering.<sup>14-18</sup>

result, Eq. (6), can be further simplified with no loss of the essential physics. We shall repeat the arguments here. We project Eq. (6) onto the basis elements of a spherical harmonic function space. We shall follow the vectorial method introduced in neutron transport theory. Since these spherical harmonic equations are only an intermediate result, we shall omit the details of their derivation, referring the interested reader to the book by Davison.<sup>22</sup> If for simplicity we momentarily neglect the induced scattering (nonlinear) terms in Eq. (6), the result is

$$\begin{aligned} & \frac{1}{c} \frac{\partial J_0}{\partial t} + \vec{\nabla}_U \cdot \vec{\nabla}_r J_1 + \mu_a' (J_0 - 4\pi B) \\ & + \mu_s (1 - 2\gamma - S_0) J_0 = 0, \end{aligned} \quad (11)$$

$$\begin{aligned} & \frac{3}{c} \frac{\partial J_1}{\partial t} + \vec{\nabla}_U \cdot \vec{\nabla}_r J_2 + 3[\mu_a' + \mu_s (1 - 2\gamma - S_1)] J_1 \\ & + U \cdot \vec{\nabla}_r J_0 = 0, \end{aligned} \quad (12)$$

$$\begin{aligned} & \frac{5}{c} \frac{\partial J_2}{\partial t} + \vec{\nabla}_U \cdot \vec{\nabla}_r J_3 + 5[\mu_a' + \mu_s (1 - 2\gamma - S_2)] J_2 \\ & + 3U \cdot \vec{\nabla}_r J_1 - U^2 \vec{\nabla}_U \cdot \vec{\nabla}_r J_1 = 0, \end{aligned} \quad (13)$$

$$\begin{aligned} & \frac{7}{c} \frac{\partial J_3}{\partial t} + \vec{\nabla}_U \cdot \vec{\nabla}_r J_4 + 7[\mu_a' + \mu_s (1 - 2\gamma - S_3)] J_3 \\ & + 5U \cdot \vec{\nabla}_r J_2 - U^2 \vec{\nabla}_U \cdot \vec{\nabla}_r J_2 = 0, \end{aligned} \quad (14)$$

$$\begin{aligned} & \frac{2n+1}{c} \frac{\partial J_n}{\partial t} + \vec{\nabla}_U \cdot \vec{\nabla}_r J_{n+1} \\ & + (2n+1)[\mu_a' + \mu_s (1 - 2\gamma)] J_n \\ & + (2n-1) U \cdot \vec{\nabla}_r J_{n-1} - U^2 \vec{\nabla}_U \cdot \vec{\nabla}_r J_{n-1} = 0, \quad n \geq 4. \end{aligned} \quad (15)$$

In these equations the vector  $\vec{U}$  is in the direction  $\vec{\Omega}$  and has an arbitrary magnitude  $U$ . The functions  $J_n$  are defined as

$$J_n = \frac{U^n}{2n+1} \sum_{m=-n}^n A_{nm} I_{nm}(\vec{r}, \nu, t) Y_{nm}(\vec{\Omega}), \quad (16)$$



where the  $I_{nm}(\vec{r}, \nu, t)$  are the coefficients of an expansion of the specific intensity in surface harmonics according to

$$I(\vec{r}, \nu, \vec{\Omega}, t) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{nm} I_{nm}(\vec{r}, \nu, t) Y_{nm}(\vec{\Omega}). \quad (17)$$

Here the surface harmonics are defined in the usual way:

$$Y_{nm}(\vec{\Omega}) = P_n^{|m|}(\cos \theta) e^{im\phi}, \quad (18)$$

where the  $P_n^m(z)$  are the associated Legendre functions and the constants  $A_{nm}$  are normalization coefficients

$$A_{nm} = 4\pi \left[ \int_{4\pi} d\vec{\Omega} Y_{nm}(\vec{\Omega}) Y_{nm}^*(\vec{\Omega}) \right]^{-1} = \frac{(2n+1)(n-|m|)!}{(n+|m|)!} \quad (19)$$

with the superscript \* on  $Y_{nm}(\vec{\Omega})$  indicating the complex conjugate. Due to the biorthogonality relationship between the surface harmonics and their complex conjugates, one has an explicit expression for  $I_{nm}(\vec{r}, \nu, t)$  in terms of the specific intensity, i.e.,

$$I_{nm}(\vec{r}, \nu, t) = \int_{4\pi} d\vec{\Omega} Y_{nm}^*(\vec{\Omega}) I(\vec{r}, \nu, \vec{\Omega}, t). \quad (20)$$

Since  $J_n$  consists of  $2n+1$  angular components (see Eq. (16)), Eq. (15) represents in general  $2n+1$  relationships between the components of  $J_{n-1}$ ,  $J_n$ , and  $J_{n+1}$ . In special cases, symmetry considerations may reduce the number of nonzero components of  $J_n$ , and the number of relationships in Eq. (15) is reduced accordingly. For example, in plane parallel problems the specific intensity is independent of the azimuthal angle  $\phi$  and hence  $J_n$  consists of only one nonzero component.

