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An Analysis of the Rayleigh-Taylor Problem of Superposed Fluids

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An Analysis of the Rayleigh-Taylor Problem of Superposed Fluids

The Inviscid Incompressible Irrotational Case; Steady-State Solutions

LOS ALAMOS NATL. LAB. LIBS.
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by

George H. Pimbley, Jr.

CONTENTS

Abstract 1

I. Introduction 1

II. Formulation of the Problem 3

III. Branches of Steady-State Solutions 5

IV. Branches of Solutions by Perturbation 12

A. Sequence of Linear Perturbation Problems Solved 12

B. Proof of Convergence of the Perturbation Series 13

V. Stability Properties 16

A. Stability of the Trivial Solutions 16

B. Instability of the Branch Solutions 19

VI. Arc-Length Formulation 23

Stability Analysis 26

VII. Summary 31

References 33

AN ANALYSIS OF THE RAYLEIGH-TAYLOR
PROBLEM OF SUPERPOSED FLUIDS

by

George H. Pimbley, Jr.

ABSTRACT

The problem of the shape of the accelerated interface between a light and a heavy fluid is introduced using the velocity potential. It is then shown that the problem admits steady-state solutions that form continuous branches bifurcating from a trivial solution, quite a usual picture with nonlinear problems. With a fixed periodicity, these branches of solutions are unstable if rectilinear coordinates are used. However, if one of the variables is allowed to be arc length along the interface, with a fixed period in the arc length, there exists a branch of stable steady-state solutions.

The corresponding interface shapes are studied, and would seem to be of interest. When stable they represent possible initial persisting configurations.

It is difficult to state how this work relates to previous work cited in the references by other authors. The author has attempted an approach that is more fundamental, in his opinion, than that of previous workers.

I. INTRODUCTION

Consider the following ordinary differential equation with the initial condition

$$u_t + au + bu^3 = 0, \qquad u(0) = u_o, \tag{1.1}$$

where a and b are real parameters. This problem has the following unique solution

$$u(t) = u_o \frac{\sqrt{a}}{\sqrt{-bu_o^2 + (a+bu_o^2)e^{2at}}} . \tag{1.2}$$

If the only thing we want is the solution, we are done. We proceed further, however, and consider the steady state of the problem in Eq. (1.1) by assuming that $u_t \equiv 0$.

$$bu^3 = -a\,u. \tag{1.3}$$

This is a very simple nonlinear eigenvalue problem with eigenvalue parameter a. The trivial solution $u \equiv 0$ exists, and a bifurcated nontrivial solution emanating from the bifurcation point $a = 0$, which branches to the left if $b > 0$ (Fig. 1).

By inspecting the solution [Eq.(1.2)] of Eqs. (1.1) we see that if $a \geq 0$, $u(t) \to 0$ as $t \to \infty$, whatever the initial condition u_o. If, however, $a < 0$, then $u(t) \to \pm\sqrt{\frac{-a}{b}}$. But $u = \pm\sqrt{\frac{-a}{b}}$ is the bifurcated nontrivial branch of the steady-state problem in Eq. (1.3). The solution tends to this branch whatever the initial condition $u_o \neq 0$, if $a < 0$.

We say that the trivial solution of Eq. (1.3) is stable for $a \geq 0$ and unstable for $a < 0$; the bifurcated solution $u = \pm\sqrt{\frac{-a}{b}}$ of Eq. (1.3) is stable

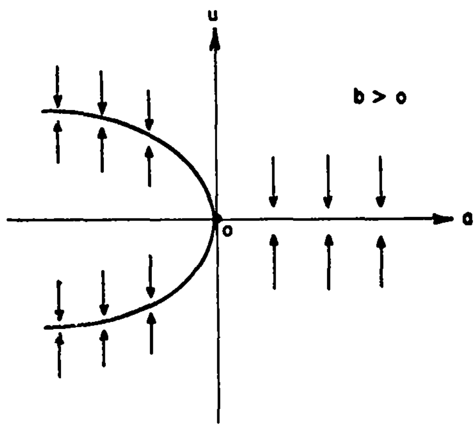

Fig. 1. Left bifurcating branch with exchange of stability.

for $a < 0$. There is an interchange of stabilities as one passes from $a > 0$ to $a < 0$.

Now if $b < 0$ in Eqs. (1.1), we have a different state of affairs. Simply putting $t \to -t$ and $a \to -a$ produces the rightward bifurcating branch shown in Fig. 2. The direction of evolution of the solutions is reversed. For $a < 0$, $u(t) \to \infty$ whatever the initial condition $u_o \neq 0$. For $a > 0$, $u(t) \to \infty$ whatever the initial condition u_o such that $a + bu_o^2 < 0$, whereas $u(t) \to 0$ as $t \to \infty$ whatever the initial condition with $a + bu_o^2 > 0$. Thus, the trivial solution is unstable if $a \leq 0$, but stable if $a > 0$. However, the right bifurcating solutions of Eq. (1.3), namely $u = \pm\sqrt{\frac{a}{-b}}$, are unstable.

The important role of the steady-state solutions of Eqs. (1.1) should be appreciated. If they are stable, the time-evolutionary solutions [Eq. (1.2)] of Eqs. (1.1) tend to them for various values of u_o, whereas if they are unstable they still play a part in defining regions of radically different behavior of the solutions. We can expect steady-state solutions to be important in general time evolutionary problems. Because steady-state solutions are more easily obtained, there is some justification in obtaining them first. This is the viewpoint we adopt here for the Rayleigh-Taylor problem of superposed fluids.

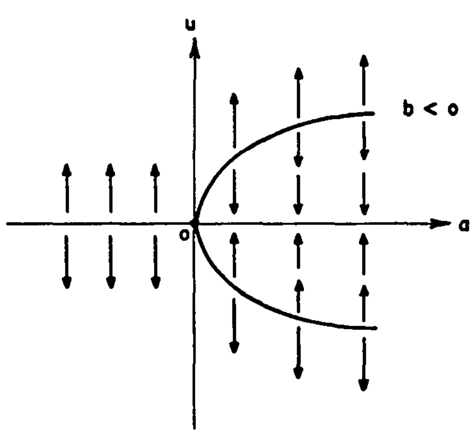

Fig. 2. Right bifurcating branch with no exchange of stabilities.

With the Rayleigh-Taylor problem, the objects of interest are the interfaces between horizontally stratified layers of fluid, where each layer has a density differing from the density of neighboring layers. In the most simple case, which we consider, there are two layers and two fluids of densities ρ_1 and ρ_2.

There also is interest in spherically stratified layers of fluid of differing densities.[1] For now we restrict this treatment to the planar case. Success with the latter is a necessary prologue to a consideration of the spherical case. Interest in the spherical case does influence some choices we must make in treating the planar case, however.

Our assembly of horizontally stratified layers is subjected to a uniform acceleration g perpendicular to the horizontal interfaces of the fluids at rest. The object of the theory is that of studying how initially planar interfaces are perturbed when we apply a uniform acceleration. Experiments have shown that a plane interface is unstable when the acceleration is applied in the direction from the lighter fluid to the heavier fluid, whence the name "Taylor Instability."[2,3]

Actually, however, there may exist interfacial configurations, or shapes, that are nonplanar and that may be stable under such an acceleration. There may be some stability in the problem and it may be a misnomer to call it "Taylor Instability." The search for such interfacial shapes does involve the steady-state problem.

In Sec. II, we derive the form of this Taylor problem which we shall use in the sequel. In Sec. III, we prove the existence of steady-states and study the resulting interfacial shapes. The author believes these steady states to have been hitherto unknown. In Sec. IV, we derive the perturbation series for these solutions and prove that the series converge. In Sec. V, we study stability and see that with rectilinear coordinates the steady states we have obtained are unstable, but not unconditionally so.

Then, in Sec. VI, we reformulate the problem in terms of arc length of the linear interface, and find that with a fixed arc-length period there is a branch of stable steady-state solutions.

A Summary is included at the end of the report.

II. FORMULATION OF THE PROBLEM

We restrict our considerations to the case of inviscid, incompressible fluids, and consider all flows to be irrotational. Also we restrict to the case of two layers of fluid, one heavy layer of density ρ_1 and one light layer of density ρ_2. In Fig. 3 we illustrate the situation by placing the heavy layer on top, the configuration in which one expects instability of a plane interface if the applied acceleration (-)g is gravity. The equation of the interface is given as $y = \eta(x,t)$. Each fluid is considered to extend to infinity in the vertical direction, a mathematical abstraction, employed in Taylor's work[4] and by many subsequent authors.

The situation can be regarded in another sense (Fig. 4) with the heavy fluid beneath the light fluid, and the entire assembly given a downward acceleration -g. It seems more reasonable to many people to portray the heavy infinite fluid on the bottom.

Yet another abstraction we make, due to the physicists Emmons, Chang, and Watson,[2] is that of

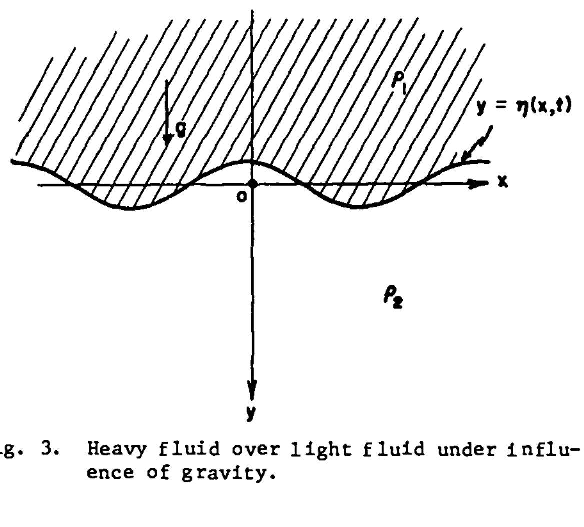

Fig. 3. Heavy fluid over light fluid under influence of gravity.

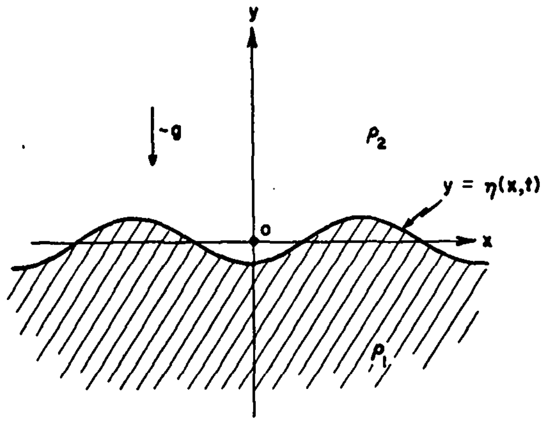

Fig. 4. Light fluid over heavy fluid accelerated downwards.

setting $\rho_2 = 0$. Thus we consider the light fluid as being so light, compared with the heavy fluid, that we can neglect its density. This enables the consideration, with proper interface conditions, of the hydrodynemics problem only in the heavy fluid.

In Eulerian variables, we have the hydrodynemics equations

$$u_t + uu_x + vu_y = -\frac{1}{\rho} p_x$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho} p_y + g \qquad (2.1)$$

in the two-dimensional region occupied by the heavy fluid. Since $\rho_1 = \rho = \text{const}$ (because we restrict to the incompressible case), ρ is considered known in the system given in Eqs. (2.1). These are equations to be solved for horizontal and vertical components of velocity (u,v). The pressure p in the fluid is not known, however, so that Eqs. (2.1) have more unknowns than equations. To determine fully the flow, we need the equation of mass conservation which, under our assumption of incompressibility, assumes the form

$$u_x + v_y = 0. \qquad (2.2)$$

Then Eqs. (2.1) and (2.2) form a closed system for the unknown quantities u, v, p to be solved with initial conditions, and yet to be considered boundary conditions.

Our treatment of the hydrodynamics of this problem closely follows that appearing in Chap. I of Stoker's book Water Waves.[5]

We restate that we are considering the inviscid case. Viscosity does not appear in Eqs. (2.1). For viscous fluids, we should need the full Navier-Stokes system (Ref. 6, Chap. 6).

As fully explained by Stoker (Ref. 5, Chap. I), an inviscid fluid, once at rest, flows irrotationally, and we have made the assumption of irrotational flow. In three dimensions, the curl of the velocity field vanishes throughout the fluid. Our two-dimensional analog of this is

$$v_x - u_y = 0, \tag{2.3}$$

which, together with Eq. (2.2) comprises the familiar Cauchy-Riemann system. Then we are enabled to define a velocity potential $\phi(x,y,t)$ by the line integral

$$\phi(x,y,t) = -\int_{x_o,y_o}^{x,y} [u(\hat{x},\hat{y},t)d\hat{x} + v(\hat{x},\hat{y},t)d\hat{y}], \tag{2.4}$$

which, by virtue of Eq. (2.3) and Green's Theorem (Ref. 7, p. 252) in the plane, is independent of the path of integration. By taking path segments parallel to the coordinate axes, one can become convinced that

$$u = -\phi_x, \text{ and } v = -\phi_y, \tag{2.5}$$

i.e., $(u,v) = -\text{grad}\ \phi$. When Eqs. (2.5) are introduced into Eq. (2.2), we obtain

$$\phi_{xx} + \phi_{yy} = 0. \tag{2.6}$$

Thus the velocity potential is a harmonic function interior to the fluid flow domain.

Further consequences stem from the irrotational character of the flow. Making use of Eqs. (2.1), (2.3), and (2.5), and the fact that ρ = const, we see that

$$-\text{grad}\phi_t + \frac{1}{2}\,\text{grad}(u^2 + v^2) + \text{grad}\,\frac{p}{\rho} = \text{grad}(gy),$$

where of course we use the differential operator $\text{grad} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$. A first integral of this latter expression is Bernoulli's Law

$$-\phi_t + \frac{1}{2}(u^2 + v^2) + \frac{p}{\rho} - gy = C(t), \tag{2.7}$$

where C(t) is an arbitrary function of t that can be taken as $C(t) \equiv 0$ with no loss of generality as is seen below.

We had stated that the basic hydrodynamic equations [Eqs. (2.1) and (2.2)] were to be solved for u, v, and p relative to initial conditions and yet to be derived boundary conditions. We now see that we are in a position to replace this basic problem with that of solving Laplace's equation [Eq. (2.6)] for the velocity potential ϕ, finding the velocities (u,v) from Eqs. (2.5), and using Bernoulli's Law [Eq. (2.7)] to get the pressure p. Of course we have yet to state what disposition is to be made with the initial and boundary conditions.

In Eq. (2.7) we can take $C(t) \equiv 0$ because we are interested in the pressure gradient; it is the latter which enters into Eqs. (2.1). The choice of C(t) does not affect the pressure gradient or the fluid flow.

Consider a boundary surface for the fluid, expressed by the equation $\xi(x,y;t) = 0$. Taking the total derivative with respect to t, we get

$$\frac{d\xi}{dt} = u\xi_x + v\xi_y + \xi_t = 0, \tag{2.8}$$

which holds on the bounding surface. Now (ξ_x,ξ_y) are the components of the vector normal to the surface. Taking into account Eqs. (2.5), we get

$$\frac{\partial\phi}{\partial n} = -\frac{u\xi_x + v\xi_y}{\sqrt{\xi_x^2 + \xi_y^2}} = +\frac{\xi_t}{\sqrt{\xi_x^2 + \xi_y^2}} = -v_n, \tag{2.9}$$

where $\frac{\partial}{\partial n}$, as usual, means differentiation along the normal to the surface, and v_n denotes the common velocity of fluid and boundary surface in the direction normal to the surface.

In the case of a fixed boundary, this gives the boundary condition

$$\frac{\partial\phi}{\partial n} = 0 \tag{2.10}$$

on the surface.

Most important for our problem is the case of a free surface such as occurs in Figs. 3 or 4 if we consider the density of the light fluid to vanish: $\rho_2 = 0$. This is a surface where the pressure p is prescribed, but the form of the surface is not specified a priori. If we say that

$$y = \eta(x,t) \tag{2.11}$$

is the form of the free surface, then its equation is written $\xi = y - \eta(x,t) = 0$, and we see from Eq. (2.8) and Eqs. (2.5) that

$$\phi_x \eta_x - \phi_y - \eta_t = 0 \tag{2.12}$$

on the surface. Because the pressure p is prescribed on the free surface, we must also have, from Bernoulli's Law [Eq. (2.7)], where we take $C(t) \equiv 0$,

$$-g\eta - \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + \frac{p}{\rho} = 0 \tag{2.13}$$

on the surface.

If we assume, as we do in this Taylor-Rayleigh problem, that the pressure on the free surface is directly proportional to the curvature K of the surface, the constant of proportionality being T, the surface tension, then Eq. (2.13) is written

$$-g\eta - \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) = \frac{T}{\rho}\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}}, \quad y = \eta(x,t) \tag{2.14}$$

on the surface.

Equations (2.6), (2.12), and (2.14), together with the vanishing of the velocities as $y \to -\infty$ in Figs. 3 or 4, comprise a substitute initial value problem for the problem consisting of Eqs. (2.1) and (2.2). This is the form of the problem considered by Emmons, Chang, and Watson[2] and later by Kiang.[8] Moreover, it is the specialization of the equations of Birkhoff[9] to the case where the smaller density is neglected.

Given initial conditions $\phi(x,y,0)$ harmonic, and $\eta(x,0)$, the initial value problem is thus specified. As with most initial value problems, however, further boundary conditions must be posed if we wish to restrict the spatial domain.

With reference to the problem of the stability of the interfaces between spherically stratified layers of fluid of differing densities, where all phenomena are periodic in the polar and azimuthal coordinate variables, it is of interest in our planar problem to consider the case where $\phi(x,y,t)$ and $\eta(x,t)$ are periodic in the variable x with given base period Λ.

It might be desired to consider interface stability problems for stratified horizontal layers of fluid in a tank of finite dimensions. Thus we must devise boundary conditions at a fixed wall. Equation (2.10) gives the proper boundary condition for the velocity potential ϕ. Posing a condition at a rigid wall for $\eta(x,t)$ requires the presence of viscosity in the problem, however. Thus with our inviscid assumption, we are prevented apparently from considering interface problems in finite tanks.

The author has considered boundary conditions of the form $\frac{\partial\eta}{\partial n} + \alpha\eta = 0$ on rigid walls, but with little physical justification in the inviscid case. It will turn out that the periodic condition upon $\eta(x,t)$ is equivalent to having the condition $\frac{\partial\eta}{\partial x} = 0$ on vertical walls separated by a distance Λ.

III. BRANCHES OF STEADY-STATE SOLUTIONS

Let us put down in orderly fashion the initial-boundary value problem derived in the previous section.

$$\text{D.E.:} \quad \phi_{xx} + \phi_{yy} = 0, \quad y < \eta(x,t) \tag{3.1}$$

$$\text{BC's} \begin{cases} \eta_t - \eta_x\phi_x + \phi_y = 0, \quad y = \eta(x,t) \\ \phi_t - \frac{1}{2}(\phi_x^2 + \phi_y^2) + g\eta + \frac{T}{\rho}\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} = 0, \\ \qquad y = \eta(x,t) \\ \phi_x, \phi_y \to 0 \text{ as } y \to -\infty \\ \phi(x+\Lambda,y,t) = \phi(x,y,t) \\ \phi_x(x+\Lambda,y,t) = \phi_x(x,y,t) \\ \eta(x+\Lambda,t) = \eta(x,t) \\ \eta_x(x+\Lambda,t) = \eta_x(x,t) \end{cases}$$

I.C.'s $\phi(x,y,0)$ given harmonic

$\eta(x,0)$ given

$\rho = \rho_1$ = density of heavy fluid

T = surface tension

-g = applied acceleration

See Fig. 5.

Because time does not occur explicity in Eqs. 3.1, the periodicity behavior we seek in the solutions can be found by posing the following two-point mixed boundary conditions.

$$\phi(0,y,t) = \phi(\Lambda,y,t),$$
$$\phi_x(0,y,t) = \phi_x(\Lambda,y,t),$$
$$\eta(0,t) = \eta(\Lambda,t),$$

and

$$\eta_x(0,t) = \eta_x(\Lambda,t).$$

We have here an unusual initial value problem. Boundary conditions are imposed along the locus $y = \eta(x,t)$, which is unknown until the problem is solved.

