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INDUCED CURRENTS IN A HEMISPHERICAL SHELL

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ABSTRACT

A mathematical study of the induced currents in a hemispherical shell has been made to serve as background and interpretive material for the experiments of Fitzhugh and Rosen (LA-521). Ed.

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Induced Currents in a Hemispherical Shell.

Chester Snow

1 Object

This problem originates in the picture of a spherically symmetric wave-front of detonation giving the neighboring material a high electrical conductivity κ, which dies out soon after the wave passes. Considering the radius $a(t)$ of the spherical wave-front as fairly well known from detonation theory, the electromagnetic effects of this thin shell moving in an applied magnetic field may be calculated if R_0 is a known constant or known function of the time. The resistance of a cube of the conducting material whose edges have a length equal to the radial thickness Δr of the shell is $R_0 = 1/(\kappa\Delta r)$. Although the effective thickness Δr might actually be an inch, the shell will be treated as infinitely thin as far as concerns the currents induced in it and their magnetic field. In fact the current distribution has been previously formulated in terms of integrals involving the unknown function of the time R_0, taking into account

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also the mutual interaction between the currents in the shell and those induced in an inner (perfectly conducting) sphere. When R_0 is known this gives an explicit formula for the current in the pick-up coil before the metal sphere begins to move as well as later. With large scale, when the implosion proper (of the metal sphere) begins, the pick-up current is not the sum of the two effects that would be produced separately by sphere and shell. There is in addition a term of mutual action which may contribute 3 to 5% to the total record when the shell is expanding. It may contribute 30 to 50% if, as is now supposed, we have to consider a converging wave of detonation.

The photographic record of an alternating current signal received in the pick-up coil in Rosen's experiments are capable (in principle) of determining R_0. In these experiments a thick hemi-spherical shell of explosive is detonated on the outside as symmetrically as possible so that it is supposed to produce a converging wave front of detonation similar to that in implo[illegible] shots except that it is hemispherical. No magnetic field is applied, but instead an exciting circuit is caused to oscillate at [illegible] Kilocycles/sec. This circuit [illegible] coaxial with the pick-up coil and with the axis of symmetry of the thick shell of explosive. Both coils [illegible] are electrically shielded, the shielding metals being cut so they could carry no currents circulating around the axis. It is assumed that the motion of the hypothetical, thin, conducting shell has no effect comparable with that of its mere position in the alternating field of the primary exciting current.

From these views and experiments arises the mathematical problem of finding the distribution of current in a thin hemispherical shell (which is stationary) under the influence of the primary current but also in the presence of the secondary current in the pick-up coil. The latter current, being wholly induced, farther from the shell, and outside it, will have a smaller influence than the primary current upon the current distribution in the shell. However there is no reason to ignore it, as the presence of the secondary does not complicate the problem in any essential manner. If the current distribution in the shell can be found in the presence of one alternating current only, the effect of two or more simultaneous currents oscillating with the same frequency but different phases is found by superposition — due effect of phases being fully taken into account by the use of complex vector potentials.

2. The Two Circuit Equations

The instantaneous primary and secondary currents are the real parts of $I_1 e^{ipt}$ and $I_2 e^{ipt}$ where p is 2π times the frequency and the complex constants I_1 and I_2 are to be found. These circuits will be considered as having all their turns N_1 or N_2 concentrated at the mean turn in each case, these being circles coaxial with the x-axis in the planes x_1, x_2 with radii ρ_1, ρ_2. The traces of these circles in the meridian half-plane ~~are~~ the points P_1 and P_2 in fig 1. It is convenient to refer to any point by its rectangular coordinates x, ρ or in same breath by its polar coordinates r, θ with the same axis and origin.

The conducting shell has for its meridian section the thin quarter-circle AA_0A' of fig 1 of radius a

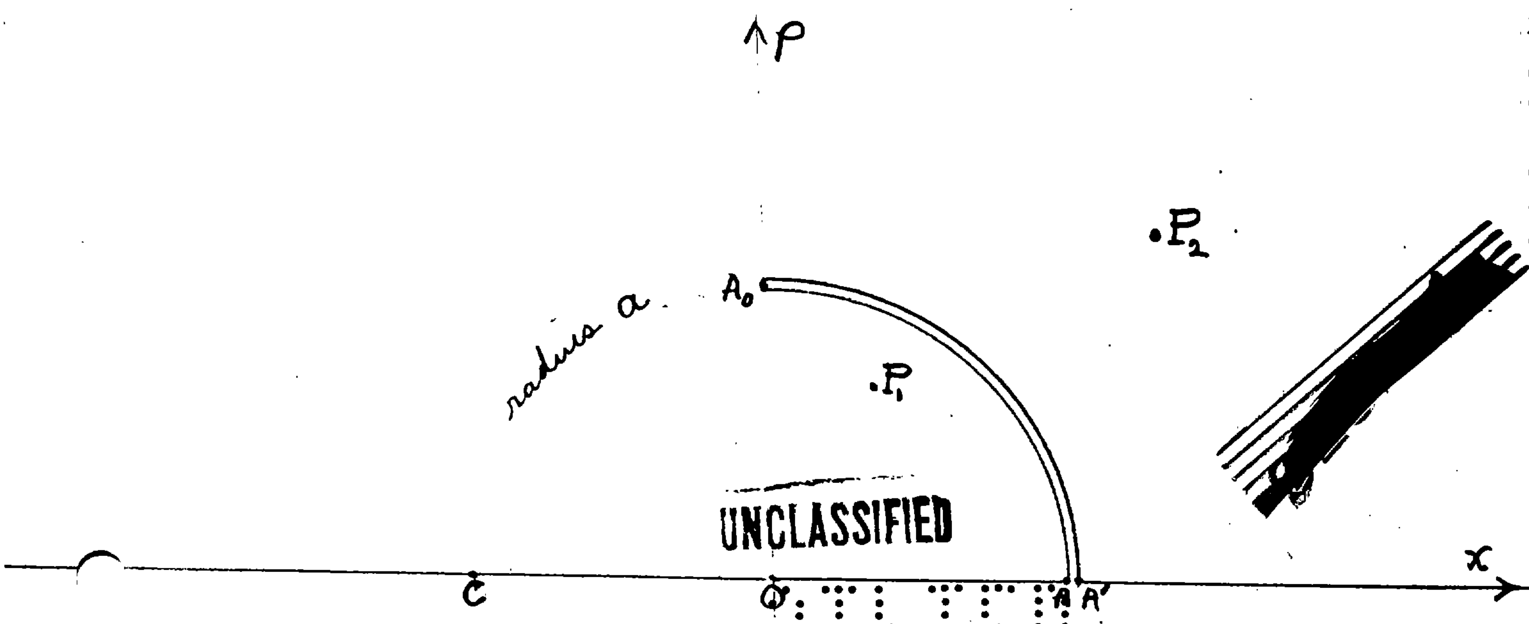

Fig. 1 The x, ρ half-plane and section of hemi-spherical shell.

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The resistance, self-inductance, and capacitance in the two circuits are R_1, L_1, C_1 and R_2, L_2, C_3. These and their mutual inductance M_{12} are all known for the particular frequency $p/2\pi$. An emf $V\cos pt$ = real part of Ve^{ipt} is applied in the primary, V being a known positive constant. The currents in the shell produce magnetic flux through the primary and secondary circuits (in the positive x direction) whose instantaneous values may be denoted by the real parts of $\phi_{13}e^{ipt}$ and $\phi_{23}e^{ipt}$.

The steady periodic state being established, the two elementary circuit equations to determine I_1 and I_2 are

$$\left[R_1 + i\left(pL_1 - \frac{1}{pC_1}\right)\right]I_1 + ipM_{12}I_2 + ip\phi_{13} = V$$

$$ipM_{12}I_1 + \left[R_2 + i\left(pL_2 - \frac{1}{pC_2}\right)\right]I_2 + ip\phi_{23} = 0$$

The constants ϕ_{13} and ϕ_{23} are found (eq (35) below) to be of the form

$$\phi_{13} = -2\pi N_1^2 \rho_1 U_{11} I_1 - 2\pi N_1 N_2 \sqrt{\rho_1\rho_2}\, U_{12} I_2$$

$$\phi_{23} = -2\pi N_1 N_2 \sqrt{\rho_1\rho_2}\, U_{12} I_1 - 2\pi N_2 \rho_2 U_{22} I_2$$

[redacted]

where the three constants U_{11}, U_{22} and U_{12} are special values of a dimensionless, symmetric, function of two points $U(r,\theta; r_1,\theta_1) = U(r_1,\theta_1; r,\theta)$

In equation (31) this is found to consist of a real and imaginary part,

$$U = U^{\infty}(r,\theta;r_1,\theta_1) + i\,\frac{2R_0}{\pi a p}\,V(r,\theta;r_1,\theta_1)$$

(for small values of R_0/ap),
and a method is offered for computing the real functions U^{∞} and V. The case of a shell with infinite conductivity ($R_0 = 0$) makes U a real function, here designated by U^{∞}. A direct formula for its evaluation is found; the computation of the real function V is more laborious.
The meaning of the subscripts is

$$V_{12} = V(r_1,\theta_1;r_2,\theta_2) \text{ etc}$$

With this relation the two circuit equations become

$$\left[R_1+R_1'+i\left(p(L_1-L_1')-\frac{1}{pC_1}\right)\right]I_1+\left[R_{12}'+ip(M_{12}-M_{12}')\right]I_2=V \qquad (1_a)$$

$$\left[R_{12}'+ip(M_{12}-M_{12}')\right]I_1+\left[R_2+R_2'+i\left(p(L_2-L_2')-\frac{1}{pC_2}\right)\right]I_2=0 \qquad (1_b)$$

where

$$\left.\begin{aligned} R_1' &\equiv 4R_0N_1^2\frac{\rho_1}{a}V_{11} & L_1' &\equiv 2\pi N_1^2\rho_1U_{11}^{\infty}\\ R_2' &\equiv 4R_0N_2^2\frac{\rho_2}{a}V_{22} \quad \text{and} & L_2' &\equiv 2\pi N_2^2\rho_2U_{22}^{\infty}\\ R_{12}' &\equiv 4R_0N_1N_2\frac{\sqrt{\rho_1\rho_2}}{a}V_{12} & M_{12}' &\equiv 2\pi N_1N_2\sqrt{\rho_1\rho_2}\,U_{12}^{\infty}\end{aligned}\right\} \quad (2)$$

Electromagnetic cgs units are used in the mathematical equations, so these are the units which apply in the preceeding equations. To use them with practical units, remembering that $\hat{V}$ and V are dimensionless, it is only neccessary to multiply the definitions of L_1', L_2' and M_{12}' by 10^{-9} to convert them to henries. Then R_0 and all the initial coil constants may be taken in practical units.