Now, in Eq. (12) we replace  $\mu_s(1-2\gamma-S_1)$  by just  $\mu_s$ , since  $S_1+2\gamma$  is of order  $\alpha$  and  $\gamma$  and hence to lowest order  $\mu_s(1-2\gamma-S_1) = \mu_s$ . By similar arguments, we replace  $\mu_s(1-2\gamma-S_2)$  in Eq. (13) by  $9\mu_s/10$ , and in Eqs. (14) and (15) we replace  $\mu_s(1-2\gamma-S_3)$  and  $\mu_s(1-2\gamma)$  in each case by  $\mu_s$ . We note, however, that we cannot make a similar simplification in Eq. (11) since  $(1-2\gamma-S_0)$  is of order  $\alpha$  and  $\gamma$ , rather than of order unity (or  $9/10$ ) as are the similar terms in Eqs. (12) through (15). Introducing these simplifications

into Eqs. (12) through (15) we find that Eqs. (11) through (15) are the spherical harmonic projections of the equation of transfer:

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\vec{r}, \nu, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I(\vec{r}, \nu, \vec{\Omega}, t) &= \mu_a [B(\nu) - I(\nu, \vec{\Omega})] \\ &- \mu_s I(\nu, \vec{\Omega}) + \frac{3\mu_s}{16\pi} \int_{4\pi} d\vec{\Omega}' [1 + (\vec{\Omega}' \cdot \vec{\Omega})^2] I(\nu, \vec{\Omega}') \\ &+ \frac{\mu_s}{4\pi} \int_{4\pi} d\vec{\Omega}' \left[ \alpha \nu^2 \frac{\partial^2}{\partial \nu^2} + (\gamma - 2\alpha) \nu \frac{\partial}{\partial \nu} + \gamma \right] I(\nu, \vec{\Omega}'). \end{aligned} \quad (21)$$

To Eq. (21) we need add the contribution of the nonlinear induced scattering terms in Eq. (6). Since these terms are of order  $\gamma$ , they can be neglected in all but the zeroth angular moment of the equation of transfer, just as we neglected all terms of order  $\alpha$  and  $\gamma$  in the linear analysis just completed except in the zeroth angular moment relationship, Eq. (11). This implies the replacement in the equation of transfer:

$$\begin{aligned} I(\nu, \vec{\Omega}) (1 - \nu \frac{\partial}{\partial \nu}) \int_{4\pi} d\vec{\Omega}' \left[ 1 - (\vec{\Omega}' \cdot \vec{\Omega}) + (\vec{\Omega}' \cdot \vec{\Omega}')^2 \right. \\ \left. - (\vec{\Omega}' \cdot \vec{\Omega}')^3 \right] I(\nu, \vec{\Omega}') \\ \longrightarrow \frac{1}{4\pi} \int_{4\pi} d\vec{\Omega}' I(\nu, \vec{\Omega}') \int_{4\pi} d\vec{\Omega} \left[ 1 - (\vec{\Omega} \cdot \vec{\Omega}') + (\vec{\Omega} \cdot \vec{\Omega}')^2 \right. \\ \left. - (\vec{\Omega} \cdot \vec{\Omega}')^3 \right] \left( 1 - \nu \frac{\partial}{\partial \nu} \right) I(\nu, \vec{\Omega}'). \end{aligned} \quad (22)$$

Thus, the full form of Eq. (21), including the effects of induced scattering, is

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\vec{r}, \nu, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I(\vec{r}, \nu, \vec{\Omega}, t) &= \mu_a [B(\nu) - I(\nu, \vec{\Omega})] \\ &- \mu_s I(\nu, \vec{\Omega}) + \frac{3\mu_s}{16\pi} \int_{4\pi} d\vec{\Omega}' [1 + (\vec{\Omega}' \cdot \vec{\Omega}')^2] I(\nu, \vec{\Omega}') \\ &+ \frac{\mu_s}{4\pi} \int_{4\pi} d\vec{\Omega}' \left[ \alpha \nu^2 \frac{\partial^2}{\partial \nu^2} + (\gamma - 2\alpha) \nu \frac{\partial}{\partial \nu} + \gamma \right] I(\nu, \vec{\Omega}') \\ &- \frac{3\mu_s}{64\pi^2} \frac{c^2}{h\nu^3} \gamma \int_{4\pi} d\vec{\Omega}' I(\nu, \vec{\Omega}') \int_{4\pi} d\vec{\Omega} \left[ 1 - (\vec{\Omega}' \cdot \vec{\Omega}') \right. \\ &\left. + (\vec{\Omega}' \cdot \vec{\Omega}')^2 - (\vec{\Omega}' \cdot \vec{\Omega}')^3 \right] (1 - \nu \frac{\partial}{\partial \nu}) I(\nu, \vec{\Omega}'), \end{aligned} \quad (23)$$

