LA-6191-MS

Informal Report

c.3

UC-34
Reporting Date: December 1975
Issued: January 1976

Stability of the Flow in a Shock Layer

by

George N. White

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87545

UNITED STATES
ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION
CONTRACT W-7405-ENG. 36

In the interest of prompt distribution, this report was not edited by the Technical Information staff.

Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22161
Price: Printed Copy $4.50 Microfiche $2.25

STABILITY OF THE FLOW IN A SHOCK LAYER

by

George N. White

ABSTRACT

The interior stability of an isothermal shock is studied as an initial value problem for inviscid one- and two-dimensional perturbations and as an initial-boundary value problem for viscous one-dimensional perturbations. The initial steady velocity profile is linear with finite width. Results, obtained from energy integral estimates, indicate stability at low Mach numbers and instability for large Mach numbers.

I. INTRODUCTION

The structure of strong, steady shocks has been investigated for many physical processes, atomic and nuclear. The emphasis has been on shocks in gases,[1] usually with low density. Particle transport has been included in some approximation. For dense gases, liquids, or solids, the strong shock structure problem has received less attention although many numerical schemes for treating nonsteady shocks are in use. Most of the schemes are based on the idealization of a discontinuity or on a nonphysical dissipation mechanism and not on reproducing the structural details that the physics would show. The adequacy of the numerical shock schemes depends on whether the physical shock thickness is small compared to the other lengths in the problem. If that is the situation, the details in the shock are unimportant. With the present uncertain state of knowledge of transport coefficients[2] and reaction rates in dense materials, it may be questionable whether detailed calculations of shock structure could have a useful level of credibility.

There is, in addition, the question of whether very strong steady shocks can be internally unstable to the degree that their thickness cannot always be predicted on the basis of a steady analysis. Such internal shock instability has been investigated in only two circumstances to my knowledge; collisionless plasmas,[3] not of interest for dense matter, and Morduchow's and Paulley's treatment[4] of a viscous, heat-conducting perfect gas. The latter attacks the problem by a normal mode analysis, but carries out details for weak shocks only, and finds stability.

This study estimates the rate of growth of an initial disturbance in the shock layer by means of energy integrals. The results show stability for initial Mach number close to unity and instability developing for larger Mach numbers. The equations are simplified by assuming isothermal flow and a linear unperturbed velocity profile, established by a mechanism not included in the basic equations. The effect of viscosity is seen to be stabilizing although estimates of a critical Reynold's number are difficult. The major defect of the approach taken is that the process establishing the shock is not permitted to influence the perturbations. This deficiency will hopefully be remedied in further work.

II. THE STEADY SHOCK STRUCTURE

The equations for isothermal one-dimensional gas flow are

$$\rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x}\right) = -\,c_o^2\,\frac{\partial \rho}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + u\,\frac{\partial \rho}{\partial x} + \rho\,\frac{\partial u}{\partial x} = 0. \qquad (2.1)$$

For steady flow Eq. (2.1) becomes

$$\rho u \frac{du}{dx} + c_o^2 \frac{d\rho}{dx} = 0 \qquad \frac{d(\rho u)}{dx} = 0 \tag{2.2}$$

with integrals

$$\begin{aligned} &\rho u = \rho_o u_o = \rho_1 u_1 \\ &\rho_o u_o (u-u_o) + c_o^2 (\rho-\rho_o) = 0. \end{aligned} \tag{2.3}$$

If the equations are made dimensionless with respect to the initial values of velocity and density, u_o and ρ_o, Eqs. (2.3) are

$$\rho u = 1 , \quad u + \frac{\rho}{M_o^2} = 1 + \frac{1}{M_o^2} . \tag{2.4}$$

The ranges of u and ρ are then

$$1 > u > \frac{1}{M_o^2} , \; M_o^2 > \rho > 1. \tag{2.5}$$

Since the velocity dependence in the shock is assumed to be linear and must satisfy the jump conditions, Eq. (2.5), the velocity profile is

$$u = \frac{1}{2} \left[\left(1 + \frac{1}{M_o^2}\right) - x \left(1 - \frac{1}{M_o^2}\right) \right], \quad -1 < x < 1 \tag{2.6}$$

where x is dimensionless with respect to the shock half-width. The flow is taken to be from left to right. For compactness, variables N and M are defined so that Eq. (2.6) can be written

$$u = N - Mx \quad . \tag{2.7}$$

III. THE BASIC PERTURBATION EQUATIONS

The assumption of isothermal flow allows the motion to be described without the energy conservation equation. The remaining equations are those for momentum and mass

$$\begin{aligned} &\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x^j} \right) = - \frac{\partial p}{\partial x^i} + \frac{\partial \tau_{ij}}{\partial x^j} \\ &\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x^j} + \rho \frac{\partial u_j}{\partial x^j} = 0 \quad . \end{aligned} \tag{3.1}$$

Consider the unperturbed flow (ρ, u_i, p, τ_{ij}) and a second flow ($\tilde{\rho}$, $\tilde{u}_i$, $\tilde{p}_i$, $\tilde{\tau}_{ij}$) and their difference or perturbed flow

$$\begin{aligned} &\rho' = \tilde{\rho} - \rho, \quad u_i' = \tilde{u}_i - u_i, \quad p_i' = \tilde{p} - p, \\ &\tau_{ij}' = \tilde{\tau}_{ij} - \tau_{ij} . \end{aligned} \tag{3.2}$$

