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THEORY OF von NEUMANN'S METHOD OF TREATING SHOCKS

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ABSTRACT

The method of allowing fluctuations in the mechanical solution of hydrodynamic problems to take the place of the entropy increase in the shock, is analysed statistically. The model corresponds to the thermal behavior of a substance of anomalously low specific heat, and the effect of the pseudo thermal motion is greater than for any real substance. Therefore the model is a good approximation only when the contribution of the thermal pressure is negligible. Curves are given from which this contribution can be estimated. Other complicated features arise when the mass intervals are not equal and in the case of radial motion. A typical case, taken from I.B.M. calculations, is discussed by way of illustration.

THEORY OF von NEUMANN'S METHOD OF TREATING SHOCKS

I. INTRODUCTION

It is well known that in hydrodynamic problems involving compressible media there may exist shock waves, i.e., places at which the velocity, pressure and density are practically discontinuous, and at which the equations of Euler do not hold. The reason for this is that the Euler equations assume the changes in the material to be reversible, whereas at a shock wave the gradients become so large that the dissipative effects (viscosity, heat conduction) become important. Indeed, in the limit usually considered, in which the dissipative terms are small in the units appropriate to the problem, the shock is a sharp discontinuity and the gradients are infinite.

Therefore it would in general not make sense to assume Euler's equations to hold even across the shock. Von Neumann has pointed out, however, that the situation is different if one uses, instead of the differential equations, the approximate difference equations which are the basis of one mechanical method of treating the equations. In conditions where a shock would form, we know there exists no solution of the differential equations. Any solution of the difference equations will approximate to a solution of the differential equations only if the changes of all functions over one interval are small, hence the difference equations cannot be expected to have any solution of this kind at a shock. There are, however, solutions of oscillatory behavior containing fluctuations with periods of the order of the interval size. These, according to von Neumann, can be regarded as a model of the increase of entropy in the shock, and indeed the fluctuations thus obtained represent the heat motion of the shocked material.

It is evident that this model of the heat motion is very crude, and that it does not represent correctly the thermal behavior of any reasonable substance. However, there are many cases of interest in which the influence of temperature on the equation of state is negligible, and in these cases one may expect that the error introduced by

the model may not be serious.

The purpose of the present report is to study the difference between the model and an actual substance in more quantitative detail and to derive criteria that may serve to estimate the error in individual cases.

II. BASIC EQUATIONS

In one case of a one-dimensional problem the differential equations are

$$\ddot{y} = -\frac{1}{\rho_o}\frac{\partial p}{\partial x}\ , \quad p = p(v)\ , \quad v = \frac{\partial y}{\partial x} \tag{2.1}$$

where $y(t)$ is the position at time t, of the point which would be at x if the material had normal density, p is the pressure, which is assumed to be a unique function of the specific volume v (reversibility). More precisely, v stands for the ratio of the volume to the normal volume. The normal density is ρ_o.

The difference equation is obtained from this by choosing time intervals δ and space intervals Δ, so that after neglecting higher than second powers of the interval sizes, (2.1) becomes:

$$\left.\begin{aligned} \frac{y_{n+1,m} + y_{n-1,m} - 2y_{n,m}}{\delta^2} &= -\frac{1}{\rho_o \Delta^2}\left(p_{n,m+\frac{1}{2}} - p_{n,m-\frac{1}{2}}\right) \\ v_{n,m} &= \frac{1}{\Delta}\left(y_{n,m+\frac{1}{2}} - y_{n,m-\frac{1}{2}}\right) \end{aligned}\right\} \tag{2.2}$$

where n, m label the time and space intervals:

$$t_n = n\delta + \text{const} \tag{2.3}$$

$$x_m = m\Delta + \text{const.}$$

We apply these equations to the state of affairs we expect behind a shock, where there will be irregular fluctuations superimposed on a "macroscopic" or mean motion. If our

interval sizes are chosen correctly, the mean quantities vary little over one interval and over a few intervals we may regard all macroscopic quantities as uniform.

As to the "atomic" motion, or fluctuations, we shall assume that the amplitude is small. The limitations introduced by this assumption will be discussed later. We can then write

$$y_{n,m} = \bar{y}_{n,m} + \eta_{n,m} \qquad (2.4)$$

where the bar denotes the "mean" position and $\eta_{n,m}$ is small. Then

$$p_{n,m} = p(\bar{v}) + (\overline{dp/dv}) \ \frac{1}{\Delta} \left(\eta_{n,m+\frac{1}{2}} - \eta_{n,m-\frac{1}{2}}\right) \qquad (2.5)$$

Then (2.2) becomes:

$$\frac{\eta_{n+1,m} + \eta_{n-1,m} - 2\eta_{n,m}}{\delta^2} = - \frac{1}{\rho_o \Delta^2} \left(\frac{dp}{dv}\right)_{\bar{v}} (\eta_{n,m+1} + \eta_{n,m-1} - 2\eta_{n,m}) \qquad (2.6)$$

Solutions of this equation can be written in the form

$$\eta_{n,m} = A \, e^{i(\phi_n + \psi_m)} \qquad (2.7)$$

where, A, ϕ, ψ are constants. Inserting this in (2.6), we have

$$(1 - \cos \phi) = - \frac{\delta^2}{\rho_o \Delta^2} \left(\frac{dp}{dv}\right)_{\bar{v}} (1 - \cos \psi) \qquad (2.8)$$

It is well known that the factor

$$\ell^2 = \frac{\delta^2}{\rho_o \Delta^2} \left(- \frac{dp}{dv}\right)_{\bar{v}} \qquad (2.9)$$

must be less than unity in order that the step-by-step solution of (2.2) be possible. Indeed, it is evident from (2.8) that if $\ell^2 > 1$, ϕ is imaginary near $\psi = \pi$, and hence there are disturbances which will grow exponentially with time, making the system unstable.

Otherwise there will be N frequencies, where N is the number of space intervals in the region under consideration. The values of ψ belonging to these will be spread uniformly over the interval $-\pi$ to π.

If the linear equations (2.6) were rigorous, all these oscillations would be independent. Since, however, the correct equation (2.2) does contain terms of higher degree, there will be a certain amount of coupling between different oscillations which, given enough time, must produce some kind of statistical equilibrium.