which is a simplified, but a priori just as accurate, form of Fraser's result, Eq. (6). In particular, Eq. (23) contains far fewer scattering terms than does Eq. (6), and the terms which account for energy transfer in the scattering interaction, i.e., those proportional to  $\alpha$  and  $\gamma$ , are isotropic in Eq. (23), whereas they are angularly dependent in Eq. (6). Both of these facts should make Eq. (23) much easier to solve, either analytically or numerically, than Eq. (6). A significant property of Eq. (23) is that it gives the proper equilibrium solution, namely, a Planck distribution at temperature T as given by Eq. (2). This can be verified by direct substitution. There are also indications that Eq. (23) is more accurate than Eq. (6), at least in the low temperature limit. Several authors<sup>23-27</sup> have used the diffusion, or Eddington, approximation to Eq. (6) to describe photon transport through cold ( $\alpha = 0$ ) electrons. All found a physically incorrect behavior, namely, a slight increase in frequency for some photons due to scattering. On the other hand, Eq. (23), or its diffusion approximation, gives the proper behavior of only "downscattering" from cold electrons.<sup>21,28</sup>

Let us form the first two angular moments of Eq. (23). Integration of Eq. (23) over all solid angle yields

$$\begin{aligned} \frac{1}{c} \frac{\partial E(\vec{r}, \nu, t)}{\partial t} + \vec{\nabla} \cdot \vec{F}(\vec{r}, \nu, t) &= \mu'_a [4\pi B(\nu) - E(\nu)] \\ + \mu_s \left[ \alpha \nu^2 \frac{\partial^2 E}{\partial \nu^2} + (\gamma - 2\alpha) \nu \frac{\partial E}{\partial \nu} + \gamma E \right] \\ - \frac{3\mu_s c^2}{16\pi h\nu^3} \gamma \left[ E(1 - \nu \frac{\partial}{\partial \nu}) E - \vec{F} \cdot (1 - \nu \frac{\partial}{\partial \nu}) \vec{F} \right. \\ \left. + \hat{P} \cdot (1 - \nu \frac{\partial}{\partial \nu}) \hat{P} - \tilde{Q} \cdot (1 - \nu \frac{\partial}{\partial \nu}) \tilde{Q} \right], \end{aligned} \quad (24)$$

whereas multiplication of Eq. (23) by  $\vec{\Omega}$  prior to integration over all solid angle gives

$$\frac{1}{c} \frac{\partial \vec{F}(\vec{r}, \nu, t)}{\partial t} + \vec{\nabla} \cdot \hat{P}(\vec{r}, \nu, t) + (\mu'_a + \mu_s) \vec{F}(\nu) = 0. \quad (25)$$

Here we have defined the first four angular moments of the specific intensity as

$$E(\vec{r}, \nu, t) = \int_{4\pi} d\vec{\Omega} I(\vec{r}, \nu, \vec{\Omega}, t), \quad (26)$$

$$\vec{F}(\vec{r}, \nu, t) = \int_{4\pi} d\vec{\Omega} \vec{\Omega} I(\vec{r}, \nu, \vec{\Omega}, t), \quad (27)$$

$$\hat{P}(\vec{r}, \nu, t) = \int_{4\pi} d\vec{\Omega} \vec{\Omega} \vec{\Omega} I(\vec{r}, \nu, \vec{\Omega}, t), \quad (28)$$

$$\tilde{Q}(\vec{r}, \nu, t) = \int_{4\pi} d\vec{\Omega} \vec{\Omega} \vec{\Omega} \vec{\Omega} I(\vec{r}, \nu, \vec{\Omega}, t). \quad (29)$$

Physically, E is the product of the speed of light and the energy density,  $\vec{F}$  is the radiative flux, and  $\hat{P}$  is the pressure tensor.  $\tilde{Q}$  has no simple physical interpretation.

