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ON THE MILNE PROBLEM FOR A LARGE PLANE SLAB WITH CONSTANT SOURCE AND ANISOTROPIC SCATTERING

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ABSTRACT

Formulae are derived for the asymptotic neutron density and the current emerging from an infinite plane slab (thickness large compared to the mean free path) which sustains a uniform production of neutrons. The slab is assumed to be weakly capturing and to scatter neutrons in accordance with the law: $(1/4\pi)(1+3f_1\mu)$ where f_1 is a constant, μ is the cosine of the angle of scattering. Expressions for the asymptotic neutron density in the slab and the emerging current in the limiting cases $f_1 = 0$ and/or no capture are also given.

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ON THE MILNE PROBLEM FOR A LARGE PLANE SLAB WITH CONSTANT SOURCE AND ANISOTROPIC SCATTERING

We consider the following problem: an infinite plane slab of material of half-thickness d bounded by vacuum on both sides contains a source of neutrons uniformly distributed throughout the slab. The neutrons are scattered anisotropically[1] without change of energy and also suffer weak capture. The half-thickness d is assumed large compared to the scattering mean free path. We wish to obtain expressions for a) the asymptotic neutron density inside the slab and b) the neutron current leaving either face.

Knowledge of (a) and (b) is useful for various problems; e.g., it gives an upper bound to the thermal utilization of a unit cell in a plane pile where the moderator has the same dimensions, mean free paths, etc., as the slab. The moderator gives rise to a roughly constant source of thermal neutrons. The uranium part of the unit cell strongly absorbs thermal neutrons and approaches vacuum which is equivalent to a black absorber. The actual thermal utilization is somewhat less than the "black" utilization.

The transport equation governing the distribution of neutrons in the slab in the case of linear scattering and with constant neutron production is (cf. Fig. 1):

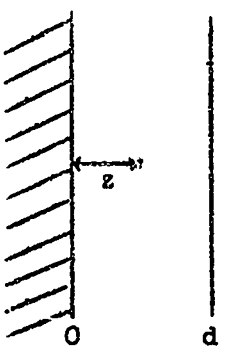

Fig. 1

$$\mu \frac{\partial \psi}{\partial z}(z,\mu) + \psi(z,\mu) = (1/2\sigma)\left[\psi_0(z) + 3f_1\mu\psi_1(z)\right] + q_0/2 \qquad (1)$$

In Eq. (1) the origin of the z-axis is taken on one face of the slab, μ is the cosine of the angle between the direction of the neutron and the positive z-axis, $\psi(z,\mu)d\mu$ is the number of neutrons per unit volume at the point z with direction cosine between μ and $\mu + d\mu$, σ is the ratio of the scattering mean free path to the total mean free path,

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1) We work with linear scattering, i.e., the scattering function is assumed to be expressible in terms of the zero and first harmonics; the generalization to a higher number of harmonics is possible but of no interest at present.

q_o is the number of neutrons produced per unit volume (assumed constant), f_1 is a measure of the deviation of the scattering function from isotropy, and finally $\psi_o(z)$ and $\psi_1(z)$ are the zero and first moments of the neutron distribution function, namely:

$$\psi_o(z) = \int_{-1}^{1} \psi(z,\mu)d\mu \tag{1a}$$

$$\psi_1(z) = \int_{-1}^{1} \mu\psi(z,\mu)d\mu \tag{1b}$$

The quantity $\psi_o(z)$ is the neutron density and the negative of $\psi_1(z)$ is the neutron current. In the above, the total mean free path is taken as unit of length and the neutron velocity is set equal to one. Eq. (1) is to be solved subject to the boundary conditions:

$$\psi(0,\mu) = 0 \quad \text{for} \quad \mu > 0 \tag{A}$$

$$(\partial\psi_o/\partial z)_{asym} = 0 \quad \text{at} \quad z = d \tag{B}$$

Condition (A) follows because the vacuum does not return any neutrons, condition (B) because of the symmetry of the problem. The use of the asymptotic part of the solution for $\psi_o(z)$ in (B) is valid as long as $d \gg 1$, the condition becoming more severe, the stronger the capture.