It is, however, possible to obtain some preliminary information useful in solving the steady-state case without knowing the form of $y = \eta(x,t)$. We ask: Does the initial value problem in Eq. (3.1) have any steady-state solutions, i.e., solutions such that all time derivatives vanish? If so, the time-dependent problem [Eqs. (3.1)] will have solutions tending to these steady-state solutions provided they are stable in some sense. There will be time-dependent solutions tending away from the steady-state solutions if they are at all unstable. Thus unstable steady-state solutions can be useful in classifying the initial data.

In this report, therefore, we consider mainly the problem [Eqs. (3.1)] with vanishing time derivatives.

Restricting to this steady-state case, we prove a result, obtained by K. Gustavson and J. Wolkowisky at the Los Alamos Scientific Laboratory in 1971, to the effect that the velocity potential $\phi(x,y)$ = const.

Theorem 1: In Eqs. (3.1), let $\phi_t = \eta_t = 0$. Then $\phi(x,y)$ = const.

Proof: The interface equation $y = \eta(x)$ is represented in vector form as $(x,\eta(x))$, which has a parameterization with respect to x. We refer to Fig. 6 where the interface curve and the lines $y = 0$ and $y = \Lambda$ form the boundary ∂D of the flow region D. The tangent to the interface curve is the vector $(1,\eta_x(x))$ again parameterized with x, while the outward-going normal (out of the nontrivial fluid) is then given by $(-\eta_x(x),1)$. Hence $\frac{\partial \phi}{\partial \vec{n}} = -\eta_x \phi_x + \phi_y = 0$ along the interface curve by the first interface condition [cf Eqs. (3.1)].

Using Green's half-formula (Ref. 10, p.23), we can write

$$\int_D |\text{grad } \phi|^2 dA = \int_{\partial D} \phi \frac{\partial \phi}{\partial \vec{n}} ds - \int_D \phi \nabla^2 \phi dA$$

$$= \int_{\text{inter-face}} \phi \frac{\partial \phi}{\partial \vec{n}} ds + \int_{\text{"top"}} \phi [\text{grad } \phi] \cdot (0,-1) ds \pm \int_{\text{sides}} \phi \phi_x ds = 0,$$

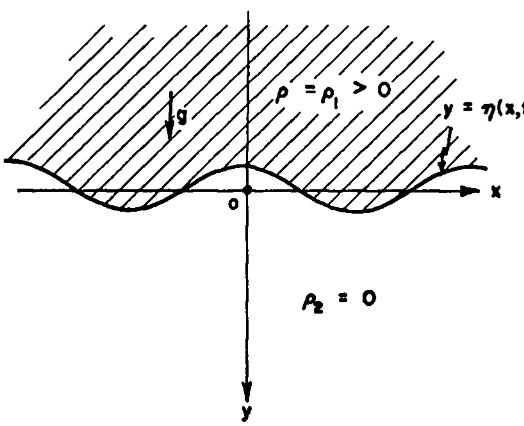

Fig. 5. Physical arrangement for problem described by Eqs. (3.1).

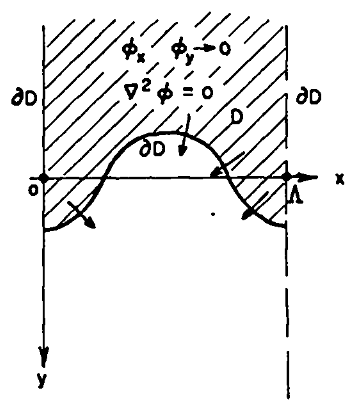

Fig. 6. Illustration in proof of Theorem 1.

since $\nabla^2\phi = 0$ in D, $\frac{\partial\phi}{\partial\vec{n}} = 0$ on the interface, grad $\phi = \phi_y \to 0$ on a horizontal line truncating D, which is moved to $-\infty$ (and which is designated "top"), and by periodicity, [cf Eqs. (3.1)]. Thus grad $\phi = 0$ in D, and $\phi(x,y)$ = const in D. This proves the theorem.

Of course $\phi(x,y)$ = const implies that steady states of Eqs. (3.1) will be quiescent states, that is, the fluid does not move in the flow region. This is natural because we have assumed the flow to be irrotational.

The two-dimensional problem in the steady-state case now reduces to a boundary value problem for a nonlinear ordinary differential equation (see Eqs. (3.1) with $\eta_t \equiv \phi_t \equiv 0$).

$$\frac{T}{\rho}\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} + g\eta = 0,$$

$$\eta(0) = \eta(\Lambda), \text{ and } \eta_x(0) = \eta_x(\Lambda). \qquad (3.2)$$

This problem results from the second interface condition, namely Bernoulli's law, and can be seen to involve Euler's differential equation for the buckled beam[11,12] in a new setting (as was pointed out by Dr. Norman Bazley).

One way to study a two-point boundary value problem is to employ a "shooting method." Let us put the differential equation in the form

$$\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} + \omega^2\eta = 0, \qquad \omega = \sqrt{\frac{g\rho}{T}} \qquad (3.3)$$

and consider the initial value problem: $\eta(0)$, $\eta_x(0)$ given.

To get a first integral, we multiply the differential equation through by η_x.

$$\frac{\eta_x\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} + \omega^2\eta\eta_x = 0,$$

whence by quadrature we have

$$-\frac{1}{\sqrt{1+\eta_x^2}} + \frac{\omega^2}{2}\eta^2 = C, \qquad (3.4)$$

where C is an arbitrary constant. Equation (3.4) describes integral curves in the η, η_x phase plane. We portray these curves for different values of the constant C in Fig. 7. Note that curves (d) and (e) are symmetrically divided and are drawn as solid lines. Drawn as dotted curves are the phase plane characteristics for the differential equation

$$-\frac{\eta_{xx}}{(1+\eta_x^2)^{\frac{3}{2}}} + \omega^2\eta = 0, \qquad (3.5)$$

which will also be useful.

In each case in Fig. 7, the maximum of the solution is given on the horizontal axis at the right extremity. We have

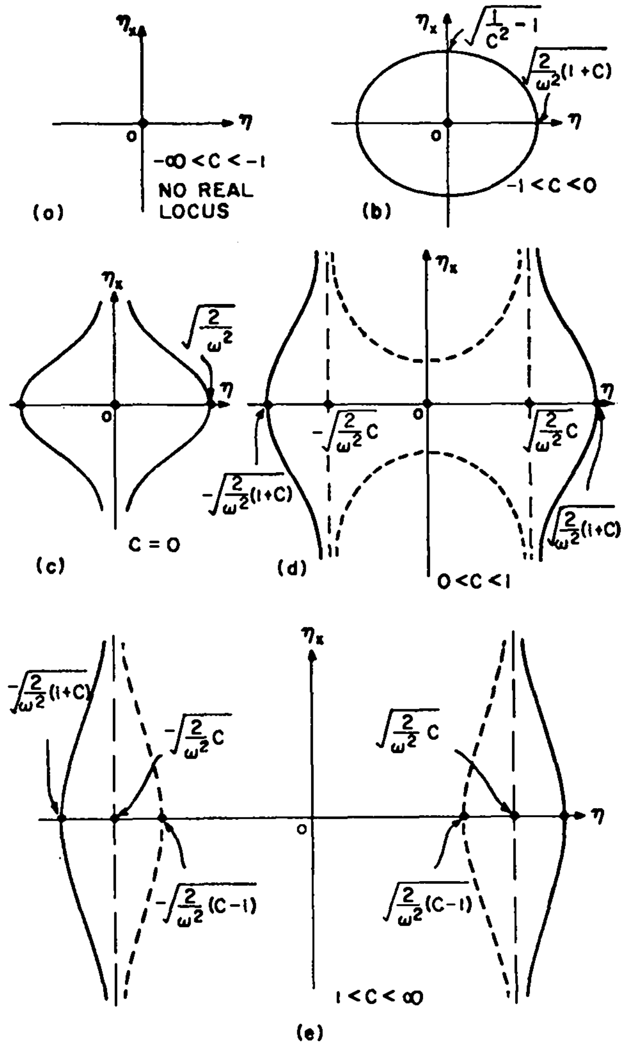

Fig. 7. Phase plane diagrams illustrating solutions of Eq. (3.4).

$$\eta_o = \max_{0 \le x \le \Lambda} \eta(x) = \sqrt{\frac{2}{\omega^2}(1 + C)}. \qquad (3.6)$$

We fix the phase of the interface function by using $\eta(0) = \eta_o$ and $\eta_x(0) = 0$ as initial conditions with Eq. (3.3). Evolution of the curves in Fig. 7 is clockwise in each case.

In Eq. (3.6) we can solve for C, and define a new number h.

$$C = -1 + \frac{1}{2}\omega^2\eta_o^2 = -1 + 2h, \qquad 0 \le h \le 1.$$

Next we compute the period or x-spread (or wavelength) of one cycle in the phase plane for the integral curve. From Eq. (3.4) we can write

$$\eta_x^2 = -1 + \frac{1}{\left[\frac{\omega^2}{2}\eta^2 - C\right]^2}$$

$$= \frac{1 - \left[\frac{\omega^2}{2}\eta^2 - C\right]^2}{\left[\frac{\omega^2}{2}\eta^2 - C\right]^2}$$

whence

$$\frac{dx}{d\eta} = \frac{\left[\frac{\omega^2}{2}\eta^2 - C\right]}{\pm\sqrt{1 - \left[\frac{\omega^2}{2}\eta^2 - C\right]^2}}.$$

We compute the period as follows (see Fig. 8).

$$P = 4\int_0^{\sqrt{\frac{2(1+C)}{\omega^2}}} \frac{\left[\frac{\omega^2}{2}\eta^2 - C\right]d\eta}{\sqrt{1 - \left[\frac{\omega^2}{2}\eta^2 - C\right]^2}}$$

$$= 4\int_0^{\eta_o} \frac{\left[\frac{\omega^2\eta^2}{2} + 1 - \frac{\omega^2\eta_o^2}{2}\right]d\eta}{\sqrt{1 - \left[\frac{\omega^2\eta^2}{2} + 1 - \frac{\omega^2\eta_o^2}{2}\right]^2}}$$

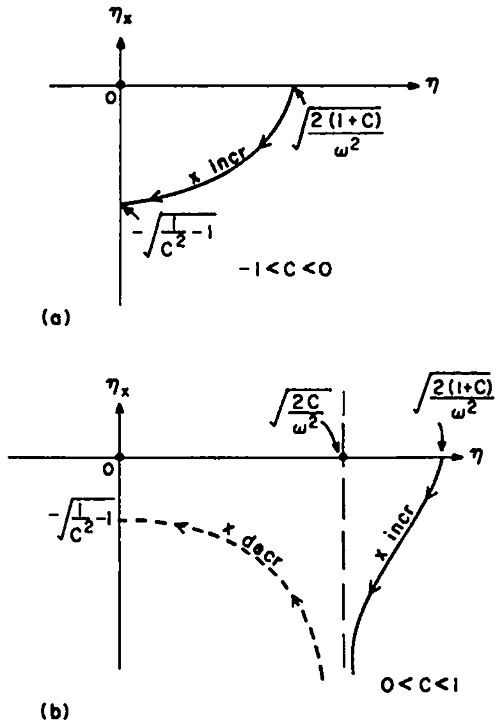

Fig. 8. Illustration helpful in period computation.

$$= 4\int_0^{\eta_o} \frac{\left[\frac{\omega^2}{2}(\eta^2 - \eta_o^2) + 1\right]d\eta}{\sqrt{1 - \left[\frac{\omega^2}{2}(\eta^2 - \eta_o^2) + 1\right]^2}}$$

$$= 4\int_0^{\eta_o} \frac{\left[1 - \frac{\omega^2\eta_o^2}{2}\left(1 - \frac{\eta^2}{\eta_o^2}\right)\right]d\eta}{\sqrt{1 - \left[1 - \frac{\omega^2\eta_o^2}{2}\left(1 - \frac{\eta^2}{\eta_o^2}\right)\right]^2}}$$

$$= 4\eta_o\int_0^1 \frac{\left[1 - \frac{\omega^2\eta_o^2}{2}(1 - y^2)\right]dy}{\sqrt{1 - \left[1 - \frac{\omega^2\eta_o^2}{2}(1 - y^2)\right]^2}},$$

with $y = \frac{\eta}{\eta_o}$

$$= 4\eta_o \int_0^1 \frac{\left[1 - \frac{\omega^2\eta_o^2}{2}(1 - y^2)\right] dy}{\sqrt{\omega^2\eta_o^2(1 - y^2) - \frac{\omega^4\eta_o^4}{4}(1 - y^2)^2}},$$

put $y = \sin\theta$

with $dy = \cos\theta \, d\theta$

$$= \frac{4}{\omega}\int_0^{\frac{\pi}{2}} \frac{\left[1 - \frac{\omega^2\eta_o^2}{2}\cos^2\theta\right]\cos\theta d\theta}{\sqrt{\cos^2\theta - \frac{1}{4}\omega^2\eta_o^2\cos^4\theta}}, \qquad h = \tfrac{1}{4}\omega^2\eta_o^2 = \tfrac{1}{2}(C + 1)$$

$$= \frac{4}{\omega}\int_0^{\frac{\pi}{2}} \frac{[1 - 2h\cos^2\theta]d\theta}{\sqrt{1 - h\cos^2\theta}} = \frac{4}{\omega} f(h), \qquad 0 < h \le \tfrac{1}{2}$$

$$= -\frac{4}{\omega}\int_0^{\cos^{-1}\frac{1}{\sqrt{2h}}} \frac{[2h\cos^2\theta - 1]}{\sqrt{1 - h\cos^2\theta}} d\theta$$

$$+ \frac{4}{\omega}\int_{\cos^{-1}\frac{1}{\sqrt{2h}}}^{\frac{\pi}{2}} \frac{[1 - 2h\cos^2\theta]}{\sqrt{1 - h\cos^2\theta}} d\theta, \quad \tfrac{1}{2} < h < 1$$

$$= -\frac{4}{\omega} f_1(h) + \frac{4}{\omega} f_2(h), \qquad \frac{1}{2} < h < 1. \qquad (3.7)$$

Note that for $0 \le h \le \frac{1}{2}$, $f(h) \in \downarrow$, while for $\frac{1}{2} < h < 1$, $f_1(h) \in \uparrow$, and $f_2(h) \in \downarrow$. Of course in $\frac{1}{2} < h < 1$, $f(h) = -f_1(h) + f_2(h)$. Thus, over $0 \le h < 1$ we have $f(h) \in \downarrow$ (see Fig. 9).

We wish to match periods. We seek solutions of the boundary value problem [Eqs. (3.2)] of periods respectively

$$\Lambda, \frac{\Lambda}{2}, \frac{\Lambda}{3}, \text{ etc.}$$

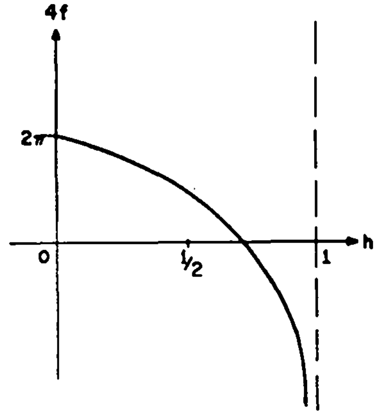

Fig. 9. Illustration suggesting shape of the function f(h).

Thus we wish to solve the equation

$$\frac{\Lambda}{m} = \frac{4}{\omega} f(h)$$

for h, putting m = 1, 2, 3, ... This is done graphically by noting intersections of $\omega\Lambda$ with $4mf(h)$ (see Fig. 10).

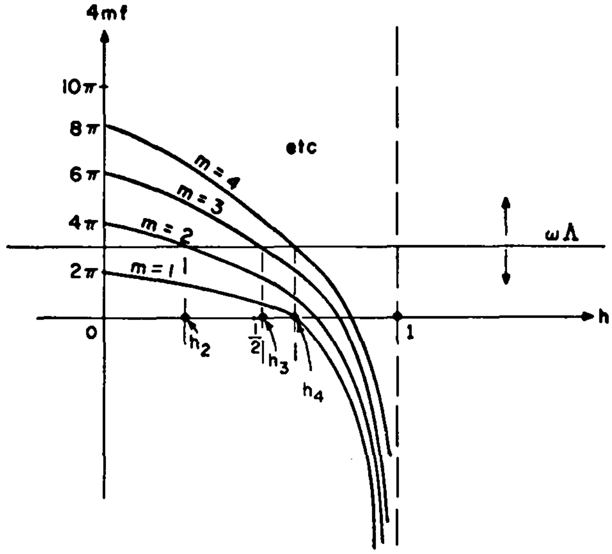

Fig. 10. Intersections yielding solutions of the boundary value problem in Eqs. (3.2).

The resulting $h_i = \frac{1}{2}(C_i + 1)$ values are then used in the initial data

$$\eta(0) = \eta_o = \sqrt{\frac{4h_i}{\omega^2}} = \sqrt{\frac{2(C+1)}{\omega^2}}, \quad \eta_x(0) = 0.$$

These initial data are then used with Eq. (3.3) to yield branches of solutions of the boundary value problem [Eqs. (3.2)].

The solution branches appear as suggested in Fig. 11, where we plot the applied acceleration g

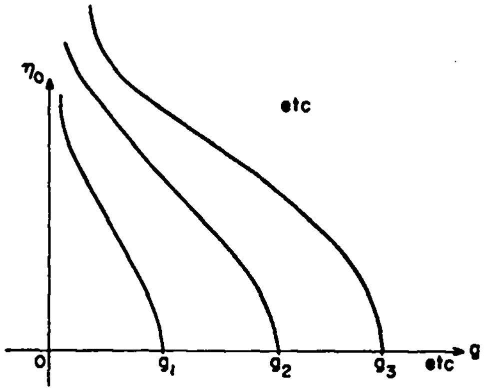

Fig. 11. Bifurcation of solutions of problem in Eqs. (3.2) and dependence of their norma upon the paremeter g.

against the sup norm $\eta_o = \max_{0 \le x \le \Lambda} \eta(x)$ of the solution. They form continuous loci in a function space as g varies and thus as $\omega = \sqrt{\frac{g\rho}{T}}$ varies in Fig. 10, and as the intersections vary in Fig. 10. The nontrivial solutions bifurcate from the trivial solution $\eta(x) \equiv 0$ at the g values (primary bifurcation points).

$$g_m = \frac{T}{\rho}\left(\frac{2m\pi}{\Lambda}\right)^2 \qquad m = 1, 2, 3, \dots .$$

Since $\eta_o = \max_{0 \le x \le \Lambda} \eta(x) = \sqrt{\frac{4h}{2}}$, and since for all intersections in Fig. 10, $h_1^{\omega} < 1$, we have $\eta_o \to \infty$ as $\omega \to 0$, or as $g \to 0$. Thus every branch of solutions becomes infinite in norm as $g \to 0$, $g > 0$.

The phase plane curves labeled (e) in Fig. 7, defined for $h > 1$, result in loci of solutions for $\omega < 0$. We may treat them later for completeness.

It is of interest to investigate the functions that make up these branches of solutions of the steady-state problem with a view to their shape in η as a function of $x \in [0,\Lambda]$. We can do this by examining the curves of Fig. 7.

For $-1 < C < 0$, or $0 < h < \frac{1}{2}$, Fig. 7b applies. This oval phase plane characteristic pertains to values of ω that yield h_i values with $0 < h_i < \frac{1}{2}$ in Fig. 10. The corresponding range of g values, for the first branch, is

$$\frac{T}{\rho}\left(\frac{4f(\frac{1}{2})}{\Lambda}\right)^2 < g < \frac{T}{\rho}\left(\frac{2\pi}{\Lambda}\right)^2 = g_1,$$

where

$$4f(h) = 4\int_0^{\frac{\pi}{2}} \frac{1-2h\cos^2\theta}{\sqrt{1-h\cos^2\theta}}\, d\theta > 0$$

is the function plotted in Fig. 9. In this range the function near $g = g_1$ is almost proportional to $\cos\frac{2\pi}{\Lambda}x$, differing from it by what will be seen to be the higher terms of a trigonometric series (see Fig. 12a).

When

$$g = \frac{T}{\rho}\left(\frac{4f(\frac{1}{2})}{\Lambda}\right)^2 < g_1,$$

it is seen from Fig. 7c that zeros occur with infinite slope (see Fig. 12b).