The expression for V is derived on the assumption that R_0/ap is so small that terms in its square ~~may~~ be neglected in comparison with terms in R_0/ap. From the ~~large~~ difference between electromagnetic effects observed with copper shells from those observed ~~in~~ shots it is thought that this assumption is good.

The inductive, or geometrical, constants L_1' L_2', M_{12}' depend only upon the positions of the points P_1 and P_2 and the radius a of the shell, but involve no material constants. The same is true of V_{11}, V_{22}, V_{12} so the resistances R_1', R_2', and R_{12}' are all proportional to the unknown material constant $R_0 = 1/\kappa\Delta r$.

Consequently as soon as the positions and coil-constants and the shell radius a are known the amplitude of the signal received in the pick-up coil (which is the absolute value $|I_2|$ of the complex constant I_2) could be ~~[illegible]~~ for several values of R_0 and a graph drawn plotting $|I_2|$ against R_0. The value of R_0 where the ordinate of this curve reaches the observed value of $|I_s|$ is the R of the shell.

It is to be expected that the functions V^∞ and V will turn out to be roughly of the same order of magnitude in the sense that ρ_1/a and a/ρ_2 are of the same order. This phrase is rather loose and will be applied when say $\rho_1/a \leq 1/$[illegible] and $a/\rho_2 \leq 1/2$. (In one of Rosen's thin copper hemispheres $a = 2.5''$, $\rho_1 = 13/16''$ and $\rho_2 = 9.5''$).

3. The Static Experiments.

The static experiments with various metallic hemispherical shells require for their theory only eq (1g) in which $|I_1| = 1$
The thing determined is the amplitude $|I_2|$ in the secondary when $|I_1|=1$ under two conditions (without and with the shell).
In the first case R'_{12}, R'_2, M'_{12}, and L'_2 are zero, and if the amplitude $|I_2|$ had been measured in amperes, this would give a check upon the correctness of the initial coil-constants R_2, L_2, M_{12} and C_2, but nothing more.

In the second case all four constants R'_{12}, R'_2, L'_2 and M'_{12} are in eq (1g) but only R'_{12} and R'_2 vary with the material constant R_0 which is known. This measurement of $|I_2|$ would be practicable for the determination of R_0 (if it were unknown), only if R_0 is so large that the two resistances R'_{12} and R'_2 are of appreciable importance compared with the other terms in eq (1g). If they are not important this means that the curve of $|I_2|$ against R_0 is so flat as to be practically useless.

To say whether or not this is the case would require that computations be made of V_{22}, V_{12}, U^{∞}_{22} and U^{∞}_{12} for this particular configuration.

The thinnest copper hemisphere used had a radial thickness $\Delta r = .015'' = .0381$ cm. The resistivity of copper at 20°C is about $1.72 \cdot 10^{-6}$ ohm-cm so that $R_0 = .000045$ ohms. It might appear at first sight that this indicates the shells to be perfect conductors as far as this method is concerned. However there is in eq(2) a large factor N_1 (= 45 turns) in the formula for R_{12}'. The corresponding term in the formula for R_2' is N_2^2 where $N_2 = 3$. But this term appears added to R_2 which is also very small. Before saying that R_2' is of no importance it would be necessary to consider ~~the~~ relative ~~magnitude~~ of the term $p(L_2 - L_2') - \frac{1}{pC_2}$ in which it must be remembered that L_2' and M_{12}' are to be multiplied by 10^{-9}.

Until such computations are made all we can say is that we share the general feeling that the shells are, for this purpose, practically perfect conductors, which means that the observed decrease of $|I_2|$ ~~(of about 88%)~~ which was ~~produced~~ by inserting the shell is practically all to be accounted for by the appearance of the constants L_2' and M_{12}' in eq(1a), which depend upon V^∞ and are independent of material constants. If so these experiments would serve only as a check upon the computations of L_2' and M_{12}'.

~~If the experimental difficulty of obtaining shells of materials~~ ~~with sufficiently large R_0 could be overcome then the method~~ ought to be a practicable method of substitution, by which the R_0 of a detonation wave-front could be found by comparison. This would avoid the need for any but elementary mathematics and computation.

The only alternative seems to be the mathematical problem followed by a serious job of computation which however is fairly straightforward.

The dynamical effects are so different from the static ones that it seems safe to conclude that R_0 for the explosive wave is large enough to make $|I_2|$ sensitive to its changes. This of course may be quite compatible with the assumption that R_0/ap is small since $p = 4 \times 10^6$.

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4. The Boundary Problem

In this section consider the shell in the presence of only one oscillating current whose trace may be any point P_1 not on the shell. Since the effect of capacity currents have been eliminated by electrical shielding, the only component of current in the shell which need be considered is ~~that~~ circulating around the axis. ~~The volume density~~ of this component may be called J_ϕ. Since the only component of the vector potential will then be A_ϕ it is written A.

The law of conduction is

$$J_\phi = -\kappa D_t A$$

or

$$j_\phi \equiv J_\phi \Delta r = -(\kappa \Delta r) D_t A \equiv -\frac{1}{R_0} D_t A$$

where j_ϕ is the equivalent surface density of current when we consider the shell infinitely thin. In that case the vector potential is continuous at the conducting arc $r = a$, $\pi/2 > \theta > 0$. Its normal derivatives have a discontinuity there which is a measure of j_ϕ in accordance with the circuital relation

$$H_\theta(a+0, \theta) - H_\theta(a-0, \theta) = 4\pi j_\phi$$

Since the magnetic field H is the curl of A the

Tangential component is given by

$$H_\theta(r,\theta) = -D_r A(r,\theta) - A(r,\theta)/r$$

The circuital relation is therefore

$$[D_r A(r,\theta)]_{r=a+0} - [D_r A(r,\theta)]_{r=a-0} = -4\pi j_\phi$$

This with the law of conduction gives the boundary condition at the conducting arc

$$[D_r A(r,\theta)]_{r=a+0} - [D_r A(r,\theta)]_{r=a-0} = \frac{4\pi}{R_0} D_t A(a,\theta)$$

These are real, instantaneous relations. In the case of a steady periodic state the ~~instantaneous~~ vector potential may be taken as the real part of Ae^{ipt} so that $D_t A = ipA$

With this understanding from here on, A will denote a complex function of position, independent of the time.

At all ordinary points it satisfies the partial differential equation

$$\left(D_x^2 + D_\rho^2 + \frac{1}{\rho} D_\rho - \frac{1}{\rho^2}\right)A = 0 = \left(D_x^2 + D_\rho^2 - \frac{3}{4\rho^2}\right)\sqrt{\rho}A$$

which is the same as

$$r D_r\left[r D_r(\sqrt{\rho}A)\right] + \left(D_\theta^2 - \frac{3}{4\sin^2\theta}\right)\sqrt{\rho}A = 0$$

Also $A \to 0$, like ρ when $\rho \to 0$

and $A \to 0$, like $\sin\theta/r^2$ when $r = \infty$.

The one remaining condition describes the nature of the singularity, or manner in which A becomes infinite when the variable point $P(r,\theta)$ approaches the fixed source-point $P_1(r_1,\theta_1)$. This will appear after some preliminaries which should be brought in at this point. They result in the replacement of the dependent function A by a function U.

Since the frequency is in the radio range (as distinguished from the optical range) the quasi-stationary equations of the electromagnetic field apply. This means that when a current changes we ignore the time taken for the change of field to be propogated to any point in space and therefore compute instantaneous fields at all distant points by the stationary formulas. (The phase relationships are properly accounted for by use of the complex potential).

It is known that the stationary vector potential at any point $P(x,\rho)$ (or $P(r,\theta)$) due to a unit steady current, (circulating around the x axis in the positive sense) in a circle of radius ρ_1, coaxial with the x-axis in the plane x_1 is

$$2\sqrt{\frac{\rho_1}{\rho}}\,Q_{1/2}\left(1+\frac{D^2}{2\rho\rho_1}\right)=2\sqrt{\frac{\rho_1}{\rho}}\,Q_{1/2}\left(\frac{\frac{r^2+r_1^2}{2rr_1}-\cos\theta\cos\theta_1}{\sin\theta\sin\theta_1}\right)$$

where the distance, measured in the meridian plane of fig 1, between the two points P and P_1 is D.

$$D^2 = (x-x_1)^2 + (\rho-\rho_1)^2.$$

$Q_{1/2}(z)$ ~~is the~~ the accepted symbol for the second kind of Legendre function of z with parameter $1/2$. This function of the argument indicated is always positive, except that it vanishes when one of the points moves to the x axis or to infinity. It is always finite except when the points approach coincidence in which case it goes to $+\infty$, and in such a manner that

$$Q_{1/2}\left(1+\frac{D^2}{2\rho\rho_1}\right) = \log 1/D \text{ + terms which are finite when } D=0.$$

The mutual inductance M between two parallel circles whose traces are the points P_1, P_2 is given by

$$\frac{M}{4\pi\sqrt{\rho_1\rho_2}} = Q_{1/2}\left(1+\frac{D_{12}^2}{4\rho_1\rho_2}\right) = 2(K-E)/\kappa - \kappa K$$

where the modulus of the complete elliptic integrals K and E is given by

$$\kappa^2 = \frac{4\rho_1\rho_2}{D_{12}^2+4\rho_1\rho_2} = \frac{4\rho_1\rho_2}{(x_1-x_2)^2+(\rho_1+\rho_2)^2}$$

There are tables available for getting numerical values of M directly without the use of elliptic integrals.