For strong amplitudes the oscillation, where the terms of different degrees in the amplitude are comparable, it is clear from dimensional arguments that the "mean free path" of the oscillations, i.e., the distance a wave travels before equilibrium has essentially been established, is of the order of the interval size, with a numerical factor which, by analogy with the problem of waves in crystal lattices, one would expect to be larger than unity. As the mean amplitude is reduced, and the coupling is weaker, the mean free path increases further, and one would thus expect that for some appreciable distance behind the shock the oscillation caused by the shock may not be in equilibrium. This effect extends over a greater distance for weaker shocks.

III. THERMODYNAMIC RELATIONS FOR SMALL ℓ

We have seen above that we may expect statistical equilibrium to be established except very closely behind the shock. To this equilibrium ordinary thermodynamics is not immediately applicable, since equation (2.6) is a difference equation in time as well as in space, and hence energy conservation does not hold in the usual form.

If, however, $\ell \ll 1$, the time interval is negligible in comparison with the space interval, and the equations are then essentially differential equations in time. They are closely analogous to those for a one-dimensional Borr-von Karman

lattice. Since the equations are linear, and there is a restoring force for each degree of freedom, it is clear that, in equilibrium the thermal energy will be kT per degree of freedom, or

$$E_{th} = \frac{1}{\rho_o \Delta} kT \qquad (3.1)$$

per unit mass, where $1/\Delta$ is the number of degrees of freedome per unit length, k Boltzmann's constant, and T the temperature.

The unit of temperature is here arbitrary, since we cannot measure the temperature of this fictitious system by bringing it into thermal contact with any other physical system, hence only the product kT has a definite meaning. For convenience we choose our units of temperature in such a way that

$$k = \rho_o \Delta \qquad (3.2)$$

equal to the mass per interval, hence

$$E_{th} = T \qquad (3.3)$$

As in any other oscillating system, this energy is on the average half kinetic and half potential. Hence for the kinetic energy per unit mass

$$\frac{1}{2} \quad \overline{u^2} = \frac{1}{2} T$$

or

$$u^2 = T \qquad (3.4)$$

where

$$u = \partial \eta / \partial t \qquad (3.5)$$

is the velocity associated with the fluctuations, and the bar denotes the statistical average.

(3.4) may be regarded as a definition of T and allows one to estimate T in any individual case. The potential energy per unit mass is

$$E(v) = \frac{1}{\rho_o} \int_v^1 p dv \qquad (3.6)$$

For small deviations, to second order inclusive,

$$E(v) = E(\bar{v}) + \frac{1}{\rho_o} p(\bar{v}) (\bar{v}-v) - \frac{1}{2} \frac{1}{\rho_o} \left(\frac{dp}{dv} \right)_{\bar{v}} (\bar{v} - v)^2 \qquad (3.7)$$

On the statistical average, the first term gives the potential energy of the mean density without temperature, the second term vanishes, and the last represents the potential part of the thermal energy, hence:

$$\frac{1}{2\rho_o} \left(-\frac{dp}{dv} \right)_{\bar{v}} \overline{(v-\bar{v})^2} = \frac{1}{2} T \qquad (3.8)$$

or

$$\overline{(v-\bar{v})^2} = \rho_o T \left| \frac{dp}{dv} \right|^{-1} \qquad (3.9)$$

Since dp/dv is known, and T can be found from (3.4) this relation can be treated. However, the equality of kinetic and potential energy, in the time average, holds for any harmonic motion whether in equilibrium or not. Hence this test merely verifies that the amplitudes of oscillation are weak enough to make the motion essentially harmonic.

We can now find the pressure caused by the heat motion. By expansion to second order

$$p = p(\bar{v}) + \frac{dp}{dv} (v-\bar{v}) + \frac{1}{2} \frac{dp^2}{dv^2} (v-\bar{v})^2$$

On the average, the first term is the pressure due to the mean density without temperature, the second term vanishes the last is the thermal pressure. Hence

$$p_{th} = \frac{1}{2} \left(\frac{d^2p}{dv^2} \right)_{\bar{v}} (\bar{v} - v)^2 = -\frac{1}{2} (\bar{v}-v)^2 \frac{d}{dv} \left| \frac{dp}{dv} \right| \qquad (3.11)$$

(The negative sign comes from the fact that dp/dv is necessarily negative.) Using (3.9)

$$p_{th} = -\frac{1}{2} \rho_o T \frac{d}{dv} \left(\log \left| \frac{dp}{dv} \right| \right) \qquad (3.12)$$

The same relation can be derived in a different way:

In general,

$$p = -\frac{\partial F}{\partial V} = -\rho_o \frac{\partial F}{\partial v} \qquad (3.13)$$

where F is the free energy per unit mass, and V the volume per unit mass. F is a sum over the different degrees of freedom, and for each degree of freedom,

$$F_i = -kT \log \omega_i + \text{const} \qquad (3.14)$$

where ω_i is its frequency (in radians per sec) and the constant may depend on the temperature but not on v. Hence, using (3.2),

$$p_{th} = -\frac{\rho_o^2 T\Delta}{M} \sum_i \frac{\partial \log \omega_i}{\partial v} , \qquad (3.15)$$

the sum to extend over all degrees of freedom belonging to the region considered, which has a total mass M. The frequencies are to be found from (2.8) with

$$\omega = \frac{1}{\delta} \phi \qquad (3.16)$$

If ℓ is small the right-hand side of (2.8) is always small, hence ϕ is a small angle and the left-hand side can be replaced by $\frac{1}{2}\phi^2$. Hence each of the frequencies is proportional to $\sqrt{|dp/dv|}$, and

$$\omega_i = f_i \sqrt{|dp/dv|} \qquad (3.17)$$

where f_i depends on the wave number ψ, but not on v. Hence all terms of the sum in (3.15) are equal, and the number of terms is equal to the number of intervals in a mass M, i.e. to

$$\frac{M}{\rho_o \Delta} \qquad (3.18)$$

Inserting this in (3.15) we obtain again (3.12)

This derivation is of interest since it shows the dependence of the result on the relation (3.17) which is not correct unless ℓ is small.