For

$$g < \frac{T}{\rho}\left(\frac{4f(\frac{1}{2})}{\Lambda}\right)^2 ,$$

the phase plane characteristic that pertains is Fig. 7d. Here the inner part of the phase plane cycle is not actually a solution of Eq. (3.3), but rather of Eq. (3.5). Equation (3.5) would be the differential equation derived from Eqs. (3.1) if Bernoulli's Law, the second interface condition, possessed a curvature term in which the root carries the minus sign. Such is the case when the arc length

$$s = -\int_0^x \sqrt{1+\eta_x^2}\, dx$$

of the curve $y = \eta(x)$ evolves in a direction opposite to that of x (Ref. 13, pp. 207-209). The result is shown in Fig. 12c, and the function $\eta(x)$ is double valued in places.

It should be possible to compute the value of $g < \frac{T}{\rho}\left(\frac{4f(\frac{1}{2})}{\Lambda}\right)^2$ where the bubble pinches off (see Fig. 12d). With reference to Figs. 12d, 7d, and the functions $f_2(h)$ and $f_1(h)$ defined by Eq. (3.7) and which represent, respectively, the x-length in which $\eta(x)$ is governed by Eq. (3.3) and the negative x-length governed by Eq. (3.5) in a quarter cycle, it seems clear that such a g value should give us an h-intercept in Fig. 10 (lower curve), which is a solution of the equation $f_2(h) = 2f_1(h)$. Because in the interval $\frac{1}{2} \le h < 1$, $f_1(h) \in \uparrow$ is positive with $\lim_{h\to 1} f(h) = \infty$ and $f(\frac{1}{2}) = 0$, whereas $f_2(h) \in \downarrow$ is

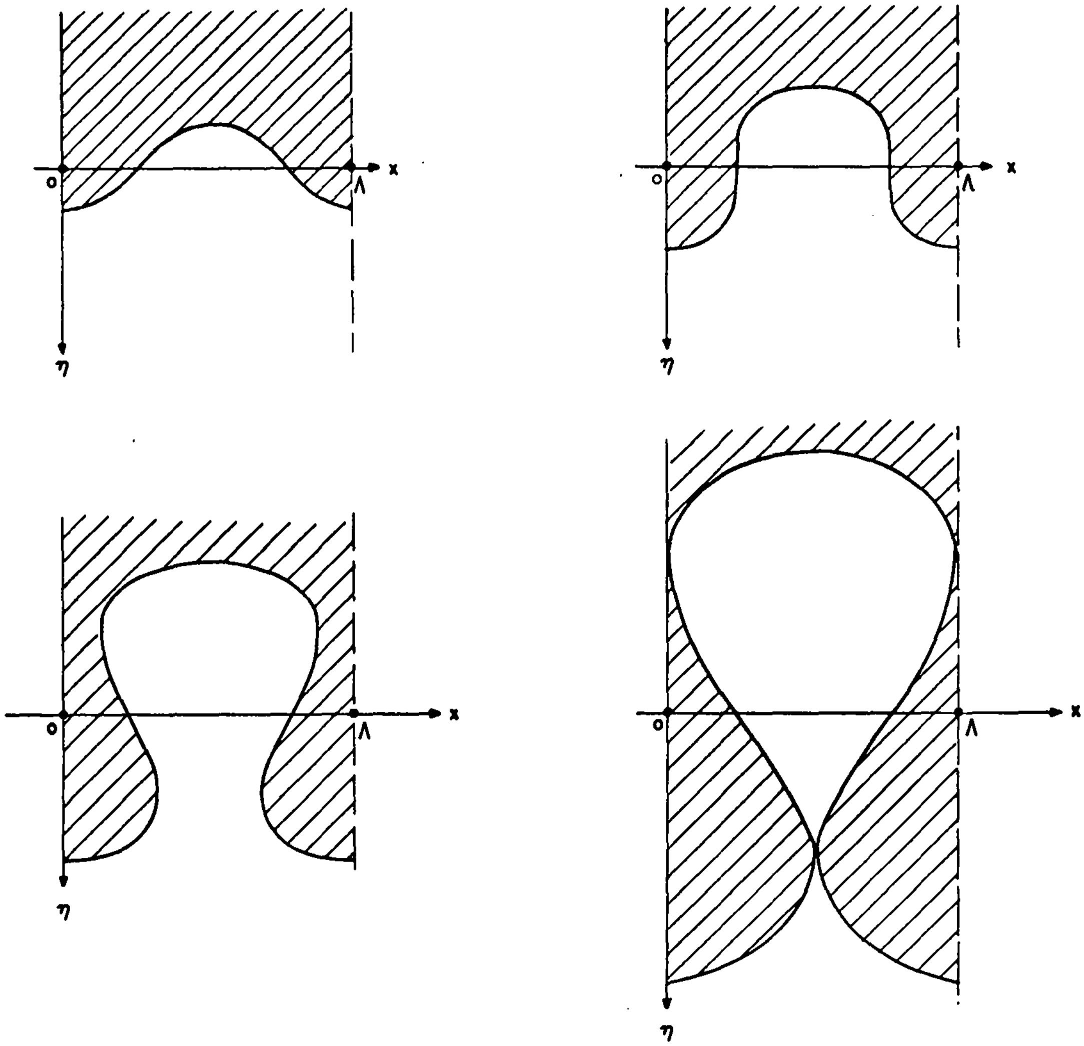

Fig. 12. Illustrations suggesting interface shapes along the first branch of bifurcated solutions, as g decreases.

positive and bounded, the equation $f_2(h) = 2f_1(h)$ is uniquely solvable for an h-value in the interval. If we let h* be this value, then the pinch-off occurs at

$$g^* = \frac{T}{\rho}\left(\frac{4f(h^*)}{\Lambda}\right)^2 < g_m.$$

For $g < g^*$ the branch solutions have no evident meaning, the functions being multiple valued wherever defined and are even defined outside of the interval $[0,\Lambda]$. They may indicate a rising bubble however.

If any interface forms were to correspond to Fig. 7e, they must appear as in Fig. 13 with points of complete discontinuity. This would be for $\omega < 0$, or g imaginary and could hardly be physical.

We summarize Sec. III as follows:

Theorem 2: The steady-state problem of Superposed Flow under inviscid, incompressible, irrotational conditions, and with periodic boundary conditions, and under the assumption that one density can be neglected, (i.e., Eqs. (3.1) where we set $\phi_t \equiv \eta_t \equiv 0$) reduces to Eqs. (3.2) for the interface shape alone. There exists a sequence $\{g_m\}$, m = 1, 2, 3, of primary bifurcation points, where $g_m = \frac{T}{\rho}\left(\frac{2m\pi}{\Lambda}\right)^2$, Λ being the assumed base period. Equations (3.2) have the trivial solution $\eta \equiv 0$ whatever the value of the acceleration g. At each bifurcation point g_m, a continuous branch of nontrivial solutions appears, bifurcating to the left. These branches represent interface shapes for given values of g. The evolution of these shapes is portrayed in Fig. 12.

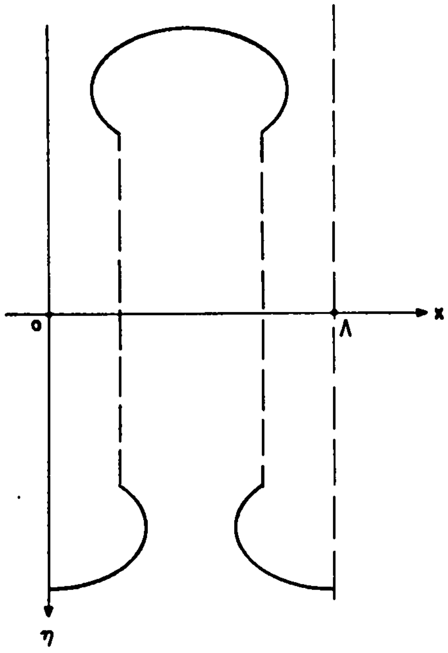

Fig. 13. Hypothetical interface corresponding phase diagram in Fig. 7c.

IV. BRANCHES OF SOLUTIONS BY PERTURBATION

Mainly because we shall need perturbation series solutions in our stability analyses, let us undertake now to find perturbation solutions of the boundary value problem [Eqs. (3.2)].

With a yet-to-be-determined parameter ε, we formally substitute the following series into Eqs. (3.2)

$$\eta(x) = \varepsilon\eta_1(x) + \varepsilon^2\eta_2(x) + \varepsilon^3\eta_3(x) + \dots \qquad (4.1a)$$

and

$$g = g_m + g_{1m}\varepsilon + g_{2m}\varepsilon^2 + g_{3m}\varepsilon^3 + \dots \quad . \qquad (4.1b)$$

Inserting the series into the differential equation, equating the coefficients of the powers of ε to zero, and attaching the boundary conditions at each step we get the following succession of linear problems, remembering that we fix the phase, i.e., $\eta(0) = \max\limits_{0 \le x \le \Lambda} \eta(x)$. Also, we employ the representation $(1 + \eta_x^2)^{-\frac{3}{2}} = 1 - \frac{3}{2}\eta_x^2 + \frac{3\cdot 5}{2\cdot 4}\eta_x^4 - \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\eta_x^6 + \dots$.

A. Sequence of Linear Perturbation Problems Solved

First-Order Problem

$$\eta_{1xx} + \frac{g_m\rho}{T}\eta_1 = 0,$$

$$\eta_1(0) = \eta_1(\Lambda), \text{ and } \eta_{1x}(0) = \eta_{1x}(\Lambda). \qquad (4.2)$$

This problem has the solution $g_m = \frac{T}{\rho}\left(\frac{2m\pi}{\Lambda}\right)^2$, $\eta_1(x) = \sqrt{\frac{2}{\Lambda}}\cos\frac{2m\pi}{\Lambda}x$, where the normalized eigenfunction $\eta_1(x)$ is unique.

We assume $\varepsilon\eta_1(x)$, yielded in the first-order problem, to comprise the entire component of the expansion [Eq. (4.1a)] in the linear eigenspace M_m associated with g_m. That is, if we consider the problem [Eqs. (4.2)] in the Hilbert Space $L_2(0,\Lambda)$ (Ref. 14, pp. 57-73), the one-dimensional linear eigenspace $M_m \subset L_2(0,\Lambda)$ is spanned by the unique normalized eigenfunction $\eta_1(x)$. Moreover, we have the decomposition $L_2(0,\Lambda) = M_m \oplus M_m^{\perp}$, where $\oplus$ indicates the direct sum (Ref. 14, p. 183) and where for the situation in Eqs. (4.2) and in the sequel, $M_m^{\perp}$ coincides with the range of the operator $(\cdot)_{xx} + \frac{g_m\rho}{T}\cdot$ with the given boundary conditions. Then the remaining terms in Eq. (4.1a), after the first, are assumed to be in $M_m^{\perp}$. There is no loss of generality in making these assumptions.

Second-Order Problem

$$\eta_{2xx} + \frac{g_m\rho}{T}\eta_2 = -g_{1m}\frac{\rho}{T}\eta_1(x),$$

$$\eta_2(0) = \eta_2(\Lambda), \text{ and } \eta_{2x}(0) = \eta_{2x}(\Lambda). \qquad (4.3)$$

Since $\eta_1(x) \in M_m$, the eigenspace, it is not in $M_m^{\perp}$ which is the range of the operator on the left in Eq. (4.3). No solution of Eq. (4.2) can exist, therefore, unless $g_{1m} = 0$. Moreover, since we assume that $\eta_2 \in M_m^{\perp}$, we must infer that $\eta_2(x) \equiv 0$. We refer the reader to the Fredholm alternative theorem (Ref. 14, p. 161), which holds in Eq. (4.3) and in the sequel because these linear boundary value problems are readily converted into Fredholm integral equations of the second kind.

Third-Order Problem

$$\eta_{3xx} + \frac{g_m\rho}{T}\eta_3 = -g_m\frac{\rho}{T}\eta_1(x) + \frac{3}{2}\eta_{1x}^2(x)\eta_{1xx}(x)$$

$$= -g_{2m}\frac{\rho}{T}\cos\frac{2m\pi}{\Lambda}x$$

$$-\frac{3}{8}\left(\frac{2}{\Lambda}\right)^{\frac{3}{2}}\left(\frac{2m\pi}{\Lambda}\right)^4\left(\cos\frac{2m\pi}{\Lambda}x - \cos\frac{6m\pi}{\Lambda}x\right),$$

$$\eta_3(0) = \eta_3(\Lambda), \text{ and } \eta_{3x}(0) = \eta_{3x}(\Lambda). \tag{4.4}$$

Again, we want the expression on the right in Eqs. (4.4) to be in the range of the operator on the left with boundary conditions. This is true only if

$$g_{2m} = -\frac{3}{8}\frac{T}{\rho}\left(\frac{2m\pi}{\Lambda}\right)^4 \frac{2}{\Lambda},$$

which leaves the term on the right proportional to $\cos\frac{6m\pi}{\Lambda}x$. Then the only solution of Eqs. (4.4) in the subspace $M_m^\perp$ is

$$\eta_3(x) = \frac{3}{64}\left(\frac{2}{\Lambda}\right)^{\frac{3}{2}}\left(\frac{2m\pi}{\Lambda}\right)^2 \cos\frac{6m\pi}{\Lambda}x.$$

Fourth-Order Problem

$$\eta_{4xx} + \frac{g_m\rho}{T}\eta_4 = -g_{3m}\frac{\rho}{T}\eta_1 - g_{2m}\frac{\rho}{T}\eta_2 + \frac{3}{2}\eta_{1x}^2\eta_{2xx}$$

$$+ 3\eta_{1x}\eta_{2x}\eta_{1xx}$$

$$\eta_4(0) = \eta_4(\Lambda), \quad \eta_{4x}(0) = \eta_{4x}(\Lambda). \tag{4.5}$$

Because $\eta_2(x) \equiv 0$, we see that $g_{3m} = 0$ is necessary in order that the right side be in $M_m^\perp$ rather than M_m. Consequently, $\eta_4(x) \equiv 0$ is the only solution in $M_m^\perp$.

We have thus generated fourth-order perturbation solutions

$$\eta(x) = \sqrt{\frac{2}{\Lambda}}\,\varepsilon\cos\frac{2m\pi}{\Lambda}x$$

$$-\frac{3\varepsilon^3}{64}\cdot\left(\frac{2}{\Lambda}\right)^{\frac{3}{2}}\left(\frac{2m\pi}{\Lambda}\right)^2\cos\frac{6m\pi}{\Lambda}x + O(\varepsilon^5),$$

and

$$g = \frac{T}{\rho}\left(\frac{2m\pi}{\Lambda}\right)^2 - \frac{3}{8}\frac{T}{\rho}\left(\frac{2m\pi}{\Lambda}\right)^4\frac{2}{\Lambda}\varepsilon^2 + O(\varepsilon^4). \tag{4.6}$$

The expansions in Eqs. (4.6) represent the m'th branch of eigenfunctions of Eqs. (3.2) as far as they converge.

It is seen that the expansion paremeter ε is simply the norm of the projection of $\eta(x)$ on M_m. As such, the expansion parameter varies from branch to branch.

B. Proof of Convergence of the Perturbation Series

Let us rewrite Eqs. (3.2) as follows.

$$\eta_{xx} + \frac{g\rho}{T}\eta = \eta_{xx}\left\{1 - \frac{1}{(1+\eta_x^2)^{\frac{3}{2}}}\right\}$$

$$= \eta_{xx}\left\{\frac{3}{2}\eta_x^2 - \frac{3\cdot5}{2\cdot4}\eta_x^4 + \frac{3\cdot5\cdot7}{2\cdot4\cdot6}\eta_x^6 - \dots\right\},$$

$$|\eta_x| < 1, \quad \eta(0) = \eta(\Lambda), \text{ and } \eta_x(0) = \eta_x(\Lambda), \tag{4.7}$$

and set

$$\eta(x,\varepsilon) = \varepsilon\eta_1(x) + \varepsilon^3 V(x,\varepsilon)$$

$$g(\varepsilon) = g_m + \varepsilon^2 f(\varepsilon), \tag{4.8}$$

where we recall the form arrived at in Eqs. (4.6). Here $\eta_1(x) = \sqrt{\frac{2}{\Lambda}}\cos\frac{2m\pi}{\Lambda}x$ and $g_m = \frac{T}{\rho}\left(\frac{2m\pi}{\Lambda}\right)^2$. We also recall that $\varepsilon\eta_1(x)$ comprises the orthogonal projection of $\eta(x,\varepsilon) \in L_2(0,\Lambda)$ into M_m, and that we therefore seek $V(x,\varepsilon) \in M_m^\perp$. Here M_m was of course the linear eigenspace spanned by $\eta_1(x) = \sqrt{\frac{2}{\Lambda}}\cos\frac{2m\pi}{\Lambda}x$, and $M_m^\perp$, the orthogonal complement, coincides with the range of the operator [cf discussion below Eqs. (4.2)].

Substituting Eqs. (4.8) into Eq. (4.7), we obtain

$$V_{xx} + \frac{\rho g_m}{T}V = -\frac{\rho}{T}f\eta_1 - \varepsilon^2\frac{\rho}{T}fV$$

$$+\frac{1}{\varepsilon^2}(\eta_1 + \varepsilon^2 V)_{xx}\left\{1 - \frac{1}{[1+\varepsilon^2(\eta_1+\varepsilon^2 V)_x^2]^{\frac{3}{2}}}\right\},$$

$$V(0) = V(\Lambda), \text{ and } V_x(0) = V_x(\Lambda), \tag{4.9}$$

as a boundary value problem for the function $V(x,\varepsilon)$. Equations (4.9) have a form that suggests the following sequence of iterative boundary value problems as a means of solution.

$$V_{xx}^{n+1} + \frac{\rho g_m}{T}V^{n+1} = -\frac{\rho}{T}f^{n+1}\eta_1 - \varepsilon^2\frac{\rho}{T}f^nV^n$$

$$+\frac{1}{\varepsilon^2}(\eta_1 + \varepsilon^2 V^n)_{xx}\left\{1 - \frac{1}{[1+\varepsilon^2(\eta_1+\varepsilon^2 V^n)_x^2]^{\frac{3}{2}}}\right\}$$

$$= -\frac{\rho}{T}f^{n+1}\eta_1 - \varepsilon^2\frac{\rho}{T}f^nV^n$$

$$+ (\eta_1 + \varepsilon^2 V^n)_{xx} \left\{ \frac{3}{2} (\eta_1 + \varepsilon^2 V^n)_x^2 \right.$$

$$\left. - \varepsilon^2 \frac{3\cdot 5}{2\cdot 4} (\eta_1 + \varepsilon^2 V^n)_x^4 + \ldots \right\},$$

$$V^{n+1}(0) = V^{n+1}(\Lambda), \text{ and } V_x^{n+1}(0)$$

$$= V_x^{n+1}(\Lambda), \tag{4.10}$$

where V^n, f^n are the n'th iterates, and the process is started with $V^o \equiv 0$, $f^o = 0$. At each step, existence of a solution requires that the right-hand side be in $M_m^\perp$, and we seek the solution V^{n+1} in $M_m^\perp$. We wish to show that this process converges to the limit function $V(x,\varepsilon)\varepsilon M_m^\perp$, which solves the problem of Eqs. (4.9), and which, substituted into Eqs. (4.8), provides the solution of Eqs. (4.7).

