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APPROXIMATE TREATMENT OF EXPONENTIAL SHOCK AND RAREFACTION WAVES

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ABSTRACT

It is shown how approximate solutions for the exponential shock and rarefaction waves may be obtained with a minimum of effort. Many such solutions are needed for a discussion of the radiation and mixing phenomena associated with the explosion of the gadget. Figs. 1 to 3 contain some results on the isothermal rarefaction wave.

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APPROXIMATE TREATMENT OF EXPONENTIAL SHOCK AND RAREFACTION WAVES

The exponential shock and rarefaction waves were treated rigorously by Serber in LA-14 for arbitrary γ (γ is ratio of specific heats at constant pressure and volume). Further in LA-10, Frankel and Davis give the results of a series of numerical integrations for various shock and rarefaction waves. The numerical integrations carried out in LA-10 -- especially for the rarefaction wave -- were quite laborious and for many discussions of the hydrodynamics of the explosion, in particular in connection with radiation and mixing phenomena, it is necessary to get a wide assortment of shock and rarefaction solutions not previously obtained. It is the purpose of this note to discuss briefly how approximate solutions for exponential shock and rarefaction waves may be obtained and to give some results.

We consider first the rarefaction wave (we use Serber's notation (cf. LA-14) throughout); we write:

$$X = X_{-1}\, e^{\alpha t}\, \phi(\xi) \tag{1}$$

$$p = p_{-1}\, e^{2\alpha t}\, f(\xi) \tag{2}$$

$$\xi = x/X_{-1}\, e^{\alpha t} \tag{3}$$

where X, x are the Eulerian and Lagrangian positions respectively, p the pressure, t the time and X_{-1}, p_{-1}, and α are constants. The head of the rarefaction wave is taken at $\xi = -1$, and the unit of pressure is chosen so that $f(-1) = 1$. The boundary conditions at the head of the wave are that the displacement is zero, the density normal, and the rarefaction wave moves with the velocity of sound; these conditions imply that

$$\phi(-1) = -1 \tag{4}$$

$$\phi'(-1) = 1 \tag{5}$$

$$\frac{p_{-1}}{\rho_{-1} X_{-1}^2 \alpha^2} = \frac{1}{\gamma_{-1}}$$

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In (6) ρ_{-1} is the uncompressed density of the core and γ_{-1} is the γ of the core. The equations of motion and energy conservation become:

$$\xi^2 \phi'' + \phi - \xi\phi' = -(1/\gamma_{-1}) f' \tag{7}$$

$$(2f - \xi f')\phi' - \gamma_{-1}\xi f\phi'' = 2 \tag{8}$$

In (8) use has also been made of the fact that the pressure is continuous across the head of the rarefaction wave.

If one examines (7) and (8), one finds that there is a sheaf-type singularity at $\xi = -1$, i.e., one coefficient in the power series expansion of say, ϕ, is ~~not~~ arbitrary, so that one can get any of an infinite number of solutions depending on the choice of its value. Moreover, since all solutions emerge from the same point $\xi = -1$, it is somewhat difficult to stay on a particular solution if one starts integrating from $\xi = -1$. The more stable method of integration would be to start at $\xi = 0$; but since (7) and (8) constitute a third-order differential equation for ϕ, which at most can be reduced to a second-order equation, and since the boundary conditions are all given at $\xi = -1$, this would become too complicated. However, if one examines the solutions in LA-10, one notices that the temperature never changes more than about 5 to 10%. We therefore assume that the temperature is constant throughout the rarefaction wave; this is equivalent to the assumption that $\gamma_{-1} = 1$ and Eqs. (7) and (8) become:

$$\xi^2 \phi'' + \phi - \xi\phi' = -f' \tag{9}$$

$$f\phi' = 1 \tag{10}$$

Inserting (10) into (9) yields the second-order equation:

$$\phi''(\xi^2 - 1/\phi'^2) + \phi - \xi\phi' = 0 \tag{11}$$

Eq. (11) can be converted into a first-order equation by treating ϕ as the independent variable and $\chi = [\xi\phi' - \phi]$ as the dependent variable; we get:

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$$\frac{d\chi}{d\phi} = \frac{\chi(\phi+\chi)}{(\phi+\chi)^2 - 1} \tag{12}$$

Solutions of Eq. (12) are obtained by assuming a value of ϕ at $\xi = 0$ which we denote by ϕ_0; this determines χ at $\xi = 0$, namely as $(-\phi_0)$ (since ϕ' is finite at $\xi = 0$). Eq. (12) is then integrated from $\phi = \phi_0$ to $\phi = -1$ and $\phi'(\xi)$ is computed from the relation:

$$\phi'(\xi) = \exp \int_{-1}^{\phi(\xi)} \frac{\chi d\phi}{[(\phi+\chi)^2 - 1]} \tag{13}$$

These integrations go very rapidly and some results are presented in Figs. 1 to 3. Figs. 1 and 2 give ϕ and ϕ' as functions of ξ for various assumed ϕ_0's. Fig. 3 is a graph of $\phi'(0)$ as a function of ϕ_0. The solution which is to be used depends on the tamper density; e.g., the solution $\phi_0 = .4741$ fits on to a shock wave with $\gamma = 1$ for equal core and tamper densities. It is interesting to note that the .4741 solution leads to a density at the interface of .45 instead of .44 as in Serber's rigorous treatment.

The exponential shock wave developed in the tamper may be treated approximately in a very simple fashion. Again using Serber's notation (cf. LA-14), we write:

$$X = X_1 e^{\alpha t} \phi(\xi) \tag{14}$$

$$p = p_1 e^{2\alpha t} f(\xi) \tag{15}$$

$$\xi = x/X_1 e^{\alpha t} \tag{16}$$

The point $\xi = 1$ corresponds to the head of the shock wave and there $\phi(1) = 1$, $\phi'(1) = (\gamma-1)/(\gamma+1)$, and $f(1) = 1$. Also, the Hugoniot condition relating the shock velocity to the pressure at the front requires that $p_1/\rho_1 = [2/(\gamma+1)] \alpha^2 X_1^2$ (ρ_1 is the tamper density). The pressure and entropy (energy) equations now become:

$$\xi^2 \phi'' + \phi - \xi\phi' = -[2/(\gamma+1)] f' \tag{17}$$

$$f\phi'^{\gamma} = \left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma} \xi^2 \tag{18}$$

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Since f does not vary much from the front to the back of the shock wave, let us assume that it is constant (and equal to its value at the front) in Eq. (18). Consequently:

$$\phi' = \left(\frac{\gamma-1}{\gamma+1}\right)\xi^{2/\gamma} \quad \text{and} \quad \phi = \frac{\gamma}{(\gamma+2)}\,\frac{(\gamma-1)}{(\gamma+1)}\,\xi^{(2+\gamma)/\gamma} + \frac{2(2\gamma+1)}{(\gamma+2)(\gamma+1)} \tag{19}$$

Substituting the above expressions for ϕ and its derivatives into (17), we find for f (using the fact that $f(1) = 1$):

$$f = \frac{3\gamma+1}{\gamma+1} - \left(\frac{2\gamma+1}{\gamma+2}\right)\xi - \frac{(\gamma-1)}{(\gamma+1)(\gamma+2)}\,\xi^{(2+\gamma)/\gamma} \tag{20}$$

Eqs. (19) and (20) constitute an approximate solution for the shock wave. To test the accuracy of the approximation, we evaluate $\phi(0)$ and $f(0)$ for $\gamma = 1.4$, 1.67 and compare with the rigorous values given in LA-10. We get:

$\phi(0) = .931$, $f(0) = 2.17$ for $\gamma = 1.4$

$\phi(0) = .886$, $f(0) = 2.25$ for $\gamma = 1.67$

The rigorous values are:

$\phi(0) = .944$, $f(0) = 2.20$ for $\gamma = 1.4$

$\phi(0) = .907$, $f(0) = 2.31$ for $\gamma = 1.67$

The agreement is quite good and moreover it is relatively easy to obtain increasingly better approximations. For example, the next approximation to the specific volume (i.e., ϕ') -- found by substituting (20) into (18) -- yields very accurate results.