The second flow also satisfies the basic equations

$$\tilde{\rho} \left(\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x^j} \right) = - \frac{\partial \tilde{p}}{\partial x^i} + \frac{\partial \tilde{\tau}_{ij}}{\partial x^j} \quad . \tag{3.3}$$

Subtraction of Eq. (3.1) from Eq. (3.3) gives equations for the perturbation quantities

$$\begin{aligned} &\rho \left(\frac{\partial u_i'}{\partial t} + u_j \frac{\partial u_i'}{\partial x^j} + u_j \frac{\partial \tilde{u}_i}{\partial x^j} \right) + \rho' \left(\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x^j} \right) \\ &\qquad = - \frac{\partial p'}{\partial x^i} + \frac{\partial \tau_{ij}'}{\partial x^j} \\ &\frac{\partial \rho'}{\partial t} + u_j' \frac{\partial \tilde{\rho}}{\partial x^j} + \left(\tilde{u}_j - u_j' \frac{\partial \rho'}{\partial x^j} \right) + \rho' \frac{\partial \tilde{u}_j}{\partial x^j} \\ &\qquad + \left(\tilde{\rho} - \rho' \right) \frac{\partial u_j'}{\partial x^j} = 0. \end{aligned} \tag{3.4}$$

If the perturbed quantities are small and their products can be neglected, the linear equations are

$$\begin{aligned} &\rho \left(\frac{\partial u_i'}{\partial t} + u_j \frac{\partial u_i'}{\partial x^j} \right) + \rho u_j' \frac{\partial u_i}{\partial x^j} + \rho' \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x^j} \right) \\ &\qquad = - \frac{\partial p'}{\partial x^i} + \frac{\partial \tau_{ij}'}{\partial x^j} \\ &\frac{\partial \rho'}{\partial t} + u_j \frac{\partial \rho'}{\partial x^j} + u_j' \frac{\partial \rho}{\partial x^j} + \rho' \frac{\partial u_j}{\partial x^j} + \rho \frac{\partial u_j'}{\partial x^j} = 0. \end{aligned} \tag{3.5}$$

In two dimensions and with dimensionless variables

$$u_1 = U(x), \quad u_2 = 0, \quad \rho = \frac{1}{U}, \quad Re = \frac{wu_o}{\nu_o} , \tag{3.6}$$

the perturbation equations are

$$\rho\left(\frac{\partial u_1'}{\partial t} + U\frac{\partial u_1'}{\partial x}\right) + \rho u_1' \frac{\partial U}{\partial x} + \rho' U \frac{\partial U}{\partial x}$$

$$= -\frac{1}{M_o^2}\frac{\partial \rho'}{\partial x} + \frac{\partial \tau_{11}'}{\partial x} + \frac{\partial \tau_{12}'}{\partial y}, \tag{3.7}$$

$$\rho\left(\frac{\partial u_2'}{\partial t} + U\frac{\partial u_2'}{\partial x}\right) = -\frac{1}{M_o^2}\frac{\partial \rho'}{\partial y} + \frac{\partial \tau_{21}'}{\partial x} + \frac{\partial \tau_{22}'}{\partial y},$$

$$\frac{\partial \rho'}{\partial t} + U\frac{\partial \rho'}{\partial x} + u_1'\frac{\partial \rho}{\partial x} + \rho'\frac{\partial U}{\partial x} + \rho\left(\frac{\partial u_1'}{\partial x} + \frac{\partial u_2'}{\partial y}\right) = 0.$$

When viscosity effects are treated, the bulk viscosity will be assumed zero so that

$$\tau_{ij}' = \frac{1}{Re}\left(\frac{\partial u_i'}{\partial x^j} + \frac{\partial u_j'}{\partial x^i}\right). \tag{3.8}$$

IV. INVISCID ONE-DIMENSIONAL INSTABILITY

A particularly simple situation exists if the flow is inviscid and one-dimensional. The system of equations is hyperbolic with low order terms,

$$\begin{pmatrix} \frac{\partial u'}{\partial t} \\ \frac{\partial \rho'}{\partial t} \end{pmatrix} + \begin{pmatrix} U & \frac{U}{M_o^2} \\ \frac{1}{U} & U \end{pmatrix}\begin{pmatrix} \frac{\partial u'}{\partial x} \\ \frac{\partial \rho'}{\partial x} \end{pmatrix} = M \begin{pmatrix} 1 & U^2 \\ -\frac{1}{U^2} & 1 \end{pmatrix}\begin{pmatrix} u' \\ \rho' \end{pmatrix} \tag{4.1}$$

or

$$\frac{\partial u'}{\partial t} + A\frac{\partial u'}{\partial x} = Bu'. \tag{4.2}$$

An energy principle for Eq. (4.2) is obtained by forming the scalar product of $v = Pu'$ and Eq. (4.2) after the matrix A has been symmetrized by a similarity transformation P. This is most easily done by diagonalizing A. The diagonal matrix is

$$\Lambda(x) = \begin{pmatrix} U + \frac{1}{M_o} & 0 \\ 0 & U - \frac{1}{M_o} \end{pmatrix} = PAP^{-1}. \tag{4.3}$$

The matrixes P and P^{-1} are

$$P = \begin{pmatrix} 1 & \frac{U}{M_o} \\ 1 & -\frac{U}{M_o} \end{pmatrix}, \quad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ \frac{M_o}{U} & -\frac{M_o}{U} \end{pmatrix}. \tag{4.4}$$