The mutual inductance M_{12} appearing in eq (1a) (1c) is

$$M_{12} = N_1 N_2 M \; 2\,Q_{1/2}\left(\frac{\frac{\sin^2\gamma_{12}}{2\gamma_{12}}-\cos\theta_1\cos\theta_2}{\sin\theta_1\sin\theta_2}\right)$$

The function $Q_{1/2}\left(1+\frac{D^2}{2\rho\rho_1}\right)$ is a symmetric function of the two points which is the fundamental function for the solution of potential problems of this type in which the boundary conditions are given on surfaces of revolution. There are an infinite number of orthogonal coordinate systems (α, β) which are "separable" coordinates for the potential equation. This means that α and β are such conjugate functions of x and ρ that the ~~potential~~ differential equation for the potential (of which that for A is a special case) may be reduced to ordinary differential equations.

In each of these systems $Q_{1/2}$ has a so-called addition-theorem, which is the canonical expansion in normal functions of $Q_{1/2}$, this being the nucleus of the integral equation which is the key to the solution of such problems.

With polar coordinates this canonical expansion is

$$Q_{1/2}\left(\frac{\frac{r^2+r_1^2}{2rr_1}-\cos\theta\cos\theta_1}{\sin\theta\sin\theta_1}\right)=\pi\sqrt{\sin\theta.\sin\theta_1}\sum_{n=1}^{\infty}\left(\frac{r}{r_1}\right)^{n+1/2}\frac{P_n^1(\cos\theta)P_n^1(\cos\theta_1)}{n(n+1)}$$

when $0 \leq r \leq r_1$, these being interchanged otherwise.

The set of functions $P_n^1(\cos\theta)$ are associated Legendre functions these being a complete set of normal functions for a spherical surface but not for a hemi-spherical one.

The boundary problem here of interest can be formulated so neatly ~~in~~ terms of polar coordinates as to tempt one to try for its solution with them. Results are disappointing for the expansion just written is not an expansion in normal functions for this hemispherical problem. One is driven to search for a coordinate system (α, β) in which such an expansion for $Q_{1/2}$ is obtainable.

The total complex vector potential produced at any point $P(r, \theta)$ by the complex current I_1 in the primary and the currents which it induces in the shell may be written

$$A(r,\theta) = N_1 I_1 \sqrt{\frac{\rho_1}{\rho}}\, G(r,\theta; r_1,\theta_1) \qquad (3)$$

and this (Green's) function G may be taken in the form

$$G(r,\theta; r_1,\theta_1) = 2\,Q_{1/2}\left(1 + \frac{D^2}{2\rho\rho_1}\right) - U(r,\theta; r_1,\theta_1) \qquad (4)$$

Since we regard the source-point $P_1(r_1, \theta_1)$ as fixed anywhere but on the conducting arc we may suppress its coordinates and write $G(r,\theta)$ and $U(r,\theta)$ where the point $P(r,\theta)$ ~~ranges freely~~.

From the partial differential equations derived above for the vector potential A it is evident that $G(r,\theta)$ is determined by the following equations

$$\left. \begin{aligned} &\left(\mathcal{D}_x^2 + \mathcal{D}_\rho^2 - \frac{3}{4\rho^2}\right) G = 0 \\ \text{that is} \quad &r\,\mathcal{D}_r (r \mathcal{D}_r G) + \mathcal{D}_\theta^2 G - \frac{3G}{4\sin^2\theta} = 0 \end{aligned} \right\} \qquad (5_a)$$

$$\left. \begin{aligned} &G \to 0 \text{ like } \rho^{3/2} \text{ when } \rho \to 0 \\ &G \to 0 \text{ like } (\sin\theta/r)^{3/2} \text{ when } r \to \infty \end{aligned} \right\} \qquad (5_b)$$

$$G(a+0,\theta) = G(a-0,\theta) \quad \text{for } \pi \geq \theta \geq 0 \qquad (5_c)$$

$$\mathcal{D}_r G(r,\theta)_{a+0} - \mathcal{D}_r G(r,\theta)_{a-0} = 0 \quad \text{for } \pi \geq \theta > \frac{\pi}{2} \qquad (5_d)$$

$$\mathcal{D}_r G(r,\theta)_{a+0} - \mathcal{D}_r G(r,\theta)_{a-0} = \frac{4\pi i p}{R_0} G(a,\theta) \quad \text{for } \pi/2 > \theta \geq 0 \qquad (5_e)$$

$$G \to \infty \text{ like } 2\log 1/D \text{ when } P \to P_1 \qquad (5_f)$$

Since the Q-functions in eq(4) prescribes the nature of the singularity for the Green's function, it is evident that $U(r,\theta)$ must be finite and continuous for all positions of the point P including the case where it becomes identical with the fixed source-point P_1. Hence the equations to determine U, are the same as the above with the omission of the last and the replacement of (5_e) by

$$U(a,\theta) - \frac{R_0}{4\pi i p}\left[\mathcal{D} U - \mathcal{D} U\right] = 2\,Q_{1/2}\left(\frac{\frac{a^2+r_1^2}{2ar_1} - \cos\theta\cos\theta_1}{\sin\theta\sin\theta_1}\right) \quad \text{for } \frac{\pi}{2} > \theta \geq 0 \qquad (5_e')$$

This boundary condition is the only place through which the source P_1 enters in the determination of V or G. Through it they will depend upon (r_1, θ_1) and the reciprocal theorem foretells their general ~~[illegible]~~ ~~[illegible]~~ dependence.

Its proof does not depend upon the shape of the conducting arc which could be any curve. The particular values of R_0 do not enter the proof: it could be real or complex, zero or infinite (if infinite the boundary condition at $r = \infty$ needs an obvious relaxation to permit the ~~current~~ from the source to escape to infinity)

To prove it, make a cut along the conducting arc and consider both sides of this cut as part of the external boundary of the x-ρ half-plane, the remainder of the external boundary being the entire x-axis and a semi-circle with center and the origin and radius which ultimately becomes infinite. Insert two internal boundaries in the shape of two circles each of radius ϵ, one with center at the source P_1 the other with center at any other point P_2 distinct from P_1, neither being on the cut.

Green's integral transformation may be written

$$\iint (u\nabla^2 v - v\nabla^2 u)\,dx\,d\rho = -\int \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds$$

where $\nabla^2 = D_x^2 + D_\rho^2$ and the surface integral is taken over

the half-plane with the external and internal boundaries specified. The line integral is around the complete boundary with the normal pointing inward. At points within the complete boundary u and v together with their first and second derivatives must be finite and continuous. These conditions are satisfied by
$u(x,\rho) = G(x,\rho; x_1,\rho_1)$ and $v(x,\rho) = G(x,\rho; x_2,\rho_2)$
In the limit $\epsilon = 0$ this transformation gives

$$G(x_1,\rho_1; x_2,\rho_2) = G(x_2,\rho_2; x_1,\rho_1)$$

every one of the six equations of the system (5) being necessary for the proof. Since the Q-function in (4) is obviously symmetric, the significant form of the reciprocal theorem is that U is unaltered by interchange of source-point and receiving points, that is

$$U(r,\theta; r_1,\theta_1) = U(r_1,\theta_1; r,\theta) \qquad (6a)$$

Although the proof assumed that neither P nor P_1 is on the cut, it is true when one but not both are on it, (from continuity).

A second relation, the inversion theorem, is quite independent of the reciprocal theorem. It also is independent of the particular value of R_0, but does depend upon the fact that the cut is some part of the arc of the semi-circle, $r=a$. It need not be the quarter-circle of fig 1 but could be the arc $\theta_0 > \theta > 0$ where θ_0 is any angle less than π.

It is easily verified that all the equations which determine $U(r,\theta)$ including ($5_e'$) retain the same form in the variables r', θ when we make the substitution $r = a^2/r'$. Two points $P(r,\theta)$ and $P'(r',\theta')$ are inverse points (or images one of the other) by reflection in the semi-circular mirror $r = a$, if

$$rr' = a^2 \quad \text{and} \quad \theta' = \theta.$$

Consequently if P_1 is any fixed point not on the cut the inversion theorem states that it induces such a current distribution on the shell, that the value of U at every point P is the same as at the image P' of P or

$$U(r,\theta;r_1,\theta_1) = U\left(\frac{a^2}{r},\theta;r_1,\theta_1\right). \qquad (6_e)$$

Applying one theorem after the other shows that the value of $U(r,\theta)$ at all points in the half-plane is unaltered by moving any source-point P_1 to its image P_1'.

These two theorems account for the preference in studying the symmetrical function U rather than the vector potential A. By their combined use it is evident that the boundary problem for U may be stated in the following simpler form.

Let $P_1(r_1,\theta_1)$ be any point not on the cut.

A function $V(r,\theta)$ is required only inside the semi-circle $r=a$, $\pi \geq \theta \geq 0$ which is determined by

$$\left.\begin{aligned} &\left(D_x^2 + D_\rho^2 - \frac{3}{4\rho^2}\right)V = 0 \\ \text{that is}\quad &r D_r(r D_r V) + D_\theta^2 V - \frac{3V}{4\sin^2\theta} = 0 \end{aligned}\right\} \qquad (7a)$$

$$V \to 0 \text{ like } \sin^{3/2}\theta \text{ when } \sin\theta \to 0 \qquad (7b)$$

$$D_r U(r,\theta) \to 0 \text{ when } r \to a-0 \text{ if } \pi \geq \theta > \pi/2 \qquad (7c)$$

$$U(a-0,\theta) + \frac{R_0}{2\pi i p} D_r U(r,\theta)_{r=a-0} = 2Q_{1/2}\left(\frac{\frac{a^2+r_1^2}{2ar_1} - \cos\theta\cos\theta_1}{\sin\theta\sin\theta_1}\right) \text{ for } \frac{\pi}{2} > \theta \geq 0 \qquad (7d)$$

The partial differential equation with rectangular coordinates (x,ρ) has been retained because it will be found to be the simpler ~~form~~ to transform into ~~[illegible]~~ (α,β).