Following the procedure outlined in Reports MT-5[2] and MT-26[3], we take the Laplace transform of both sides of Eq. (1). We get:

$$\phi(s,\mu)\left[1 + s\mu\right] = \frac{1}{2\sigma}\left[\phi_o(s) + 3f_1\mu\phi_1(s)\right] + \frac{q_o}{2s} + \mu\psi(0,\mu) \tag{2}$$

where

$$\phi(s,\mu) = \int_0^{\infty} \psi(z,\mu)\, e^{-sz}\, dz$$

$$\phi_o(s) = \int_{-1}^{1} \phi(s,\mu)d\mu, \qquad \phi_1(s) = \int_{-1}^{1} \mu\phi(s,\mu)d\mu$$

Integrating both sides of (2) over $d\mu$ from -1 to 1, we find:

2) (BM-110; MT-5) G. Placzek and W. Seidel - "Milne's Problem in Transport Theory".

3) (BM-225; MT-26) C. Mark - "Milne's Problem for Anisotropic Scattering".

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$$s\phi_1(s) = (1/\sigma - 1)\,\phi_0(s) + q_0/s + \int_{-1}^{0} \mu\psi(0,\mu)\,d\mu \qquad (3)$$

where we have made use of boundary condition (A). Substituting for $\phi_1(s)$ into (2), dividing by $(1+s\mu)$ and integrating over $d\mu$ from -1 to 1, Eq. (2) is transformed into:

$$\phi_0(s)\left[1 - \frac{\text{arth } s}{\sigma s} + \frac{3f_1(\sigma-1)}{\sigma^2 s^2}\left(1 - \frac{\text{arth } s}{s}\right)\right] = \frac{\sigma q_0}{s}\left[\frac{3f_1}{\sigma^2 s^2}\left(1 - \frac{\text{arth } s}{s}\right) + \frac{\text{arth } s}{\sigma s}\right]$$
$$+ g_+(s) + \frac{3f_1 g_+(0)}{\sigma s^2}\left(1 - \frac{\text{arth } s}{s}\right) \qquad (4)$$

In Eq. (4) we have written:

$$g_+(s) = \int_{-1}^{0} \frac{\mu\psi(0,\mu)}{(1+s\mu)}\,d\mu \qquad (5)$$

so that $g_+(0) = \int_{-1}^{0} \mu\psi(0,\mu)d\mu$; $g_+(0)$ represents the negative of the neutron current flowing into vacuum.

Proceeding further along the lines of MT-26[3] we rewrite Eq. (4) in the form:

$$\left[s\phi_0(s) + \sigma q_0\right] K(s) = (3f_1/\sigma s^2)\,H(s)\left[sg_+(0) + \sigma q_0\right] + \left[sg_+(s) + \sigma q_0\right] \qquad (6)$$

where

$$K(s) = 1 - \frac{\text{arth } s}{\sigma s} + \frac{3f_1(\sigma-1)}{\sigma^2 s^2}\left(1 - \frac{\text{arth } s}{s}\right)$$

$$H(s) = \left(1 - \frac{\text{arth } s}{s}\right)$$

Expressing $H(s)$ in terms of $K(s)$:

$$H(s) = \frac{\sigma^2 s^2\,K(s)}{\sigma^2 s^2 + 3(\sigma-1)f_1} - \frac{\sigma(\sigma-1)\,s^2}{\sigma s^2 + 3(\sigma-1)f_1}$$

we may transform Eq. (6) into:

$$\Phi(s)\,K(s) = G(s) \qquad (7)$$

where

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$$\Phi(s) = \left[s\phi_0(s) + \sigma q_0\right]\left[\sigma s^2 + 3(\sigma-1)f_1\right] - 3f_1\sigma\left[sg_+(0) + \sigma q_0\right] \tag{7a}$$

$$G(s) = \left[sg_+(s) + \sigma q_0\right]\left[\sigma s^2 + 3(\sigma-1)f_1\right] - 3f_1(\sigma-1)\left[sg_+(0) + \sigma q_0\right] \tag{7b}$$

In Eq. (7) $\Phi_0(s)$ is analytic in the plane $R(s) > \nu$ (where ν is the positive root of $K(s)$), $G(s)$ is analytic in the half-plane $R(s) < 1$ and $K(s)$ is analytic in the strip $|R(s)| < 1$. Just as in the case of no capture, it can be shown[4] that in the strip $K(s)$ has just two zeros, namely, $\pm\nu$, and approaches unity as $|s| \to \infty$. We may therefore adopt the usual device of defining a function:

$$\tau(s) = \left(\frac{s^2 - 1}{s^2 - \nu^2}\right) K(s) \tag{8}$$

The function $\log \tau(s)$ is analytic and single-valued in the strip $|R(s)| < 1$, provided a particular determination of the logarithm is chosen[5], and approaches zero as $|s| \to \infty$ in the strip. The usual decomposition then follows:

$$\tau(s) = \tau_+(s)/\tau_-(s) \tag{8a}$$

where

$$\tau_+(s) = \exp\left[\frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} \frac{\log \tau(u)du}{u - s}\right]$$

$$\tau_-(s) = \exp\left[\frac{1}{2\pi i}\int_{\beta - i\infty}^{\beta + i\infty} \frac{\log \tau(u)du}{u - s}\right]$$

and $|Re(s)| < \beta$ with $\nu < \beta < 1$. Introducing (8) and (8a) into (7) yields:

$$\Phi(s)\,\frac{(s^2 - \nu^2)}{(s+1)}\,\frac{1}{\tau_-(s)} = \frac{(s-1)}{\tau_+(s)}\,G(s) \tag{9}$$