We may reasonably surmise that, for our present purpose, ℓ can be regarded as small as long as (3.17) is substantially correct. To test this, the solutions of (2.8) have been plotted in Fig. 1 against ℓ for different angles ϕ. It is seen that all lines are straight in good approximation up to $\ell = 1/2$ and in view of (2.9) this means that in this range all frequencies are proportional to $\sqrt{|dp/dv|}$. For $\ell = 1/2$ only the uppermost curves begin to bend, and without further investigation it is not possible to estimate to what extent this would affect our conclusions.

IV. HUGONIOT RELATIONS

Consider now a shock, running into material at rest. (This causes no essential loss of generality). Let v_1 be the specific volume, p_1 the pressure ahead of the shock, and assume there is no thermal motion ahead of the shock. Then the first Hugoniot relation is

$$\bar{u} = U(1 - \bar{v}/v_1) \qquad (4.1)$$

where $\bar{u}$ is the mean velocity and $\bar{v}$ the mean specific volume behind the shock, U the shock velocity. This relation only expresses conservation of material and must be satisfied automatically in our model.

The other two relations:

$$p - p_1 = \rho_0 U^2 \frac{v_1 - \bar{v}}{v_1^2} \qquad (4.2)$$

and

$$\rho_0 (E - E_1) = \frac{1}{2} (v_1 - \bar{v}) (p + p_1) \qquad (4.3)$$

will now also contain the thermal pressure and energy:

$$p(\bar{v}) - p_1 + p_{th} = \frac{\rho_0 U^2}{v_1^2} (v_1 - \bar{v}) \qquad (4.4$$

$$\rho_0 \left[E(\bar{v}) - E_1 + E_{th} \right] = \frac{1}{2} \left[p(\bar{v}) + p(v_1) + p_{th} \right] (v_1 - \bar{v}) \qquad (4.5)$$

It is convenient to write g for the ratio

$$g = \rho_o E_{th}/p_{th} \tag{4.6}$$

Then, by (3.3) and (3.12):

$$g = \frac{2}{\left|\frac{d}{dv} \log\left|\frac{dp}{dv}\right|\right|} \tag{4.7}$$

and with this abbreviation, we can solve (4.5) for p_{th} :

$$p_{th} = \frac{\frac{1}{2}(p+p_1)(v_1-\bar{v}) - \rho_o(E(\bar{v}) - E_1)}{g - \frac{1}{2}(v_1-\bar{v})} \tag{4.8}$$

and

$$\frac{\rho_o U^2 (v_1-\bar{v})}{v_1^2} = \frac{g\left[p(\bar{v})-p_1\right] + p_1(v_1-\bar{v}) - \rho_o\left[E(\bar{v}) - E_1\right]}{g - \frac{1}{2}(v_1-\bar{v})} \tag{4.9}$$

The thermodynamic properties of the model are unimportant as long as (4.8) is small compared to the pressure $p(\bar{v})$. At the same time (4.8) allows one to estimate the amplitude of fluctuations to be expected behind a shock of given strength.

For this purpose one may either compare the average pressure with the pressure belonging to the average volume $\bar{v}$ or use the mean square velocity fluctuation, which, using (4.6) and (3.3), (3.4) is

$$\overline{u_{th}^2} = g\, p_{th} \tag{4.10}$$

For strong shocks it is evident from (4.8) that there is a limiting compression ratio, which cannot be exceeded even for an infinite pressure and this is given by the condition

$$g = \frac{1}{2}(v_1-\bar{v}) \tag{4.11}$$

As g depends on the volume in the final state, it is most convenient to express this limiting compression by giving the highest volume v_1 for which a given $\bar{v}$ can be reached in a single shock:

$$v_1 = \bar{v} + 2g(\bar{v}) \qquad (4.12)$$

Fig. 2 shows g as a function of the density for tuballoy using the equation of state of Metropolis (LA-208).

Fig. 3 shows the limiting compression ratio for the same data, from Eq.(4.12).

Fig. 4 shows the thermal pressure in the same case, starting from normal density and from material at twice normal density. For comparison, the "cold" pressure $p(v)$ is also shown, as well as the correct thermal pressure behind a shock with normal material, as calculated by Keller.

It is evident from this figure that, whenever the thermal pressure amounts to an appreciable contribution to the cold pressure, it is considerably in excess of the true value.

Table I lists the thermal pressures for tuballoy for various values of v_1 and $\bar{v}$.

From these values, the mean amplitude of the fluctuations was obtained by the formula

$$\overline{\Delta v^2} = p_{th}\, g \left|\frac{dp}{dv}\right|^{-1} \qquad (4.13)$$

which follows immediately from (3.9), (3.3) and (4.6). Since, for a harmonic oscillation, the root mean square amplitude is $1/\sqrt{2}$ times the maximum amplitude, we can define a minimum volume that would be reached for harmonic oscillation of the same $\overline{\Delta v^2}$. This is

$$v_{min} = \bar{v} - \sqrt{2\,\overline{\Delta v^2}} \qquad (4.14)$$

This quantity, which is convenient for practical purposes, is shown in Fig. 5.

V. OTHER EFFECTS

The analysis given in the previous sections in some ways is still very idealized, and we want to discuss a few effects that have been neglected.

(a). The fact has already been referred to that the discussion applies only to small ℓ, and probably in practice is $\ell < 1/2$. This probably covers all applications of practical interest.

(b). Moreover, we have assumed that the oscillations are always in statistical equilibrium. In fact, all oscillations arise at the shock front, and it will take them a finite time to get into equilibrium. This means that there will be a region behind the shock front in which there is no equilibrium. The extent of this region is inversely proportional to the temperatures since the establishment of equilibrium depends on the coupling between different degrees of freedom by the terms of higher order in the amplitude of the oscillations. This effect may cause errors if the extent of the non-equilibrium region is comparable to the distance over which the dynamical variables change appreciably.

(c). In the discussion given above, we have assumed harmonic oscillations, which is correct only for small amplitudes. When the "thermal" pressure exceeds the "cold" pressure this is no longer justified. As a result the thermodynamic properties of the system may differ from our description at high temperatures and in particular the limiting compression may be appreciably affected. On the other hand the limit of applicability of the method will remain unchanged since this refers to the condition that the thermal pressure is negligible which means small amplitude.