If Riemann variables v are defined by

$$v = Pu = \begin{pmatrix} u' + \frac{U}{M_o}\rho' \\ u' - \frac{U}{M_o}\rho' \end{pmatrix}, \tag{4.5}$$

Eq. (4.2) is

$$\frac{\partial v}{\partial t} + \Lambda\frac{\partial v}{\partial x} + \left(P\frac{\partial P^{-1}}{\partial t} + \Lambda P\frac{\partial P^{-1}}{\partial x}\right) v = PBP^{-1}v. \tag{4.6}$$

In component form the equations are

$$\begin{pmatrix} \frac{\partial v_1}{\partial t} \\ \frac{\partial v_2}{\partial t} \end{pmatrix} + \begin{pmatrix} U + \frac{1}{M_o} & 0 \\ 0 & U - \frac{1}{M_o} \end{pmatrix}\begin{pmatrix} \frac{\partial v_1}{\partial x} \\ \frac{\partial v_2}{\partial x} \end{pmatrix} = \frac{M}{2}$$

$$\begin{pmatrix} 1 + M_oU - \frac{2}{M_oU} & 1 - M_oU \\ 1 + M_oU & 1 - \left(M_oU - \frac{2}{M_oU}\right) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{4.7}$$

or in matrix form

$$\frac{\partial v}{\partial t} + \Lambda\frac{\partial v}{\partial x} = \hat{B}v. \tag{4.8}$$

The energy principle can now be constructed by taking the inner product of v with Eq. (4.8). The inner product is the obvious one

$$(u,v) = u_1v_1 + u_2v_2 \tag{4.9}$$

and gives with Eq. (4.8)

$$\frac{\partial}{\partial t}(v,v) + \frac{\partial}{\partial x}(v,\Lambda v) = \left(v, \left[\frac{\partial\Lambda}{\partial x} + 2\hat{B}\right] v\right) \tag{4.10}$$

where

$$\frac{\partial\Lambda}{\partial x} + 2\hat{B} = M \begin{pmatrix} M_oU - \frac{2}{M_oU} & 1 - M_oU \\ 1 + M_oU & -\left(M_oU - \frac{2}{M_o}\right) \end{pmatrix} . \tag{4.11}$$

Consider now an interval $D < x < C$ contained in the shock region and having a domain of dependence also in the shock region. The boundaries in (x,t) space of the domain of dependence of $[P,C]$ are the initial segment $[A,B]$ and the characteristics. Integration of Eq. (4.10) over the domain of dependence and use of Green's theorem gives

$$\int_D^C (v,v)dx - \int_A^B (v,v)dx = \iint \left(v, \left[\frac{\partial\Lambda}{\partial x} + 2\hat{B}\right] v\right) dx\, dt$$
$$- \int_D^A \left[(v,\Lambda v)n_1 + (v,v)n_2\right] ds$$
$$- \int_B^C \left[(v,\Lambda v)n_2 + (v,v)n_2\right] ds \quad . \tag{4.12}$$

The two integrals along the characteristics are positive.[5] The energy at time t is

$$E(t) = \int_D^C (v,v)dx \tag{4.13}$$

and at $t = 0$

$$E(0) = \int_A^B (v,v)dx \quad . \tag{4.14}$$

The energy inequality is then

$$E(t) - E(0) < \iint \left(v, \left[\frac{\partial\Lambda}{\partial x} + 2\hat{B}\right] v\right) dx\, dt \tag{4.15}$$

The integrand is a quadratic form for which the symmetric part of the matrix $\left(\frac{\partial\Lambda}{\partial x} + 2\hat{B}\right)/M$ has eigenvalues satisfying

$$\lambda^2 = \left(M_oU - \frac{2}{M_oU}\right)^2 + 1. \tag{4.16}$$

Regardless of the size of M_o, one of the eigenvalues, λ, is always positive and greater than one. The largest value of λ occurs at the outlet boundary of the shock region for $U = \frac{1}{M_o^2}$. The inequality in terms of λ_{max} is

$$E(t) = E(0) < M\lambda_{max} \int_o^t E(t)dt \tag{4.17}$$

or from Gronwall's Lemma

$$E(t) < E(0)e^{M\lambda_{max}t} . \tag{4.18}$$

As a function of M_o, the growth rate $M\lambda_{max}$ goes to zero as $M_o \to 1$ and for $M_o >> 1, M\lambda_{max} \propto M_o$. This would imply stability for weak shocks and a degree of instability depending on the accuracy of the estimate for strong shocks.

The characteristics give a clear picture of the flow in the shock layer. They are given by

$$x = \frac{\left(N \pm \frac{1}{M_o}\right)}{M}\left(1 - e^{-M(t-t_o)}\right) + x_o e^{-M(t-t_o)} . \tag{4.19}$$

The positive characteristics all continue through the layer whereas at the sonic point, $x_s = M_o - 1/M_o + 1$, the negative characteristic is a vertical asymptote for both the supersonic and subsonic negative characteristics. Points to the left of x_s have a domain of dependence also to the left of x_s. Points to the right of x_s have a domain of dependence stretching both left and right of x_s. The domain of influence, of course, can only extend downstream of the shock layer.