5. Cyclidic Coordinates

Let $z = x + i\rho = re^{i\theta}$ be a complex variable, and make a cut in its half-plane along the circular arc, $r = a$, from $\theta = 0$ to $\theta = \theta_0$, where the constant θ_0 is any angle between zero and π. There is no appreciable complication ~~in~~ in this generalization of the quadrantal arc of fig 1. We may later place $\theta_0 = \pi/2$.

The half-plane thus cut may be represented conformally upon a semi-infinite strip ~~of~~ the plane of ~~a~~ complex variable $\omega = \alpha + i\beta$. We take this strip as $0 < \alpha < \pi$ and $0 < \beta < \infty$. The conformance is shown by similar lettering of points in figures ~~2a~~ and 2b.

The equation of transformation is

$$z = -a\left(\frac{\sin\omega - i\sinh\gamma}{\sin\omega + i\sinh\gamma}\right) \tag{8}$$

from which
$$\frac{dz}{d\omega} = -i\,\frac{2a\sinh\gamma\cos\omega}{(\sin\omega + i\sinh\gamma)^2}$$

whence
$$\frac{1}{h} \equiv \left|\frac{dz}{d\omega}\right| = \frac{2a\sinh\gamma\sqrt{\cosh^2\beta - \sin^2\alpha}}{[\cosh(\beta-\gamma) + \cos\alpha][\cosh(\beta-\gamma) - \cos\alpha]}$$

that is
$$\sqrt{dx^2 + d\rho^2} = \frac{\sqrt{d\alpha^2 + d\beta^2}}{h} = \frac{2a\sinh\gamma\sqrt{\cosh^2\beta - \sin^2\alpha}\,\sqrt{d\alpha^2 + d\beta^2}}{[\cosh(\beta-\gamma) + \cos\alpha][\cosh(\beta+\gamma) - \cos\alpha]} \tag{9}$$

The conformance breaks down where $\frac{dz}{d\omega}$ is zero or infinite.

It is zero when $\beta = 0$, $\alpha = \frac{\pi}{2}$ which corresponds to the end A_0 of the cut. It is also zero when $\beta \to \infty$ so the point C at the left end of the semi-circle is carried to infinity in the w-strip. It is infinite when $\beta = \gamma$ and $\alpha \doteq \pi$ so that the infinite semi-circle in the z-half-plane [illegible] to the point B on the boundary of the w-strip.

To make the point A_0 at the end of the cut, (which is $z = ae^{i\theta_0}$) correspond to the point $\alpha = \pi/2$, $\beta = 0$ the positive real constant γ must be so chosen as to bring this about. Placing $w = \pi/2$ and $z = ae^{i\theta_0}$, [illegible] becomes

$$a e^{i\theta_0} = -a\left(\frac{1-i\sinh\gamma}{1+i\sinh\gamma}\right) = a\left(\frac{\sinh^2\gamma - 1 + i\,2\sinh\gamma}{\cosh^2\gamma}\right)$$

so

$$\cos\theta_0 = \frac{\sinh^2\gamma - 1}{\cosh^2\gamma} \quad \text{and} \quad \sin\theta_0 = \frac{2\sinh\gamma}{\cosh^2\gamma}$$

or

$$\sinh\gamma = \cot\theta_0/2, \quad \cosh\gamma = 1/\sin\theta_0/2 \quad \text{and} \quad \tanh\gamma = \cos\theta_0/2 \qquad (10)$$

We later take $\theta_0 = \pi/2$ so $\sinh\gamma = 1$, $\gamma = .881$, $\cosh^2\gamma = 2$.

Resolving the second member of (8) into real and imaginary parts and equating the first to x, the second to $i\rho$ gives the two real equations expressing the rectangular coordinates in terms of α and β

$$x = \frac{a(\sinh^2\gamma - \sinh^2\beta - \sin^2\alpha)}{[\cosh(\beta-\gamma)+\cos\alpha][\cosh(\beta+\gamma)-\cos\alpha]} \qquad (11_a)$$

$$\rho = \frac{2a\sinh\gamma\cosh\beta\sin\alpha}{[\cosh(\beta-\gamma)+\cos\alpha][\cosh(\beta+\gamma)-\cos\alpha]} \qquad (11_b)$$

If amplitudes and angles are equated in eq (8) ~~this~~ gives the real equations expressing the polar coordinates in terms of α and β

$$r = a\sqrt{\frac{[\cosh(\beta-\gamma)-\cos\alpha]\cdot[\cosh(\beta+\gamma)+\cos\alpha]}{[\cosh(\beta-\gamma)+\cos\alpha]\cdot[\cosh(\beta+\gamma)-\cos\alpha]}} \qquad (12a)$$

$$\left.\begin{aligned} \cos\theta &= \frac{\sinh^2\gamma-\sinh^2\beta-\sin^2\alpha}{\sqrt{[\cosh^2(\beta-\gamma)-\cos^2\alpha]\cdot[\cosh^2(\beta+\gamma)-\cos^2\alpha]}} \\ \sin\theta &= \frac{2\sinh\gamma\cosh\beta\sin\alpha}{\sqrt{[\cosh^2(\beta-\gamma)-\cos^2\alpha]\cdot[\cosh^2(\beta+\gamma)-\cos^2\alpha]}} \end{aligned}\right\} \qquad (12b)$$

These equations show that a pair of inverse points in the z-plane are carried into two points of the w-strip which are equidistant from the base $\beta = 0$ and from the vertical bisector $\alpha = \pi/2$. If one of them is $P(\alpha, \beta)$ where $0<\alpha<\pi/2$ its inverse or image is $P'(\pi-\alpha, \beta)$. Any ~~function of $\mu = \cos\alpha$ [illegible]~~ be an even function of μ, will have the same value at two points which are images of each other.

Any given point will actually be described by assigning numerical values to any its polar coordinates. It is but a few minutes work to find the numerical values of its coordinates α, β. The steps are perhaps more simple if we first find the numerical values of its polar coordinates r', θ' referred to the point C as origin, so

$$\left.\begin{aligned} r'\cos\theta' &= r\cos\theta + a = x + a \\ r'\sin\theta' &= r\sin\theta = \rho \end{aligned}\right\} \quad (13)$$

It is then found that

$$\left.\begin{aligned} \frac{r'\cos\alpha}{2a\sinh\gamma} &= \frac{\cos\theta' - r'/2a}{\sinh\beta} \\ \frac{r'\sin\alpha}{2a\sinh\gamma} &= \frac{\sin\theta'}{\cosh\beta} \end{aligned}\right\} \quad (14_a)$$

which may also be written

$$\left.\begin{aligned} \frac{r'\cosh\beta}{2a\sinh\gamma} &= \frac{\sin\theta'}{\sin\alpha} \\ \frac{r'\sinh\beta}{2a\sinh\gamma} &= \frac{\cos\theta' - r'/2a}{\cos\alpha} \end{aligned}\right\} \quad (14_b)$$

Between the two latter, β may be eliminated by use of the relation $\cosh^2\beta - \sinh^2\beta = 1$, and this gives

$$\frac{r'^2\sin^4\alpha}{4a^2} - \sin^2\alpha\left\{\frac{r'^2}{4a^2} + \sinh^2\gamma\left[\sin^2\theta' + \left(\cos\theta' - \frac{r'}{2a}\right)^2\right]\right\} + \sinh^2\gamma\sin^2\theta' = 0 \quad (15)$$

Solving for $\sin^2\alpha$ gives

$$2\sin^2\alpha = 1 + \left(\frac{2a\sinh\gamma}{r'}\right)^2\left\{\sin^2\theta' + \left(\cos\theta' - \frac{r'}{2a}\right)^2 - \sqrt{\left[\left(\frac{r'}{2a\sinh\gamma} + \sin\theta'\right)^2 + \left(\cos\theta' - \frac{r'}{2a}\right)^2\right]\cdot\left[\left(\frac{r'}{2a\sinh\gamma} - \sin\theta'\right)^2 + \left(\cos\theta' - \frac{r'}{2a}\right)^2\right]}\right\} \quad (16)$$

This determines α when $\frac{r'}{a}$, θ', and γ are numerically given, after which β is found by the first of (14b). In this manner the coordinates (α_1, β_1) of the trace P_1 of the primary circuit may be found, and similarly (α_2, β_2) of the secondary. Reference to figures (2a) and (2b) shows that β_1 and β_2 will both be less than γ while $\cos\alpha_1$ will be positive and $\cos\alpha_2$ negative.

To trace the correspondence between the z-plane and the w-strip we may first find the ~~equation~~ of the locus of a curve which is a member of the family α = constant. From eq (15), after placing $\sinh\gamma = \cot\theta_0/2$ this is found to be

$$r' = \left(\frac{2a\cot^2\theta_0/2}{\cot^2\theta_0/2 - \cos^2\alpha}\right)\cos\theta'\left\{1 - \frac{\cos\alpha}{\cot\theta_0/2}\sqrt{1-\left(\cot^2\theta_0/2-\cos^2\alpha\right)\tan^2\theta'}\right\} \qquad (17a)$$

This is the equation of the meridian curves of a family of confocal, cyclids of revolution. It inverts into a family of one-sheeted, confocal, hyperboloids of revolution. Each member is a curve beginning at the point C and ending perpendicularly on the cut. If the constant parameter α of the curve is less than $\pi/2$ it ends on the inside of the cut, if greater than $\pi/2$ on the outside. If $\alpha = \pi/2$ it is the arc of the semi-circle $r = a$ which supplements the cut. This one is the only member of the family drawn in fig 2a.