The Eddington approximation follows immediately from Eqs. (24) and (25). We assume that the specific intensity is almost isotropic and hence can be represented by the first two terms in a surface harmonic expansion, i.e.,

$$I(\vec{r}, \nu, \vec{\Omega}, t) = \frac{1}{4\pi} E(\vec{r}, \nu, t) + \frac{3}{4\pi} \vec{\Omega} \cdot \vec{F}(\vec{r}, \nu, t). \quad (30)$$

Equation (30) allows one to express  $\hat{P}$  and  $\tilde{Q}$  in terms of E and  $\vec{F}$ . Equations (24) and (25) then reduce to:<sup>21</sup>

$$\begin{aligned} \frac{1}{c} \frac{\partial E(\vec{r}, \nu, t)}{\partial t} + \vec{\nabla} \cdot \vec{F}(\vec{r}, \nu, t) &= \mu'_a [4\pi B(\nu) - E(\nu)] \\ + \mu_s \left[ \alpha \nu^2 \frac{\partial^2 E}{\partial \nu^2} + (\gamma - 2\alpha) \nu \frac{\partial E}{\partial \nu} + \gamma E \right] \\ - \frac{\mu_s c^2 \gamma}{4\pi h\nu^3} \left[ E^2 - \nu E \frac{\partial E}{\partial \nu} - \frac{6}{5} \vec{F} \cdot \vec{F} + \frac{6}{5} \nu \vec{F} \cdot \frac{\partial \vec{F}}{\partial \nu} \right], \end{aligned} \quad (31)$$

$$\frac{1}{c} \frac{\partial \vec{F}(\vec{r}, \nu, t)}{\partial t} + \frac{1}{3} \vec{\nabla} E(\vec{r}, \nu, t) + (\mu'_a + \mu_s) \vec{F}(\nu) = 0, \quad (32)$$

which are the Eddington equations including scattering and temporal effects. Since the intensity representation, Eq. (30), is only strictly correct when  $|\vec{F}| \ll E$ , the quadratic terms in  $\vec{F}$  in Eq. (31) can consistently be neglected.

Of course, one can envision less direct methods of relating  $\hat{P}$  and  $\tilde{Q}$  to E and  $\vec{F}$  than through the use of Eq. (30). Two such schemes have recently been suggested<sup>29-31</sup> which lead to

modifications of the diffusion equations, (30) and (31). Radiative transfer codes based on the transport equation (23), the Eddington equations, (24) and (25), or the modified Eddington equations just mentioned, are presently available at the Air Force Weapons Laboratory; Gulf General Atomic Incorporated; Los Alamos Scientific Laboratory; and Systems, Science, and Software. In all cases the frequency variable is handled by the usual group method. For the scattering terms one must cast the derivative terms in the frequency variable into finite difference form in terms of the group intensities. While there is no unique way to do this, several different schemes seem to give satisfactory results.<sup>29,32</sup>

We now summarize the advantages and disadvantages of the differential model of Compton and inverse Compton scattering. The major advantage of this model possesses is its simplicity. All of the information contained in the three variable function  $\sigma_s(v' \rightarrow v, \mu)$  has been transformed into a few very simple coefficients in a differential operator. Further, the energy exchange terms in the equation of transfer resulting from this model are isotropic. The disadvantages are two. In the first place, the occurrence of derivatives with respect to frequency in the equation of transfer can lead to stability problems in a numerical solution. Accordingly, more care is required in devising an iterative scheme to handle the frequency group index. Secondly, the differential model of scattering is known to be inaccurate for high frequencies, i.e., for those values of  $v$  such that  $hv/mc^2$  cannot be considered to be much less than one. In practice, the cutoff in frequency above which accurate results cannot be expected is probably around  $hv \approx 50$  kev ( $\gamma \approx 0.1$ ). Further, if the spectrum is not a smooth function of frequency, inaccuracies will occur for even lower frequencies.

#### 4. AN INTEGRAL MODEL OF SCATTERING

In this section we consider an alternative to the differential model of scattering which does not have the disadvantages of that model. This alternative uses directly the scattering kernel  $\sigma_s(v \rightarrow v', \mu)$  as computed by Cantwell.<sup>13</sup> That

is, with this scattering kernel available, all of the data needed in Eq. (1) are known and in principle one needs only to cast Eq. (1) into group form and numerically solve the resulting equations. However, it is desirable and possible to simplify Eq. (1) before forming the group equations. We now consider such a simplification.