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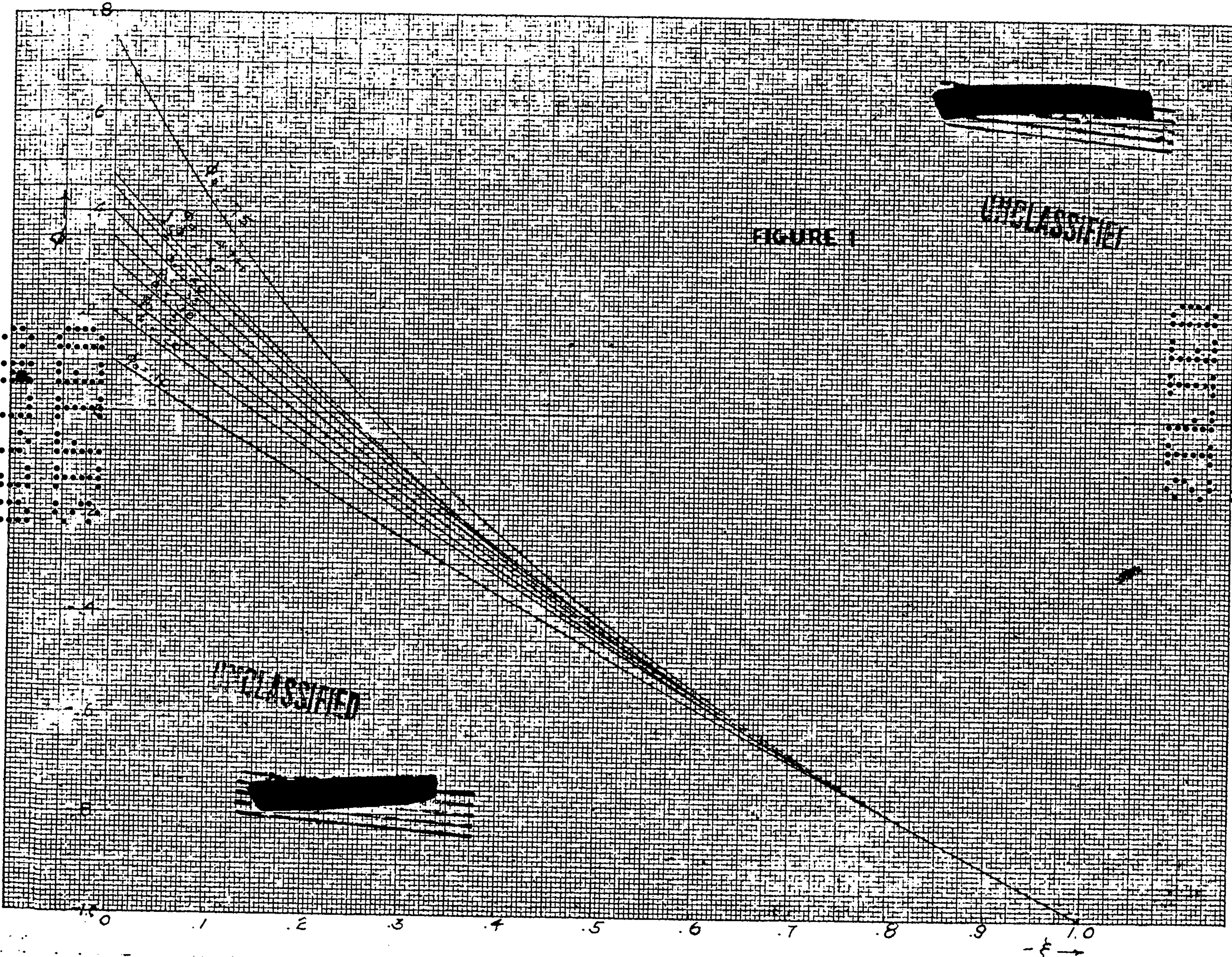