An obvious question at this point is whether the initial value approach is the best one. Why not impose boundary conditions on the disturbance either on the edges of the shock layer or at infinity as is done for shear and boundary layer stability? If the shock layer were infinite in width, as is the case for constant viscosity, boundary conditions at infinity would be natural. For nonlinear viscosity

(Von Neumann-Richtmyer pseudo-viscosity for example) it is possible to have a finite width as has been assumed for this investigation. It is difficult to see how in a physical way boundary conditions on the disturbance could be applied in the interior of the flow. If homogeneous boundary conditions at infinity are chosen, the standard treatment involves assuming a time dependence of the form $e^{\alpha t}$. An eigenvalue problem for α results, and the stability is determined by $\mathrm{Re}[\alpha]$. There can be difficulties with this normal mode analysis as pointed out by K. M. Case.[6]

V. INVISCID TWO-DIMENSIONAL INSTABILITY

The two-dimensional perturbation equations are

$$\begin{pmatrix} \frac{\partial u_1'}{\partial t} \\ \frac{\partial u_2'}{\partial t} \\ \frac{\partial \rho'}{\partial t} \end{pmatrix} + \begin{pmatrix} U & 0 & \frac{U}{M_o^2} \\ 0 & U & 0 \\ \frac{1}{U} & 0 & U \end{pmatrix} \begin{pmatrix} \frac{\partial u_1'}{\partial x} \\ \frac{\partial u_2'}{\partial x} \\ \frac{\partial \rho'}{\partial x} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{U}{M_o^2} \\ 0 & \frac{1}{U} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial u_1'}{\partial y} \\ \frac{\partial u_2'}{\partial y} \\ \frac{\partial \rho'}{\partial y} \end{pmatrix} = M \begin{pmatrix} 1 & 0 & U^2 \\ 0 & 0 & 0 \\ -\frac{1}{U^2} & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \rho' \end{pmatrix} \tag{5.1}$$

or

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = MCu \, . \tag{5.2}$$

The characteristic condition for the first order system is

$$|\lambda I + \mu A + \nu B| = 0. \tag{5.3}$$

Expanded Eq. (5.3) is

$$(\lambda + \mu U)^3 - (\lambda + \mu U)\left(\frac{\mu^2+\nu^2}{M_o^2}\right) = 0. \tag{5.4}$$

Solutions of Eq. (5.4) for λ are

$$\lambda = -\mu U \pm \frac{\sqrt{\mu^2+\nu^2}}{M_o} \, , \; -\mu U. \tag{5.5}$$

For real μ,ν, λ is real, and the system is totally hyperbolic. The bicharacteristics or rays are given by

$$\begin{gathered} \frac{dx}{dt} = U \\ \left(\frac{dx}{dt} - U\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{1}{M_o^2} \end{gathered} \tag{5.6}$$

which are the equations for particle paths and Monge cone for small disturbances.

Finding a useful energy principle for Eq. (5.2) is not so direct as in the one-dimensional case. Then the matrix A could be diagonalized and was therefore symmetric. If A is not symmetric

$$(u,Au_x) = (u,Au)_x - (u,A_xu) - (u_x,Au). \tag{5.7}$$

If A is symmetric

$$(u,Au_x) = \frac{1}{2}\left[(u,Au)_x - (u,A_xu)\right] \, . \tag{5.8}$$

For Eq. (5.8) Green's theorem converts the divergence term to a surface integral and the term (u,A_xu) is a quadratic form that can be estimated. In the two-dimensional case, Eq. (5.2), both A and B must be symmetric or symmetrizable by the same transformation for the energy principle to work. Although the basic nonlinear hydrodynamic equations are symmetric hyperbolic (the matrixes can be simultaneously symmetrized), their linearization is not always symmetrizable.[7] Fortunately the matrixes A and B of Eq. (5.2) can be simultaneously symmetrized by the positive definite transformation.

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{U^2}{M_o^2} \end{pmatrix} . \tag{5.9}$$

The conditions under which two matrices can be simultaneously symmetrized do not appear to be known. For a single matrix A, it is known that when the matrix has distinct real eigenvectors, a symmetrizing positive definite matrix H can be constructed from the matrix T, which diagonalizes A by a similarity transformation, in the simple manner

$$H = T^*T. \tag{5.10}$$

This process was applied to the matrix A of Eq. (5.2). Application of H so determined, to the matrix B of Eq. (5.2) almost symmetrized B. A small change to the form, Eq. (5.9), resulted in a matrix symmetrizing both A and B. In the one-dimensional problem, A was diagonalized and symmetrized by a similarity transformation, P, and the dependent variable u was transformed to v by P. The matrix H symmetrizes A and B by the single matrix multiplication

$$(HA) = (HA)^* , \qquad (HB) = (HB)^* , \tag{5.11}$$

not by a similarity transformation. H is positive definite, however, and is used to form a new inner product

$$(u,Hv) . \tag{5.12}$$

The energy is then

$$\int (u,Hu)\, dx\, dy . \tag{5.13}$$

Transformation of Eq. (5.2) by H in this way and forming the inner product with u gives

$$\left(u,H \frac{\partial u}{\partial t}\right) + \left(u,HA \frac{\partial u}{\partial x}\right) + \left(u,HB \frac{\partial u}{\partial y}\right) = M(u,HCu) . \tag{5.14}$$