As in the differential model of scattering, it is easily argued that a small error is introduced if the energy exchange terms are retained only in the zeroth angular moment of Eq. (1). Thus all of the energy exchange terms in Eq. (1) can be replaced with their angular averages. This leads to the equation of transfer, suppressing the algebra,

$$\begin{aligned} \frac{1}{c} \frac{\partial I(\vec{r}, \nu, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \nabla I(\vec{r}, \nu, \vec{\Omega}, t) &= \mu'_e(\nu) [B(\nu) - I(\nu, \vec{\Omega})] \\ &- \sigma_s(\nu) I(\nu, \vec{\Omega}) + \frac{1}{4\pi} \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \sigma_{s0}(\nu' \rightarrow \nu) \\ &\quad \times \frac{\nu}{\nu'} I(\nu', \vec{\Omega}') \\ &+ \frac{\mu_s}{16\pi} \int_{4\pi} d\vec{\Omega}' [3(\vec{\Omega} \cdot \vec{\Omega}')^2 - 1] I(\nu, \vec{\Omega}') \\ &- \frac{1}{4\pi} \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \int_{4\pi} d\vec{\Omega}'' \sigma_s(\nu \rightarrow \nu', \vec{\Omega}' \cdot \vec{\Omega}'') \\ &\quad \times I(\nu, \vec{\Omega}'') \frac{c^2}{2h\nu'^3} I(\nu', \vec{\Omega}') \\ &+ \frac{1}{4\pi} \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \int_{4\pi} d\vec{\Omega}'' \sigma_s(\nu' \rightarrow \nu, \vec{\Omega}' \cdot \vec{\Omega}'') \\ &\quad \times I(\nu', \vec{\Omega}'') \frac{c^2}{2h\nu'\nu^2} I(\nu, \vec{\Omega}'). \end{aligned} \quad (33)$$

Here  $\sigma_{s0}(\nu' \rightarrow \nu)$  is defined by the Legendre polynomial expansion of the scattering kernel

$$\sigma_s(\nu' \rightarrow \nu, \mu) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) \sigma_{sn}(\nu' \rightarrow \nu) P_n(\mu), \quad (34)$$

and hence

$$\sigma_{s0}(\nu' \rightarrow \nu) = 2\pi \int_{-1}^1 d\mu \sigma_s(\nu' \rightarrow \nu, \mu). \quad (35)$$

The other new symbol appearing in Eq. (33) is  $\sigma_s(\nu)$ . It is given by

$$\begin{aligned}\sigma_s(\nu) &= \int_0^\infty d\nu' \sigma_{so}(\nu \rightarrow \nu') \\ &= 2\pi \int_0^\infty d\nu' \int_{-1}^1 d\mu \sigma_s(\nu \rightarrow \nu', \mu).\end{aligned}\quad (36)$$

If for the moment we neglect the induced scattering terms, i.e., those quadratic in the intensity, a great simplification has been effected in going from Eq. (1) to Eq. (33) from the computational point of view. A function of three variables,  $\sigma_s(\nu' \rightarrow \nu, \mu)$ , occurring in Eq. (1) has been replaced in Eq. (33) with a function of two variables  $\sigma_{so}(\nu' \rightarrow \nu)$ . Unfortunately, however, the function of three variables still occurs in the induced scattering terms of Eq. (33).

There are at least two ways to rectify this situation, both of which are less than satisfactory. In the first method, we expand  $\sigma_s(\nu' \rightarrow \nu, \mu)$  in the induced scattering terms in Eq. (33) according to Eq. (34), retaining the  $n = 0$  term, but replacing all of the higher terms with the corresponding terms from the differential model result, Eq. (23). This leads to the replacement of Eq. (33) by

$$\begin{aligned}& \frac{1}{c} \frac{\partial I(\vec{r}, \vec{\nu}, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I(\vec{r}, \vec{\nu}, \vec{\Omega}, t) \\ &= \mu'_s(\nu) [B(\nu) - I(\nu, \vec{\Omega})] \\ &+ \sigma_s(\nu) I(\nu, \vec{\Omega}) + \frac{1}{4\pi} \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \sigma_{so}(\nu' \rightarrow \nu) \\ &\quad \times \frac{\nu}{\nu'} I(\nu', \vec{\Omega}') \\ &+ \frac{\mu_s}{16\pi} \int_{4\pi} d\vec{\Omega}' [3(\vec{\Omega} \cdot \vec{\Omega}')^2 - 1] I(\nu, \vec{\Omega}') \\ &+ \frac{1}{16\pi^2} \int_0^\infty d\nu' \frac{c^2}{2h\nu'} \left[ \frac{1}{\nu'^2} \sigma_{so}(\nu' \rightarrow \nu) \right. \\ &- \left. \frac{1}{\nu'^2} \sigma_{so}(\nu \rightarrow \nu') \right] \int_{4\pi} d\vec{\Omega}' I(\nu', \vec{\Omega}') \int_{4\pi} d\vec{\Omega}'' I(\nu, \vec{\Omega}'') \\ &+ \frac{3\mu_s}{64\pi^2} \frac{c^2}{h\nu^3} \gamma \int_{4\pi} d\vec{\Omega}' I(\nu, \vec{\Omega}') \int_{4\pi} d\vec{\Omega}'' \left[ \frac{1}{3} + (\vec{\Omega}' \cdot \vec{\Omega}'') \right. \\ &- \left. (\vec{\Omega}' \cdot \vec{\Omega}'')^2 + (\vec{\Omega}' \cdot \vec{\Omega}'')^3 \right] (1 - \nu \frac{\partial}{\partial \nu}) I(\nu, \vec{\Omega}').\end{aligned}\quad (37)$$

Equation (37) has the desired property that the three-variable function  $\sigma_s(\nu' \rightarrow \nu, \mu)$  does not appear, but this was achieved at the expense of introducing the disadvantages of the differential

model, as discussed in the last section, into the induced scattering terms.