(d). Conduction of Energy. In the physical applications of most interest, heat conduction is usually negligible and an accurate model therefore ought to give adiabatic changes in the state of matter everywhere except at the shock front. In our

model, there exists a finite heat conductivity carried by the sound waves and limited mainly by the second-order terms which limit the free run of such waves. The "mean free path" of the waves is dimensionally of the order of the interval size, multiplied by a function of temperature which is large for low temperatures. Hence, one may expect some error due to this effect for weak shocks. However, as the heat content is then small, this will not lead to appreciable errors. In any case, for any shock strength this effect can be made negligible by a generous choice of the number of intervals used.

(e). Changing interval size. It is often convenient in calculations not to make all intervals equal but to use groups of smaller intervals in regions where more structure is required. Consider the boundary between two such groups. Physically, the two parts of the material ought to be in equilibrium for equal temperature, i.e., for equal energy per unit mass. This does not, however, in the model correspond to equality of temperatures as defined by (3.3) since the latter is measured on a conventional scale. In the model, the two groups of points are in equilibrium if the energy per degree of freedom is the same, which means <u>different</u> energies per unit mass. What we have done corresponds, in effect, to using different values of Avogadio's number in different parts of the material.

Suppose, for example, that a shock is moving through a range where the interval size is Δ, and that a short distance behind the shock it is reduced to $\Delta/2$. Then, as soon as the disturbances have had time to travel back to the region with the smaller intervals, an equilibrium will be approached in which the energy per interval is the same, so that the temperature at the small intervals will be twice as high as in the larger ones.

VI. APPLICATION TO SPHERICAL PROBLEMS

An important class of problems concerns motion with spherical symmetry. Then we have, in place of (2.1)

$$\ddot{R} = -\frac{1}{\rho_o}\, 3R^2 \frac{\partial p}{\partial M}\,;\; v = \frac{\partial R^3}{\partial M} \tag{6.1}$$

where R is the actual distance from the center, and the independent variable M is the mass contained between the point under consideration and the center. If we not obtain the linear equations for small disturbances, we have to bear in mind that the mass contribution to the free energy comes from oscillations with wave lengths of the order of the interval size, and that, for any reasonable choice of interval, this is small compared to the distance from the center. Hence if we write again

$$R = \bar{R} + \eta \tag{6.2}$$

where $\bar{R}$ represents the undisturbed motion, the variation of η is much more rapid than that of $\bar{R}$. Hence we find:

$$\ddot{\eta} = -\frac{1}{\rho_o}\, 3R^2 \frac{dp}{dv} \cdot 3R^2 \frac{\partial^2 \eta}{\partial M^2} \tag{6.3}$$

Here R^2 should be regarded as locally constant. If the calculation is carried out with constant intervals in mass, this leads to an equation of the type of (2.6). It is still true that upon a change in volume each frequency is proportional to $\left|dp/dv\right|^{1/2}$ and hence the formulae of Section III still apply.

However, in addition the frequencies vary with R, the position of the mass element, and hence we find a dependence of free energy on position. This means a radial force G per unit mass

$$G = -\partial F/\partial R \tag{6.4}$$

Using (3.14)

$$G = - \frac{\rho_o \Lambda}{M} \; T \sum_i \frac{\partial \log \omega_i}{\partial R}$$

and, in view of (6.3)

$$\frac{\partial \log \omega_i}{\partial R} = \frac{4}{R}$$

so that

$$G = - 4T/R \qquad (6.5)$$

Using (4.6) and (3.3):

$$G = - \frac{4 g}{R \rho_o} \; p_{th} \qquad (6.6)$$

In the numerical example illustrated in Fig. 2, g is of the order of 1/4 over a considerable region, hence

$$\rho_o \, G \sim - p_{th}/R \qquad (6.7)$$

If the true pressure varies, for example, as the inverse radius, the true pressure gradient is

$$- p/R$$

and hence the relative error in the acceleration is of the order of p_{th}/p, i.e., the same as the relative error in the pressure itself.

VII. COMPARISON WITH I.B.M. CALCULATIONS

As an illustration of the above relations, typical data were taken from I.B.M. Problem 10, in which a collapsing shell hits a preassembled sphere, the shock returning into the shell being treated by the von Neumann method. One would expect this case to give fairly clean-cut results, since the region considered uses a constant interval size so that no complications arise from changes in the interval, and also since, over the shell, the radius varies by a small factor so that the radial effects described above are not likely to be serious.

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It was found that no clear picture of the fluctuations is obtained by plotting the dynamic quantities against mass at constant time, but that very regular curves result for the volume at a given mass point as a function of time. This is not surprising since the condition $\ell < 1$ means that, for an oscillation with a wave length comparable to the space interval the preiod must necessarily be rather larger than one time interval.

Fig. 6 shows the specific volume for a number of mass points as a function of time. At time zero the impact between the moving and the stationary shell had already taken place and the shock had been carried by analytical calculation to about mass point No. 30. The interface was at point 23. Hence the curves for mass points 23, 24, 26, which are shown in the figure, show initially no fluctuations, but they later acquire some thermal motion from their neighbors. Mass point 30 is just about being shocked when the I.B.M. calculation starts. All following points pass the shock in conditions to which our theory is applicable. One can see that the oscillations of volume are fairly harmonic, and that, in particular, the first minimum ought to represent the mean volume after the shock minus the maximum amplitude, and the curves of Fig. 5 should therefore be applicable.

By reading the volumes ahead of the shock and the first minimum we can then obtain the average volume behind the shock:

Mass Point No.	32	35	40	45
V_1	.70	.69	.69	.71
V_{min}	.44	.42	.42	.43
V	.62	.60	.60	.61

The last line is taken from Fig. 5 by interpolation. The mean volumes behind the shock obtained in this way are in reasonable agreement with averages obtained from Fig. 6 by inspection, with the exception of mass point 32 for which it appears

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that the average by inspection is somewhat lower ; this may be due to the fact that this mass point, being too near to the "cold" compressed region, is losing amplitude by contact with its neighbors, so that our theory overestimates its fluctuations. For the rest, the predictions are well borne out.