Each iterate in Eq. (4.10) is given in terms of the previous iterate by the mapping

$$\{u,e\} = T_\varepsilon\{V,f\}, \tag{4.11}$$

where u is given as the solution, unique in $M_m^\perp$, of the problem

$$u_{xx} + \frac{\rho g_m}{T} u = -\frac{\rho}{T} e\,\eta_1 - \varepsilon^2 \frac{\rho}{T} fV$$

$$+ (\eta_1 + \varepsilon^2 V)_{xx} \left\{ \frac{3}{2} (\eta_1 + \varepsilon^2 V)_x^2 \right.$$

$$\left. - \varepsilon^2 \frac{3\cdot 5}{2\cdot 4} (\eta_1 + \varepsilon^2 V)_x^4 + \ldots \right\},$$

$$u(0) = u(\Lambda), \text{ and } u_x(0) = u_x(\Lambda), \tag{4.12}$$

and $e(\varepsilon)$ is determined from the condition for the existence of a solution u of Eqs. (4.12) for the given $V\varepsilon M_m^\perp$ and $f(\varepsilon)$
$\left(\text{right side of Eqs. (4.12)}, \sqrt{\frac{2}{\Lambda}} \cos \frac{2m\pi}{\Lambda} x\right) = 0$, using the inner product notation for $L_2(0,\Lambda)$. Thus,

$$e(\varepsilon) = -\varepsilon^2 f(\varepsilon)(V,\eta_1)$$

$$+ \frac{T}{\rho} \left([\eta_1 + \varepsilon^2 V]_{xx} \left\{ \frac{3}{2} [\eta_1 + \varepsilon^2 V]_x^2 \right.\right.$$

$$\left.\left. - \varepsilon^2 \frac{3\cdot 5}{2\cdot 4} [\eta_1 + \varepsilon^2 V]_x^4 + \ldots \right\},\ \eta_1 \right), \tag{4.13}$$

where, of course, $\eta_1(x) = \sqrt{\frac{2}{\Lambda}} \cos \frac{2m\pi}{\Lambda} x$.

The mapping T_ε given in Eqs. (4.11) through (4.13) carries the space $\mathcal{B}^2$ of couples

$$\mathcal{B}^2 = [\{V,f\} | V \varepsilon C^2(0,\Lambda),\ \ f \varepsilon R]$$

into itself. Here $C^2(0,\Lambda)$ denotes the space of twice continuously differentiable functions defined on the closed interval $[0,\Lambda]$ with the norm

$$\|V\|_2 = \max_{0 \le x \le \Lambda} |V(x)| + \max_{0 \le x \le \Lambda} |V_x(x)|$$

$$+ \max_{0 \le x \le \Lambda} |V_{xx}(x)|$$

$$= \|V\|_c + \|V_x\|_c + \|V_{xx}\|_c,$$

and R denotes the real numbers. We define the following norm for $\mathcal{B}^2$.

$$|||\{V,f\}||| = \|V\|_2 + |f|. \tag{4.14}$$

Let us define the "improper" Green's function $G(x,\xi)$ satisfying the problem

$$G_{xx} + \frac{g_m \rho}{T} G = -\delta(x-\xi) + \frac{2}{\Lambda} \cos \frac{2m\pi}{\Lambda} x \cos \frac{2m\pi}{\Lambda} \xi,$$

$$G(0,\xi) = G(\Lambda,\xi), \quad G_x(0,\xi) = G_x(\Lambda,\xi),$$

which can be expressed as follows.

$$G(x,\xi) = \frac{2}{\Lambda} \sum_{\substack{p=0 \\ p\neq m}}^{\infty} \frac{\cos \frac{2p\pi}{\Lambda} x \cos \frac{2p\pi}{\Lambda} \xi}{\left(\frac{2p\pi}{\Lambda}\right)^2 - \left(\frac{2m\pi}{\Lambda}\right)^2}.$$

Then Eqs. (4.12) can be written in terms of an integral operator in $M_m^\perp$.

$$u(x) = \int_0^\Lambda G(x,\xi)\ \{-\frac{\rho}{T} e\eta_1 + \varepsilon^2 \frac{\rho}{T} fV$$

$$+ (\eta_1 + \varepsilon^2 V)_{xx}\ [\tfrac{3}{2} (\eta_1 + \varepsilon^2 V)_x^2$$

$$- \varepsilon^2 \frac{3\cdot 5}{2\cdot 4} (\eta_1 + \varepsilon^2 V)_x^4 + \ldots]\}d\xi. \tag{4.15}$$

The kernel $G(x,\xi)$ is piecewise twice differentiable.

The iterations in Eqs. (4.10), defined in terms of the mapping for Eq. (4.11), converge provided the mapping of Eq. (4.11) is contracting on some set. Below we give such a set.

Let $\hat{G}$, $\hat{G}_x$, and $\hat{G}_{xx}$ be, respectively, the norms in $L_2(0,\Lambda)$ of the linear integral operators generated, respectively, by the kernels $G(x,\xi)$, $G_x(x,\xi)$, and $G_{xx}(x,\xi)$. Then, let $\bar{G} = \hat{G} + \hat{G}_x + \hat{G}_{xx}$.

Let there be given a ball $S_R \subset \mathcal{E}^2$ with radius R such that

$$R > \|\eta_1\|_2 + 6[\bar{G}\|\eta_1\|_2^3 + \frac{T}{\rho}\|\eta_1\|_2^4 + \|\eta_1\|_2^5];$$

that is

$$S_R = [\{u,e\} \in \mathcal{E}^2 | \ |||\{u,e\}||| \leq R].$$

After much calculation it turns out that there exists an $\varepsilon_o < \sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{R^2}}}$ such that for $0 < |\varepsilon| < \varepsilon_o$, $T_\varepsilon\{u,e\}$ maps S_R into itself; that is, $0 < |\varepsilon| < \varepsilon_o$ and $\{u,e\} \in S_R \Rightarrow T_\varepsilon\{u,e,\} \in S_R$.

Again, there exists an $\varepsilon_1 < \sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{R^2}}}$ such that for $0 < |\varepsilon| < \varepsilon_1$, $T_\varepsilon\{u,e\}$ is contracting S_R; that is $0 < |\varepsilon| < \varepsilon$, and $\{u_1,e_1\} \in S_R$, $\{u_2,e_2\} \in S_R$ implies that

$$|||T_\varepsilon(u_1,e_1) - T_\varepsilon(u_2,e_2)||| \leq \theta|||\{u_1,e_1\} - \{u_2,e_2\}|||,$$

where θ is some number with $0 < \theta < 1$, independent of ε for $0 < |\varepsilon| < \varepsilon_1$.

The above facts can be shown by lengthy but quite standard procedures. We can use the smallest of ε_0, ε_1.

Then we have the estimate

$$|||\{u^{n+p},e^{n+p}\} - \{u^n,e^n\}|||$$

$$\leq \sum_{q=1}^{p} |||\{u^{n+q},e^{n+q}\} - \{u^{n+q-1},e^{n+q-1}\}|||$$

$$\leq \sum_{q=1}^{p} \theta|||\{u^{n+q-1},e^{n+q-1}\} - \{u^{n+q-2},e^{n+q-2}\}|||$$

$$\leq \sum_{q=1}^{p} \theta^{n+q}|||\{u^2,e^2\} - \{u^1,e^1\}|||$$

$$\to 0 \quad \text{as} \quad n \to \infty, \tag{4.16}$$

which is the Cauchy criterion for convergence of the iterative sequence $\{u^n,e^n\}$. Hence by completeness there exists a limit pair $\{u^*,e^*\}$ which, by continuity, provides a solution of Eqs. (4.9) unique in the neighborhood of the origin $\{V,f\} = \{0,0\}$ provided by the ball S_R, and valid for $0 < |\varepsilon| < \varepsilon_o$. A solution of Eqs. (4.9) leads to a solution of Eqs. (4.7) by means of Eqs. (4.8).

In the iterative process,

$$\{u^{n+1},e^{n+1}\} = T_\varepsilon\{u^n,e^n\},$$

where e^{n+1} is defined in terms of e^n by Eq. (4.13) (with e^n for f), and where u^{n+1} is defined in terms of u^n by Eq. (4.15) (with u^n for V), which process is shown above to converge in the norm [Eq. (4.14)], <u>each iterate is an analytic function of ε</u>. The iteration starts with {0,0}, and each iterate $\{u^n,e^n\}$ has components expressed in terms of analytic u^{n-1}, e^{n-1} and ε as analytic expressions. Moreover, it can be seen that the estimate of Eq. (4.16) can be made uniform in ε in any closed subset of the disk $0 \leq |\varepsilon| < \varepsilon_o$. Thus, the iterates $\{u^n,e^n\}$ converge uniformly to the limit function on <u>any</u> closed subset of $0 \leq |\varepsilon| < \varepsilon_o$ and so the limit pair $\{u^*,e^*\}$ has components analytic in the disk $0 \leq |\varepsilon| < \varepsilon_o$. Thus u^* and e^* can each be developed in convergent ε power series in the disk.

The expressions for u^* and e^* are now inserted in Eqs. (4.8) for V,f to give the convergent ε power series expansions for $\eta(x,\varepsilon)$ and $g(\varepsilon)$.

These a priori convergent power series expansions in ε are now used as the expansions [Eqs. (4.1 a,b)] and the first few coefficients are produced by the succession of linear problems in the first part of this section.

We can summarize Sec. IV by stating the following result.

<u>Theorem 3</u>: The solutions $(g,\eta(x))$ of the boundary value problem [Eqs. (3.2)], belonging to the m'th branch, m = 1, 2, 3, ... , which represent interface shapes under steady-state conditions for given g, are analytic in a parameter $\varepsilon = (\eta(x), \eta_1(x))$. Here $\eta_1(x)$ is the normalized solution of the linearized problem stated in Eqs. (4.2). Thus, ε is the magnitude of the projection of $\eta(x)$ on the one-dimensional space spanned by $\eta_1(x)$, which we have called M_m. This analyticity holds in at least a small open neighborhood of the origin (the trivial solution). Hence the solutions $(g,\eta(x))$ on the m'th branch can be developed in the convergent power series [Eqs. (4.1)] in which the first few terms are given by Eqs. (4.6).

V. STABILITY PROPERTIES

In the preceding two sections we have proven the existence of steady-state solutions of the initial value problem [Eqs. (3.1)] that arises in the theory of superposed fluids. We have assumed, of course, that one of the densities is small enough to be ignored, and that the viscosities are zero.

These time-independent steady-state solutions can be used as initial data for Eqs. (3.1)., i.e., $\phi(x,y,0) \equiv \text{const}$, $\eta(x,0) = \eta_o(x)$, where g, $\eta_o(x)$ is a given solution of Eqs. (3.2). Then quite trivially, a steady-state solution is a solution of initial value problem in Eqs. (3.1) with these initial conditions.

The interfaces in experimentally observed fluid motions do bear some resemblance to the steady-state solutions of Sec. III, at least in the shape.[2,3] The two points of departure seem to be symmetry and motion. Experimentally observed interface shapes lack the symmetry about the horizontal axis possessed by our steady-state solutions. Moreover, experimentally observed interfaces are in motion and grow in emplitude. Presumably if experimental fluid motion were to commence with a steady-state solution as interface, this interface would persist. This would depend, however, on what we can determine here about the stability of a steady-state solution.

Steady-state solutions comprise a very particular sort of initial data to be used with the time dependent problem [Eqs. (3.1)]. More general initial data will function in a manner that must depend on its relationships to the steady-state solutions used as initial data. For example, for the first mode of branch solutions (m = 1) characterized in Secs. III and IV, how do general time dependent solutions behave if the initial data are pointwise less, or pointwise greater, than the steady-state solution? Much remains to be learned here.

In considering the stability of a steady-state solution: $\phi = \text{const}$, $\eta = \eta_o(x)$, we seek to study the behavior of the time-dependent solutions of Eqs. (3.1) with initial data in some neighborhood of the steady-state solution. The latter is stable if all time-dependent solutions tend to it that have the initial data lying in the neighborhood. It is unstable if in <u>any</u> neighborhood of the steady-state solution, some initial data exist such that the resulting time-dependent solution leaves the neighborhood.

The type of stability analysis given below is that which has been satisfying in engineering circles. It assumes that the neighborhood is sufficiently small that with good approximation we can regard the nearby time-dependent problem as a perturbation of the steady-state problem, and then we linearize the former. It has been justified in the case of the more general Navier-Stokes equations of fluid flow.[6] Our equations are derived from the Navier-Stokes equations by assuming the viscosity, and the rotation of the flow, to be zero. However, our problem is complicated by a free interface, in other words a moving boundary.

We proceed with the linear stability analysis for steady-state solutions of Eqs. (3.1). Let $\eta_o(x)$ be a steady-state interface as determined in Secs. III or IV. Of course the steady-state velocity potential under our conditions is $\phi = \text{const}$. We write the linear perturbation or variational problem as follows (see Fig. 14).

$$\text{DE:} \quad k_{xx} + k_{yy} = 0, \qquad y < \eta_o(x)$$

$$\text{BC's}\begin{cases} h_t - \eta_{ox}k_x + k_y = 0, \quad y = \eta_o(x) \\ k_t + gh - 3\,\dfrac{T}{\rho}\,\dfrac{\eta_{ox}\eta_{oxx}}{(1+\eta_{ox}^2)^{\frac{5}{2}}}\,h_x + \dfrac{T}{\rho}\,\dfrac{h_{xx}}{(1+\eta_{ox}^2)^{\frac{3}{2}}} \\ \qquad = 0, \qquad y = \eta_o(x) \\ k_x, k_y \to 0 \text{ as } y \to -\infty \\ k(t,0,y) = k(t,\Lambda,y) \\ k_x(t,0,y) = k_x(t,\Lambda,y) \\ h(t,0) = h(t,\Lambda) \qquad k(0,x,y) \quad \text{given harmonic} \\ h_x(t,0) = h_x(t,\Lambda) \qquad h(0,x) \quad \text{given} \end{cases} \tag{5.1}$$

A. Stability of The Trivial Solutions

First we study the stability, or lack of stability, of the trivial solution or plane interface. Setting $\eta_o(x) \equiv 0$ and $\phi(x,y) \equiv \text{const}$ in Eqs. (5.1),

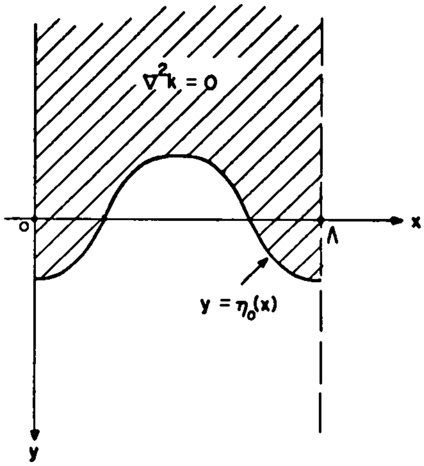

Fig. 14. Physical arrangement for discussing stability of the first branch of bifurcated interface patterns.

we get the linear problem (see Fig. 15)

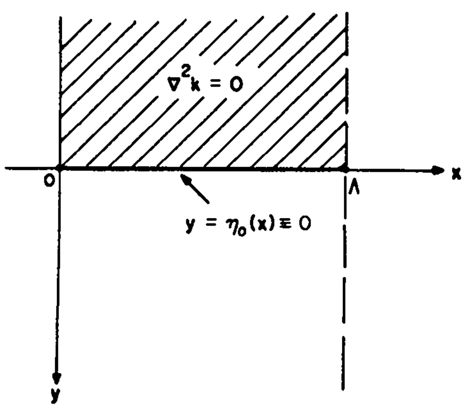

Fig. 15. Physical arrangement for discussing stability of the plane interface.

DE: $\quad k_{xx} + k_{yy} = 0, \quad y < 0$

$$
\text{BC's}\begin{cases}
h_t + k_y = 0, & y = 0 \\
k_t + gh + \frac{T}{\rho} h_{xx} = 0, & y = 0 \\
k_x,\ k_y \to 0 \text{ as } y \to -\infty & \\
k(t,0,Y) = k(t,\Lambda,y),\ h(t,0) = h(t,\Lambda) & \\
k_x(t,0,y) = k_x(t,\Lambda,y),\ h_x(t,0) = h_x(t,\Lambda) & \\
k(0,x,y) \quad \text{given harmonic} & \\
h(0,x) \quad \text{given.} &
\end{cases} \tag{5.2}
$$

Following standard procedure (Ref. 15, Chap. X) we put $k(t,x,y) = e^{\lambda t}k(x,y)$ and $h(t,x) = e^{\lambda t}h(x)$, and seek to determine λ as an eigenvalue. If we write $\hat{h} = \lambda h$ for convenience, we get a linear eigenvalue problem with λ^2 as eigenvalue

DE: $\quad k_{xx} + k_{yy} = 0, \quad y < 0$

$$
\text{BC's}\begin{cases}
\hat{h} + k_y = 0,\ y = 0 \\
\lambda^2 k + g\hat{h} + \frac{T}{\rho}\hat{h}_{xx} = 0, \quad y = 0 \\
\text{with} \\
g \neq g_m = \frac{T}{\rho}\left(\frac{2m\pi}{\Lambda}\right)^2 \\
k_x, k_y \to 0 \text{ as } y \to -\infty \\
k(0,y) = k(\Lambda,y) \\
k_x(0,y) = k_x(\Lambda,y) \\
\hat{h}(0) = \hat{h}(\Lambda), \quad \hat{h}_x(0) = \hat{h}_x(\Lambda).
\end{cases} \tag{5.3}
$$

If the harmonic function $k(x,y)$ in Eqs. (5.3) exists, it possesses a Fourier expansion as follows.

$$
k(x,y) = \sum_{n=-\infty}^{\infty} A_n(y)e^{i\nu_n x} \qquad \nu_n = \frac{2n\pi}{\Lambda}. \tag{5.4}
$$

Then from the first interface condition in Eqs. (5.3) ($\hat{h} + k = 0$ at $y = 0$), we get

$$
\hat{h}(x) = -\sum_{n=-\infty}^{\infty} A_n'(0)e^{i\nu_n x}. \tag{5.5}
$$

However, we can solve the differential equation in the second interface condition by

$$
\hat{h}_{xx} + \frac{g\rho}{T}\hat{h} = -\lambda^2 \frac{\rho}{T}\sum_{n=-\infty}^{\infty} A_n(0)e^{i\nu_n x}.
$$

Assuming the Fourier expansion $\hat{h}(x) = \sum_{n=-\infty}^{\infty} B_n e^{i\nu_n x}$, we get by the substitution,

$$
-\sum_{n=-\infty}^{\infty} B_n(\nu_n^2 - \frac{g\rho}{T})e^{i\nu_n x}
$$

$$
= -\lambda^2 \frac{\rho}{T}\sum_{n=-\infty}^{\infty} A_n(0)e^{i\nu_n x},
$$

whence, by equating coefficients [see Eq. (5.5)],

$$B_n = \frac{\lambda^2 \frac{\rho}{T} A_n(0)}{\nu_n^2 - \frac{g\rho}{T}} = -A_n'(0),$$

$$n = 0, \pm 1, \pm 2, \ldots \qquad (5.6)$$

If Eq. (5.4) represents a harmonic function, however, and if it satisfies the conditions $k_y \to 0$ as $y \to -\infty$ and $k(x,0) \neq 0$ of Eqs. (5.3), then $A_n(y) = \text{const} \cdot e^{\nu_n y}$, from which we have $A_n'(0) = \nu_n A_n(0)$. This is substituted on the right in Eq. (5.6). We observe, too, that only positive indices can be used. From Eq. (5.6),

$$\lambda^2 = -\nu_p(\nu_p^2 - \frac{g\rho}{T}) \frac{T}{\rho} = \lambda_p^2, \; p = 1, 2, 3, \ldots \qquad (5.7)$$

is a necessary condition that $A_p(0)$ be nontrivial. This gives a sequence $\{\lambda_p^2\}$ of squared eigenvalues for Eqs. (5.3) that are positive or negative according to the g value. To each eigenvalue λ_p^2 there corresponds a set of Fourier coefficients: $A_1(0) = 0, \ldots, A_{p-1}(0) = 0, A_p(0) = 1, A_{p+1}(0) = 0, \ldots, A_n(0) = 0$ for $n > p$. The corresponding eigenelement is $k_p(x,y) = e^{\nu_p y + i\nu_p x}$, $\hat{h}_p(x) = -\nu_p e^{i\nu_p x}$, or in real terms,

$$k_p(x,y) = e^{\nu_p y} \cos \nu_p x,$$

$$\hat{h}_p(x) = -\nu_p \cos \nu_p x, \quad \nu_p = \frac{2p\pi}{\Lambda},$$

$$p = 1, 2, 3, \ldots . \qquad (5.8)$$

Except for the constant term, the latter set is complete on $0 \leq x \leq \Lambda$ and the former set is complete with respect to harmonic functions that vanish as $y \to -\infty$. Hence the λ^2-spectrum of Eqs. (5.3) consists exactly of the sequence $\{\lambda_p^2\}$, and the solution of Eqs. (5.2) is

$$k(t,x,y) = \sum_{p=0}^{\infty} C_p e^{\pm\lambda_p t + \nu_p y} \cos \nu_p x$$

$$h(t,x) = \sum_{p=0}^{\infty} D_p e^{\pm\lambda_p t} \frac{\nu_p \cos \nu_p x}{|\lambda_p|}, \qquad (5.9)$$

C_p and D_p being the Fourier coefficients of expansion for the initial disturbance in terms of the eigenfunctions in Eqs. (5.8).