FIGURE 1

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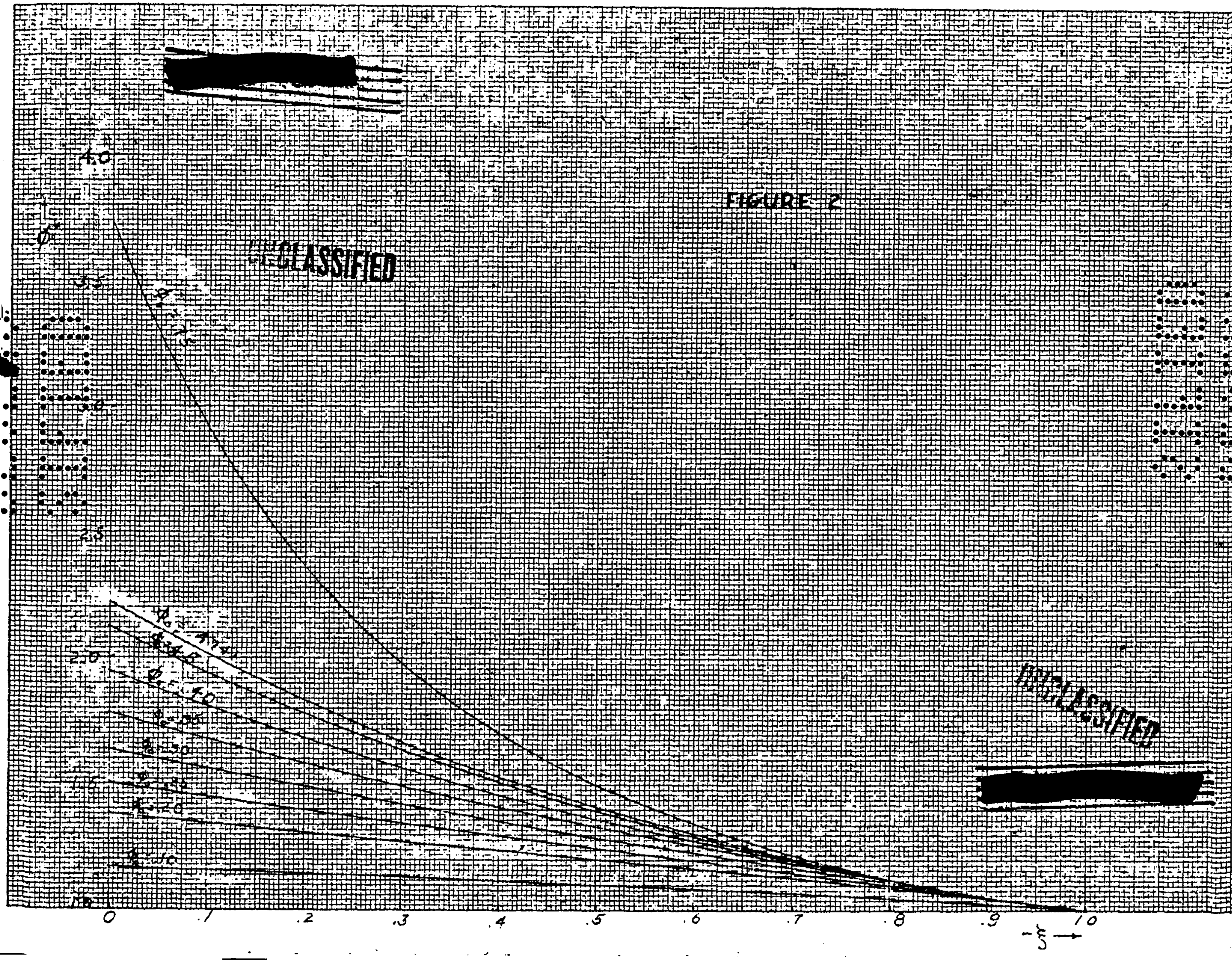