A similar test was applied to I.B.M. run A, dealing with the shock returned from the center of a solid implosion. Here the amplitudes of different mass points differ rather widely, no doubt due to the fact that "heat conduction" between different rate groups tends to establish a constant energy per interval. Since over the period of a fluctuation the density changes by an appreciable amount it is not easy to obtain averages "by inspection" but the averages derived in the above manner from the first minimum are comparable with the averages by inspection to the extent to which the latter can be defined.

TABLE I

p_{th} in megabars.

Theoretical Pressures in Model as a

Function of the Final Volume $\overline{v}$ and the Initial Volume v_1.

v_1 \ $\overline{v}$	.9091	.8333	.7692	.7142	.6667	.6250	.5556	.5	.4	.3333	.2857	.25	.2
1.0	.0063	.0331	.158	.541	1.47	3.73	26.85						
.9								15.27					
.8								3.72	88.5				
.75								1.96					
.7								.960	11.7	553			
.65								.369	5.68	55.7			
.6								.163	2.81	21.5	443		
.55									1.05	9.33	60.8		
.5									.339	4.05	22.1	139	
.45									.118	1.59	9.04	39.3	
.4										.344	2.90	13.5	233
.35										.137	.711	4.17	51.2
.3													12.1
.25													1.79

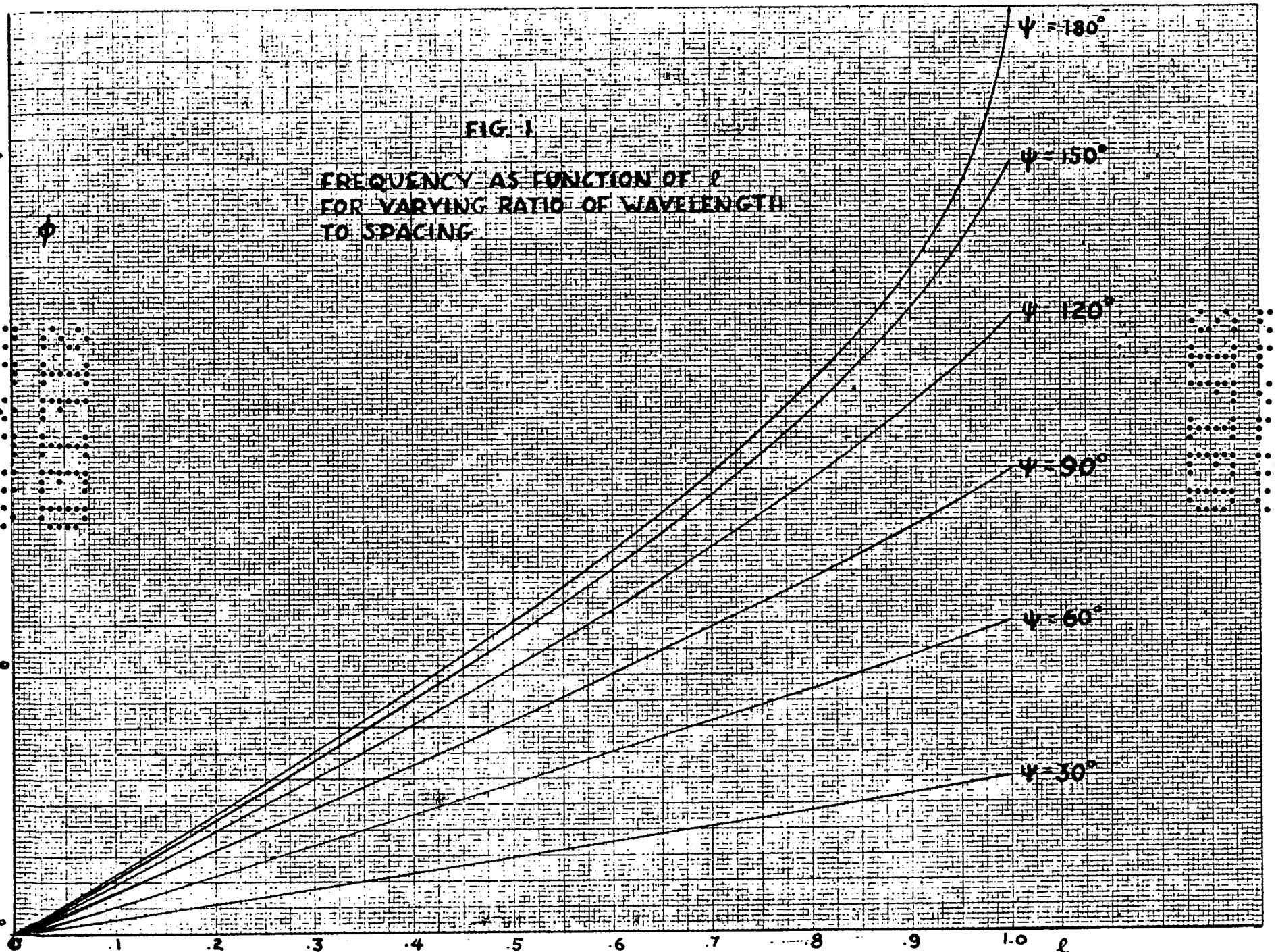

FIG. 1

FREQUENCY AS FUNCTION OF ℓ
FOR VARYING RATIO OF WAVELENGTH
TO SPACING

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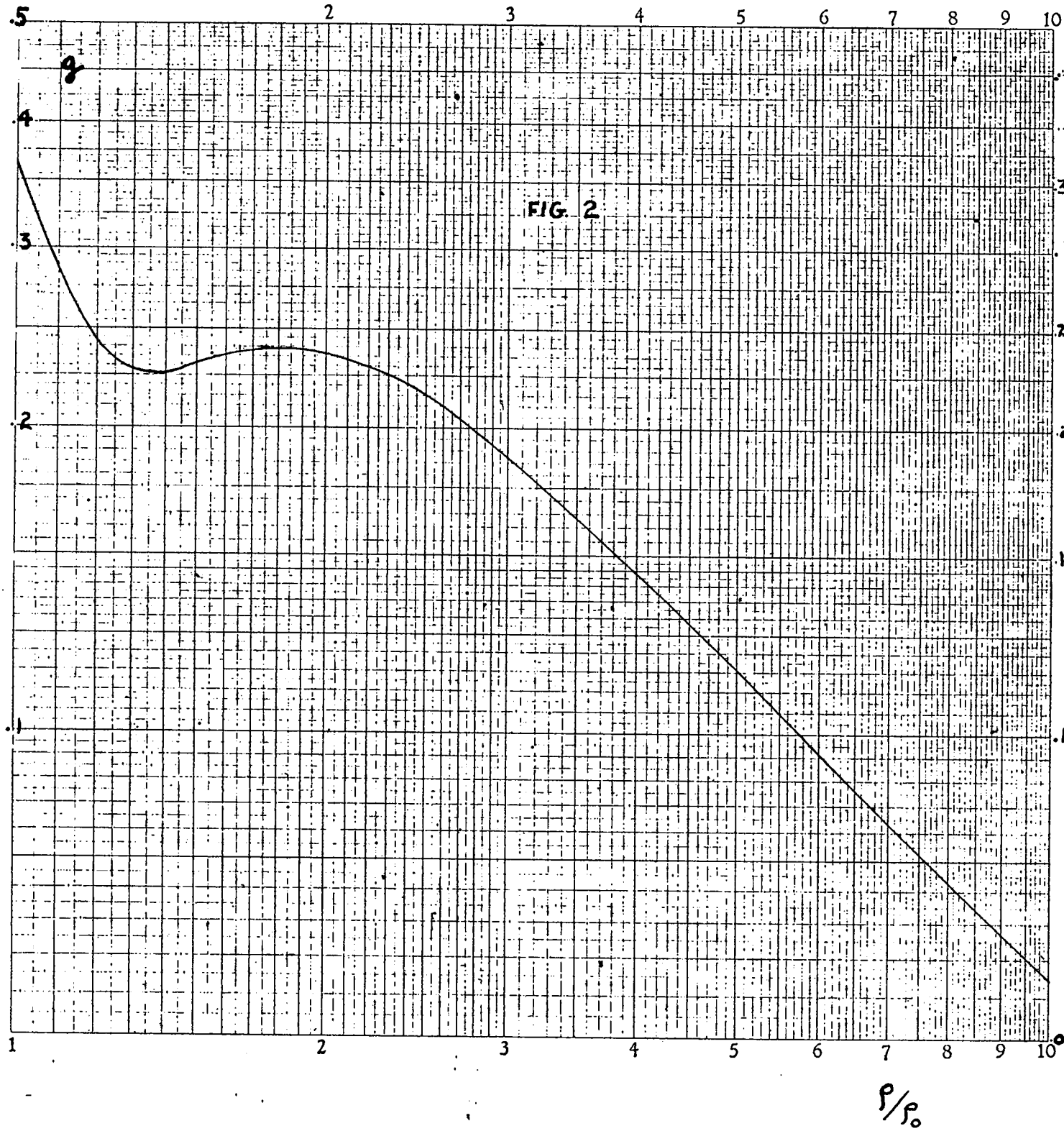

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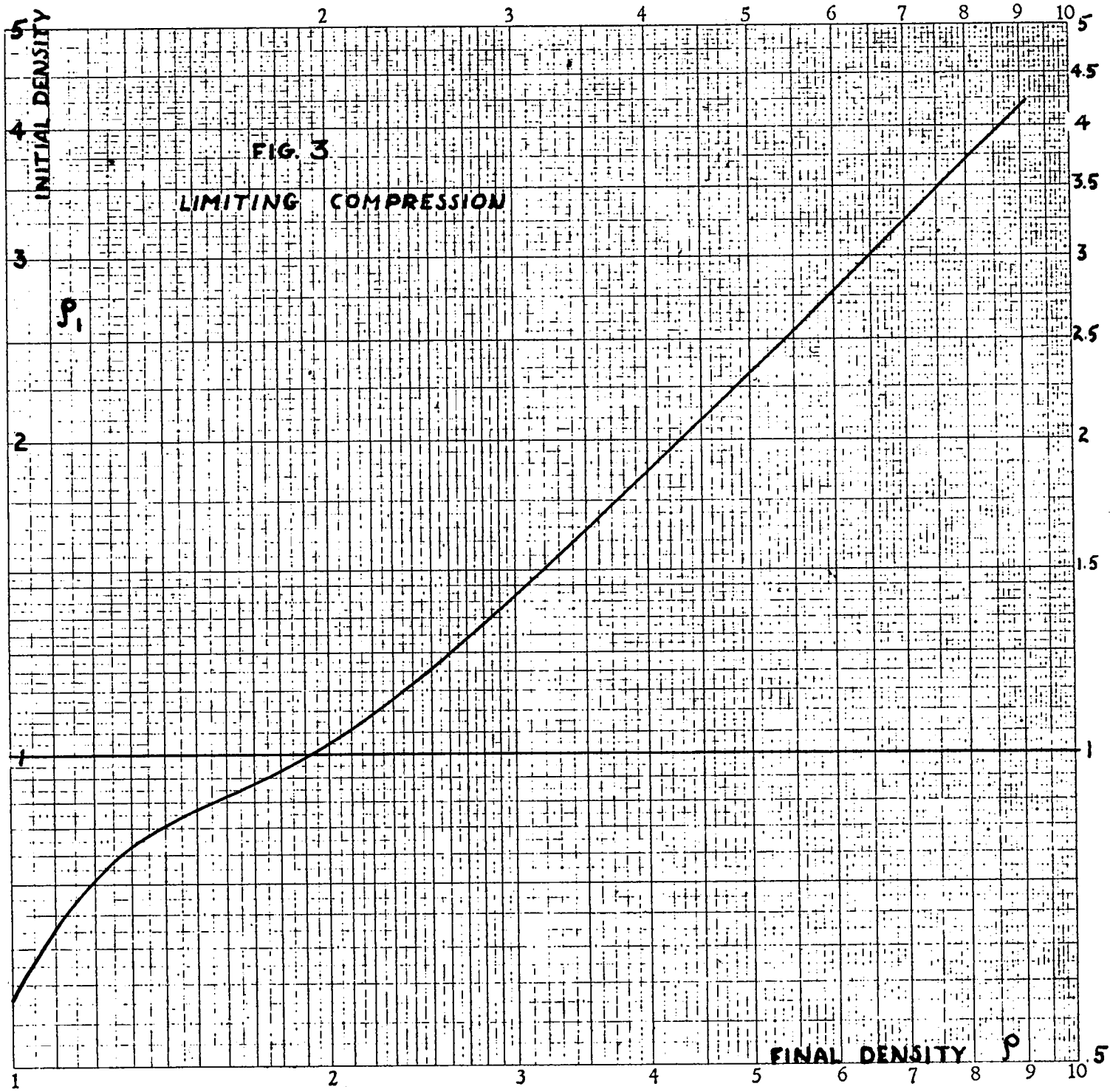


