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NOTE ON THE SHOCK COMPRESSION OF A γ-LAW GAS

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PHYSICS

ABSTRACT

The compression of a gas between a uniformly moving piston and a rigid wall is discussed in the one-dimensional case. Expressions are derived showing the relation between the density, pressure, temperature and entropy, respectively, produced behind the initial shock caused by the piston, and the values of these quantities after any number of reflections of the shock from piston or rigid wall. It is shown that the limiting value (as the number of reflections goes to infinity) of the quantity $S=pV^{\gamma}$ divided by its value behind the initial shock is given by

$$\frac{S_{\infty}}{S_{o}} = \frac{\left[\Gamma(a)\right]^{\gamma-1}}{a}, \qquad (1 < \gamma < \infty)$$

where $a=2\gamma/(\gamma-1)$, γ is the ratio of specific heats, p is the pressure and V is the specific volume of the gas.

1. INTRODUCTION

Consider the one-dimensional system consisting of a gas confined between two plane, parallel walls that are impenetrable to heat. One wall is considered to be rigid and the other to move toward the first with constant speed. The moving wall will initiate a shock in the gas. The shock will reflect at the rigid wall producing a second shock which will in turn be reflected at the moving wall, and so on. Using the Hugoniot equations for shocks, it is possible to compute the ratios of density and pressure behind the shock to their values in front of the shock after each successive reflection. It is then possible to compute the values of density and pressure behind the shock, after any number of reflections, in terms of their values behind the initial shock. Thus, the corresponding values of such thermodynamic variables as temperature and entropy can also be determined. Of particular interest is the computation of the limiting value of the entropy as the number of reflections goes to infinity. This latter quantity is derived as a function of the γ which characterizes the gas (cf 6).

2. PRESSURE RATIO AND COMPRESSION AFTER EACH SUCCESSIVE REFLECTION

Let ρ^o be the initial density of the gas and u_o the speed of the moving wall. Behind the initial shock started by the piston the density, ρ_o, material velocity, u, and pressure, p_o, are given by

$$
(1) \qquad \left.\begin{cases} \rho_o = \dfrac{\gamma+1}{\gamma-1}\,\rho^o , \\ \\ u = u_o , \\ \\ p_o = \dfrac{\gamma+1}{2}\,\rho^o u_o^2 , \end{cases}\right\} \text{(initial shock)}
$$

where γ is the ratio of specific heats of the gas. Let ρ_i, u_i, p_i be the density, material velocity and pressure, respectively, behind the shock after the i'th reflection, and let

$$
(2,a) \qquad \sigma_i = \frac{p_i}{p_{i-1}} ,
$$

$$
(2,b) \qquad \eta_i = \frac{\rho_i}{\rho_{i-1}} .
$$

The Hugoniot equations describing conservation of mass and momentum give in the present case

$$
(3) \qquad (u_{i-1} - u_i)^2 = (p_i - p_{i-1})\left(\frac{1}{\rho_{i-1}} - \frac{1}{\rho_i}\right).
$$

The boundary conditions are such that:

(a): If the shock is reflected from the rigid wall, $u_i=0$, $u_{i-1}=u_o$;

(b): If the shock is reflected from the moving wall, $u_i=u_o$, $u_{i-1}=0$.

So in either case the left hand side of (3) is u_o^2, and, substituting (2), it becomes

$$(\sigma_i - 1)(1 - \frac{1}{\eta_i}) = \frac{\rho_{i-1}}{p_{i-1}} u_o^2 .$$

Now let

$$\mu_i = \frac{\gamma - 1}{2} \frac{\rho_{i-1}}{p_{i-1}} u_o^2 ,$$

where

$$\mu_{i+1} = \frac{\eta_i}{\sigma_i} \mu_i, \quad \mu_1 = \frac{\gamma - 1}{2} \frac{\rho_o}{p_o} u_o^2 = 1,$$

and the above expression becomes

$$(\sigma_i - 1)(1 - \frac{1}{\eta_i}) = \frac{2}{\gamma - 1} \mu_i .$$

Assuming a γ-law for the gas; i.e., that the internal energy per unit mass after the i'th reflection is given by

$$E_i = \frac{1}{\gamma - 1} \frac{p_i}{\rho_i} ,$$

the Hugoniot equation for conservation of energy reduces to

$$\eta_i = \frac{\gamma - 1 + (\gamma + 1)\sigma_i}{\gamma + 1 + (\gamma - 1)\sigma_i} .$$