The left-hand side of Eq. (9) is analytic in the half-plane $Re(s) > \nu$ and the right-hand side is analytic in the half-plane $Re(s) < \beta$. Since there is a region of overlap, each side is the analytic continuation of the other. Examination of the behavior of the two sides of (9) as $|s| \to \infty$ in the plane shows that they approach infinity as $|s|^3$. By an extension of Liouville's theorem, it follows that each side may be equated to a polynomial of order three. We therefore write:

4) cf. MT-56, C. Mark - "Some Constants and Expansions Used in Applications of the Wiener-Hopf Method" (unpublished).

5) The determination $\log 1 = 0$ is chosen.

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$$\Phi(s)\,\frac{(s^2-\nu^2)}{(s+1)\,\bar{\tau}_-(s)} = C_0 + C_1 s + C_2 s^2 + C_3 s^3 = \frac{(s-1)\,G(s)}{\bar{\tau}_+(s)} \tag{10}$$

where C_0, C_1, C_2, and C_3 are constants.

The problem is now to evaluate the C's. From (10) it is evident that $C_0 = -G(0)/\bar{\tau}_+(0)$ which is zero since $G(0) = 0$ (cf. (7b)) and $\bar{\tau}_+(0)$ is finite. C_1 is also zero as can be seen by writing

$$C_1 s + C_2 s^2 + C_3 s^3 = \frac{(s-1)\,G(s)}{\bar{\tau}_+(s)}$$

differentiating both sides with respect to s and setting $s = 0$. Hence:

$$G(s) = \frac{s^2(C_2 + C_3 s)}{(s-1)}\,\bar{\tau}_+(s) \tag{11}$$

Inserting (7b) for $G(s)$ and rearranging terms, we find:

$$\left[s g_+(s) + \sigma q_0\right] = \frac{s^2(C_2 + C_3 s)\,\bar{\tau}_+(s) + 3f_1(\sigma-1)\left[s g_+(0) + \sigma q_0\right](s-1)}{(s-1)\left[\sigma s^2 + 3(\sigma-1)\,f_1\right]} \tag{12}$$

From (12) it would follow that $\left[s g_+(s) + \sigma q_0\right]$ has poles at $s = \pm\xi$ where $\xi = i\sqrt{3f_1(1-1/\sigma)}$; since this is impossible it follows that the numerator of the right-hand side of (12) must vanish for $s = \xi$ and $s = -\xi$, i.e.,

$$(C_2 + C_3\xi)\,\bar{\tau}_+(\xi) - \sigma\left[\xi g_+(0) + \sigma q_0\right](\xi - 1) = 0 \tag{13a}$$

$$(C_2 - C_3\xi)\,\bar{\tau}_+(-\xi) - \sigma\left[-\xi g_+(0) + \sigma q_0\right](-\xi - 1) = 0 \tag{13b}$$

Eqs. (13a) and (13b) yield values for C_2 and C_3, namely:

$$C_2 = \frac{\xi^2\, g_+(0)(\alpha + \beta/\xi)}{(\alpha^2 - \beta^2)} - \frac{\sigma q_0(\alpha + \beta\xi)}{(\alpha^2 - \beta^2)} \tag{14a}$$

$$C_3 = \frac{-g_+(0)(\alpha + \beta\xi)}{(\alpha^2 - \beta^2)} + \frac{\sigma q_0}{(\alpha^2 - \beta^2)}(\alpha + \beta/\xi) \tag{14b}$$

with

$$\alpha = \frac{\bar{\tau}_+(\xi) + \bar{\tau}_+(-\xi)}{2\sigma}, \qquad \beta = \frac{\bar{\tau}_+(\xi) - \bar{\tau}_+(-\xi)}{2\sigma} \tag{14c}$$

Eqs. (14a) and (14b) express C_2 and C_3 in terms of the unknown constant $g_+(0)$; this is as it should be since we still must take into account boundary condition

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(B). To determine $g_+(0)$, we find the asymptotic solution for $\psi_0(z)$ and then impose the boundary condition (B).