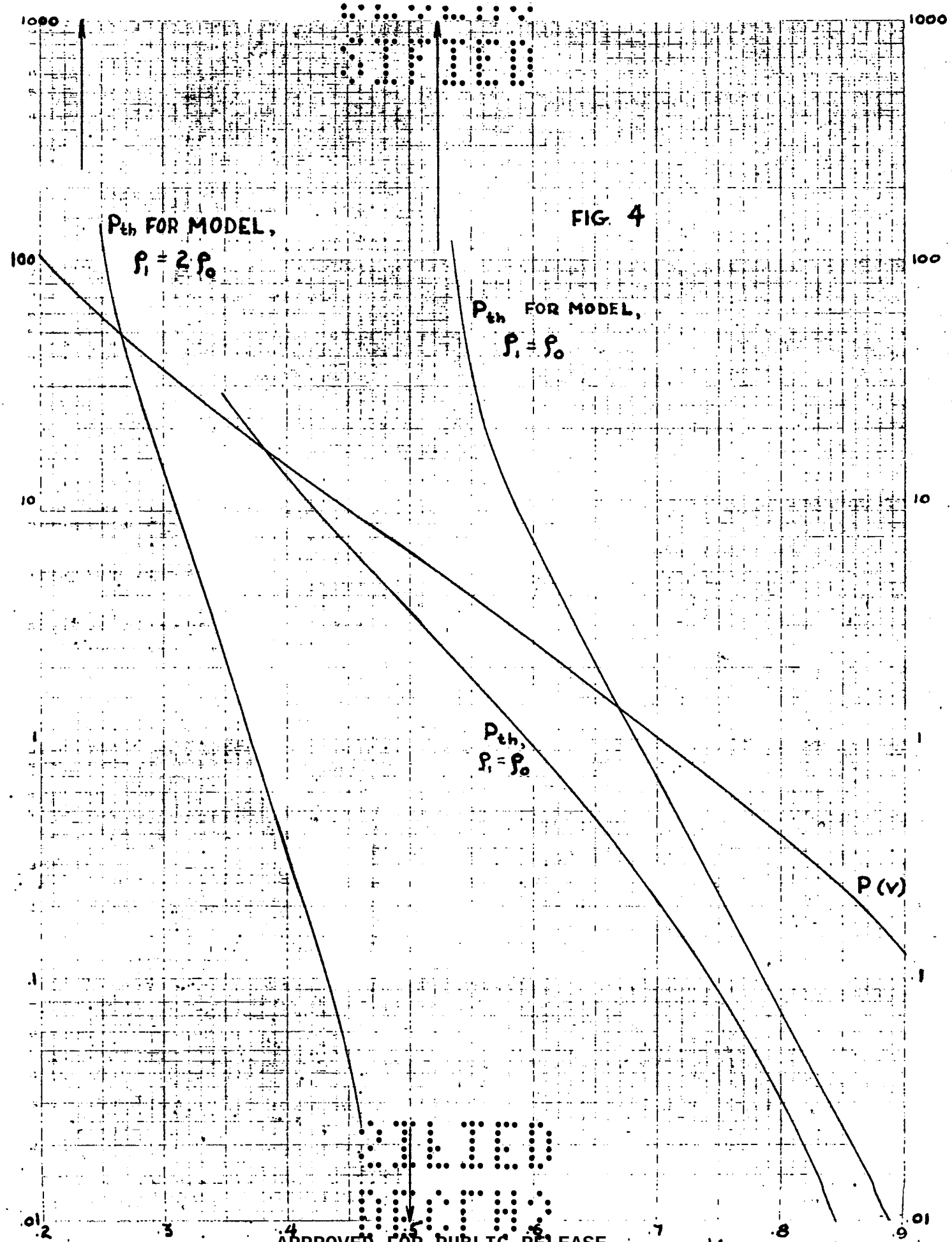
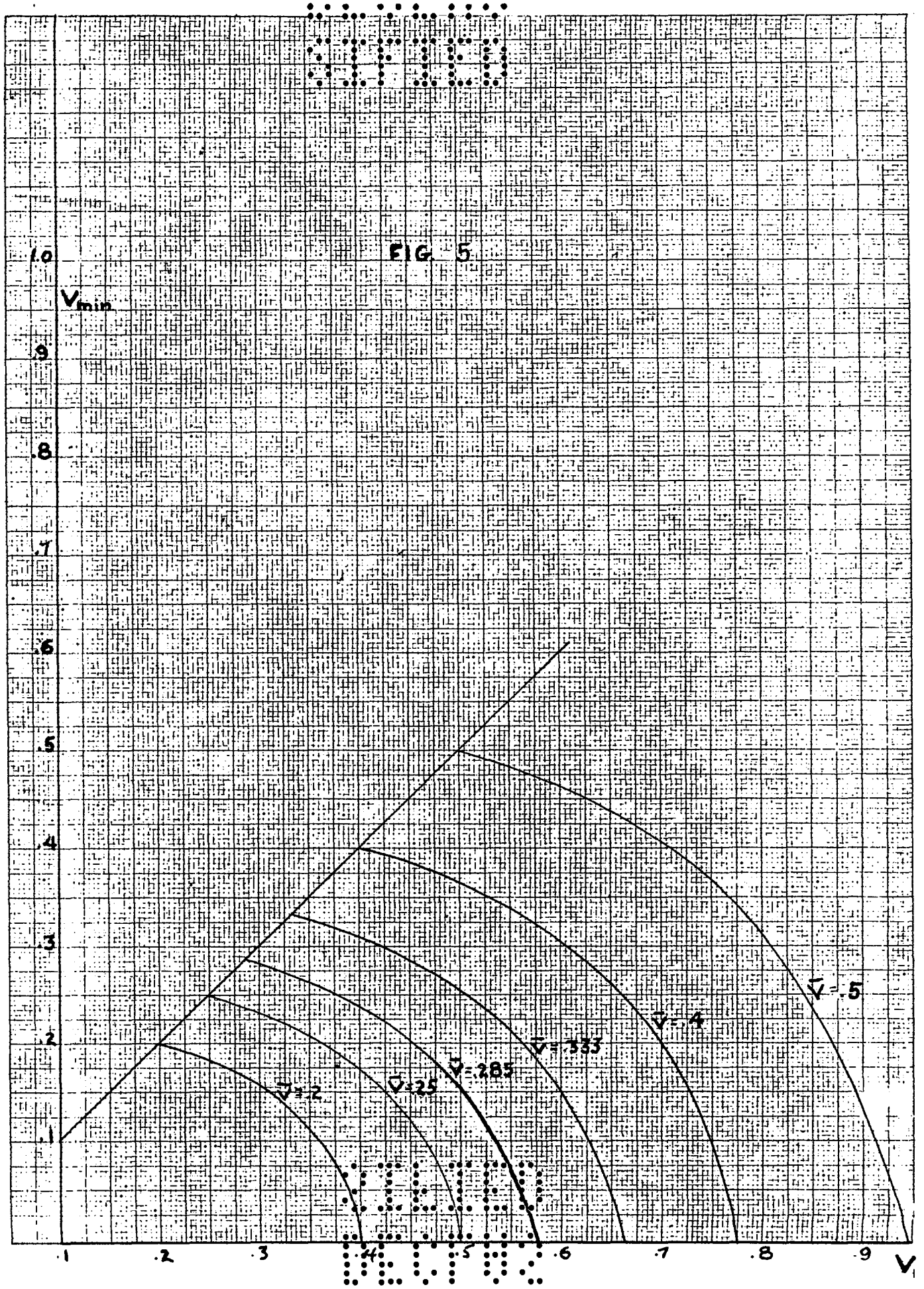