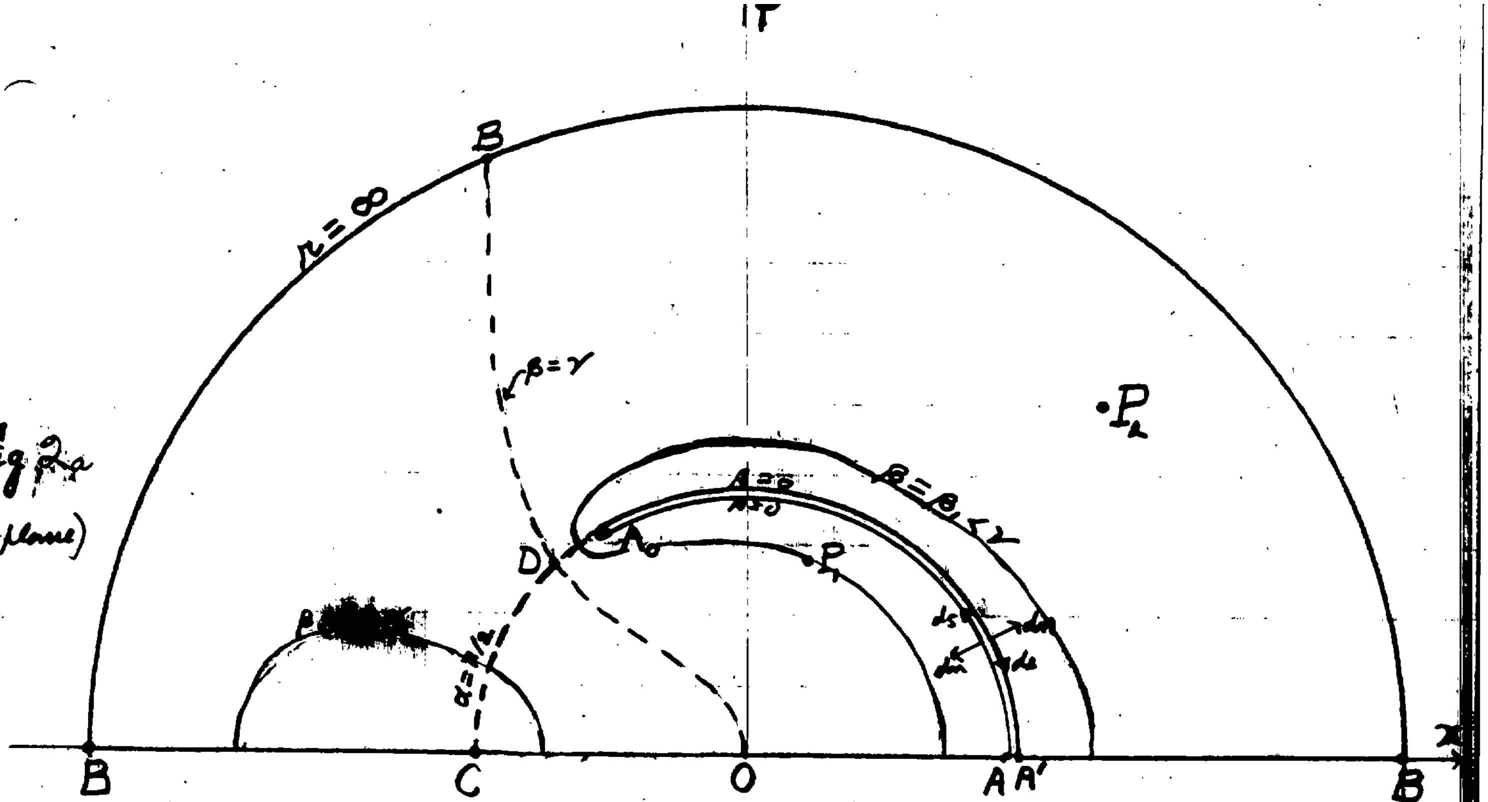


Fig 2a (z-plane)

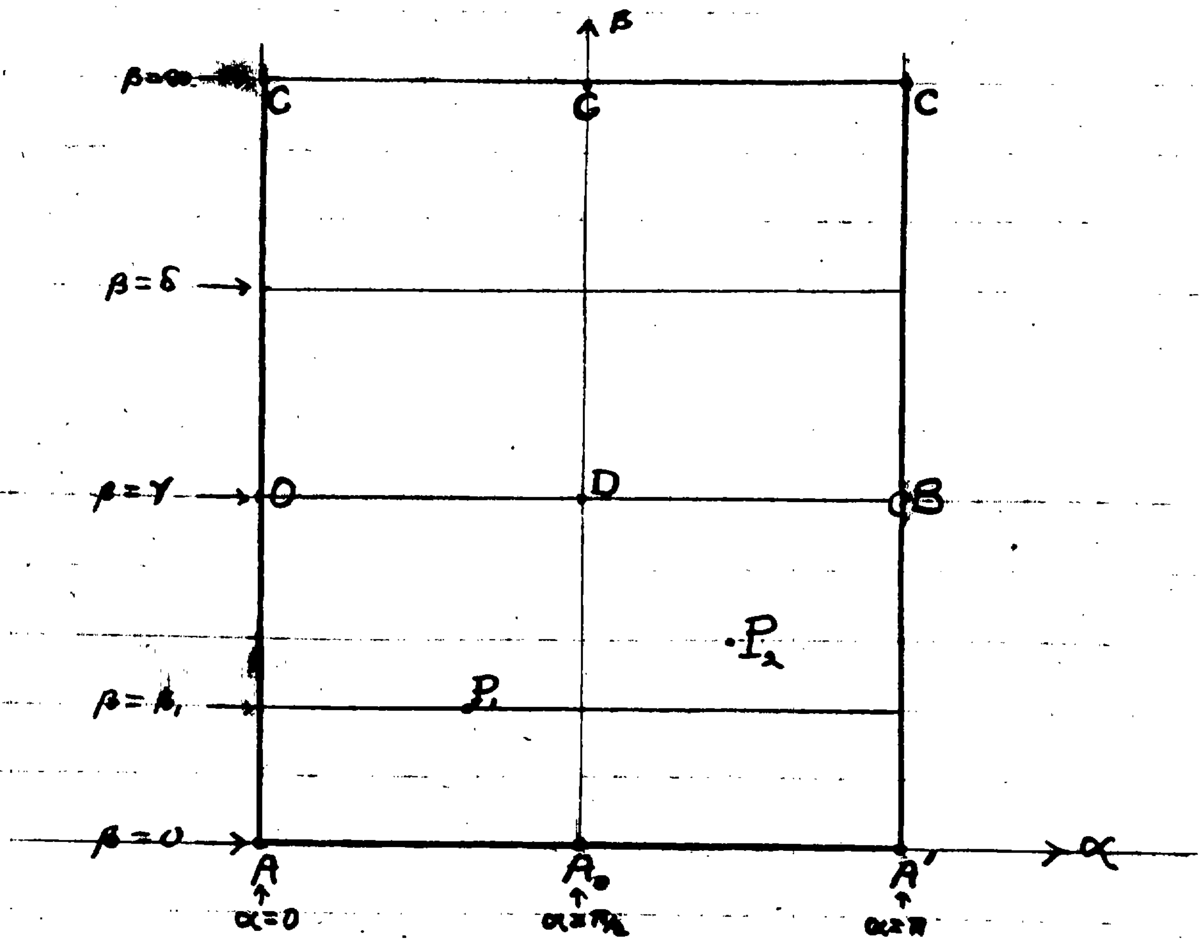


Fig 2b (α-strip)

$z = -\pi\left(\frac{\sin w - i\sinh\gamma}{\sin w + i\sinh\gamma}\right)$ where $\sinh\gamma = \cot\alpha/2$

The equation of the family of curves β = constant, orthogonal to the first family, is obtained by squaring and adding together the two equations (14a) thus eliminating α. This gives

$$r' = \frac{2a\cot^2\theta_0/_2}{\cot^2\theta_0/_2 - \sinh^2\beta}\left\{\cos\theta' \pm \frac{\sinh\beta}{\cot\theta_0/_2}\sqrt{1 - \frac{\sin^2\theta'}{\cosh^2\beta\,\sin^2\theta_0/_2}}\right\} \qquad (17_e)$$

This family of the meridian curves of confocal cyclids of revolution inverts into a family of meridian curves of confocal oblate spheroids. Every curve begins perpendicularly on the x axis, and with one exception, ends perpendicularly on that axis. The exception is the open curve $\beta = \gamma$ which begins at 0 and goes to ∞ with a vertical asymptote passing through C. All curves with $\beta > \gamma$ lie to the left of this open curve, all ~~both~~ $\beta < \gamma$ to the right. (This is where primary & secondary are located)

~~The locus in the plane of the equation~~ $\beta = \gamma$ is the infinite semi-circle BBB together with the open curve $\overline{ODB}$ whose equation is

$$r' = a\,\frac{\left[1 - \sin^2\frac{\theta_0}{2}\cdot\sin^2\theta'\right]}{\cos\theta'} \qquad (17_f)'$$

If β is small, the locus β = constant hugs the cut on both sides, and the locus $\beta = 0$ is both sides of the cut. As we go up the inside of the cut α increases from zero to the value $\pi/2$ when its edge A_0 is reached at $\theta = \theta_0$. Returning thence to the x-axis on the outside of the cut, α goes on increasing to a limit π. Two adjacent points on opposite sides of the cut in fig 2a are coincident image points, which are transformed into two widely separated points on the base line of the w-strip, where $\beta = 0$, and where their coordinates are α and $\pi - \alpha$.