To find the asymptotic solution for $\psi_0(z)$, we write down the expression for $\phi_0(s)$ given by (10) and (7a) and find the contribution to the Laplace inverse from the poles[6]. For $\phi_0(s)$ we have:

$$\phi_0(s) = -\frac{\sigma q_0}{s} + \frac{3f_1\left[sg_+(0) + \sigma q_0\right]}{s(s^2-\xi^2)} + \frac{\bar{\tau}_-(s)s(s+1)(C_2 + C_3 s)}{\sigma(s^2-\xi^2)(s^2-\nu^2)} \tag{15}$$

The contribution to $\psi_0(z)$ (the Laplace inverse of $\phi_0(s)$) from the poles is:

$$\begin{aligned}\psi_0(z)_{asym} = \frac{\sigma q_0}{(\sigma-1)} &+ \frac{C_2}{2\sigma(\nu^2-\xi^2)}\left[\bar{\tau}_-(\nu)(1+\nu)e^{\nu z} + (1-\nu)\,\bar{\tau}_-(-\nu)e^{-\nu z}\right] \\ &+ \frac{C_3\nu}{2\sigma(\nu^2-\xi^2)}\left[\bar{\tau}_-(\nu)(1+\nu)e^{\nu z} - (1-\nu)\,\bar{\tau}_-(-\nu)e^{-\nu z}\right]\end{aligned} \tag{16}$$

We may rewrite (16) in the form:

$$\psi_0(z)_{asym} = \frac{\sigma q_0}{(\sigma-1)} + C_2 A(\nu,\xi)\cosh\nu(z+\bar{z}_0) + C_3\nu A(\nu,\xi)\sinh\nu(z+\bar{z}_0) \tag{17}$$

where

$$A(\nu,\xi) = \frac{\sqrt{(1-\nu^2)\,\bar{\tau}_-(\nu)\,\bar{\tau}_-(-\nu)}}{\sigma(\nu^2-\xi^2)} \tag{17a}$$

$$\bar{z}_0 = \frac{1}{2\nu}\left\{\log\left[(1+\nu)\,\bar{\tau}_-(\nu)\right] - \log\left[(1-\nu)\,\bar{\tau}_-(-\nu)\right]\right\} \tag{17b}$$

The boundary condition (B) applied to (17) now yields:

$$C_2 \sinh\nu(d+\bar{z}_0) + C_3\nu\cosh\nu(d+\bar{z}_0) = 0 \tag{18}$$

Using the definitions (14a) and (14b) for C_2 and C_3, Eq. (18) permits us to solve for $g_+(0)$ with the result:

$$g_+(0) = \sigma q_0\,\frac{\left\{(\alpha+\beta/\xi) - (\alpha+\beta\xi)\left[\tanh\nu(d+\bar{z}_0)/\nu\right]\right\}}{\left\{(\alpha+\beta\xi) - \xi^2(\alpha+\beta/\xi)\left[\tanh\nu(d+\bar{z}_0)/\nu\right]\right\}} \tag{19}$$

It is to be recalled that $\xi = i\sqrt{3f_1(1-1/\sigma)}$.

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6) The branch-point contribution yields the non-asymptotic part of the solution.

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$$\alpha = \frac{\overline{\tau}_-(\xi) + \overline{\tau}_+(-\xi)}{2\sigma}, \qquad \beta = \frac{\overline{\tau}_+(\xi) - \overline{\tau}_+(-\xi)}{2\sigma}$$

Eqs. (17), (17a), (17b) and (19) constitute asymptotic solution for the neutron density in the slab; the negative of (19) is the current leaving either face of the slab.

If the capture is weak the various $\overline{\tau}$-functions can be expanded in powers of ν_0^2 ($\nu_0 = 1/L$ with L the ordinary diffusion length); neglecting all terms beyond ν_0^2, we obtain (cf. MT-56[4]):

$$g_+(0) = -q_0\sigma\left[\frac{\tanh\nu(d+\overline{z}_0) - \sigma\nu B/3f_1(\sigma-1)}{\nu + B\tanh\nu(d+\overline{z}_0)}\right] \tag{20}$$

where

$$B = z_0 f_1 \nu_0^2\left\{1 + \frac{\nu_0^2}{3}\left[\frac{4}{5} + (1+f_1)\frac{k}{z_0} - f_1 + 3z_0 f_1\right]\right\}$$