Thus σ_i and η_i, for any i, are completely specified by the expressions

$$(\sigma_i - 1)(1 - \frac{1}{\eta_i}) = (a-2)\mu_i \tag{4,a}$$

$$\eta_i = \frac{1+(a-1)\,\sigma_i}{a-1+\sigma_i} \tag{4,b}$$

where

$$a = \frac{2\gamma}{\gamma-1} \tag{4,c}$$

$$\mu_{i+1} = \frac{\eta_i}{\sigma_i}\mu_i\ , \qquad \mu_1 = 1\,. \tag{4,d}$$

Eliminating η_i from equations (4) there results

$$\sigma_i = \frac{1}{2}\left\{2+(a-1)\mu_i \pm \sqrt{\mu_i\left[4a+\mu_i(a-1)^2\right]}\ \right\} \tag{5}$$

The physical meaningful solution corresponds to a choice of the positive sign in front of the radical because, for example, with i=1, the other branch gives $\sigma_1=0$. The solutions for the + branch are found to be

$$\sigma_i = \frac{i+a}{i} \tag{6,a}$$

$$\begin{pmatrix} 1 < \gamma < \infty \\ \infty > a > 2 \end{pmatrix}$$

$$\eta_i = \frac{i+(a-1)}{i+1} \tag{6,b}$$

$$(i = 0,\ 1,\ 2,\ \ldots)$$

That (6) is the solution of (4) may be verified by induction as follows:

(1) For a given i, say i=1, (6,a) gives the correct result for σ_i, namely:

$$\sigma_i = 1+a = \frac{3\gamma-1}{\gamma-1} \quad .$$

(2) The equations (4) yield the following relation between σ_{i+1} and σ_i:

$$\frac{(\sigma_{i+1}-1)^2}{(\sigma_i-1)^2} = \frac{1}{\sigma_i}\,\frac{1+(a-1)\,\sigma_{i+1}}{a-1+\sigma_i} ,$$

and since (6,a) satisfies this relation it is proved that (6,a) is the correct expression for σ_i.

(3) It follows that (6,b) is correct since equations (6) satisfy (4,b)

3. PRESSURE RATIO AND COMPRESSION AFTER n REFLECTIONS

Let π_n and K_n be the pressure ratio and compression, respectively, after n reflections referred to the pressure p_o and density ρ_o behind the initial shock. Then

$$\text{(7,a)} \qquad \pi_n = \frac{p_n}{p_o} = \prod_{i=1}^{n} \sigma_i = \frac{(a+1)(a+2)\ldots(a+n)}{n!} \quad ,$$

$$(7,b) \qquad K_n = \frac{\rho_n}{\rho_o} = \prod_{i=1}^{n} \eta_i = \frac{a(a+1)\ldots(a+n-1)}{(n+1)!}, \quad (n = 1, 2, \ldots).$$

4. SHOCK SPEED AND TIME

Let m_n be the magnitude of the rate of mass flow per unit area across the shock after the n'th reflection. Then

$$m_n = \frac{|u_{n-1} - u_n|}{\frac{1}{\rho_{n-1}} - \frac{1}{\rho_n}} = \rho_{n-1} \frac{u_o}{1 - \frac{1}{\eta_n}}$$

$$= \frac{\rho_n}{\eta_n - 1} u_o .$$

Substituting (6,b) this becomes

$$m_n = \frac{u_o}{a-2}(n+1)\rho_n ,$$

or, using (1) and (7,b),

$$(8) \qquad \begin{cases} m_n = \dfrac{u_o \rho_o}{a-2} = \dfrac{a-1}{a-2} u_o \rho^o, \; n = 0, \\[2ex] m_n = \dfrac{u_o \rho_o}{a-2}(n+1) K_n = m_o (n+1) K_n, \quad n = 1, 2, \ldots . \end{cases}$$