FIGURE 2

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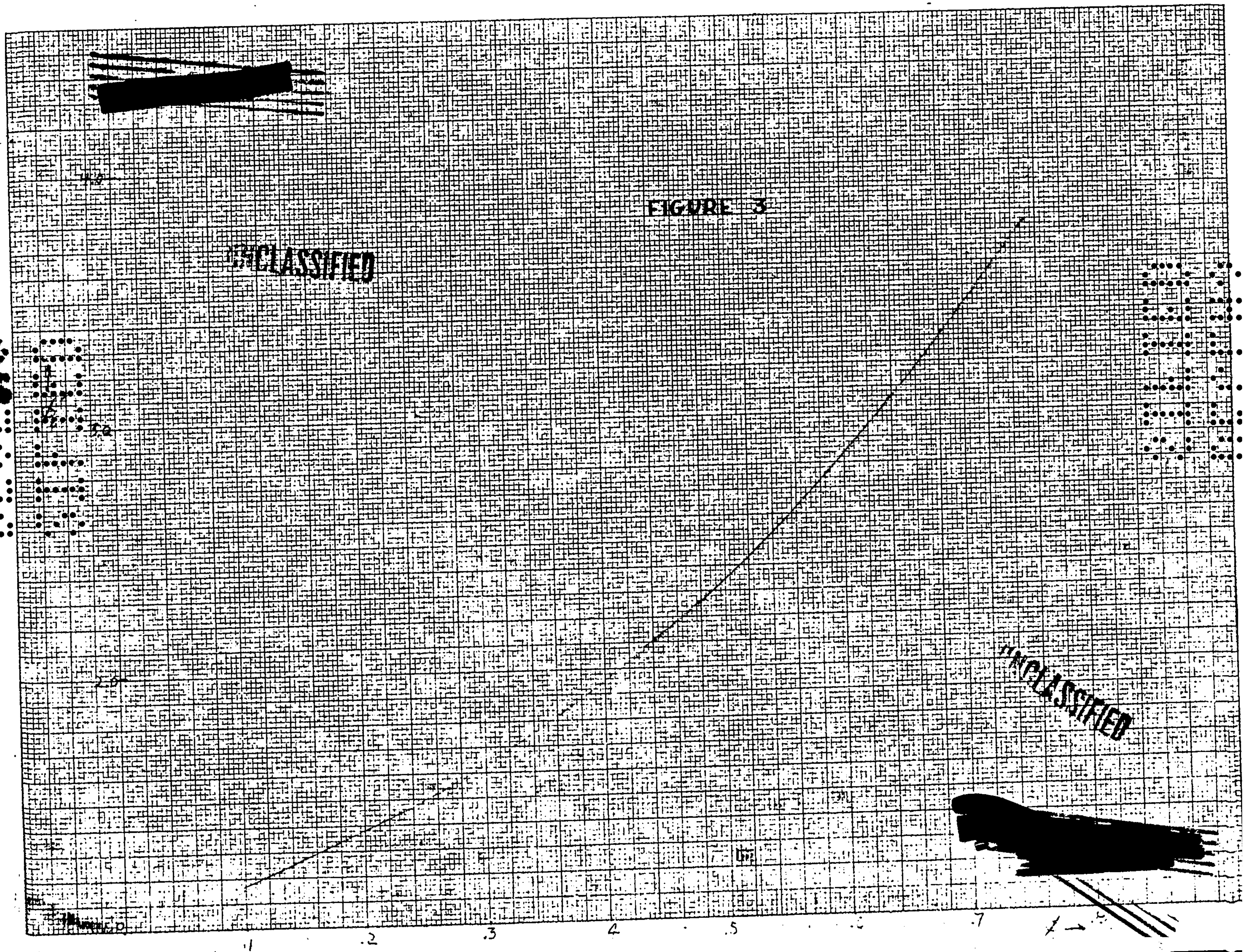

FIGURE 3

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