The symmetry of HA and HB and their dependence on x alone permits Eq. (5.14) to be written

$$\frac{\partial}{\partial t} (u,Hu) + \frac{\partial}{\partial x} (u,HAu) + \frac{\partial}{\partial y} (u,HBu) = \left(u, \left[2MHC + \frac{\partial (HA)}{\partial x}\right] u\right) . \tag{5.15}$$

Integration over a domain of dependence in the shock region between times t = 0 and t gives

$$\int \left[(u,Hu)n_t + (u,HAu)n_x + u,HBu)n_y\right] dS = \int \left(u, \left[2MHC + \frac{\partial (HA)}{\partial x}\right] u\right) dx\, dy\, dt . \tag{5.16}$$

The region of integration is a truncated conoid. The energy terms come from

$$\int (u,Hu)n_t dS = \int (u,Hu)n_t dx\, dy = E(t). \tag{5.17}$$

The result is

$$E(t)-E(0) = - \int_{\text{sides}} \left[(u,Hu)n_t + (u,HAu)n_x + (u,HBn)ny\right] dS + \int \left(u, \left[2MHC + \frac{\partial (HA)}{\partial x}\right] u\right) dx\, dy\, dt . \tag{5.18}$$

The integral over the sides or mantle of the conoid is

$$\int \left(u, \left[Hn_t + HAn_x + HBny\right] u\right) dS, \tag{5.19}$$

and this side surface must be characteristic. The characteristic condition is that the matrix in Eq. (5.19) have a determinant

$$\begin{vmatrix} n_t + Un_x & 0 & \frac{Un_x}{M_o^2} \\ 0 & n_t + Un_x & \frac{Un_y}{M_o^2} \\ \frac{Un_x}{M_o^2} & \frac{Un_y}{M_o^2} & n_t + Un_x \end{vmatrix} = 0. \tag{5.20}$$

Two equations result, as in Eq. (5.5)

$$n_t + Un_x = 0 , \qquad (n_t + Un_x)^2 = \frac{n_x^2 + n_y^2}{M_o^2} . \tag{5.21}$$

For this symmetric matrix there is a unitary transformation reducing the form in Eq. (5.19) to

$$v_1^2(n_t+Un_x) + v_2^2\left(n_t+Un_x-\frac{\sqrt{n_x^2+n_y^2}}{M_o}\right) + v_3^2\left(n_t+Un_x+\frac{\sqrt{n_x^2+n_y^2}}{M_o}\right). \tag{5.22}$$

This form will be positive or negative definite if $n_t + Un_x > 0$ or < 0. For the lower sheet of the coniod associated with a domain of dependence, the positive sign is correct. The integral Eq. (5.19) is positive and Eq. (5.18) yields the inequality

$$E(t) - E(0) \leq \int\left(u, \left[2MHC + \frac{\partial(HA)}{\partial x}\right] u\right) dx\,dy\,dt\,. \tag{5.23}$$

An upper bound for the integral in Eq. (5.23) in terms of the energy, $E = \int (u,Hu)\,dx\,dy$ is obtained again from the maximum eigenvalue of the symmetric part of the matrix

$$\left(2MC + H^{-1}\frac{\partial(HA)}{\partial x}\right)\Big/M. \tag{5.24}$$

The eigenvalues of Eq. (5.24) are given by

$$\lambda^2 = 1 + \left(U^2 - \frac{1}{U} - \frac{1}{2U^2} - \frac{1}{M_o^2}\right)^2$$
$$\lambda = -1\ . \tag{5.25}$$

The maximum λ occurs for $U = 1/M_o^2$, the outlet value. For $M_o >> 1$, $\lambda_{max} \propto M_o^4$ in this two-dimensional estimate whereas from Eq. (4.16), the one-dimensional eigenvalue is proportional M_o. The difference is the result of the new inner product in terms of H and not of the extra dimensionality. There are, indeed, no terms from HB contributing to the low order forcing terms. The new eigenvalue that appears in Eq. (5.25) is $\lambda = -1$, which indicates a stabilizing effect. The significant fact remains that the maximum eigenvalue is positive at every point in the shock layer.

VI. VISCOUS INSTABILITY

When viscosity is included, as in Eq. (3.1), the equations are not hyperbolic; the concept of domain of dependence is lost, and a different energy principle must be found. Serrin's[8] energy

$$E(t) = \int \frac{\rho}{2}\left(u_k'u_k'+\rho'^2\right) dV = \int \frac{\rho}{2}\left(u'^2+\rho'^2\right) dV, \tag{6.1}$$

which he used to prove uniqueness for viscous, compressible flows, is an obvious candidate. His arguments show that specified velocity perturbations on all boundary surfaces and specified density perturbation where the normal velocity is into the region yield a unique solution. In spite of the reservations expressed at the end of Sec. IV, u_1', ρ' will be specified zero on the boundaries of the shock layer according to the rule just stated. Multiplication of Eq. (3.4) by u_1' and ρ' followed by integration over a fixed volume with use of the boundary conditions through Green's theorem gives

$$\begin{aligned}\frac{dE}{dt} = &-\int\left[\rho\rho' u_j' \frac{\partial\tilde{\rho}}{\partial x^j} + \rho\rho'^2 \frac{\partial \tilde{u}_j}{\partial x^j} + \rho'^2\rho\frac{\partial u_j'}{\partial x^j}\right] dV \\ &-\int\left[\rho\frac{u_i'u_j'}{2}\left(\frac{\partial\tilde{u}_i}{\partial x^j}+\frac{\partial\tilde{u}_j}{\partial x^i}\right) + \rho' u_i'\left(\frac{\partial\tilde{u}_i}{\partial t} + \tilde{u}_j\frac{\partial\tilde{u}_i}{\partial x^j}\right)\right. \\ &\left. - P'\frac{\partial u_k'}{\partial x^k} + \frac{\tau_{ij}'}{2}\left(\frac{\partial u_j'}{\partial x^i}+\frac{\partial u_j'}{\partial x^i}\right)\right] dV \\ &-\int \frac{\rho}{2}\rho'^2\, u_1 n_1 dS\ .\end{aligned} \tag{6.2}$$