Let M be the total mass per unit area of the gas in the system, and L the initial distance between the walls, then

$$M = \rho^o L.$$

The time Δt_i required for the shock to reach the opposite boundary after the (i-1)st reflection; i.e., the time interval between the (i-1)st and i'th reflections is given by

$$\Delta t_i = \frac{M}{m_{i-1}} = \frac{\rho^o L}{m_{i-1}} .$$

Substituting (8) and letting

$$T = \frac{L}{u_o}$$

be the time necessary for the moving wall to traverse the distance L, the above becomes

$$(9) \qquad \begin{cases} \Delta t_i = \dfrac{a-2}{a-1} T = \dfrac{2}{\gamma+1} T & (i = 1), \\ \Delta t_i = \dfrac{\Delta t_1}{i \, K_{i-1}} & (i = 2, 3, \cdots). \end{cases}$$

The time t_n at which the n'th reflection occurs is then

$$(10) \qquad t_n = \sum_{i=1}^{n} \Delta t_i = T \frac{a-2}{a-1} \left(1 + \sum_{i=2}^{n} \frac{1}{i \, K_{i-1}}\right).$$

As a check on the formulation, it will be shown that $\lim_{n \to \infty} \frac{t_n}{T} = 1$.

Substitute (7,b) into (10) and there results

$$\frac{t_\infty}{T} = \frac{a-2}{a-1}\left(1+\sum_{i=2}^{\infty}\frac{1}{i}\prod_{j=1}^{j=i-1}\frac{j+1}{j+a-1}\right) = \frac{a-2}{a-1}\left(1+\sum_{i=2}^{\infty}\frac{(i-1)!}{a(a+1)\cdots(a+i-2)}\right).$$

Let $n = i-1$, and this becomes

$$\frac{t_\infty}{T} = \frac{a-2}{a-1}\left(1+\sum_{n=1}^{\infty}\frac{n!}{a(a+1)\cdots(a+n-1)}\right) = \frac{a-2}{a-1}F(1,1;\ a;\ 1)\ ,$$

where $F(\alpha,\beta;\ a;\ z)$ is the hypergeometric series which converges in this case for $z = 1$ since $1 < \gamma < \infty$ implies $\infty > a > 2$, and hence $\alpha + \beta - a < 0$. (cf., Whittaker and Watson, Modern Analysis , Chapter XIV). Thus

$$\frac{t_\infty}{T} = \frac{a-2}{a-1}\,\frac{\Gamma(a)\Gamma(a-2)}{\Gamma(a-1)\Gamma(a-1)} = \frac{\Gamma(a)\Gamma(a-1)}{\Gamma(a)\Gamma(a-1)} = 1$$

5. TEMPERATURE

Let

$$\theta_i = \frac{p_i}{\rho_i}\ .$$

Then the ratio of the value of this quantity after n reflections to its value behind the initial shock is given by

$$\text{(11)}\qquad T_n = \frac{\theta_n}{\theta_o} = \frac{p_n}{p_o}\frac{\rho_o}{\rho_n} = \prod_{i=1}^{n}\frac{\sigma_i}{\eta_i} = \frac{\pi_n}{\kappa_n} = \frac{(a+n)(1+n)}{a}\ .$$

6. ENTROPY

Let

$$S_i = \frac{p_i}{\rho_i^{\gamma}}$$

Then the ratio of the value of this quantity after n reflections to its value behind the initial shock is given by

$$\epsilon_n = \frac{S_n}{S_o} = \frac{p_n}{p_o}\left(\frac{\rho_o}{\rho_n}\right)^{\gamma} = \prod_{i=1}^{n} \frac{\sigma_i}{\eta_i^{\gamma}} = \frac{\pi_n}{\kappa_n^{\gamma}} . \tag{12}$$