Consequently any function of position in the z-plane which is single valued in a region which includes the cut (such as $V(r,\theta)$), will transform into a function of μ and β which is an even function of μ ($= \cos\alpha$).

6. Boundary Problem with Cyclide Coordinates

To transform the partial differential equation (7a) one finds

$$(\mathcal{D}_x^2+\mathcal{D}_\rho^2)U = h^2(\mathcal{D}_\alpha^2+\mathcal{D}_\beta^2)U$$

so that (7) becomes

$$(\mathcal{D}_\alpha^2+\mathcal{D}_\beta^2)U - \frac{3}{4h^2\rho^2}U = 0$$

Reference to (9) and (11)f shows that

$$\frac{1}{h^2\rho^2} = \frac{1}{\sin^2\alpha} - \frac{1}{\cosh^2\beta}$$

Hence $U(\alpha,\beta)$ must satisfy the equation

$$\left\{\mathcal{D}_\alpha^2+\mathcal{D}_\beta^2 + \frac{3}{4}\left(\frac{1}{\sin^2\alpha}-\frac{1}{\cosh^2\beta}\right)\right\}U = 0 \qquad (18)$$

at all points in the strip.

This has solutions of the form

$$U = u(\alpha)\,v(\beta) \quad , \text{where}$$

$$\frac{d^2}{d\alpha^2}u + \left[(n+\tfrac{1}{2})^2 - \frac{3}{4\sin^2\alpha}\right]u = 0$$

$$\frac{d^2 v}{d\beta^2} - \left[(n+\tfrac{1}{2})^2 - \frac{3}{4\cosh^2\beta}\right]v = 0$$

where n is an arbitrary constant

In the first make the double substitution

$$\mu = \cos\alpha \quad \text{and} \quad u(\alpha) = \sqrt{\tan\alpha}\; y(\mu)$$

It becomes

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dy}{d\mu}\right] + \left[n(n+1) - \frac{1}{1-\mu^2}\right] y = 0$$

which is satisfied by the associated Legendre function $P_n^1(\mu)$

The double substitution

$$\eta = i\sinh\beta \quad \text{and} \quad v = \sqrt{\cosh\beta}\; y(\eta)$$

carries the equation for v into the same form

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dy}{d\eta}\right] + \left[n(n+1) - \frac{1}{1-\eta^2}\right] y = 0$$

For this problem the elements of the solution required are the particular solutions of the form

$$U = Q^1_{2s-1}(i\sinh\beta)\, P^1_{2s-1}(\mu) . \sqrt{\tan\alpha \cosh\beta}$$

For $0 < \mu < 1$, the even functions of μ, $P^1_{2s-1}(\mu)$, $(s = 1, 2, 3, \cdots \infty)$ constitute a complete set of orthogonal functions,

$$\int_0^1 P^1_{2s-1}(\mu) P^1_{2n-1}(\mu)\, d\mu = 0 \text{ if } n \neq s, \quad = \frac{(2s-1)\,2s}{4s-1} \text{ if } n = s \qquad (19)$$

The functions $Q^1_{2s-1}(i\sinh\beta)$ are real; their definition being

$$Q^1_{2s-1}(i\sinh\beta) = \frac{\sqrt{\pi}(-1)^{s+1}}{\sqrt{1+e^{-2\beta}}} \frac{(2s)!}{\Gamma(2s+\frac{1}{2})} e^{-2s\beta} F\left(-\tfrac{1}{2}, \tfrac{3}{2}, 2s+\tfrac{1}{2}; \frac{e^{-2\beta}}{1+e^{-2\beta}}\right) \qquad (20)$$

where F is the hypergeometric function
From this, one finds

$$Q^1_{2s-1}(i0) = \sqrt{\pi}(-1)^{s+1}\frac{\Gamma(s+1/2)}{(s-1)!} \text{ and } \left[D_\beta Q^1_{2s-1}(i\sinh\beta)\right]_{\beta\to 0} = -2\sqrt{\pi}(-1)^{s+1}\cdot\frac{s!}{\Gamma(s-\frac{1}{2})} \qquad (20)$$

From the preceeding discussion it is evident that U will satisfy all but the last boundary condition if it is represented by the convergent series, valid everywhere,

$$U(\alpha,\beta) = 4\sqrt{\sin\alpha\cosh\beta}\sum_{s=1}^{\infty}\frac{(4s-1)}{(2s-1)2s}B_s\, Q^1_{2s-1}(i\sinh\beta)\, P^1_{2s-1}(\mu) \qquad (21)$$

When $r\to a-0$ while $\theta_0 > \theta \geq 0$, this means that
$\beta\to 0$ while $\pi/2 > \alpha \geq 0$, that is, the cut is approached from within, so that by eq (9)

$$dr = -\frac{2a\sinh\gamma\cdot\mu}{\cosh^2\gamma-\mu^2}\,d\beta$$

By use of (40) the last boundary condition (7d) becomes

$$\frac{R_0}{2\pi i\rho a}\sum_{s=1}^{\infty}(-1)^{s+1}(4s-1)\frac{(s-1)!}{\Gamma(s+\frac{1}{2})}B_s P^1_{2s-1}(\mu) + \frac{\sinh\gamma\cdot\mu}{\cosh^2\gamma-\mu^2}\sum_{s=1}^{\infty}(-1)^{s+1}(4s-1)\frac{\Gamma(s-\frac{1}{2})}{s!}B_s P^1_{2s-1}(\mu) =$$

$$= \frac{\sinh\gamma\cdot\mu}{\cosh^2\gamma-\mu^2}\cdot\frac{2}{\sqrt{\pi\sin\alpha}}\,Q^1_{1/2}\left(\frac{\frac{a^2+r_1^2}{2ar_1}-\cos\theta_1\cos\theta}{\sqrt{\sin\theta_1\sin\theta}}\right) \qquad (22)$$

which must be satisfied for $0<\mu\leq 1$.

The canonical expansion of the function $Q_{1/2}\left(1+\frac{\rho^2}{2rr_1}\right)$ in these coordinates is derived in a publication of the National Bureau of Standards MT 15 (1942).

We require in (22) only the case where P is on the cut, $(r=a)$ This is, (page 282 eq 21),

$$Q_{1/2}\left(\frac{\frac{a^2+r_1^2}{2ar_1}-\cos\theta_1\cos\theta}{\sin\theta_1\sin\theta}\right)=$$

$$=\frac{\sqrt{\pi\sin\alpha\sin\alpha_1\cosh\rho_1}}{2}\sum_{s=1}^{\infty}(-1)^{s+1}\frac{(4s-1)}{(2s-1)2s}\frac{\Gamma(s-\frac{1}{2})}{s!}Q^1_{2s-1}(i\sinh\rho_1)P^1_{2s-1}(\mu_1)P^1_{2s-1}(\mu) \qquad (23)$$

which is valid for P_1 any point in the z half-plane.

Hence the boundary condition (22) takes the form

$$\frac{R_0}{2\pi i p a}\sum_{s=1}^{\infty}(-1)^{s+1}(4s-1)\frac{(s-1)!}{\Gamma(s+\frac{1}{2})}B_s P^1_{2s-1}(\mu)+\frac{\sinh\gamma\cdot\mu}{\cosh^2\gamma-\mu^2}\sum_{s=1}^{\infty}(-1)^{s+1}(4s-1)\frac{\Gamma(s-\frac{1}{2})}{s!}B_s P^1_{2s-1}(\mu)=$$

$$=\frac{\sinh\gamma\cdot\mu}{\cosh^2\gamma-\mu^2}\sqrt{\sin\alpha_1\cosh\rho_1}\sum_{s=1}^{\infty}\frac{(-1)^{s+1}(4s-1)}{(2s-1)2s}\frac{\Gamma(s-\frac{1}{2})}{s!}Q^1_{2s-1}(i\sinh\rho_1)P^1_{2s-1}(\mu_1)P^1_{2s-1}(\mu) \qquad (24)$$

for $0<\mu\leq 1$

If this equation is multiplied by $P^1_{2n-1}(\mu)\,d\mu$ and integrated from $\mu=0$ to $\mu=1$, taking account of the integral formula (19), the result is an infinite system of linear equations to determine the unknown coefficients B_s

This set of equations may be written

$$\frac{2R_0}{\pi i p a} B_n + \sum_{s=1}^{\infty} (4s-1)\, b_{n,s} B_s = \sum_{s=1}^{\infty} (4s-1)\, b_{n,s} B_s^{\infty} \qquad (25)$$

where n takes in succession the values $n = 1, 2, 3, ---\infty$
The coefficients for a perfectly conducting shell ($R_0 = 0$) are the real constants

$$B_s^{\infty} = \sqrt{\sin\alpha_1 \cosh\beta_1}\, \frac{Q'_{2s-1}(i\sinh\beta_1)\, P'_{2s-1}(\mu_1)}{(2s-1)\, 2s} \qquad (26)$$

The real coefficients are pure numerics, defined by

$$b_{n,s} = b_{s,n} \equiv \frac{(-1)^{n+s}\, \Gamma(n-\frac{1}{2})\, \Gamma(s-\frac{1}{2})}{n!\; s!} \sinh\gamma \int_0^1 \frac{\mu\, P'_{2n-1}(\mu)\, P'_{2s-1}(\mu)}{\cosh^2\gamma - \mu^2}\, d\mu \qquad (27)$$

The only place where the angular extent θ_0 of the conducting arc enters these equations is in the determination of these coefficients.

For the ($\theta_0 = \pi/2$), which generates a hemi-spherical shell we take

$\sinh\gamma = 1$ and $\cosh^2\gamma = 2$.