For a problem with only shear viscosity

$$\tau_{ij}' = \frac{2}{Re} D_{ij}' = \frac{1}{Re}\left(\frac{\partial u_i'}{\partial x^j} + \frac{\partial u_j'}{\partial x^i}\right). \tag{6.3}$$

With the strainrate tensor D_{ij}, Eq. (6.2) can be written

$$\frac{dE}{dt} = -\int\Big[\rho\rho' u'_j \frac{\partial\tilde{\rho}}{\partial x^j} + \rho\rho'^2 \frac{\partial \tilde{u}_j}{\partial x^j} + \rho^2\rho' \frac{\partial u'_j}{\partial x^j}$$

$$+ \rho u'_i u'_j \tilde{D}_{ij} + \rho' u'_i \left(\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x^j}\right)$$

$$- P' D'_{kk} + \frac{2}{Re} D'_{ij} D'_{ij}\Big] dV - \int \frac{\rho}{2} \rho'^2 u_1 n_1 dS \ . \tag{6.4}$$

The incompressible version of Eq. (6.4) is just

$$\frac{1}{\rho}\frac{dE}{dt} = -\int \left[u'_i u'_j D_{ij} + \frac{2}{\rho Re} D'_{ij} D'_{ij}\right] dV \tag{6.5}$$

which contains only primed and unprimed quantities and has not been linearized. Equation (6.5) is the basis of many nonlinear stability analyses for viscous fluids (Serrin[9]).

For the shock stability problem, the nonlinear Eq. (6.4) appears formidable. The linearized form of Eq. (6.4) will therefore be used. Third and higher order terms in primed quantities being neglected Eq. (6.4) reduces to

$$\frac{dE}{dt} = -\int \Big[\rho\rho' u'_j \frac{\partial \rho}{\partial x^j} + \rho\rho'^2 \frac{\partial u_j}{\partial x^j} + \rho^2\rho' \frac{\partial u'_j}{\partial x^j}$$

$$+ \rho u'_i u'_j D_{ij} + \rho' u'_i \frac{du_i}{dt} - P' D'_{kk}$$

$$+ \frac{2}{Re} D'_{ij} D'_{ij}\Big] dV - \int \frac{\rho}{2} \rho'^2 u_1 n_1 dS \ . \tag{6.6}$$

The unprimed nonzero quantities for the initially one-dimensional steady shock are

$$u_1 = U(x), \quad \rho = U^{-1}, \quad \frac{\partial u_1}{\partial x} = \frac{dU}{dx} = -M,$$

$$D_{11} = \frac{dU}{dx} = -M \ . \tag{6.7}$$

The linearized energy equation is then

$$\frac{dE}{dt} = -\int\Big[\rho\left(u_1'^2 + \rho'^2\right)\frac{dU}{dx} + \rho\rho' u'_1 \frac{d\rho}{dx}$$

$$+ \rho^2\rho' \frac{\partial u'_j}{\partial x^j} + \rho' u'_1 U \frac{dU}{dx} - \frac{\rho'}{M_o^2} D'_{kk} \tag{6.8}$$

$$+ \frac{2}{Re} D'_{ij} D'_{ij}\Big] dV - \int \frac{\rho}{2}\cdot\rho'^2\, U n_1 dS \ .$$

According to Serrin's uniqueness theorem ρ' in the surface integral at the outlet shock boundary may not be taken zero. The integral is, however, positive, and Eq. (6.8) may be converted to an inequality

$$\frac{dE}{dt} < -\int \Big[-\rho\left(u_1'^2 + \rho'^2\right)M + \rho\rho' u'_1 \frac{d\rho}{dx} + \rho^2\rho' \frac{\partial u'_j}{\partial x^j}$$

$$- \rho' u' \, UM - \frac{\rho'}{M_o^2} D'_{kk} + \frac{2}{Re} D'_{ij} D'_{ij}\Big] dV \ . \tag{6.9}$$

If Eq. (6.9) is simplified for the one-dimensional case, it is

$$\frac{dE}{dt} < \int\Big\{u'\left[\frac{Mu'}{U} + MU\rho' - \frac{1}{M_o^2}\frac{\partial\rho'}{\partial x}\right]$$

$$+ \rho'\left[\frac{\rho' M}{U} - \frac{u' M}{U^3} - \frac{1}{U^2}\frac{\partial u'}{\partial x}\right] - \frac{2}{Re}\left(\frac{\partial u'}{\partial x}\right)^2\Big\} dV \ . \tag{6.10}$$

This inequality could have been obtained directly from the one-dimensional Eqs. (4.1) It is clear that much of the inaccuracy of any estimate will be the result of the lost flux terms.