7. LIMITING VALUE OF THE ENTROPY

Let

$$\epsilon_\infty(\gamma) = \lim_{n\to\infty} \epsilon_n .$$

Substituting (6) into (12) this becomes

$$\epsilon_\infty(\gamma) = \prod_{i=1}^{\infty} \frac{(i+a)(i+1)^{\gamma}}{i(i+a-1)^{\gamma}} .$$

Rearranging, the above may be written

$$\epsilon_\infty(\gamma) = \prod_{i=1}^{\infty} \frac{(1+\frac{a}{i})(1+\frac{1}{i})^{\gamma}}{(1+\frac{a-1}{i})^{\gamma}}$$

$$= \prod_{i=1}^{\infty}\left\{\frac{\left[(1+\frac{1}{i})^{a-1}(1+\frac{a-1}{i})^{-1}\right]^{\gamma}}{(1+\frac{1}{i})^{a}(1+\frac{a}{i})^{-1}}\;\frac{(1+\frac{1}{i})^{\gamma}(1+\frac{1}{i})^{a}}{(1+\frac{1}{i})^{(a-1)\gamma}}\right\}.$$

The factor on the right is unity since, using (4,c),

$$\gamma - (a-1)\gamma + a = 0.$$

Introduce Euler's formula for the Γ-function; namely,

$$z\Gamma(z) = \prod_{i=1}^{\infty} \left\{ (1+\tfrac{1}{i})^{z} \, (1+\tfrac{z}{i})^{-1} \right\} ,$$

(cf., Whittaker and Watson, Modern Analysis, p. 237) and there results

$$\epsilon_{\infty}(\gamma) = \frac{\left[(a-1)\Gamma(a-1)\right]^{\gamma}}{a\,\Gamma(a)} ,$$

or

(13) $$\epsilon_{\infty}(\gamma) = \frac{\left[\Gamma(a)\right]^{\gamma-1}}{a} = \frac{\left[\Gamma(a)\right]^{\frac{2}{a-2}}}{a} \qquad (a = \tfrac{2\gamma}{\gamma-1};\ 1 < \gamma < \infty)$$

The limiting values of $\epsilon_{\infty}(\gamma)$ are obtained as follows:

$$\lim_{\gamma \to 1} \epsilon_{\infty}(\gamma) = \lim_{a \to \infty} \frac{\left(\Gamma(a)\right)^{\frac{2}{a-2}}}{a}$$

$$= \lim_{a \to \infty} \frac{\left(e^{-a}\, a^{\frac{2a-1}{2}}\right)^{\frac{2}{a-2}}}{a}$$

$$= \lim_{a \to \infty} e^{-\frac{2a}{a-2}}\, a^{\frac{a+1}{a-2}}$$

$$= \lim_{a \to \infty} \ell^{-2} a$$

$$= \infty \; ;$$

$$\lim_{\gamma \to \infty} \mathcal{E}_{\infty}(\gamma) = \lim_{a \to 2} \frac{\left(\Gamma(a)\right)^{\frac{2}{a-2}}}{a} = \lim_{x \to 0} \frac{f(x)}{x+2} ,$$

where

$$f(x) = \left(\Gamma(x+2)\right)^{\frac{2}{x}} .$$

Expanding in a power series,

$$\ell n \; f(x) = \frac{2}{x} \ell n \; \Gamma(x+2) = \frac{2}{x} \left\{ \ell n \, \Gamma(2) + \frac{x \, \Gamma'(2)}{\Gamma(2)} x + \cdots \right\}$$

$$= 2 \, \Gamma'(2) + \cdots = 2(1-c) + \cdots \quad ,$$

where c = Euler's constant $\approx$ 0.5772157.

Thus

$$\lim_{\gamma \to \infty} \mathcal{E}_{\infty}(\gamma) = \lim_{x \to 0} \frac{\ell^{2(1-c)+\cdots}}{x+2}$$

$$= \frac{1}{2} \ell^{2(1-c)}$$

$$\approx 1.16488 \; .$$

A few values of $\mathcal{E}_{\infty}$ tabulated as a function of γ are shown below.

γ	1	1.5	2	3	∞
$\mathcal{E}_{\infty}(\gamma)$	∞	1.826	1.5	1.333	1.165