For $g < g_1 = \frac{\rho}{T}\nu_1^2$, $\lambda_1^2 = -\nu_1(\nu_1^2 - \frac{g\rho}{T}) \frac{T}{\rho} < 0$, and a fortiori $\lambda_2^2 < 0$, $\lambda_3^2 < 0$, etc. This gives imaginary time multipliers in Eqs. (5.9) and bounded oscillatory behavior. For $g_1 < g < g_2$, $\lambda_1^2 > 0$, but $\lambda_2^2 = -\nu_2(\nu_2^2 - \frac{g\rho}{T}) \frac{T}{\rho} < 0$ and of course $\lambda_3^2 < 0$, $\lambda_4^2 < 0$, etc. Equations (5.9) have each a growing exponential. Generally, if $g_{m-1} < g < g_m$, $\lambda_1^2 > 0, \lambda_2^2 > 0, \ldots, \lambda_{m-1}^2 > 0, \lambda_m^2 < 0, \lambda_{m+1}^2 < 0, \ldots, \lambda_n^2 < 0$ for $n > m$. Equations (5.9) have m-1 growing exponentials.

We can summarize Sec. V, so far, as follows.

Theorem 4: For $0 < g < g_1$ the trivial steady-state solution $\phi(x,y) = \text{const}$, $\eta(x) \equiv 0$ is "marginally stable" in that λ_p is pure imaginary, $p = 1, 2, 3, \ldots$. For $g_{m-1} < g < g_m$, $m = 2, 3, 4, \ldots$, the trivial solution is unstable, with "m-1 degrees of instability" in that $\lambda_1, \ldots, \lambda_{m-1}$ are positive and furnish increasing exponentials in Eqs. (5.9) whereas only λ_m, λ_{m+1}, etc., are pure imaginary. As $g > 0$ passes through the value g_{m-1}, the eigenvalues $\pm\lambda_{m-1}$, originally pure imaginary, vanish and then become a $\pm$ real pair (see Fig. 16).

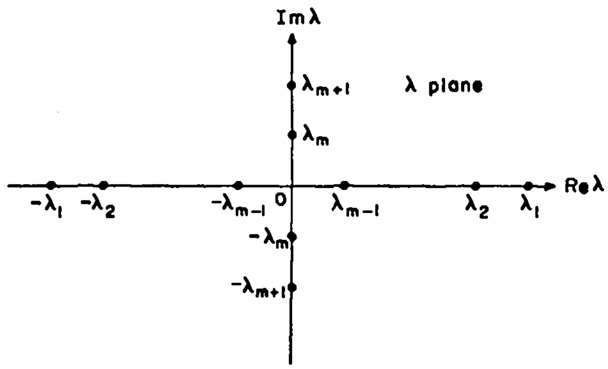

Fig. 16. Illustration suggesting configuration of eigenvalues in Theorem 4.

For $0 < g < g_1$ we have stated the condition as "marginal stability." Actually purely imaginary eigenvalues λ for Eqs. (5.3) constitute a necessary condition for stability, but this has never been shown to be generally sufficient in hydrodynamics. At least with the solutions in Eqs. (5.9) it assures no exponential growth, which seems to satisfy many authors (Ref. 15, Chap. X).

It is commonly thought that viscosity has a depressant effect on fluid motions and that, with viscosity in the problem, the eigenvalues λ_n would lie somewhere in the open left half plane for $0 < g < g_1$ rather than on the borderline imaginary axis. Then a sufficient condition for stability would be satisfied.

If $g = g_m$, then $\lambda_m = 0$, $m = 1, 2, 3, \ldots$. This is a bifurcation situation. By the analysis of Secs. III and IV, nontrivial branches of steady-state solutions break away from the trivial solution at these values.

The initial interval $0 < g < g_1$, where the trivial solution is marginally stable, results from the presence of surface tension T. Removal of surface tension appears to cause the eigenvalue spectra $\{g_m\}$ and $\{\lambda_m\}$ to degenerate to continuous spectra.

B. Instability of the Branch Solutions

Next we study the stability of the nontrivial continuous branches of steady-state solutions derived in Secs. III and IV. The m'th branch bifurcates from the trivial solution of Eqs. (3.1) (nemely, ϕ = const, $\eta \equiv 0$) at $g = g_m$.

We make use of the convergent perturbation expansions given symbolically in Eqs. (4.1) with actual evaluations of coefficients given in Eqs. (4.6). We substitute these for g and $\eta_o(x)$ in the following linear variational problem.

DE: $k_{xx} + k_{yy} = 0,\ y < \eta_o(x)$

$$\text{BC's}\begin{cases} \hat{h} - \eta_{ox}k_x + k_y = 0, \quad y = \eta_o(x) \\ \lambda^2 k + g\hat{h} - 3\,\frac{T}{\rho}\,\frac{\eta_{ox}\eta_{oxx}}{(1+\eta_{ox}^2)^{\frac{5}{2}}}\,\hat{h}_x + \frac{T}{\rho}\,\frac{\hat{h}_{xx}}{(1+\eta_{ox}^2)^{\frac{3}{2}}} = 0, \\ \qquad y = \eta_o(x) \\ k_x,\ k_y \to 0 \quad \text{as } y \to -\infty \\ k(0,y) = k(\Lambda,y) \qquad \hat{h}(0) = \hat{h}(\Lambda) \\ k_x(0,y) = k_x(\Lambda,y), \qquad \hat{h}_x(0) = \hat{h}(\Lambda) \end{cases} \tag{5.10}$$

which results when we set $k(t,x,y) = e^{\lambda t}k(x,y)$ and $h(t,x) = e^{\lambda t}h(x)$, and then $\hat{h}(x) = \lambda h(x)$ in Eqs. (5.1). We also substitute into Eqs. (5.10) the perturbation series

$$\lambda^2 = \lambda_0^2 + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \ldots,$$

$$\hat{h} = \hat{h}_0 + \varepsilon\hat{h}_1 + \varepsilon^2\hat{h}_2 + \ldots,$$

and

$$k = k_0 + \varepsilon k_1 + \varepsilon^2 k_2 + \ldots, \tag{5.11}$$

where the coefficients are to be determined, and ε has the same meaning as in Sec. IV in connection with Eqs. (4.1).

To carry out a perturbation series solution of Eqs. (5.10), it is convenient to expand the nonlinear interface conditons (BC's) in Taylor series as follows.

$$\hat{h} + k_y(x,0) - \eta_{ox}k_x(x,0) + \eta_o k_{yy}(x,0) - \eta_o\eta_{ox}k_{xy}(x,0) + \frac{1}{2}\,\eta_o^2 k_{yyy}(x,0) + \ldots = 0$$

$$\lambda^2 k(x,0) + \lambda^2\eta_o k_y(x,0) + \frac{1}{2}\,\lambda^2\eta_o^2 k_{yy}(x,0) + g\hat{h} + \frac{T}{\rho}\,\hat{h}_{xx} - \frac{3}{2}\,\frac{T}{\rho}\,\eta_{ox}^2\hat{h}_{xx} - 3\,\frac{T}{\rho}\,\eta_{ox}\eta_{oxx}\hat{h}_x + \ldots = 0. \tag{5.12}$$

This is valid because $k(x,y)$ can be extended as a harmonic function.

Substituting the series of Eqs. (4.1) and (5.11) into Eqs. (5.10) with interface conditions [Eqs. (5.12)], and equating coefficients of the powers of ε to zero, we get a succession of linear problems.

Zero'th-Order Problem

$$k_{xx} + k_{yy} = 0, \quad y < 0$$

$$\text{BC's}\begin{cases} \hat{h}_o + k_{oy}(x,0) = 0, \quad y = 0 \\ \lambda_o^2 k_o(x,0) + g_m\hat{h}_o + \frac{T}{\rho}\,\hat{h}_{oxx} = 0, \quad y = 0 \\ k_{ox},\ k_{oy} \to 0 \text{ as } y \to -\infty \\ k_o(0,y) = k_o(\Lambda,y) \qquad \hat{h}_o(0) = \hat{h}(\Lambda) \\ k_{ox}(0,y) = k_{ox}(\Lambda,y) \qquad \hat{h}_{ox}(0) = \hat{h}_{ox}(\Lambda). \end{cases} \tag{5.13}$$

This is the same problem solved above for the trivial solution, namely Eqs. (5.3), but with $g = g_m$. Thus, we have $\lambda_o^2 = 0$, $\hat{h}_o = -A\,\nu_m \cos \nu_m x$, $k_o = Ae^{\nu_m y} \cos \nu_m x$ with $\nu_m = \frac{2m\pi}{\Lambda}$, and A is an arbitrary constant.

First-Order Problem

DE: $\quad k_{1xx} + k_{1yy} = 0, \quad y < 0$

BC's
$$\begin{cases} \hat{h}_1 + k_{1y}(x,0) - \eta_{1x}k_{ox}(x,0) \\ \quad + \eta_1 k_{oyy}(x,0) = 0, \quad y = 0 \\ \alpha_1 k_o(x,0) + \lambda_o^2 \eta_1 k_{oy}(x,0) + \lambda_o^2 k_1(x,0) \\ \quad + g_m \hat{h}_1 + \frac{T}{\rho}\hat{h}_{1xx} = 0 \\ k_{1x}, k_{1y} \to 0 \text{ as } y \to -\infty \\ k_1(0,y) = k_1(\Lambda,y), \quad \hat{h}_1(0) = \hat{h}_1(\Lambda) \\ k_{1x}(0,y) = k_{1x}(\Lambda,y), \quad \hat{h}_{1x}(0) = \hat{h}_{1x}(\Lambda). \end{cases} \quad (5.14)$$

From the two interface conditions we have, substituting from the zero'th order problem, and for $\eta_1(x) = \sqrt{\frac{2}{\Lambda}} \cos \nu_m x$,

$$\begin{aligned} \hat{h}_1 + k_{1y}(x,0) &+ A\eta_{1x}\nu_m \sin \nu_m x + A\eta_1\nu_m^2 \cos \nu_m x \\ &= \hat{h}_1 + k_{1y}(x,0) - A\sqrt{\frac{2}{\Lambda}}\, \nu_m^2 \sin^2\nu_m x \\ &+ A\sqrt{\frac{2}{\Lambda}}\, \nu_m^2 \cos^2\nu_m x \\ &= \hat{h}_1 + k_{1y}(x,0) + A\sqrt{\frac{2}{\Lambda}}\, \nu_m^2 \cos 2\nu_m x, \end{aligned} \quad (5.15a)$$

and

$$\alpha_1 A \cos \nu_m x + \frac{T}{\rho}\nu_m^2 \hat{h}_1 + \frac{T}{\rho}\hat{h}_{1xx} = 0. \quad (5.15b)$$

The latter equation is a differential equation to be solved subject to the periodicity conditions of Eqs. (5.14). Because $A \cos \nu_m x$ solves the homogeneous case in Eqs. (5.15b), it is necessary in order for a solution to exist that $\alpha_1 = 0$. Recalling the decomposition $L_2(0,\Lambda) = M_m \oplus M_m^\perp$ of Sec. IV where M_m is the eigenspace associated with eigenvalue g_m, and $M_m^\perp$ is the orthogonal complement that coincides with the range of the operator $(\cdot)_{xx} + \frac{g_m\rho}{\Lambda}\cdot$ (which makes its appearance in Eq. (5.15b) because $g_m = \frac{T}{\rho}\frac{2m\pi}{\Lambda} = \frac{T}{\rho}\nu_m$), we suppose that $\hat{h}_o = -A\nu_m \cos \nu_m x$ constitutes the total component of the expansion in Eqs. (5.11) for $\hat{h}(x)$ in M_m. A is an arbitrary constant and M_m is spanned by $\cos \nu_m x$ if we fix the phase as in Eq. (3.6). Thus, we seek $\hat{h}_1$ in $M_m^\perp$, and the only solution is $\hat{h}_1 \equiv 0$.

To find k_1, we have from the first interface condition Eqs. (5.15a),

$$k_{1y}(x,0) = -A\sqrt{\frac{2}{\Lambda}}\, \nu_m^2 \cos 2\nu_m x,$$

which with the DE: $k_{1xx} + k_{1yy} = 0$ and the periodicity condition yields

$$k_1(x,y) = -\frac{A}{2}\sqrt{\frac{2}{\Lambda}}\, \nu_m e^{2\nu_m y} \cos 2\nu_m x + C(x),$$

where $C(x)$ must be harmonic, periodic, and therefore a constant.

Mainly we are interested in how λ^2 behaves when $\varepsilon \neq 0$. With $\alpha_1 = 0$ as determined above, we have as yet no information.

Second-Order Problem

DE: $\quad k_{2xx} + k_{2yy} = 0$

BC's
$$\begin{cases} \hat{h}_2 + k_{2y}(x,0) - \eta_{1x}k_{1x}(x,0) + \eta_1 k_{1yy}(x,0) \\ \quad - \eta_1\eta_{1x}k_{oxy}(x,0) + \frac{1}{2}\eta_1^2 k_{oyyy}(x,0) = 0 \\ \alpha_2 k_o(x,0) + \alpha_1 k_1(x,0) + \alpha_1\eta_1 k_{oy}(x,0) \\ \quad + \lambda_o^2 k_2(x,0) + \lambda_o^2 \eta_1 k_{1y}(x,0) \\ \quad + \lambda_o^2 \eta_2 k_{oy}(x,0) + \lambda_o^2 k_{oyy}(x,0) + \frac{T}{\rho}\nu_m^2 \hat{h}_2 \\ \quad - \frac{3}{8}\frac{T}{\rho}\frac{2}{\Lambda}\nu_m^4 \hat{h}_o + \frac{T}{\rho}\hat{h}_{2xx} \\ \quad - \frac{3}{2}\frac{T}{\rho}\eta_{1x}^2 \hat{h}_{oxx} - 3\frac{T}{\rho}\eta_{1x}\eta_{1xx}\hat{h}_{ox} = 0 \\ k_{2x}, k_{2y} \to 0 \text{ as } y \to -\infty \\ k_2(0,y) = k_2(\Lambda,y), \quad \hat{h}_2(0) = \hat{h}_2(\Lambda) \\ k_{2x}(0,y) = k_{2x}(\Lambda,y), \quad \hat{h}_{2x}(0) = \hat{h}_{2x}(\Lambda). \end{cases} \quad (5.16)$$

With $\lambda_o^2 = 0$ and $\alpha_1 = 0$ as was determined from the first-order problem, the interface conditions become

$$\begin{aligned} \hat{h}_2 + k_{2y}(x,0) &- \eta_{1x}A\sqrt{\frac{2}{\Lambda}}\, \nu_m^2 \sin 2\nu_m x \\ &- 2\eta_1 A\sqrt{\frac{2}{\Lambda}}\, \nu_m^2 \cos 2\nu_m x \\ &+ A\eta_1\eta_{1x}\nu_m^2 \sin \nu_m x + \frac{1}{2}A\eta_1^2\nu_m^3 \cos \nu_m x = 0, \end{aligned}$$

and

$$\alpha_2 \ell \cos \nu_m x + \frac{T}{\rho} \nu_m^2 \hat{h}_2 + \frac{3}{8} \frac{T}{\rho} \nu_m^5 A \cos \nu_m x + \frac{T}{\rho} \hat{h}_{2xx}$$

$$- \frac{3}{2} \frac{T}{\rho} \eta_{1x}^2 \nu_m^3 A \cos \nu_m x - 3 \frac{T}{\rho} \eta_{1x} \eta_{1xx} \nu_m^2 A \sin \nu_m x = 0.$$

It is the latter relationship that enables us to determine α_2. We rearrange it as follows.

$$\hat{h}_{2xx} + \nu_m^2 \hat{h}_2 = -\alpha_2 A \frac{\rho}{T} \cos \nu_m x - \frac{3}{8} A \frac{2}{\Lambda} \nu_m^5 \cos \nu_m x$$

$$+ \frac{3}{2} \eta_{1x}^2 \nu_m^3 A \cos \nu_m x + 3 \eta_{1x} \eta_{1xx} \nu_m^2 A \sin \nu_m x.$$

With $\eta_1(x) = \sqrt{\frac{2}{\Lambda}} \cos \nu_m x$, this becomes

$$\hat{h}_{2xx} + \nu_m^2 \hat{h}_2 = -\alpha_2 A \frac{\rho}{T} \cos \nu_m x - \frac{3}{8} A \frac{2}{\Lambda} \nu_m^5 \cos \nu_m x$$

$$+ \frac{3}{2} \nu_m^5 A \frac{2}{\Lambda} \sin^2 \nu_m x \cos \nu_m x + 3 \nu_m^5 A \frac{2}{\Lambda} \sin^2 \nu_m x \cos \nu_m x$$

$$= A(-\alpha_2 \frac{\rho}{T} + \frac{3}{4} \frac{2}{\Lambda} \nu_m^5) \cos \nu_m x - \frac{9}{8} A \frac{2}{\Lambda} \nu_m^5 \cos 3 \nu_m x. \qquad (5.17)$$

Equation (5.17) is a differential equation to be solved for $\hat{h}$ with the periodicity condition given in Eqs. (5.16). The condition for a solution to exist, of course, is that the right side will have no component in the linear eigenspace corresponding to eigenvalue ν_m, (i.e., M_m). This is the case provided that $\alpha_2 = \frac{3}{4} \frac{T}{\rho} \frac{2}{\Lambda} \nu_m^5$. Thus $\alpha_2 > 0$.

With reference to the expansions in Eqs. (5.11), we have found that $\alpha_1 = 0$ and $\alpha_2 > 0$. Then for $\epsilon \neq 0$, but small, it is clear that $\lambda^2 = \lambda_m^2 > 0$. The behavior of the eigenvalue λ_m, which determines timewise exponential growth along the m'th branch and which indicates instability, is illustrated as follows (see Fig. 17).

Thus for $\epsilon = 0$, the λ values for $n \geq m$ are on the imaginary axis of the λ plane with a double eigenvalue at the origin. As soon as $\epsilon \neq 0$, there exists an eigenvalue λ_m in the right half plane.

The above arguments indicate that the branches of nontrivial solutions [Eqs. (4.6)], m = 1, 2, ... , representing steady-state interface configurations are unstable. We have $\lambda_m^2 > 0$, and a fortiori Re $\lambda_p^2 > 0$, p = 1, 2, ... , m-1. Thus, the m'th branch has m degrees of instability, m = 1, 2, ...

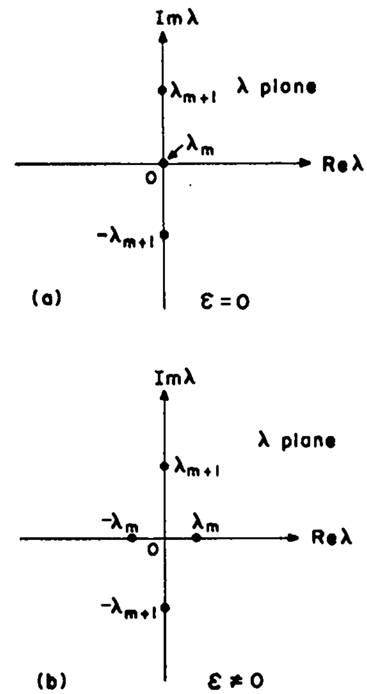

Fig. 17. Illustration suggesting configuration of eigenvalues in discussing stability of the m'th branch interfaces.

These conclusions rest on our ability to show that the expansions in Eqs. (5.11) are convergent. Expansions [Eqs. (4.1)], as we have seen, are convergent power series in ϵ in a neighborhood of $\epsilon = 0$, and thus represent analytic functions. These are the series used for the functions $\eta_o(x)$ and g in Eqs. (5.10). The question now asked is whether λ^2, $\hat{h}$, and k, appearing as unknowns in Eqs. (5.10), are also analytic functions in some neighborhood of $\epsilon = 0$, having been given that $\eta_o(x)$ and g are analytic in ϵ. If λ^2, $\hat{h}$, and k are analytic functions of ϵ in a neighborhood of $\epsilon = 0$, then certainly we may expand them in the convergent series of Eqs. (5.11).