The question is what kind of bounds can be obtained from the inequality Eq. (6.10)? Compared to the hyperbolic inequality, Eq. (6.10) is complicated by the presence of the x-derivatives of ρ' and u'; it is not a quadratic form in ρ', u'. Although the mathematical nature of the approximation is not clear, the form of the inequality, Eq. (6.10), suggests the following approach. Suppose

$$\frac{Mu'}{U} + MU\rho' - \frac{1}{M_o^2}\frac{\partial\rho'}{\partial x} = \rho u' \alpha$$

$$\rho' \frac{M}{U} - u' \frac{M}{U^3} - \frac{1}{U^2}\frac{\partial u'}{\partial x} = \rho\rho' \alpha \ . \tag{6.11}$$

If such an α can be found, the nonviscous part of the integral in Eq. (6.10) would be $2\alpha E(t)$. The differential Eqs. (6.11) are for u', ρ' satisfying homogeneous boundary conditions as described. They are also the inviscid equations minus the flux terms where time dependence is $e^{\alpha t}$. This situation is reminiscent of viscous Taylor[10] instability based on inviscid kinematics. The question asked above is changed to what relation does the α in Eq. (6.11) bear to α in the full equations? Before looking at that and before estimating the viscous terms we will see what can be said about the α in Eq. (6.11) and whether it is physically sensible.

First in Eq. (6.11) U is introduced as an independent variable. When this is done, the equations are

$$\begin{aligned} \frac{d\rho'}{dU} &= M_o^2\left(\frac{\alpha}{M} - 1\right)u' - M_o^2 U\rho' \\ \frac{du'}{dU} &= U\left(\frac{\alpha}{M} - 1\right)\rho' + \frac{u'}{U} . \end{aligned} \tag{6.12}$$

Elimination of ρ' gives the single equation

$$\frac{d^2u'}{dU^2} + \left(M_o^2 U - \frac{2}{U}\right)\frac{du'}{dU} + \left\{\frac{2}{U^2} - M_o^2\left[\left(\frac{\alpha}{M} - 1\right)^2 + 1\right]\right\}u = 0 . \tag{6.13}$$

This is of standard Strum-Liouville form with

$$\begin{aligned} &\frac{d^2u'}{dU^2} + P(U)\frac{du'}{dU} + \left[q(U) - \lambda^2\right]u' = 0 \\ &u'(1) = u'\left(\frac{1}{M_o^2}\right) = 0 . \end{aligned} \tag{6.14}$$

The problem can be made self-adjoint by introducing a new function

$$h(U) = e^{\int P} \tag{6.15}$$

to give

$$\frac{d}{dU}\left(h\frac{du'}{dU}\right) + \left[h(U)q(U) - h(U)\lambda^2\right]u' = 0 . \tag{6.16}$$

If $\lambda = ik^{1/2}$, $hq = f > 0$, $h > 0$,

$$\frac{d}{dU}\left(h\frac{du'}{dU}\right) + (f + hk)u' = 0 \tag{6.17}$$

is a regular Sturm-Liouville system with an infinite sequence of real eigenvalues $k_o < k_1 < - -$ with $\lim_{n\to\infty} k_n = \infty$. Since

$$\int u'\frac{d}{dU}\left(h\frac{du'}{dU}\right)dU + k\int u'^2 h\,dU + \int f u'^2 dU = 0,$$

$$k = \frac{\int h\left[\left(\frac{du'}{dU}\right)^2 - 2\frac{u'^2}{U^2}\right]dU}{\int hu'^2 dU} ; \tag{6.18}$$

there is a possibility of a finite number of $k_i < 0$. The number of negative k depends on the factor $\frac{2}{U^2} = q(U)$ which has a range $(2, 2M_o^4)$. For $M_o >> 1$, k will be more negative and $\lambda^2 = -k$ more positive. The growth rate α is related to λ^2 by

$$\alpha = M\left(1 \pm \sqrt{\frac{\lambda^2}{M_o^2} - 1}\right) . \tag{6.19}$$

Since

$$\lambda^2 = \frac{\int h\left[\frac{2u'^2}{U^2} - \left(\frac{du'}{dU}\right)^2\right]dU}{\int hu'^2 dU} \leq 2M_o^4 , \tag{6.20}$$

for large M_o, α is proportional to M_o. For $M_o \to 1$, $\alpha \to 0$. This is essentially the behavior found before from the one-dimensional inviscid hyperbolic analysis. The approximations made in Eq. (6.12) so far do not produce inconsistent results.

The next step is to examine the viscous integral which can be expressed in terms of α and the primed variables u', ρ'. The integral involving dissipation is

$$\int \left(\frac{\partial u'}{\partial x}\right)^2 dx = \int M^2 \left[\rho' U \left(\frac{\alpha}{M} - 1\right) + \frac{u'}{U}\right] dx. \tag{6.21}$$

This can be estimated using upper bounds for λ^2, U

$$\int \left(\frac{\partial u'}{\partial x}\right)^2 dx < 4M^2 M_o^2 E(t) \quad . \tag{6.22}$$

The energy inequality that results is

$$\frac{dE}{dt} < 2 \left[\alpha - 2 \frac{M^2 M_o^2}{Re}\right] E(t) \quad . \tag{6.23}$$

Since for $M_o >> 1$, $2\alpha \approx \sqrt{2} M_o$ in this case

$$\frac{dE}{dt} < M_o \left[\sqrt{2} - \frac{M_o}{Re}\right] E(t) \tag{6.24}$$

where $Re = w u_o \rho_o / \mu_o$. For weak shocks Eq. (6.19) and the other inviscid results show that $M \approx \alpha \approx 0$.