In applications the constant $2R_0/\pi i p a$ is ~~probably small~~ since $p = 4 \cdot 10^6$, hence terms with the square of this constant as a factor may be neglected so that writing

$$B_s = B_s^{\infty} + i\left(\frac{2R_0}{\pi p a}\right) \frac{C_s}{(4s-1)} \qquad (28)$$

the set of equations to determine the

real constants C_s, is

$$\sum_{s=1}^{\infty} b_{n,s} C_s = B_n^{\infty} \quad (n=1,2,3,\ldots\infty) \qquad (29)$$

Or, writing out a few of these equations.

$$\begin{aligned}
b_{11} C_1 + b_{12} C_2 + b_{13} C_3 + b_{14} C_4 + \cdots &= B_1^{\infty} \\
b_{21} C_1 + b_{22} C_2 + b_{23} C_3 + b_{24} C_4 + \cdots &= B_2^{\infty} \\
b_{31} C_1 + b_{32} C_2 + b_{33} C_3 + b_{34} C_4 + \cdots &= B_3^{\infty} \\
b_{41} C_1 + b_{42} C_2 + b_{43} C_3 + b_{44} C_4 + \cdots &= B_4^{\infty} \\
\cdots\cdots\cdots
\end{aligned} \qquad (29')$$

The numerical evaluation of the coefficients C_s gives the asymptotic solution of the problem for large values of [illegible].

A few of the reasons may be indicated here for believing that this solution is quite feasible and that an approximation sufficient for the applications in view may be obtained with a finite number of these real coefficients C_s possibly the first three, perhaps four [redacted] It is better to postpone a discussion of numerical methods to a later section but one or two relations may be given here

With large integer s, the functions $P'_{2s-1}(\mu)$ becomes infinite like $\sqrt{s}$. For if s is large

$$\sqrt{\sin\alpha}\, P'_{2s-1}(\mu) \; [illegible] \; (\ldots) \qquad (30_1)$$

$$\frac{\sqrt{\cosh\beta}\, Q'_{2s-1}(i\sinh\beta)}{(2s-1)\ldots 2s} \; [illegible] \; (2s+\tfrac{1}{2})\beta \; [illegible] \qquad (30_2)$$

$$[illegible]\; Q'_{2s-1}(i\sinh\beta)\, P'_{2s-1}(\mu) \; [illegible] \; \cos\left([2s-\tfrac{1}{2}] [illegible] \tfrac{\pi}{4}\right) \cdots \qquad (30_3)$$

Hence for large s

$$B_s^\infty \sim \frac{e^{-(2s+\frac{1}{2})\beta_1}}{2s} \cos\left((2s-\tfrac{1}{2})\alpha_1 - \tfrac{3\pi}{4}\right) \qquad (30d)$$

These terms are the second members of the equations (29)'

The symmetrical coefficients b_{ns} vanish like $\frac{1}{n}$ when $n \to \infty$ (s constant)

If β_1 were large the second members of (29)' would rapidly approach zero with increasing order, s.

The coordinates β_1 and β_2 for primary and secondary circuits cannot exceed γ which is .881 for the hemispherical problem. The nearer these circuits are to the shell the smaller β_1 and β_2. ~~If~~ ~~...~~ is very small β_1 is near .881, and β_2 is near this value, ~~the larger is $\frac{r_2}{a}$~~

~~As a representative example~~, take

$\beta_1 = .75$ ~~...~~ (30d) ~~gives~~ approximately, as

$$B_4^\infty \sim \frac{e^{-6.375}}{8} \cos\left(7.5\alpha_1 - \tfrac{3\pi}{4}\right) = .0002 \cos\left(7.5\alpha_1 - \tfrac{3\pi}{4}\right)$$

These are the reasons for expecting that a reasonable approximation may be obtained by neglecting all but the first three or four of the coefficients C_s and of the equation (29)'.

In the later section a method is derived for computing the numerical coefficients b_{ns}. It seems better to insert a summary at this point.

7. Summary and Bearing of [redacted]

$$U(\alpha,\beta;\alpha_1,\beta_1) = U^{\infty}(\alpha,\beta;\alpha_1,\beta_1) + i\,\frac{2R_0}{\pi p a}\,V(\alpha,\beta;\alpha_1,\beta_1) \qquad (31)$$

where U^{∞} and V are real functions [redacted] independent of R_0/ap, and may be computed by

$$U^{\infty}(\alpha,\beta;\alpha_1,\beta_1) =$$

$$= 4\sqrt{\sin\alpha\cosh\beta\,\sin\alpha_1\cosh\beta_1}\sum_{s=1}^{\infty}(4s-1)\left[\frac{Q^1_{2s-1}(i\sinh\beta)\,P^1_{2s-1}(\mu)}{(2s-1)\,2s}\right]\cdot\left[\frac{Q^1_{2s-1}(i\sinh\beta_1)\,P^1_{2s-1}(\mu_1)}{(2s-1)\,2s}\right] \qquad (32$$

$$V(\alpha,\beta;\alpha_1,\beta_1) = 4\sqrt{\sin\alpha\cosh\beta}\sum_{s=1}^{\infty}C_s\,\frac{Q^1_{2s-1}(i\sinh\beta)\,P^1_{2s-1}(\mu)}{(2s-1)\,2s} \qquad (32$$

where $\mu \equiv \cos\alpha$, and the coefficients C_s are functions of μ_1 and β_1 found as solutions of the set of equations (29).

As is indicated by (30c), three or four terms may be [redacted] for computing a sufficient approximation to these series.

By (32a) the real function U^{∞} for a perfectly conducting shell automatically satisfies the reciprocal theorem and ~~they~~ term satisfies the inversion theorem. Since V is a first approximation (with high frequency) it is not expected to satisfy the reciprocal theorem exactly, although [redacted] inversion in the conducting arc.

Up to ~~this point~~, the currents in the shell have been considered as a periodic current distribution existing in the presence of one periodic (complex) current, effectively $N_1 I_1$ in a circle whose trace is the source-point P_1. The phases of the induced currents with respect to the phase of the primary current are taken into account by the use of the complex potential in the form given in equations (3) and (4)

Consequently when both complex current sources are present the complex vector potential is not eq (3). This is to be replaced by

$$A(r,\theta) = N_1 I_1 \sqrt{\tfrac{\rho_1}{\rho}}\, G(r,\theta;\, r_1,\theta_1) + N_2 I_2 \sqrt{\tfrac{\rho_2}{\rho}}\, G(r,\theta;\, r_2,\theta_2) \qquad (33)$$

where the Greens function in each case is defined similarly to that given in (4). The evaluation of the U-function given above applies in either case. The part of each Green's function which is represented by $2Q_{1/2}$ contributes to the self and mutual inductances of the coils as if there were no shell present. The initial circuit-constants of primary and secondary had lumped inductance and capacitances which were not effective in producing an alternating field at the shell, and these constants are assumed to have been determined experimentally for the particular frequency.

The vector potential $A_3(r,\theta)$ which is produced by all currents in the shell is therefore by (33) and (4)

$$A_3(r,\theta) = -N_1 I_1 \sqrt{\tfrac{\rho_1}{\rho}}\, U(r,\theta;\, r_1,\theta_1) - N_2 I_2 \sqrt{\tfrac{\rho_2}{\rho}}\, U(r,\theta;\, r_2,\theta_2) \qquad (34)$$

They produce a flux through primary and secondary which is $\phi_{13} = 2\pi\rho_1 A_3(r_1,\theta_1)$ and $\phi_{23} = 2\pi\rho_2 A_3(r_2,\theta_2)$, that is,

$$\left.\begin{aligned}\Phi_{13} &= -2\pi N_1^2 \rho_1 U_{11} I_1 - 2\pi N_1 N_2 \sqrt{\rho_1\rho_2}\, U_{12} I_2 \\ \Phi_{23} &= -2\pi N_1 N_2 \sqrt{\rho_1\rho_2}\, U_{12} I_1 - 2\pi N_2^2 \rho_2 U_{22} I_2\end{aligned}\right\} \quad (35)$$

These, and eq(31) have been used in the discussion in section 2 so that ~~the~~ job which confronts the computer should now be clear.

The points $P_1(r_1, \theta_1)$, $P_2(r_2, \theta_2)$ (traces of primary and secondary) being numerically given, as well as the radius a of the shell, it is first necessary to find the corresponding cyclidic coordinates α_1, β_1 and α_2, β_2. This is a small amount of work as explained in equations (13) to (16).

Then U_{11}^{∞}, U_{22}^{∞} and U_{12}^{∞} are to be computed by the series (32)

Before computing V_{11}, V_{22}, V_{12} by the series (32_b) it is necessary to compute a number of the coefficient C_1, C_2, C_3, C_4, and for this the coefficients $b_{m,s}$ must be computed.

In view of the experimental application it would seem that an error of at least 10% could be tolerated in evaluating U_{11}^{∞}, U_{12}^{∞}, U_{13}^{∞}, V_{11}, V_2 and V_{12}

8. Methods for computing $b_{n,s}$ [redacted]

Since $b_{n,s} = b_{s,n}$, the first sixteen of these coefficients ~~in the square n,s~~ $= 1,2,3,4$ are known if we compute the ten in the triangle which includes the diagonal, namely $1 \leq s \leq n = 1,2,3,4$. In that case the highest value of $n+s-2$ is 6.