We can reason as follows. The differential equation and first interface condition, together with boundary conditions, in Eqs. (5.10) give $k(x,y) = G\hat{h}(x)$, where G is the integral operator formed with a Green's function. Then $\hat{h}(x) = G^{-1}k(x,\eta_o(x))$, where G^{-1} is unbounded but closed (Ref. 14, p.300). This is then inserted into the second interface condition of Eqs. (5.10) to give

$$\lambda^2\hat{h}(x) = -G^{-1}\left[g\cdot -3\,\frac{T}{\rho}\,\frac{\eta_{ox}\eta_{oxx}}{(1+\eta_{ox}^2)^{\frac{5}{2}}}\,\frac{d}{dx}\,\cdot + \frac{T}{\rho}\,\frac{1}{(1+\eta_{ox}^2)^{\frac{3}{2}}}\,\frac{d^2}{dx^2}\,\cdot \right]\hat{h}(x), \tag{5.18}$$

where on the right, in consideration of boundary conditions, we have a product of closed linear operators. We find, moreover, that the closed operator on the right in Eq. (5.18) is analytic in $\varepsilon = 0$ with domain independent of ε. Thus, we invoke theorems of Kato on holomorphic femilies of closed unsymmetric operators (Ref. 16, pp. 375-379]. There exists a neighborhood of $\varepsilon = 0$ in which eigenvalues λ^2 and eigenfunctions $\hat{h}(x)$ are analytic in ε (Ref. 16, p. 379, remark 2.9). Then one may see that $k(x,y) = G\hat{h}(x)$ is analytic near $\varepsilon = 0$. Thus we have at least an indication of proof that λ^2, $\hat{h}(x)$, and $k(x,y)$ are developable in convergent ε-power series in some neighborhood of $\varepsilon = 0$.

Then employing a priori convergent power series as expansions of Eqs. (5.11), we calculate the first few coefficients as needed, and as was done in the stability analysis.

We can summarize as follows.

Theorem 5: The m'th branch of nontrivial steady-state solutions of Eqs. (3.1), which bifurcates to the left from the trivial solutions at $g = g_m = \frac{T}{\rho}\nu_m^2$, and is represented by the convergent expansion [Eq. (4.6)], has m degrees of instability. Thus, the variational problem of Eqs. (5.10), where we let g, η_o be the expansions of Eqs. (4.6), has m eigenvalues λ with positive real parts, when $\varepsilon \neq 0$.

If we knew enough about the total collection of eigenvalues and eigenfunctions of Eqs. (5.10) to be able to write the solution in a form similar to Eqs. (5.9), these eigenvalues with positive real parts would represent modes with amplitudes growing exponentially with time.

The perturbation method, unfortunately, doesn't yield any information about the variation of the higher eigenvalues of Eqs. (5.10), nemely, λ_p, $p > m$. For $\varepsilon = 0$ they are on the imaginary axis. Because we have just shown that positive real parts of $\lambda_1, \lambda_2, \ldots, \lambda_m$ result in instability of the m'th branch, we are not unduly worried here that we have been unable to prove that λ_p, $p > m$ do not develop positive real parts when $\varepsilon \neq 0$. It appears likely that λ_p, $p > m$, does remain imaginary, however, and that λ_p, $p < m$, remains real.

From general considerations relating to nonlinear operators, λ_m, once positive for small ε [i.e., for (g,η) near $(g_m,0)$], remains positive along the entire branch. This is because $\lambda_m = 0$ would represent a bifurcation situation in which the branch would either split or cease evolving to the left. For this we can refer the reader to some of the author's earlier papers.[17-20] We studied the branches in some detail in Sec. III, however, and found no such secondary bifurcations. Hence, once these branches start out unstable, they remain unstable.

Putting Theorems 4 and 5 together we can draw the following further conclusions. For $0 < g < g_1$ the trivial solution representing the plane interface is stable. The trivial solution, on the other hand, is unstable for $g_1 < g < g_2$ since $\lambda_1 > 0$ in that interval. The first branch of nontrivial solutions, Eq. (4.6) for $m = 1$, bifurcates to the left at $g = g_1 = \frac{T}{\rho}\nu_1^2$ and exists in the interval $0<g<g_1$. Here again we have $\lambda_1 > 0$. The first branch of nontrivial solutions seems to continue in $0 < g < g_1$, the type of instability possessed by the trivial solution in $g_1 < g < g_2$. The stability possessed by the trivial solution in $0 < g < g_1$ simply disappears at $g = g_1$.

Again for $g_{m-1} < g < g_m$ the trivial solution representing the plane interface has m-1 degrees of instability, meaning that $\lambda_1, \ldots, \lambda_{m-1}$ are positive. In the next interval, $g_m < g < g_{m+1}$ we add λ_m to the collection of positive eigenvalues. The m'th branch [Eqs. (4.6)] of nontrivial solutions bifurcates to the left at $g = g_m = \frac{T}{\rho}\nu_m^2$ and continues into $g_{m-1} < g < g_m$, and indeed for all $g < g_m$, the stability properties possessed by the trivial solution in the interval $g_m < g < g_{m+1}$; namely, Eqs. (5.10) have the m eigenvalues with positive real parts when expansions [Eqs. (4.6)] are substituted for g and η_o, and there are m degrees of instability. This holds for $m = 2, 3, 4, \ldots$, etc. A consistent picture emerges of the instability of these branches.

VI. ARC-LENGTH FORMULATION

Pursuant to a suggestion by Karl Gustafson and J. Wolkowisky of the University of Colorado, and Bergen R. Suydem, LASL Group, T-6, at a LASL meeting, we reformulate the Taylor Problem of Superposed Fluids so that arc length s along the interface curve is used instead of the horizontal abscissa x as a space variable. The immediate motive for doing this is the occurrence of multiple-valued solutions of the steady-state problem as deduced in Sec. III. These functions would be single-valued if arc length were the independent variable.

Drawbacks of an arc-length formulation are the need to restrict to the two-dimensional case, and the fact that with arc length, if one desires a domain of fixed length for the problem, the actual linear dimension must shrink as more of the arc-length fixed interval is used up in deviations from the trivial solution. This last point, however, is interesting from the standpoint of a clindrical or spherical implosion geometry where the interface does actually shrink in circumference.

First let us consider Eqs. (3.2). To convert to arc-length form, we make the substitution

$$s = \int_0^x \sqrt{1 + \eta_{x'}^2}\, dx' .$$

We have

$$\eta_{xx} = \eta_{xs}\frac{ds}{dx} = \eta_{xs}\sqrt{1+\eta_x^2}$$

$$= (\eta_s \frac{ds}{dx})_s \sqrt{1+\eta_x^2}$$

$$= \eta_{ss}\,(1+\eta_x^2) + \eta_s \frac{d}{ds}\frac{ds}{dx}\sqrt{1+\eta_x^2} .$$

But for $\eta_s^2 < 1$ we can write

$$\frac{ds}{dx} = \sqrt{1+\eta_x^2} = \sqrt{1 + (\eta_s \frac{ds}{dx})^2}$$

$$= \sqrt{1 + \eta_s^2(1+\eta_x^2)} = \sqrt{1 + \eta_s^2(1+\eta_x^2(1+\eta_x^2))}$$

$$= \sqrt{\sum_{i=0}^{\infty} \eta_s^{2i}} = \frac{1}{\sqrt{1-\eta_s^2}}$$

so that

$$\frac{d}{ds}\frac{ds}{dx} = \frac{d}{ds}\frac{1}{\sqrt{1-\eta_s^2}} = \frac{\eta_s\eta_{ss}}{(1-\eta_s^2)^{\frac{3}{2}}}$$

and

$$\eta_{xx} = \frac{\eta_{ss}}{1-\eta_s^2} + \frac{\eta_2}{\sqrt{1-\eta_s^2}}\,\frac{\eta_s\eta_{ss}}{(1-\eta_s^2)^{\frac{3}{2}}}$$

$$= \frac{\eta_{ss}}{1-\eta_s^2}\left[1 + \frac{\eta_s^2}{1-\eta_s^2}\right] = \frac{\eta_{ss}}{(1-\eta_s^2)^2} .$$

Substituting into the differential equation of the problem in Eqs. (3.2), we get

$$\frac{T}{\rho}\,\frac{\eta_{ss}}{\pm(1-\eta_s^2)^2}\,(1-\eta_s^2)^{\frac{3}{2}} + g\eta = 0.$$

Thus, Eqs. (3.2) become, in arc length form,

$$\frac{\eta_{ss}}{\pm\sqrt{1-\eta_s^2}} + \omega^2\eta = 0, \qquad \omega = \sqrt{\frac{\rho g}{T}} ,$$

$$\eta(0) = \eta(\Lambda_a), \quad \text{and} \quad \eta_s(0) = \eta_s(\Lambda_a), \tag{6.1}$$

where of course Λ_a is such that

$$\Lambda = x(\Lambda_a) = \int_0^{\Lambda_a} \sqrt{1-\eta_{s'}^2}\, ds', \qquad \Lambda_a \text{ fixed},$$

$$\Lambda = \Lambda(\eta). \tag{6.2}$$

Proceeding in a manner similar to that in Sec. III, we arrive at a family of closed curves in the (η,η_s) phase plane given by

$$\mp\sqrt{1-\eta_s^2} + \frac{\omega^2\eta^2}{2} = C, \tag{6.3}$$

where C is a real paremeter. The curves of Fig. 18a are for $-1 < C < 0$, and we choose the (-) sign in Eq. (6.3) indicating that the arc length grows in direct proportion to x. The curves of Fig. 18b are for $0 < C \le 1$, and the (-) is chosen in Eq. (6.3) for $|\eta| > \sqrt{\frac{2C}{\omega}}$ while the (+) is the proper choice when $|\eta| < \sqrt{\frac{2C}{\omega}}$ indicating that arc length grows while x decreases. Figure 18b corresponds to a multiple value function of x (Ref. 13, pp.207-209).

The expression for the period of the cycles illustrated by Fig. 18 is

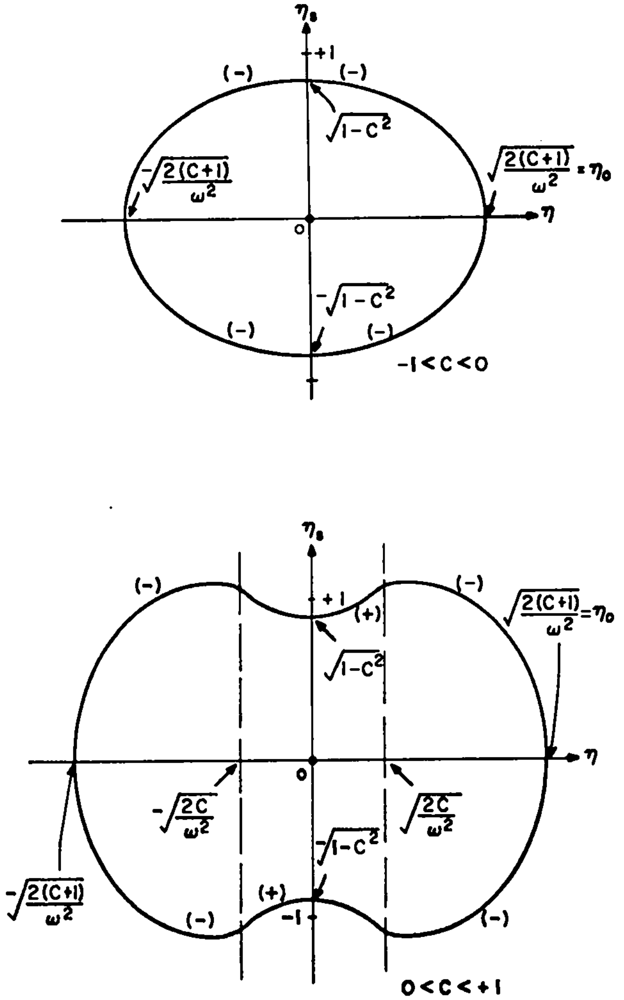

Fig. 18. Phase plane diagrams illustrating solution of Eq. (6.3).

$$P = 4\int_0^{\sqrt{\frac{2}{\omega^2}(C+1)}} \frac{d\eta}{\sqrt{1-\left(\frac{\omega^2\eta^2}{2}-C\right)^2}} = \frac{4}{\omega}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-h\cos^2\theta}}$$

$$= \frac{4}{\omega}\,\bar{f}(h)\ \varepsilon\uparrow ,$$

where, as before, we have $C = -1 + 2h$ with $h = \frac{1}{4}\omega^2\eta_o^2$.

To match periods, we solve graphically the equation $\frac{\Lambda_a}{m} = \frac{4}{\omega}\bar{f}(h)$, or $\omega\Lambda_a = 4m\bar{f}(h)$, so as to obtain those values of h which, when used in the initial data

$$\eta(0) = \frac{2\sqrt{h}}{\omega}, \qquad \eta_s(0) = 0$$

for the differential equation in Eqs. (6.1), produce solutions of Eqs. (6.1) periodic in s of period Λ_a (see Fig. 19).

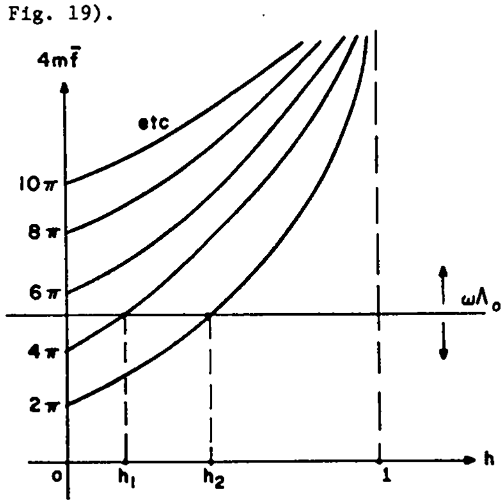

Fig. 19. Intersections yielding solutions of the boundary value problem in Eqs. (6.1).

Letting $\omega = \sqrt{\frac{g\rho}{T}}$, or g itself, vary, by inspection we get rightward bifurcating branches of solutions, $g_m = \frac{T}{\rho}\left(\frac{2m\pi}{\Lambda_a}\right)^2$ being the bifurcation points. Because these branches bifurcate from the trivial solution $\eta(s) \equiv 0$, we see from Eq. (6.2) that $\Lambda = \Lambda_a$, and the bifurcation points are the same as with the formulation of Sec. III.

We can also solve Eq. (6.1) (in the small) by using the perturbation series method of Sec. IV. The method proceeds analogously, and the convergence proof is similar. There results

$$\eta(s) = \sqrt{\frac{2}{\Lambda_a}}\ \varepsilon\cos\frac{2m\pi}{\Lambda_a}s + \frac{\varepsilon^3}{64}\left(\frac{2}{\Lambda_a}\right)^{\frac{3}{2}}\left(\frac{2m\pi}{\Lambda_a}\right)^2\cos\frac{2m\pi}{\Lambda_a}s + O(\varepsilon^5)$$

$$g = \frac{T}{\rho}\left(\frac{2m\pi}{\Lambda_a}\right)^2 + \frac{1}{8}\frac{T}{\rho}\left(\frac{2m\pi}{\Lambda_a}\right)^4\left(\frac{2}{\Lambda_a}\right)\varepsilon^2 + O(\varepsilon^4), \tag{6.4}$$

which, in contrast to Eqs. (4.6), indicates bifurcation to the right.

Evidently the rightward bifurcating solution branches remain bounded and even fall back to zero as $g \to \infty$. Indeed since $\eta(0) = \max_{0\le x\le\Lambda_a}\eta(x) = \frac{2\sqrt{h}}{\omega} = 2\frac{\sqrt{hT}}{\sqrt{g\rho}}$, and the appropriate values of h resulting from the graphical method of Fig. 19 are positive and less than unity, we have

$$\eta(0) \le 2\sqrt{\frac{T}{g\rho}} \to 0 \text{ as } g \to \infty.$$

Thus, in the limit, the initial data for the differential equation in Eqs. (6.1) becomes trivial.

We infer the interface shapes from Fig. 18. Points on the locus where $\eta_s = \pm 1$ correspond to points on the η vs x plot where $\eta_x = \pm\infty$, i.e., where the curve has a vertical tangent. For $g = \frac{T}{\rho}\left(\frac{4m\overline{f}(\frac{1}{2})}{\Lambda_a}\right)^2 > \frac{T}{\rho}\left(\frac{2m\pi}{\Lambda_a}\right)^2 = g_m$, the zeros have vertical slopes. For yet greater g, the η vs x plot becomes multiple valued, as in Fig. 12c[except that the domain $0 \le x \le \Lambda$ of the problem shrinks with constant arc-length interval $0 \le s \le \Lambda_a$ in accordance with Eq. (6.2)].

Apparently there is again a g-value where the bubble pinches off similarly to Fig. 12d, but with shrunken domain. The h intercept value where this would happen would be a solution of the equation $\overline{f}_2(h) = 2\overline{f}_1(h)$, where

$$\overline{f}_1(h) = \int_0^{\cos^{-1}\frac{1}{\sqrt{2h}}} \frac{d\theta}{\sqrt{1-h\cos^2\theta}} \quad \epsilon \uparrow,$$

$$\overline{f}_2(h) = \int_{\cos^{-1}\frac{1}{\sqrt{2h}}}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-h\cos^2\theta}},$$

are defined on $\frac{1}{2} \le h < 1$, $\overline{f}_1(h) \to \infty$ as $h \to 1$, and $\overline{f}_2(h)$ is bounded. Letting h^* be such a value ($h^* > \frac{1}{2}$), the g value is

$$g^* = \frac{T}{\rho}\left(\frac{4m\overline{f}(h^*)}{\Lambda_a}\right)^2 > \frac{T}{\rho}\left(\frac{4m\overline{f}(\frac{1}{2})}{\Lambda_a}\right)^2 > \frac{T}{\rho}\left(\frac{4m\pi}{\Lambda_a}\right)^2 = g_m.$$

It is to be noted that the preceding remarks about infinite slopes, and a bubble pinching off, relate to the η vs x plot, which is given parametrically by $\eta(s)$ and $x(s) = \int_0^s \sqrt{1-\eta_s^2}\, ds'$. The eigensolution $\eta(s)$ of Eqs. (6.1), which is a member of a rightward bifurcating continuous branch, is a twice differentiable bounded single-valued function of the arc-length s for all $g > g_m$. For $g > g^*$ it corresponds to a multiple-valued function of x that probably is devoid of any physical meaning. As a function of s, however, the branch functions are decently defined.

In the above arc-length considerations, we started directly with Eqs. (6.1), which is the arc-length analog of problem in Eqs. (3.2). It was relatively easy to convert Eqs. (3.2) to arc-length form. Now we attempt to write the basic problem [Eqs. (3.1)] in arc-length form with a view to linearization and stability analysis.

First, a preliminary. In Fig. 6, the interface shape $\eta(x)$ is represented as the lower boundary of the flow region D and is a function of the horizontal coordinate x. In the proof of Theorem 1, the vector form of the interface $[x,\eta(x)]$ is a parameterization by x. $[1,\eta_x(x)]$ is the paremeterized tangent vector along the interface, and $(-\eta_x(x),1)$ is the outward normal. The outward normal derivative of the potential ϕ is $\frac{\partial\phi}{\partial\vec{n}} = -\eta_x\phi + \phi_y$.