As a second method of eliminating  $\sigma_s(\nu' \rightarrow \nu, \mu)$  from Eq. (33), we argue as follows: In going from Eq. (1) to Eq. (33), the net effect on the scattering terms linear in the intensity was as if we had used a scattering kernel

$$\begin{aligned}\sigma_s(\nu' \rightarrow \nu, \mu) &= \frac{1}{4\pi} \sigma_0(\nu' \rightarrow \nu) \\ &+ \frac{\mu_s}{16\pi} (3\mu^2 - 1) \delta(\nu - \nu'),\end{aligned}\quad (38)$$

where, as before,  $\mu_s$  represents the Thomson scattering coefficient, and  $\delta(z)$  is the Dirac delta function. Let us, as an approximation, also use Eq. (38) as the scattering kernel in the induced scattering terms in Eq. (33). Actually, we can make a weaker assumption and use

$$\sigma_s(\nu' \rightarrow \nu, \mu) = \frac{1}{4\pi} \sigma_0(\nu' \rightarrow \nu) + f(\mu) \delta(\nu - \nu'), \quad (39)$$

where  $f(\mu)$  is an arbitrary angular function, as the kernel in Eq. (33). The term containing the Dirac delta function contributes nothing when Eq. (39) is used in Eq. (33), and we find

$$\begin{aligned}& \frac{1}{c} \frac{\partial I(\vec{r}, \vec{\nu}, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I(\vec{r}, \vec{\nu}, \vec{\Omega}, t) = \mu'_s(\nu) [B(\nu) - I(\nu, \vec{\Omega})] \\ &- \sigma_s(\nu) I(\nu, \vec{\Omega}) + \frac{1}{4\pi} \int_0^\infty d\nu' \int_{4\pi} d\vec{\Omega}' \sigma_{so}(\nu' \rightarrow \nu) \\ &\quad \times \frac{\nu}{\nu'} I(\nu', \vec{\Omega}') \\ &+ \frac{\mu_s}{16\pi} \int_{4\pi} d\vec{\Omega}' [3(\vec{\Omega} \cdot \vec{\Omega}')^2 - 1] I(\nu, \vec{\Omega}') \\ &+ \frac{1}{16\pi^2} \int_0^\infty d\nu' \frac{c^2}{2h\nu'} \left[ \frac{1}{\nu'^2} \sigma_{so}(\nu' \rightarrow \nu) \right. \\ &- \left. \frac{1}{\nu'^2} \sigma_{so}(\nu \rightarrow \nu') \right] \int_{4\pi} d\vec{\Omega}' I(\nu', \vec{\Omega}') \int_{4\pi} d\vec{\Omega}'' I(\nu, \vec{\Omega}''),\end{aligned}\quad (40)$$

as the equation of transfer. In going from Eq. (33) to Eq. (40), we have made a significant approximation in the induced scattering terms and it is reasonable to inquire under what circumstances this might introduce a nonnegligible error. For high energy photons (say  $h\nu > 3kT$ ), it is well-

known that the induced scattering terms in the equation of transfer contribute very little and hence their treatment is relatively unimportant. For dilute radiation ( $I \ll B$ ), the quadratic nature of the induced terms also makes their contribution to the equation of transfer small. Finally, for almost isotropic radiation, only the zeroth angular moment of the scattering kernel, which we have treated rigorously in Eq. (40), is significant. Hence a nonnegligible error may have been introduced into Eq. (40) in the case of low energy photons belonging to an intense, highly anisotropic, specific intensity. Equation (40) also follows from Eq. (37) by neglecting those terms with a coefficient  $\gamma$ . This suggests that the terms we are discussing are small in any event. Equation (37) is an improvement over Eq. (40), and Eq. (33) is a further improvement over Eq. (37). The "rigorous" description, Eq. (1), is of course the most accurate. My feeling is that Eq. (40) should be adequate in the majority of practical problems dealing with radiant energy transfer, but there may be instances for which one of the more precise equations (1), (33), or (37), is required. In the remainder of this discussion, we shall use Eq. (40) as the equation of transfer. Making use of  $E(\vec{r}, \nu, t)$  as defined by Eq. (26), we can rewrite Eq. (40) in the less imposing form:

$$\begin{aligned} & \frac{1}{c} \frac{\partial I(\vec{r}, \nu, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I(\vec{r}, \nu, \vec{\Omega}, t) \\ &= \mu'_a(\nu) B(\nu) - \sigma(\nu) I(\nu, \vec{\Omega}) \\ &+ \frac{1}{4\pi} \int_0^\infty d\nu' \sigma_{so}(\nu' \rightarrow \nu) \frac{\nu}{\nu'} E(\nu') \\ &+ \frac{\mu_s}{16\pi} \int_{4\pi} d\vec{\Omega}' [3(\vec{\Omega} \cdot \vec{\Omega}')^2 - 1] I(\nu, \vec{\Omega}') \\ &+ \frac{1}{16\pi^2} E(\nu) \int_0^\infty d\nu' \frac{c^2}{2h\nu'} \left[ \frac{1}{\nu^2} \sigma_{so}(\nu' \rightarrow \nu) \right. \\ &\left. - \frac{1}{\nu'^2} \sigma_{so}(\nu \rightarrow \nu') \right] E(\nu'), \end{aligned} \quad (41)$$

where we have defined

$$\sigma(\nu) \equiv \sigma_s(\nu) + \mu'_a(\nu). \quad (42)$$

It should be noted that Eq. (40) or (41) gives the proper equilibrium distribution, namely, a Planck distribution at temperature  $T$  as given by Eq. (2).

These equations also give the correct Thomson scattering limit. As both  $\alpha$  and  $\gamma$  approach zero, we have

$$\sigma_{so}(\nu' \rightarrow \nu) \rightarrow \mu_s \delta(\nu - \nu'), \quad (43)$$

$$\sigma_{so}(\nu) \rightarrow \mu_s; \quad \sigma(\nu) \rightarrow \mu_s + \mu'_a(\nu). \quad (44)$$

Then Eq. (40) or (41) reduces to

$$\begin{aligned} & \frac{1}{c} \frac{\partial I(\vec{r}, \nu, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I(\vec{r}, \nu, \vec{\Omega}, t) = \mu'_a(\nu) [B(\nu) - I(\nu, \vec{\Omega})] \\ & - \mu_s I(\nu, \vec{\Omega}) + \frac{3\mu_s}{16\pi} \int_{4\pi} d\vec{\Omega}' [1 + (\vec{\Omega} \cdot \vec{\Omega}')^2] I(\nu, \vec{\Omega}'), \end{aligned} \quad (45)$$

which is the rigorous, in the sense of Eq. (1), equation of transfer for Thomson scattering.

One can very easily derive the diffusion, or Eddington, approximation corresponding to Eq. (41). Forming the first two angular moments of this equation, we find

$$\begin{aligned} & \frac{1}{c} \frac{\partial E(\vec{r}, \nu, t)}{\partial t} + \vec{\nabla} \cdot \vec{F}(\vec{r}, \nu, t) = 4\pi \mu'_a B(\nu) - \sigma(\nu) E(\nu) \\ & + \int_0^\infty d\nu' \sigma_{so}(\nu' \rightarrow \nu) \frac{\nu}{\nu'} E(\nu') + \frac{E(\nu)}{4\pi} \int_0^\infty d\nu' \frac{c^2}{2h\nu'} \\ & \times \left[ \frac{1}{\nu^2} \sigma_{so}(\nu' \rightarrow \nu) - \frac{1}{\nu'^2} \sigma_{so}(\nu \rightarrow \nu') \right] E(\nu'), \end{aligned} \quad (46)$$

$$\frac{1}{c} \frac{\partial \vec{F}(\vec{r}, \nu, t)}{\partial t} + \vec{\nabla} \cdot \hat{P}(\vec{r}, \nu, t) + \sigma(\nu) \vec{F}(\nu) = 0, \quad (47)$$

where  $E$ ,  $\vec{F}$ , and  $\hat{P}$  are defined by Eqs. (26) through (28). Use of Eq. (30) as the specific intensity representation in Eqs. (46) and (47) leads immediately to the diffusion approximation. Equation (46) is unaffected, and Eq. (47) becomes

$$\frac{1}{c} \frac{\partial \vec{F}(\vec{r}, \nu, t)}{\partial t} + \frac{1}{3} \vec{\nabla} E(\vec{r}, \nu, t) + \sigma(\nu) \vec{F}(\nu) = 0, \quad (48)$$

which is just the Eddington relationship between the radiative flux and the energy density. As discussed in the previous section, other methods for eliminating  $\hat{P}(\vec{r}, \nu, t)$  from Eq. (47) have been suggested.<sup>29-31</sup>