FIG. 4

FIG. 5



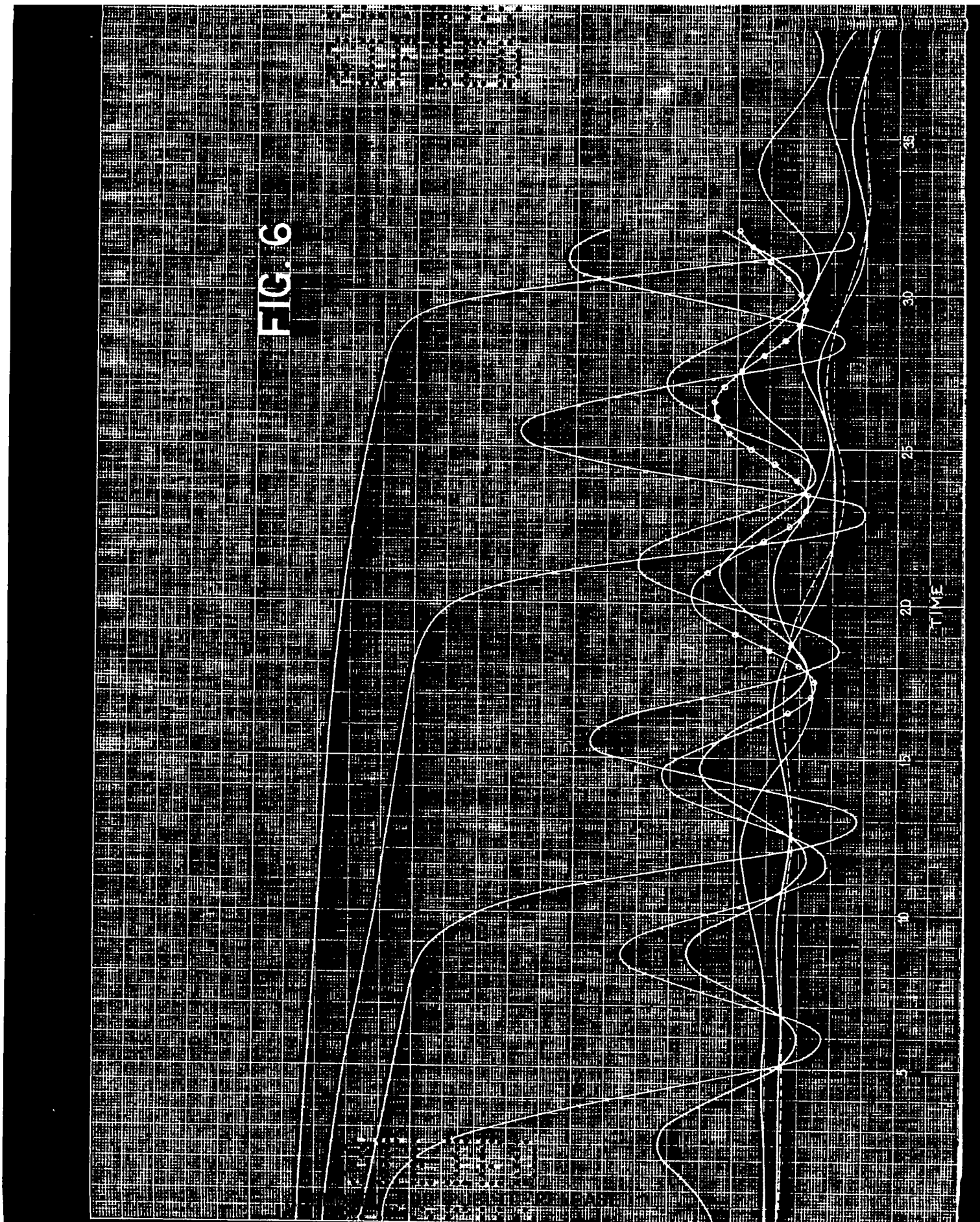


FIG. 6