In arc-length form, Eqs.(3.1) become (see Fig.20)

$$\text{DE:}\quad \phi_{xx} + \phi_{yy} = 0, \quad y < \eta(s),$$

$$\text{BC's}\begin{cases} \eta_t(t,s) - \dfrac{\eta_s(t,s)}{\sqrt{1-\eta^2(t,s)}}\ \phi_x(t,x(s),\ \eta(t,s)) \\ \quad + \phi_y(t,x(s),\ \eta(t,s)) = 0 \\ \quad \underline{\text{or}}\ \eta_t + \dfrac{\partial\phi}{\partial\vec{n}} = 0 \\ \phi_t(t,x(s),\ \eta(t,s)) - \frac{1}{2}[\phi_x^2(t,x(s),\ \eta(t,s)) \\ \quad + \phi_y^2(t,x(s),\ \eta(t,s))] \\ \quad + g\eta(t,x) + \dfrac{T}{\rho}\dfrac{\eta_{ss}(t,s)}{\sqrt{1-\eta_s^2(t,s)}} = 0 \\ \phi_x,\ \phi_y \to 0 \text{ as } y \to -\infty \\ \eta(t,s+\Lambda_a) = \eta(t,s),\quad \phi(t,x(s+\Lambda_a),y) = \phi(t,x(s),y) \\ \eta_s(t,s+\Lambda_a) = \eta_s(t,s),\quad \phi_x(t,x(s+\Lambda_a),y) = \phi_x(t,x(s),y). \end{cases} \tag{6.5}$$

Initial conditions: $\phi(0,x,y)$ harmonic, and $\eta(0,s)$ both given,

$$x(s) = \int_0^s \sqrt{1-\eta_{s'}^2}\ ds'$$

$$\Lambda = x(\Lambda_a) = \int_0^{\Lambda_a} \sqrt{1-\eta_{s'}^2}\ ds'.$$

The latter equation relates Λ, the variable period in x, with Λ_a, the fixed period in s.

With the steady-state case we put $\eta_t = \phi_t \equiv 0$ where again, by the Gustafson-Wolkowisky reasoning of Theorem 1, we have $\phi(x,y) = \text{const}$. The steady-state problem thus reduces to Eq. (6.1) for a single autonomous differential equation. Right bifurcating solutions $\eta(s)$ are given by Eq. (6.3), where $\varepsilon = (\cos \frac{2m\pi}{\Lambda_a} s, \eta(s))$, where we use the scalar product of $L_2(0,\Lambda_a)$. The paremetric representation of the interface is given then by $(x(s), \eta(s))$ where $x(s) = \int_0^s \sqrt{1 - \eta_{s'}^2}\, ds'$.

Stability Analysis

We now linearize Eqs. (6.5) to produce the variational problem centered at a steady-state solution: $\phi(x,y) = \text{const}$, $\eta = \eta_o(s)$. We have (see Fig. 21)

DE: $k_{xx} + k_{yy} = 0, \qquad y < \eta_o(s)$

$$
\text{BC's}\begin{cases}
h_t(t,s) - \dfrac{\eta_{os}(s)}{\sqrt{1-\eta_{os}^2(s)}}\, k_x(t,x(s),\eta_o(s)) \\
\qquad + k_y(t,x(s),\eta_o(s)) = 0 \\
\underline{\text{or}}\ h_t(t,s) + \dfrac{\partial}{\partial \vec{n}} k(t,x(s),\eta_o(s)) = 0, \\
\qquad \vec{n} = \text{outward normal.} \\
k_t(t,x(s),\eta_o(s)) + gh(t,s) + \dfrac{T}{\rho}\dfrac{h_{ss}(t,s)}{\sqrt{1-\eta_{os}^2(s)}} \\
\qquad + \dfrac{T}{\rho}\dfrac{\eta_{os}(s)\eta_{oss}(s)}{(1-\eta_{os}^2(s))^{\frac{3}{2}}}\, h_s(t,s) = 0 \\
k_x,\ k_y \to 0 \text{ as } y \to -\infty \\
h(t,0) = h(t,\Lambda_a) \\
h_s(t,0) = h_s(t,\Lambda_a) \\
k(t,0,y) = k(t,x(\Lambda_a),y) \\
k_x(t,0,y) = k_x(t,x(\Lambda_a),y).
\end{cases}
\tag{6.6}
$$

We propose to find out initially if the first right-bifurcating branch is stable. To this end we expand the interface conditions of Eqs. (6.6) in Taylor series

$$
h_t + k_y(t,x(s),0) - \frac{\eta_{os}(s)}{\sqrt{1-\eta_{os}^2}}\, k_x(t,x(s),0)
$$

$$
+ \eta_o(s)k_{yy}(t,x(s),0)
$$

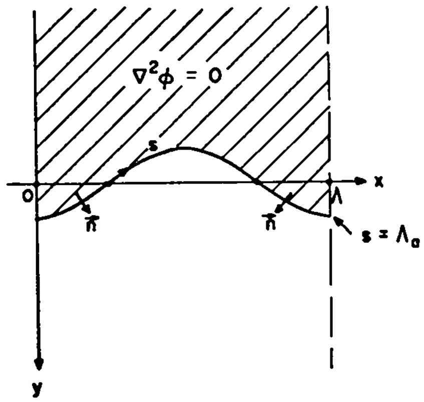

Fig. 20. Illustration helpful in converting the problem of Eqs. 3.1 to arc-length form.

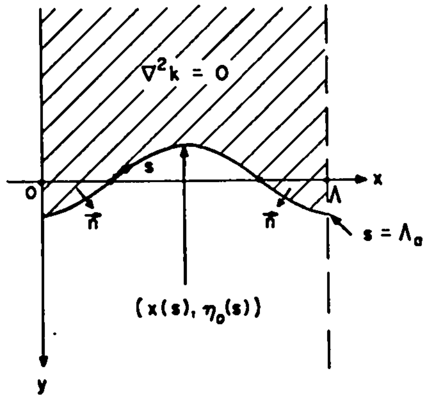

Fig. 21. Physical arrangement for discussing stability of the first branch interface in arc-length form.

$$
- \eta_o(s)\frac{\eta_{os}(s)}{\sqrt{1-\eta_{os}^2}}\, k_{xy}(t,x(s),0)
$$

$$
+ \tfrac{1}{2}\eta_o^2(s)k_{yyy}(t,x(s),0) + \ldots = 0
$$

$$
k_t(t,x(s),0) + gh(t,s) + \frac{T}{\rho}\, h_{ss}(t,s)
$$

$$
+ \eta_o(s)k_{ty}(t,x(s),0)
$$

$$
+ \tfrac{1}{2}\eta_o^2(s)k_{tyy}(t,x(s),0) + \tfrac{1}{2}\frac{T}{\rho}\,\eta_{os}^2(s)h_{ss}(t,s)
$$

$$
+ \frac{T}{\rho}\,\eta_{os}(s)\eta_{oss}(s)h_s(t,s) + \ldots = 0.
$$

This is justified for a harmonic function $k(t,x,y)$ with a harmonic extension across the interface $y = \eta_o(s)$.

Putting $h(t,s) = e^{\lambda t}h(s)$ and $k(t,x,y) = e^{\lambda t}k(x,y)$ as is standard procedure, and setting $\hat{h}(s) = \lambda h(s)$, we get an eigenvalue problem for $\hat{h}(s)$, $k(x,y)$, with eigenvalue paremeter λ^2.

$$\text{DE:}\quad k_{xx} + k_{yy} = 0, \quad y < 0$$

$$\text{BC's}\begin{cases} \hat{h}(s) + k_y(x(s),0) - \eta_{os}(s)k_x(x(s),0) \\ \quad + \eta_o(s)k_{yy}(x(s),0) \\ -\eta_o(s)\eta_{os}(s)k_{xy}(x(s),0) + \tfrac{1}{2}\eta_o^2(s)k_{yyy}(x(s),0) \\ -\tfrac{1}{2}\eta_{os}^3 k_x(x(s),0) - \tfrac{1}{2}\eta_o(s)\eta_{os}^3(s)k_{xy}(x(s),0) \\ \qquad\qquad + \dots = 0 \\ \lambda^2 k(x(s),0) + g\hat{h}(s) + \frac{T}{\rho}\hat{h}_{ss}(s) + \lambda^2\eta_o(s)k_y(x(s),0) \\ +\tfrac{1}{2}\lambda^2\eta_o^2(s)k_{yy}(x(s),0) + \tfrac{1}{2}\frac{T}{\rho}\eta_{os}^2(s)\hat{h}_{ss}(s) \\ +\frac{T}{\rho}\eta_{os}(s)\eta_{oss}(s)\hat{h}_s(s) + \dots = 0 \\ k_x,\ k_y \to 0 \text{ as } y \to -\infty, \end{cases} \tag{6.7}$$

with periodicity conditions

$$\hat{h}(0) = \hat{h}(\Lambda_a), \qquad k(0,y) = k(\Lambda,y),$$

$$\hat{h}_s(s) = \hat{h}_s(\Lambda_a), \text{ and } \quad k_x(0,y) = k_x(\Lambda,y),$$

where

$$\Lambda = x(\Lambda_a) = \int_0^{\Lambda_a} \sqrt{1 - \eta_{os'}^2}\, ds' \le \Lambda_a.$$

We propose to solve Eqs. (6.7) by means of a perturbation method, but we must first express $x(s) = \int_0^s \sqrt{1 - \eta_{os'}^2}\, ds'$ by means of a perturbation series based on the series already found for $\eta_o(s)$, nemely, Eqs. (6.3).

$$x(s) = \int_0^s [1 - \tfrac{1}{2}\eta_{os'}^2 - \tfrac{1}{8}\eta_{os'}^4 - \dots]\, ds'$$

$$= s - \frac{\varepsilon^2}{2}\left(\frac{2}{\Lambda_a}\right)\left(\frac{2\pi}{\Lambda_a}\right)^2 \int_0^s \sin^2 \frac{2\pi}{\Lambda_a} s' ds'$$

$$- \frac{\varepsilon^4}{64}\left(\frac{2}{\Lambda_a}\right)^2\left(\frac{2\pi}{\Lambda_a}\right)^3\left(\frac{6\pi}{\Lambda_a}\right)\int_0^s \sin\frac{2\pi}{\Lambda_a}s' \sin\frac{6\pi}{\Lambda_a}s' ds'$$

$$- \frac{\varepsilon^4}{8}\left(\frac{2}{\Lambda_a}\right)^2\left(\frac{2\pi}{\Lambda_a}\right)^4\int_0^s \sin^4\frac{2\pi}{\Lambda_a} s' ds' - \dots,$$

$$= s - \varepsilon^2 x_2(s) - \varepsilon^4 x_4(s) - \dots,$$

with known or determinable coefficients $x_i(s)$. In particular, we have

$$\Lambda = x(\Lambda_a) = \Lambda_a - \frac{\varepsilon^2}{2}\left(\frac{2\pi}{\Lambda_a}\right)^2 - \frac{3\varepsilon^2}{16}\frac{1}{\Lambda_a}\left(\frac{2\pi}{\Lambda_a}\right)^4 - \dots .$$

Then we can make the following expansion.

$$k_y(x(s),0) = k_y(s,0) - k_{xy}(s,0)x_2(s)\varepsilon^2 - \dots,$$

again valid if k_y is a harmonic function harmonically extended across the interface.

We now take this linear variational expanded boundary value problem, and substitute the perturbation series

$$\lambda^2 = \lambda_o^2 + \alpha_1\varepsilon + \alpha_2\varepsilon^2 + \dots,$$

$$\hat{h} = \hat{h}_o + \varepsilon\hat{h}_1 + \varepsilon^2\hat{h}_2 + \dots,$$

and

$$k = k_o + \varepsilon k_1 + \varepsilon^2 k_2 + \dots, \tag{6.8}$$

together with $x(s) = s - \varepsilon^2 x_2(s) - \varepsilon^4 x_4(s) - \dots$, and the series in Eqs. (6.4 a,b), which we have proved to be convergent. Here ε has the seme meaning as before.

Because for the trivial solution the arc-length formulation is the same as that in rectilinear coordinate x, we can infer from Sec. V that $\lambda_o^2 = 0$ $\hat{h}_o = -\frac{2\pi}{\Lambda_a} A \cos\frac{2\pi}{\Lambda_a} s$, and

$$k_o = Ae^{\frac{2\pi}{\Lambda_a}y} \cos\frac{2\pi}{\Lambda_a} x.$$

We again get the following linear problems.

First-Order Problem

DE: $k_{1xx} + k_{1yy} = 0, \quad y < 0$

BC's
$$\begin{cases} \hat{h}_1(s) + k_{1y}(s,0) - \eta_{1s}(s)k_{ox}(s,0) \\ \qquad + \eta_1(s)k_{oyy}(s,0) = 0 \\ \alpha_1 k_o(s,0) + g_1\hat{h}_1(s) + \frac{T}{\rho}\hat{h}_{1ss}(s) = 0 \\ \text{(where we do not include terms multiplied by } \lambda_o^2 = 0) \\ k_{1x}, k_{1y} \to 0 \text{ as } y \to -\infty \\ \hat{h}_1(0) = \hat{h}_1(\Lambda_a), \quad k_1(0,y) = k_1(\Lambda,y) \\ \hat{h}_{1s}(0) = \hat{h}_{1s}(\Lambda_a), \quad k_{1x}(0,y) = k_{1x}(\Lambda,y). \quad (6.9) \end{cases}$$

Now, $k_o(s,0) = A\cos\frac{2\pi}{\Lambda_a}s$. Hence since $g_1 = \frac{T}{\rho}\left(\frac{2\pi}{\Lambda_a}\right)^2$, the condition that $\hat{h}_1(s)$ exists is that $\alpha_1 = 0$. Using the convention of Sec. IV and V that $\hat{h}_o = -\frac{2\pi}{\Lambda_a}A\cos\frac{2\pi}{\Lambda_a}s$ constitute the entire component of expansion [Eqs. (6.8)] for $\hat{h}$ in the null space M_1 of the operator $(\cdot)_{ss} + \frac{g_1\rho}{T}(\cdot)$, we seek $\hat{h}_1$ in $M_1^{\perp}$. Then since $\alpha_1 = 0$, the solution for $\hat{h}_1$ unique in $M_1^{\perp}$ is $\hat{h}_1 \equiv 0$.

We find k_1 from the first interface condition of Eqs. (6.9) and Laplace's equation. We have

$$\begin{aligned} k_{1y}(s,0) &= -\left[\sqrt{\frac{2}{\Lambda_a}}\left(\frac{2\pi}{\Lambda_a}\right)\sin\frac{2\pi}{\Lambda_a}s\right]\left[-A\frac{2\pi}{\Lambda_a}\sin\frac{2\pi}{\Lambda_a}s\right] \\ &\quad - \left[\sqrt{\frac{2}{\Lambda_a}}\cos\frac{2\pi}{\Lambda_a}s\right]\left[A\left(\frac{2\pi}{\Lambda_a}\right)^2\cos\frac{2\pi}{\Lambda_a}s\right] \\ &= \sqrt{\frac{2}{\Lambda_a}}A\left(\frac{2\pi}{\Lambda_a}\right)^2\cos\frac{4\pi}{\Lambda_a}s, \end{aligned}$$

and also

$$k_{1y}(x,0) = \sqrt{\frac{2}{\Lambda_a}}A\left(\frac{2\pi}{\Lambda_a}\right)^2\cos\frac{4\pi}{\Lambda_a}x, \qquad (6.10)$$

using the same form. Then we solve the boundary value problem for $k_1(x,y)$ using the Laplace equation with boundary data [Eq. (6.10)], k_x, $k_y \to 0$ as $y \to -\infty$, and periodicity conditions in x.

Second-Order Problem

DE: $k_{2xx} + k_{2yy} = 0, \quad y < 0$

BC's
$$\begin{cases} \hat{h}_2(s) + k_{2y}(s,0) - \eta_{1s}(s)k_{1x}(s,0) + \eta_1(s)k_{1yy}(s,0) \\ \qquad -k_{oxy}(s,0)x_2(s) - \eta_1(s)\eta_{1s}(s)k_{oxy}(s,0) \\ \qquad + \tfrac{1}{2}\eta_1^2(s)k_{oyyy}(s,0) = 0 \\ \alpha_2 k_o(s,0) + g_1\hat{h}_2(s) + g_{12}\hat{h}_o(s) + \frac{T}{\rho}\hat{h}_{2ss}(s) \\ \qquad + \tfrac{1}{2}\frac{T}{\rho}\eta_{1s}^2(s)\hat{h}_{oss}(s) \\ \qquad + \frac{T}{\rho}\eta_{1s}(s)\eta_{1ss}(s)\hat{h}_{os}(s) = 0 \\ k_{2x}, k_{2y} \to 0 \text{ as } y \to -\infty \qquad (6.11) \end{cases}$$

(where we have not included terms multiplied by $\lambda_o^2 = \alpha_1 = 0$),

$$\hat{h}_2(0) = \hat{h}_2(\Lambda_a),$$

$$\hat{h}_{2s}(0) = \hat{h}_{2s}(\Lambda_a),$$

$$k_2(0,y) = k_2(\Lambda_a,y),$$

and

$$k_{2s}(0,y) = k_{2s}(\Lambda_a,y).$$

It is the second interface condition that we use to determine α_2. Rearranging and substituting the following previously determined expressions [see Eqs. (6.4)]

$$g_1 = \frac{T}{\rho}\left(\frac{2\pi}{\Lambda_a}\right)^2, \quad \eta_1(s) = \sqrt{\frac{2}{\Lambda_a}}\cos\frac{2\pi}{\Lambda_a}s,$$

$$\hat{h}_o = -\frac{2\pi}{\Lambda_a}A\cos\frac{2\pi}{\Lambda_a}s,$$

$$g_{12} = \frac{1}{8}\frac{T}{\rho}\left(\frac{2}{\Lambda_a}\right)\left(\frac{2\pi}{\Lambda_a}\right)^4, \quad k_o(x,y) = Ae^{\frac{2\pi}{\Lambda_a}y}\cos\frac{2\pi}{\Lambda}x,$$

we have

$$\hat{h}_{2ss} + \left(\frac{2\pi}{\Lambda_a}\right)^2 \hat{h}_2 =$$

$$- \frac{1}{8}\left(\frac{2}{\Lambda_a}\right)\left(\frac{2\pi}{\Lambda_a}\right)^4 \hat{h}_o - \alpha_2 \frac{\rho}{T} A \cos \frac{2\pi}{\Lambda_a} s$$

$$- \frac{1}{2}\left(\frac{2}{\Lambda_a}\right)\left(\frac{2\pi}{\Lambda_a}\right)^4 \left(\sin^2 \frac{2\pi}{\Lambda_a} s\right)\hat{h}_{oss}$$

$$- \frac{2}{\Lambda_a}\left(\frac{2\pi}{\Lambda_a}\right)^3 \left(\sin \frac{2\pi}{\Lambda_a} s \cos \frac{2\pi}{\Lambda_a} s\right)\hat{h}_{os}$$

$$= A \left[\frac{1}{8}\left(\frac{2}{\Lambda_a}\right)\left(\frac{2\pi}{\Lambda_a}\right)^5 - \alpha_2 \frac{\rho}{T} - \frac{3}{8}\left(\frac{2}{\Lambda_a}\right)\left(\frac{2\pi}{\Lambda_a}\right)^5\right] \cos \frac{2\pi}{\Lambda_a} s$$

$$+ \frac{3}{8}\left(\frac{2}{\Lambda_a}\right)\left(\frac{2\pi}{\Lambda_a}\right)^5 A \cos \frac{6\pi}{\Lambda_a} s \qquad (6.12)$$

to be solved with the periodic boundary conditions of Eqs. (6.11). The last reduction is accomplished with common trigonometric identities.

The necessary and sufficient condition that $\hat{h}_2$ exists is that the right side of Eq. (6.12) has no component in M_1, the null space of $(\cdot)_{ss} + \left(\frac{2\pi}{\Lambda_a}\right)^2 (\cdot)$. This means that we must have $\alpha_2 = -\frac{1}{4}\frac{T}{\rho}\left(\frac{2}{\Lambda_a}\right)\left(\frac{2\pi}{\Lambda_a}\right)^5 < 0$.

Thus, with $\lambda_o^2 = 0$, $\alpha_1 = 0$, but $\alpha_2 < 0$, we infer from Eqs. (6.8) that, at least for small enough $\varepsilon \neq 0$, $\lambda_1^2 < 0$. λ_1^2 is, of course, the lowest (in magnitude) or first eigenvalue of the variational problem [Eqs. (6.7)], with the first branch of Eqs. (6.1) substituted for $\eta_o(s)$, i.e., Eqs. (6.4) with m = 1. A fortiori we should have $\lambda_k^2 < 0$, k = 2, 3, ..., provided these numbers stay real.

How do we know that the higher-order eigenvalues λ_k^2, $k \geq 2$, of the variational problem [Eqs. (6.7)] stay real when we pass from $\varepsilon = 0$ to $\varepsilon \neq 0$?