In any group treatment of the frequency variable, the integrals over frequency in the scattering terms become sums, and the scattering kernels become scattering matrices. Since the determination of these scattering matrices is at the heart of any numerical treatment of the integral formulation of scattering, let us develop the group equation corresponding to Eq. (41). The scattering matrices will fall out as by-products of the derivation of these group equations. Similar considerations apply to the moment equations, (46) and (47). While the derivation of group equations<sup>33</sup> is somewhat arbitrary, we follow here the usual practice of simply integrating Eq. (41) over the frequencies defining the  $i^{\text{th}}$  group. That is, we consider all photons with frequencies between  $\nu_{n-1}$  and  $\nu_n$  to belong to the  $n^{\text{th}}$  group. If there are  $N$  total groups, then  $\nu_0 = 0$  and  $\nu_N = \infty$ . We define the  $n^{\text{th}}$  group specific intensity  $I_n$  as

$$I_n(\vec{r}, \vec{\Omega}, t) = \int_{\nu_{n-1}}^{\nu_n} d\nu I(\vec{r}, \nu, \vec{\Omega}, t). \quad (49)$$

Integration of Eq. (41) over the  $n^{\text{th}}$  group yields

$$\begin{aligned} \frac{1}{c} \frac{\partial I_n(\vec{r}, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \nabla I_n(\vec{r}, \vec{\Omega}, t) &= S_n - \sigma_n I_n(\vec{\Omega}) \\ + \frac{1}{4\pi} \sum_{m=1}^N A_{nm} E_m + \frac{\mu_s}{16\pi} \int_{4\pi} d\vec{\Omega}' [3(\vec{\Omega} \cdot \vec{\Omega}')^2 - 1] I_n(\vec{\Omega}') \\ + \frac{1}{16\pi^2} E_n \sum_{m=1}^N B_{nm} E_m, \quad 1 \leq n \leq N, \end{aligned} \quad (50)$$

where we have defined

$$S_n = \int_{\nu_{n-1}}^{\nu_n} d\nu \mu_a(\nu) B(\nu), \quad (51)$$

$$\sigma_n = \frac{\int_{\nu_{n-1}}^{\nu_n} d\nu \sigma(\nu) I(\nu, \vec{\Omega})}{\int_{\nu_{n-1}}^{\nu_n} d\nu I(\nu, \vec{\Omega})}, \quad (52)$$

$$A_{nm} = \frac{\int_{\nu_{n-1}}^{\nu_n} d\nu \int_{\nu_{m-1}}^{\nu_m} d\nu' \sigma_{go}(\nu' \rightarrow \nu) \frac{\nu}{\nu'} E(\nu')}{\int_{\nu_{m-1}}^{\nu_m} d\nu E(\nu)}, \quad (53)$$

$$B_{nm} = \frac{\int_{\nu_{n-1}}^{\nu_n} d\nu E(\nu) \int_{\nu_{m-1}}^{\nu_m} d\nu' \frac{c^2}{2h\nu'} \left[ \frac{1}{\nu^2} \sigma_{go}(\nu' \rightarrow \nu) - \frac{1}{\nu'^2} \sigma_{go}(\nu \rightarrow \nu') \right] E(\nu')}{\int_{\nu_{n-1}}^{\nu_n} d\nu E(\nu) \int_{\nu_{m-1}}^{\nu_m} d\nu' E(\nu')} \quad (54)$$

Equation (50) is exact in the sense that if the exact specific intensity is used to compute the group constants defined by Eqs. (51) through (54), Eq. (50) will give exact results for the group intensities. It should be noted that all of the group constants are functions of space and time, and, in addition,  $\sigma_n$  is a function of  $\vec{\Omega}$ . In practice, of course, the exact specific intensity cannot be used to calculate the group constants since it is unknown. An approximation to  $I(\nu, \vec{\Omega})$  and  $E(\nu)$  is used, and generally this approximation, for example, a Planck function at the local temperature, is independent of  $\vec{\Omega}$ . In practice, then,  $\sigma_n$  is a function only of  $\vec{r}$  and  $t$ , as are the scattering matrices of  $A$  and  $B$ .

The advantage of this integral formulation of scattering over the differential model is accuracy. The integral formulation treats high energy photons with an accuracy which the differential model cannot achieve. Its only disadvantage is complexity. One is forced to deal with at least a two-dimensional array,  $\sigma_{go}(\nu' \rightarrow \nu)$ , and perhaps one of three dimensions,  $\sigma_{ra}(\nu' \rightarrow \nu, \mu)$ . However, the fact that Cantwell<sup>13</sup> has computed these arrays makes this method attractive. To my knowledge, there are no existing computer codes which use this integral formulation of Compton and inverse Compton scattering.

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