The question remains whether α as defined by Eq. (6.11) and the estimates based on it, particularly for $M_o >> 1$, are good enough for the final inequality Eq. (6.23) to be useful. The full inviscid equations do not reduce to a standard Sturm-Liouville problem for which the eigenvalues have a known distribution as was the circumstance for Eq. (6.11) so that a numerical solution of the full viscous equations is probably necessary to determine the validity of the viscous estimates. Before a numerical treatment is deserved, model equations with more realistic physics should be settled upon.

VII. CONCLUSIONS

On the basis of simplified compressible flow equations the energy estimates for inviscid shock layer growth rates in one and two dimensions show similar instability with the rate increasing with the initial Mach number, M_o, and the rate tending to zero as $M_o \to 1$. The effect of viscosity in the one-dimensional case is to reduce the instability. If Re is sufficiently large for a given M_o, the perturbations would be damped. Since the growth rates are based on upper bounds for various quantities, there is reason to question any numerical values for the rates that could be obtained by prescribing specifically the initial quantities M_o, c_o, ρ_o, w, μ_o .

The assumption, common throughout, that w, the shock width, remains constant is probably not so questionable as the omission of the processes producing that width from the basic equations. In some way the model used supposes that there are two length scales, one determining the width, and the other shorter scale associated with the perturbations and that for small perturbations there is no coupling between the processes. The fact that there can be more than one length scale connected with shock structure is not is doubt. For ionizing shocks[1] calculations show as many as four internal scales with the strength of the shock determining their importance.

The practical question behind this investigation is if shocks can be internally unstable, what is the final effect on a flow with shocks and is it significant? The particular perturbations admitted here, confined to the shock layer, certainly indicate a degree of instability depending on the shock strength. Such perturbations are, however, not the ones most likely to occur in realistic, time-dependent situations, although they could be injected in a steady shock. The most likely perturbations would exist ahead of the shock, would flow into it, be amplified in the shock layer, flow downstream and be damped. If this picture is correct, the final effect would be to broaden the shock by distance over which amplified perturbations subside. This assumes that the perturbations on the average do not increase the initial Mach number. If the perturbations contained sufficient energy to increase M_o the shock speed would adjust but width of the shock could still be affected. In many respects it appears that the problem of interior instability of a shock encountering disturbances in the inflow is similar to that of a shock running into a region with suitable scales of turbulence or inhomogeneity. For the turbulent problem there is some theoretical and experimental evidence[10,11] that for weak shocks the width of the shock layer can be significantly increased. The effects of shock strength and scale of turbulence have not been considered to my knowledge.

In summary, for the simplified model assumed, internal shock instability appears likely. More realistic models are needed to determine whether the growth rates can be high enough to have practical significance. At large Mach numbers, experimental evidence of the effects of instability on the shock structure would be very difficult to obtain. At lower Mach numbers, experiments should be possible but it might prove difficult to determine whether widening, if it occurred, was attributable predominantly to instability.

REFERENCES

1. R. A. Gross, "The Physics of Strong Shock Waves in Gases," Rendiconti S.I.F. XLVII Scuola Internationali de Fysici Enrico Fermi, in Physics of High Energy Density (Academic Press, New York, 1971).

2. J. A. McLennan, "Theory of Viscosity at High Pressures," Los Alamos Scientific Laboratory internal document, September 1968.

   J. A. McLennan, "Viscosity in Aluminum at High Pressures," Los Alamos Scientific Laboratory internal document, July 1969.

3. D. A. Tidman and N. A. Krall, Shock Waves in Collisionless Plasmas, (Wiley-Interscience, New York, 1971).

4. M. Morduchow and A. Paulley, "Stability of Normal Shock Waves with Viscosity and Heat Conduction," Phys. Fluids 14, No. 2, 323-331 (1971).

5. R. Courant and D. Hilbert, Methods of Mathematical Physics Vol. II, (Interscience Pub., New York, 1962).

6. K. M. Case, "Hydrodynamic Stability and the Initial Value Problem," AMS Proceedings of Symposia in Appl. Math. Vol. XIII, "Hydrodynamic Instability," 1962.

7. D. Gottlieb and B. Gustafson, "On the Navier-Stokes Equations with Constant Total Temperature," ICASE report 75-12 (June 1975), Universities Space Research Administration, NASA's Langley Research Center, Hampton, VA.

8. J. Serrin, "On the Uniqueness of Compressible Fluid Motion," Arch. Rat. Mech. Anal. 3, 271-288 (1959).

9. J. Serrin, "On the Stability of Viscous Fluid Motions," Arch. Rat. Mech Anal. 3, 1-13 (1959).

10. J. Miles and J. Dienes, "Taylor Instability in a Viscous Liquid," Phys. Fluids 9, No. 12 (1966).

11. J. E. F. Williams and M. S. Howe, "On the Possibility of Turbulent Thickening of Weak Shock Waves," J. Fluid Mech. 58, Part 3, 461-480 (1973).

12. S. C. Crow, "Distortion of Sonic Bangs by Atmospheric Turbulence," J. Fluid Mech. 37, Part 3, 529-563 (1969).