We propose to find out by inserting, instead of the first expansion in Eqs. (6.8) into Eqs. (6.7) the expansion

$$\lambda^2 = \lambda_m^2 + \beta_1\varepsilon + \beta_2\varepsilon^2 + \dots , \qquad (6.13)$$

where $\lambda_m^2 = -\nu_m(\nu_m^2 - \frac{g_1\rho}{T})\frac{T}{\rho}$, $m \geq 2$, is the m'th eigenvalue for the variational problem with $\eta(s) \equiv 0$. When we do this, and equate coefficients of the same powers of ε in all the expressions that make up Eqs. (6.7), we see that the n'th order perturbation problem has the following form.

DE: $\quad k_{nxx} + k_{nyy} = 0$.

$$\text{BC's} \begin{cases} \hat{h}_n(s) + k_{ny}(s,0) + F(s) = 0 \\ \beta_n k_o(s,0) + \lambda_m^2 k_n(s,0) + g_1\hat{h}_n(s) \\ \quad + \frac{T}{\rho}\hat{h}_{nss}(s) + G(s) = 0 \\ k_{nx},\ k_{ny} \to 0 \text{ as } y \to -\infty \\ k_n(0,y) = k_n(\Lambda,y), \quad \hat{h}_n(0) = \hat{h}_n(\Lambda_a) \\ k_{nx}(0,y) = k_{nx}(\Lambda,y), \quad \hat{h}_{ns}(0) = \hat{h}_{ns}(\Lambda_a) \\ \text{where } \Lambda = x(\Lambda_a). \end{cases} \qquad (6.14)$$

Here F(s) and G(s) represent lower-order terms that are supposedly known when we are studying the n'th order problem. Also, we take as an induction assumption that F(s) and G(s) contain only real terms. We know that $k_o = Ae^{\nu_m y}\cos \nu_m s$, $\hat{h}_o(s) = -\nu_{am}A \cos \nu_{am}s$, where $\nu_m = \frac{2m\pi}{\Lambda}$, $\nu_{am} = \frac{2m\pi}{\Lambda_a}$, therefore we can start such an induction. Because these are the solutions of Eqs. (5.3), we have $\Lambda_a = \Lambda$.

We wish to determine if the coefficients β_n as determined by successive problems [Eqs. (6.14)] are indeed real.

Assume that $\hat{h}_n(s)$ above is known. For example, we could let it be $-\nu_m \cos \nu_{am}s$. We can then solve the boundary value problem

$$k_{nxx} + k_{nyy} = 0,\ y < 0,$$

$$k_{ny} = -\hat{h}_n(s) - F(s), \qquad k_x,\ k_y \to 0, \text{ as } y \to -\infty$$

and periodicity with period Λ. The result is

$$k_n(x,y) = -\frac{1}{\nu} e^{\nu_m y}\hat{h}_n(s) + \bar{k}(x,y),$$

where $\bar{k}(x,y)$ is a known harmonic function with $\bar{k}_y(s,0) = F(x)$ and

$$k_n(s,0) = -\frac{1}{\nu_{am}}\hat{h}_n(s) + \bar{k}(s,0).$$

We now have the D.E. and the first interface condition satisfied. For the second interface condition, substituting we get

$$+ \nu_{em}(\nu_{am}^2 - \frac{g_1\rho}{T}) \frac{T}{\rho} \cdot \frac{1}{\nu_{em}} \hat{h}_n(s) + \beta_n k_o(s,0)$$

$$+ g_1\hat{h}_n(s) + \frac{T}{\rho} \hat{h}_{nss}(s) + \lambda_n^2\overline{k}(s,0) + G(s) = 0$$

or

$$\frac{T}{\rho} \hat{h}_{nss}(s) + g_m\hat{h}_n(s) = -\beta_n k_o(s,0) - \lambda_m^2\overline{k}(s,0) - G(s).$$

A solution of the homogeneous equation is then

$$\hat{h}_n(s) = A \cos \nu_{am} s = k_o(s,0).$$

The necessary conditon for $\hat{h}_n(s)$ to exist then gives

$$\beta_n = \frac{-(\lambda_m^2\overline{k}(s,0) + G(s), k_o(s,0))}{(k_o, k_o)},$$

which is a real number.

We thus see that the coefficients in the expansion of Eq. (6.13) must be real, so that the higher-order eigenvalues of Eqs. (6.7) do stay real in the perturbation process. Then since $\lambda_m^2 < 0$ for $m \geq 2$, $\pm\lambda_m$ are pure imaginary.

We can summarize the results of this section as follows.

Theorem 6: Equations (6.5), which are Eqs. (3.1) but with arc length along the interface $\eta(x)$ used as an independent variable instead of x, and with fixed periodicity Λ_a along this arc-length variable, has continuous branches of nontrivial solutions bifurcating from the trivial solution at the discrete g values $g_m = \frac{T}{\rho}\left(\frac{2m\pi}{\Lambda_a}\right)^2$, m = 1, 2, 3, ... The m'th branch bifurcates at $g = g_m$. The bifurcation of the branches is to the right. The first branch, which starts at $g = g_1$, is stable to small perturbations. The trivial solution is stable in the interval $(0,g_1)$, and the first branch continues the seme stability into the region $g > g_1$. The shapes of the interface are as indicated in Fig. 12 except that the periodicity is $\Lambda = x(\Lambda_a)$. All solutions are single valued, however, as a function of the arc-length variable s. The branches return to the trivial solution as $g \to \infty$.

The stability of the first branch, as mentioned in the theorem, is in the sense that, with the solution of the variational problem given as sums of terms of the type

$$k(t,x,y) = e^{\lambda t}k(x,y),\ h(t,s) = e^{\lambda t}h(s),$$

the constants λ are conjugate imaginary numbers. This is "marginal stability" for which see the comment in connection with Fig. 16. (See Fig. 22)

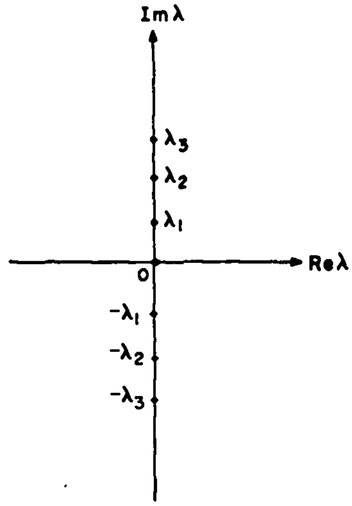

Fig. 22. Suggested eigenvalue configuration in the case of marginal stability.

We worked out these rightward-facing branches by using the curves of Fig. 19, and we know there is no secondary bifurcation, i.e., splitting or dividing. Thus, there can be no point on the branches where Eqs. (6.7) have a vanishing eigenvalue $\lambda^2 = 0$. (Otherwise we should have a bifurcation situation.) Hence on the first branch, since we start with $\lambda_1^2 < 0$, this condition persists and the branch remains stable (see Fig. 23).

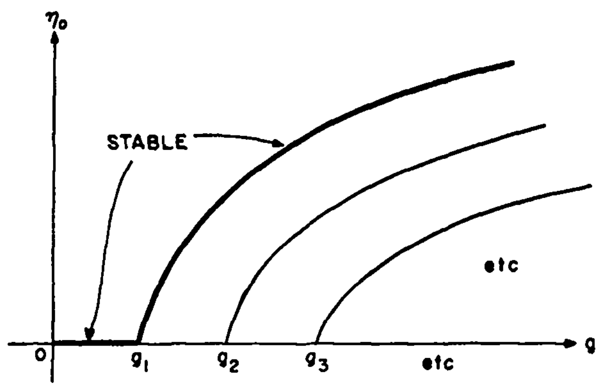

Fig. 23. Illustration of branches of suggested interface solutions for the arc-length paremetrized problem in Eqs. (6.1).

It can be shown that the second branch is stable except for one positive eigenvalue; thus $\lambda_1 > 0$, but $\lambda_2 < 0$, $\lambda_3 < 0$, and so on. Thus, the second branch has the same stability properties for

$g > g_2$ as the trivial solution does in $g_1 < g < g_2$. Similarly, this is true for the m'th branch.

This stability is with respect to perturbations of the seme base period Λ_a of arc length, or some quotient thereof by an integer. In other words, all considerations are with respect to the Hilbert space $L_2(0,\Lambda_a)$.

In a cylindrically or spherically symmetric assembly, we have a base period Λ_a imposed on any shell-like interface by the periodicity of the polar angle. Any perturbation is therefore in $L_2(0,\Lambda_a)$. This is a strictly geometrical consideration.

Thus does the arc-length formulation, with a fixed length Λ_a of arc in a period, differ from the formulation in terms of a rectilinear coordinate x with a fixed x-length Λ as period. We have branches of nontrivial solutions going to the right rather than to the left, and we have branches that are altogether more stable, as has been described. In fact, for the arc-length formulation, the first branch is actually stable to perturbations of period Λ_a, while for the rectilinear coordinate formulations, one already has one degree of instability to such perturbations on the first branch.

Similar, but easier, arguments could be used in Sec. V to show that λ_k^2, for $k > m$, stay real and negative when one passes (up the m'th branch) from $\varepsilon = 0$ to $\varepsilon \neq 0$. This is a point about which we expressed doubts in Sec. V, but which we were able to resolve in Sec. VI by using Eqs. (6.14).

Sattinger[21] has shown by topological methods for a certain type of nonlinear operator equation that nontrivial solutions bifurcating above criticality, i.e., $g = g_1$, are stable, whereas nontrivial solutions bifurcating below criticality are unstable. Sattinger's operator equation is not sufficiently general to cover our problems where compactness seems to be lacking. Yet it is interesting to observe that we have the seme result with our x-formulation vs our s-formulation.

VII. SUMMARY

It seems appropriate now to collect the results produced in this report for the Taylor Problem of Superposed Fluids, and then to discourse on further results that are needed and the possibility of getting them.

The results have been stated in the text as Theorems. These will be restated, with references to equations in the text where needed.

Theorem 1: In Eqs. (3.1), let $\phi_t = \eta_t = 0$ so that we have the steady-state problem. Then $\phi(x,y) \equiv$ const in the flow region of the heavy fluid. (We recall that the light fluid has vanishingly small density, as assumed in Sec. II, and the flow is inviscid, incompressible, and irrotational.)

Theorem 2: The steady-state problem, with periodic boundary conditons, reduces to Eqs. (3.2) for the interface shape alone. There exists a sequence of primary bifurcation points $\{g_m\}$, m = 1, 2, 3, ... , where $g_m = \frac{T}{\rho}\left(\frac{2m\pi}{\Lambda}\right)^2$, Λ being the assumed base period. Equations (3.2) have the trivial solution whatever the applied acceleration g. At each bifurcation point g_m, a continuous branch of nontrivial solutions appears, bifurcating to the left. These branches of solutions represent interface shapes for given values of g.

The evolution of these interface shapes on the first branch is portrayed in Fig. 12. The other branches evolve similarly except that the periods are integral divisors of the base period Λ.

The term "bifurcation" used in Theorem 2 is standard. The usage goes back to Poincare. See the heuristic Sec. V in the author's Lecture Notes (Ref. 20, p. 43). See also the rich bibliography provided in those notes (Ref. 20, p. 114-119), and the references in certain other papers of the author (Refs. 17 through 19).

Continuing, we have

Theorem 3: The bifurcated nontrivial solutions $(g,\eta(x))$ of Eqs. (3.2), belonging to the m'th branch m = 1, 2, 3, ... , and which represent interface shapes under steady-state conditons for given g, are analytic in terms of parameter $\varepsilon = (\eta(x),\eta_1(x))$. Here $\eta_1(x)$ is the normalized solution of linearized Eqs. (4.2), and $(\cdot,\cdot)$ is the inner product of the space $L_2(0,\Lambda)$. This parameter ε is the magnitude of the projection of $\eta(x)$ on the one-dimensional subspace spanned by $\eta_1(x)$. The analyticity in ε holds in some small neighborhood of the trivial solution. Then the solutions $(g,\eta(x))$ on the m'th branch can be developed in the convergent power series [Eqs. (4.1)] in which the first few coefficients are given by Eqs. (4.6).

Theorem 4: For $0 < g < g_1$, the trivial steady-state solution, $\phi(x,y) = \text{const}$, $\eta(x) \equiv 0$, is "marginally stable" in that λ_p is pure imaginary, $p = 1, 2, \ldots$. Here $\{\lambda_p\}$ is the discrete eigenvalue sequence of the linear variational problem [Eqs. (5.3)]. For $g_{m-1} < g < g_m$, $m = 2, 3, 4, \ldots$, the trivial solution is unstable with "m-1 degrees of instability" in that $\lambda_1, \ldots, \lambda_{m-1}$ are positive and furnish increasing exponential terms in Eqs. (5.9). Only λ_m, λ_{m+1}, etc., are pure imaginary. As g passes through the value g_{m-1}, the eigenvalues $\pm\lambda_{m-1}$, originally a conjugate imaginary pair, becomes a $\pm$ real pair.

Theorem 5: The m'th branch of nontrivial, leftward bifurcating steady-state solutions, which starts at $g = g_m = \frac{T}{\rho} \nu_m^2$, and is represented by the convergent expansions [Eqs. (4.6)], has "m degrees of instability." Thus, linear variational Eqs. (5.10), in which we let g and η_o be the expansions [Eqs. (4.6)], have m positive eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ when $\varepsilon \neq 0$, whereas the remaining eigenvalues λ_p are pure imaginary.

Section V gives the meaning of "marginally stable" in Theorem 4. In the same theorem, we note that the initial interval $0 < g \leq g_1$ of stability for the trivial solution (the plane interface) is caused by the presence of surface tension T.

In Theorem 5, we have indicated real or pure imaginary eigenvalues $\{\lambda_p\}$ as contrasted with the statement in Sec. V. We can get this refinement by appealing to the arguments of Sec. VI relative to Eqs. (6.14).

The last theorem summarizes Sec. VI in which all of the previous considerations are recast in terms of an arc-length variable. We use arc length along the one-dimensional interface as one of the space variables, replacing x. This enables us to evolve branches of eigenfunctions that will represent interface shapes which are single-valued functions. With x as independent variable, the double-valued functions, which eventually appear in the branches, are shown in Fig. 12. With arc length as variable, these can be specified as single-valued functions. If we specify a fixed periodicity in the arc-length variable, nemely Λ_a, we find, however, that the x-periodicity, namely Λ, shrinks as we evolve the branches; we have Λ expressed by Eq. (6.2). Yet this situation may be analogous to that which arises in a spherically symmetric or cylindrically symmetric geometry. Here a space period Λ_a is actually imposed by the geometry along the arc length of the interface.

With the arc-length formulation, we have the following.

Theorem 6: The problem in Eqs. (6.5), which are Eqs. (3.1) reformulated in terms of arc length and with fixed periodicity along this arc-length variable, has continuous branches of nontrivial solutions bifurcating from the trivial solution at the discrete values $g = g_m = \frac{T}{\rho}\left(\frac{2m\pi}{\Lambda_a}\right)^2$, $m = 1, 2, 3, \ldots$. The m'th branch bifurcates at $g = g_m$. In contrast with Theorem 2, the bifurcation of these branches is to the right. The first branch that starts at $g = g_1$ is stable to small perturbations of period Λ_a, or integer division thereof, Λ_a being the base period. The trivial solution is stable in the interval $(0,g_1)$, and the first branch assumes and continues this stability into the region $g > g_1$ (there is an "exchange of stabilities"). As functions of x, the shapes of the interfaces are as indicated in Fig. 12a through d, except that the periodicity is $\Lambda = x(\Lambda_a)$. All solutions are single valued in terms of the arc-length variables. The branches return to the trivial solution as $g \to \infty$.

One can envisage two main lines in which this work will be extended: steady state and time evolutionary. We have herein treated the steady-state problem of Superposed Flow under the assumptions of no viscosity, no compressibility, and no rotational flow. The steady-state problem should be treated with viscosity. Further, the problem should be treated with rigid wall boundary conditions, as is possible with viscosity. Multiple interface problems could be considered in the steady state. The steady-state cylindrically and spherically symmetric problems should certainly merit attention.

Again, the time evolutionary problem is of prime importance, and though we dealt with the steady-state case in this report, we did not do so apart from the time-evolutionary problem. In the introduction we tried to show how the steady state is a part of the time evolutionary problem by a simple exemple. When we studied stability, we were taking up the time-evolutionary problem, if in a special way. We have found solution branches which the time-varying solution is either attracted to, or is repelled from, as the case may be. Even a solution branch with "m degrees of instability" is attractive for initial data in subspaces of co-dimension m.

Our chief ambition is to solve and describe the solutions of the time-evolutionary problem, and to do so in a satisfying way. Several avenues present themselves. We shall ultimately handle this problem.

With reference to this report, we must decide which formulation is worth extending in the time-wise sense: the rectilinear variable formulation, or the arc-length variable, fixed arc-length period, formulation. This may await the study of the spherically symmetric steady-state case.

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REFERENCES

1. G. I. Bell, "Taylor Instability on Cylinders and Spheres in the Small Amplitude Approximation," LA-1321, Los Alamos Scientific Laboratory Report (1957).

2. H. W. Emmons, C. T. Chang, and B. C. Watson, "Taylor Instability of Finite Surface Waves," J. Fluid Mech. 7, 177 (1960).

3. D. J. Lewis, "The Instability of Liquid Surfaces when Accelerated in a Direction Perpendicular to Their Planes, II," Proc. Roy. Soc. A202, 81 (1950).

4. G. I. Taylor, "The Instability of Liquid Surfaces when Accelerated in a Direction Perpendicular to their Planes, I.," Proc. Roy. Soc. A201, 192 (1950).

5. J. J. Stoker, Water Waves (Interscience Publishing Co., New York, New York 1957), Chap. 1, p. 3.

6. O. A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow (Gordon and Breach, 2nd Ed., New York, New York, 1969), Chap.6, p.181.

7. R. Courant and D. Hilbert, Methods of Mathematical Physics II (Interscience Publishers, New York, New York 1962), Chap. IV, p. 252.

8. R. L. Kiang, "Nonlinear Theory of Inviscid Taylor Instability near the Cutoff Wavenumber," Phys. Fluids 12, 1333 (1969).

9. G. Birkhoff, "Taylor Instability and Laminar Mixing," LA-1862, Los Alamos Scientific Laboratory Report (1954); Appendixes, LA-1927 (1955).

10. K. O. Friedrichs, Methods of Mathematical Physics (New York University Notes, 1954), pg 23.

11. M. S. Berger, "On Von Karman's Equations and the Buckling of a Thin Elastic Plate," Commun. Pure Appl. Math. 20, 687 (1967).

12. L. Euler, Additementum I de curvis elasticis, Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes, Lausanne & Geneva, 1744.

13. G. E. F. Sherwood and A. E. Taylor, Calculus (Prentice-Hall, Inc., Englewood Cliffs, N. J., 3rd Ed., 1958), Chap. VII, pp. 207-209.

14. F. Riesz, and B. Sz.-Nagy, Functional Analysis (Frederick Ungar Publishing Co., New York, 1955).

15. S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability (Clarendon Press, Oxford, 1968), Chap. X, p. 428.

16. T. Kato, Perturbation Theory for Linear Operators (Springer-Verlag, New York, 1966), pp. 375-379.

17. G. H. Pimbley, Jr., "The Eigenvalue Problem for Sublinear Hammerstein Operators with Oscillation Kernels," J. Math. & Mech.12, 577 (1963).

18. G. H. Pimbley, Jr., "Positive Solutions of a Quadratic Integral Equation," Arch. Ratl. Mech. Anal. 24, 107 (1967).

19. G. H. Pimbley, Jr., "Two Conditions Preventing Secondary Bifurcation for Eigenfunctions of Sublinear Hammerstein Operators," J. Math. & Mech. 18, 787 (1969). Corrigendum, Indiana U. Math. J. 20, 337 (1970).

20. G. H. Pimbley, Jr., "Eigenfunction Branches of Nonlinear Operators and their Bifurcations," Springer-Verlag, in Lecture Notes in Mathematics, 104, Sec. 6 (1969). Corrigendum, Math. Zeitschrift 115, 231 (1970).

21. D. H. Sattinger, "Stability of Bifurcating Solutions by Leray-Schauder Degree," Arch. Ratl. Mech. Anal. 43, 154 (1971).

J. B. Keller and I. Kolodner, "Instability of Liquid Surfaces and the Formation of Drops," J. Appl. Phys. 25, 918 (1954).

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