$$\overline{z}_0 = \frac{z_0}{(1-f_1)}\left\{1 + \frac{\nu_0^2}{3}\left[\frac{1}{(1-f_1)} - \frac{f_1}{5} + \frac{2f_1 k}{z_0} + z_0^2\left(\frac{1}{1-f_1} - 1 - f_1 + 2f_1^2\right)\right]\right\}$$

$$\nu = \nu_0\sqrt{1-f_1}$$

with

$$z_0 = .71044\ \ldots$$

$$k = .17226\ \ldots$$

It is interesting to consider the limiting case of isotropic scattering: allowing f_1 and therefore ξ to approach zero, $g_+(0)$ becomes

$$g_+(0) = -q_0\sigma\left[\frac{\tanh\nu_0\left(d + \overline{z}_0^{(isot)}\right)}{\nu_0} - 1 - \frac{\overline{\tau}_+'^{(isot)}(0)}{\overline{\tau}_+^{(isot)}(0)}\right] \tag{21}$$

In Eq. (21), $\overline{z}_0^{(isot)}$ is defined by:

$$\overline{z}_0^{(isot)} = \frac{1}{2\nu_0}\left\{\log\left[(1+\nu_0)\,\overline{\tau}_-^{(isot)}(\nu_0)\right] - \log\left[(1-\nu_0)\,\overline{\tau}_-^{(isot)}(-\nu_0)\right]\right\} \tag{22}$$

To evaluate $\overline{z}_0^{(isot)}$ it is necessary to know the definition of $\overline{\tau}_-^{(isot)}(s)$ and $\overline{\tau}_+^{isot}(s)$ $\left(\text{since } \overline{\tau}_-^{(isot)}(-s) \equiv 1/\overline{\tau}_+^{(isot)}(s)\right)$; we have:

$$\overline{\tau}_-^{(isot)}(s) = \exp\left[\frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty}\frac{\log\overline{\tau}^{(isot)}(u)\,du}{u-s}\right] \tag{23a}$$

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$$\bar{\tau}_+^{(isot)}(s) = \exp\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{\log \bar{\tau}^{(isot)}(u)du}{u-s}\right] \qquad (23b)$$

where

$$\bar{\tau}^{(isot)}(s) = \frac{s^2-1}{s^2-\nu_o^2}\left(1-\frac{\operatorname{arth} s}{\sigma s}\right) \qquad (23c)$$

The term $\bar{\tau}_+'^{(isot)}(0)/\bar{\tau}_+^{(isot)}(0)$ is the logarithmic derivative of $\bar{\tau}_+^{isot}(s)$ evaluated at $s=0$ and can be found from (23b). For weak capture, if we retain terms up to order ν_o^2, (22) reduces to (this can be seen directly from (20)):

$$g_+(0) = -q_o\sigma\left[\frac{\tanh \nu_o\left(d+\bar{z}_o^{(isot)}\right)}{\nu_o} - z_o - \frac{k}{3}\nu_o^2\right] \qquad (24)$$

where

$$\bar{z}_o^{(isot)} \cong z_o\left(1+\frac{\nu_o^2}{3}\right) \qquad (24a)$$

It is just as simple to find the asymptotic solution for $\psi_o(z)$ in the limiting case of isotropic scattering; we arrive at the expression:

$$\psi_o(z)_{asym} = (\sigma/\sigma-1)\, q_o + A(\nu_o)\cosh \nu_o(d-z) \qquad (25)$$

where

$$A(\nu_o) = \frac{-q_o\sigma^2}{\cosh\left[\nu_o\left(d+\bar{z}_o^{(isot)}\right)\right]}\sqrt{\frac{2(1-\nu_o^2)}{\left[1-\sigma(1-\nu_o^2)\right]\left[\sigma-1\right]}} \qquad (25a)$$

and $\bar{z}_o^{isot}$ is defined by (22). For weak capture $\bar{z}_o^{(isot)}$ is defined more explicitly by (24a).

One final limiting case is worth mentioning; the case of isotropic scattering and zero capture. The current, of course, becomes $(q_o d)$ while the asymptotic neutron density is:

$$\psi_o^{(isot;no\ capture)}(z)_{asym} = \frac{-3q_o z^2}{2} + A\left(z+\bar{z}_o^{(isot;no\ capture)}\right) \qquad (26)$$

where

$$\left\{\begin{array}{l} \cancel{A = +3q_o d} \text{ [redacted]} \\ \bar{z}_o^{(isot;no\ capture)} = z_o + \frac{1}{2d}\left(z_o^2 - 1/15\right) \end{array}\right.$$

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