To compute them in finite terms begin with the polynomial

$$\frac{\Gamma(n-\frac{1}{2})}{n!} P^{1}_{2n-1}(\mu) = 2\sqrt{1-\mu^2} \sum_{t=0}^{n-1} \frac{(-1)^t (1-\mu^2)^t \, \Gamma(t+n+1/2)}{t!\,(t+1)!\,(n-1-t)!} \qquad (36)$$

Write the same equation with n replaced by s and multiply the two. Rearranging the product of the two polynomials on the right in ascending powers of $(1-\mu^2)$ we get

$$\frac{\Gamma(n+\frac{1}{2})\,\Gamma(s-\frac{1}{2})}{n! \quad s!} P^{1}_{2n-1}(\mu) \cdot P^{1}_{2s-1}(\mu) = 4\,\frac{\Gamma(n+\frac{1}{2})\,\Gamma(s+\frac{1}{2})}{(n-1)!\,(s-1)!} \sum_{K=0}^{n+s-2} (-1)^K D^{n,s}_K (1-\mu^2)^{K+1} \qquad (37)$$

where

$$D^{n,s}_K = D^{s,n}_K =$$

$$= \sum_{t=0}^{K} \frac{1}{t!\,(t+1)!\,(K-t)!\,(K-t+1)!} \left[\frac{(n-1)!\;\Gamma(t+n+\frac{1}{2})}{(n-1-t)!\;\Gamma(n+\frac{1}{2})} \; \frac{(s-1)!\;\Gamma(K-t+s+\frac{1}{2})}{(s-1-[K-t])!\;\Gamma(s+\frac{1}{2})} \right] \qquad (38)$$

The K^{th} coefficient $D^{n,s}_K$ (consists of) ~~has~~ $(K+1)$ terms which are symmetrical and rational functions of n and s

$$D_0^{n,s} = 1$$

$$D_1^{n,s} = \frac{1}{2}\left[(n-2)(n+\tfrac{1}{2}) + (s-2)(s+\tfrac{1}{2})\right]$$

$$D_2^{n,s} = \frac{1}{12}\left[(n-2)(n-3)(n+\tfrac{1}{2})(n+\tfrac{3}{2}) + 3(n-1)(n+\tfrac{1}{2})(s-1)(s+\tfrac{1}{2}) + (s-2)(s-3)(s+\tfrac{1}{2})(s+\tfrac{3}{2})\right]$$

$$D_3^{n,s} = \frac{1}{3!\,4!}\Big\{(n-1)(n-2)(n+\tfrac{1}{2})(n+\tfrac{3}{2})\left[(n-3)(n+\tfrac{5}{2}) + 6(s-1)(s+\tfrac{1}{2})\right]$$
$$+ (s-1)(s-2)(s+\tfrac{1}{2})(s+\tfrac{3}{2})\left[(s-3)(s+\tfrac{5}{2}) + 6(n-1)(n+\tfrac{1}{2})\right]\Big\}$$

$$D_4^{n,s} = \frac{1}{4!\,5!}\Big\{(n-1)(n-2)(n-3)(n-4)(n+\tfrac{1}{2})(n+\tfrac{3}{2})(n+\tfrac{5}{2})(n+\tfrac{7}{2})$$
+ same term with n replaced by s
$$+ 10(n-1)(n-2)(n-3)(n+\tfrac{1}{2})(n+\tfrac{3}{2})(n+\tfrac{5}{2})(s-1)(s+\tfrac{1}{2})$$
+ same term with n and s interchanged
$$+ 20(n-1)(n-2)(n+\tfrac{1}{2})(n+\tfrac{3}{2})(s-1)(s-2)(s+\tfrac{1}{2})(s+\tfrac{3}{2})\Big\}$$

$$D_5^{n,s} = \frac{1}{5!\,6!}\Big\{(n-1)(n-2)(n-3)(n-4)(n-5)(n+\tfrac{1}{2})(n+\tfrac{3}{2})(n+\tfrac{5}{2})(n+\tfrac{7}{2})(n+\tfrac{9}{2})$$
+ same function of s
$$+ 15(n-1)(n-2)(n-3)(n-4)(n+\tfrac{1}{2})(n+\tfrac{3}{2})(n+\tfrac{5}{2})(n+\tfrac{7}{2})(s-1)(s+\tfrac{1}{2})$$
+ same term with n and s interchanged
$$+ 50(n-1)(n-2)(n-3)(n+\tfrac{1}{2})(n+\tfrac{3}{2})(n+\tfrac{5}{2})(s-1)(s-2)(s+\tfrac{1}{2})(s+\tfrac{3}{2})$$
+ same term with n and s interchanged $\Big\}$

$$D_6^{n,s} =$$

$$= \frac{1}{6!\,7!}\Big\{(n-1)(n-2)\cdots(n-6)(n+\tfrac{1}{2})(n+\tfrac{3}{2})\cdots\cdots(n+\tfrac{11}{2})$$

+ same function of s

$$+21(n-1)(n-2)\cdots(n-5)(n+\tfrac{1}{2})(n+\tfrac{3}{2})\cdots(n+\tfrac{9}{2})(s-1)(s+\tfrac{1}{2})$$

+ same with n and s interchanged

$$+105(n-1)(n-2)\cdots(n-4)(n+\tfrac{1}{2})(n+\tfrac{3}{2})\cdots(n+\tfrac{7}{2})(s-1)(s-2)(s+\tfrac{1}{2})(s+\tfrac{3}{2})$$

+ same with n and s interchanged

$$+175(n-1)(n-2)(n-3)(n+\tfrac{1}{2})(n+\tfrac{3}{2})(n+\tfrac{5}{2})(s-1)(s-2)(s-3)(s+\tfrac{1}{2})(s+\tfrac{3}{2})(s+\tfrac{5}{2})\Big\}$$

etc, (using $\dfrac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\;n!} = \dfrac{1.3.5\ldots(2n-1)}{2.4.6\cdots 2n}$ where $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

and $z\Gamma(z) = \Gamma(z+1)$)

If eq(37) is multiplied by $(-1)^{n+s}\mu\,d\mu/(2-\mu^2)$ and integrated from zero to 1, the first member becomes $b_{n,s}$ as defined in (27) for the hemispherical shell ($\sinh\gamma = 1$) The result is

$$b_{n,s} = (-1)^{n+s}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(s+\frac{1}{2})}{(n-1)!\,(s-1)!}\sum_{K=0}^{n+s-2}(-1)^K D_K^{n,s}\, S_K \qquad (39)$$

where

$$S_K = 4\int_0^1 \frac{(1-\mu^2)^{K+1}\mu}{2-\mu^2}\,d\mu = 2\int_0^1 \frac{z^{K+1}\,dz}{1+z} = \Psi\left(\frac{K+2}{2}+\frac{1}{2}\right) - \Psi\left(\frac{K+2}{2}\right)$$

where $\Psi(z) \equiv \frac{d}{dz}\log\Gamma(z)$

The S_K may be computed by

$$\left.\begin{aligned} S_0 &= 2 - \log_n 4 \\ S_{2n} &= \frac{2}{2n+1} + \sum_{t=1}^{n} \frac{\;}{t(2t-1)} - \log_n 4 \\ S_{2n-1} &= \log_n 4 - \sum_{t=1}^{n} \frac{1}{t(2t-1)} \end{aligned}\right\} \qquad (40)$$

Or (since $\log 2 = \sum_{t=1}^{\infty} \frac{(-1)^{t+1}}{t}$),

$$S_K = 2\left[\sum_{t=1}^{K+1} \frac{(-1)^{t+1}}{t} - \log 2\right] = -2 \sum_{t=K+2}^{\infty} \frac{(-1)^{t+1}}{t}$$

By use of the recurrence-relation

$S_{K+1} = \frac{2}{K+2} - S_K$, the eq (39) may be written

$$\frac{4}{\pi} b_{m,s} = (-1)^{m+s} \left[\frac{1\cdot 3\cdot 5\cdots(2m-1)\cdot 1\cdot 3\cdot 5\cdots(2s-1)}{2^{m+s-2}\cdot(m-1)!\,(s-1)!}\right]\cdot$$

$$\cdot\Big\{ 2 + D_1^{ms} + \frac{5}{3} D_2^{ms} + \frac{7}{6} D_3^{ms} + \frac{47}{30} D_4^{ms} + \frac{37}{30} D_5^{ms} + \frac{319}{210} D_6^{ms} + \cdots$$
$$- \left[1 + D_1^{ms} + D_2^{ms} + D_3^{ms} + D_4^{ms} + D_5^{ms} + D_6^{ms} + \cdots\right] \log_n 4 \Big\} \qquad (41)$$

where $\log_n 4 = 1.386\,294\,361$ and $D_K^{ms} = 0$ if $m+s-2 > K$

This eliminates S_K for computing D_K up to $K=6$

In the next section is described the numerical application of the asymptotic solution to experimental data. This was made by the computing group under P. Whitman. It applies to two given points P_1 and P_2 (primary and secondary), and to a given shell-radius a. The latter was selected at a particular instant on the scope-trace recording pick-up current I_2 on the basis of a known velocity of the front of the converging detonation-wave. Since the coefficients b_{ns} are independent of these data, and appear in other connections, the first sixteen thus found are tabulated here, or rather $4b_{ns}/\pi$

$$\frac{4}{\pi} b_{n,s}$$

	$n=1$	$n=2$	$n=3$	$n=4$	
$s=1$	+.61371	-.19625	-.00440	-.00794	
$s=2$	-.19625	+.19364	-.05839	+.00119	
$s=3$	-.00440	-.05839	+.08219	-.02769	
$s=4$	-.00794	+.00119	-.02769	+.04473	

For computing $P^1_{2s-1}(\mu)$ where $\mu = \cos\alpha$

$$P^1_1(\mu) = \sin\alpha$$
$$P^1_3(\mu) = \frac{3}{2}\sin\alpha\,(5\mu^2-1)$$
$$P^1_5(\mu) = \frac{105}{8}\sin\alpha\,(3\mu^4-2\mu^2+\tfrac{1}{7})$$

$$P^1_7(\mu) = \frac{231}{16}\sin\alpha\,(13\mu^6-15\mu^4+\tfrac{45}{11}\mu^2-\tfrac{5}{33})$$

The recurrence-relation is

$$P^1_{2s+1}(\mu) = -\left(\frac{4s+1}{4s-3}\right)P^1_{2s-3}(\mu) + \left[(4s+1)\mu^2 - \frac{8s^2-4s-3}{4s-3}\right]\frac{(4s-1)\,P^1_{2s-1}(\mu)}{(2s-1)(2s)} \qquad (42)$$

The functions $Q^1_{2s-1}(i\sinh\beta)$ satisfy the same relation provided that μ is replaced by $i\sinh\beta$. To compute them in finite terms let the positive acute angle ϕ (in radians) be determined by

$$\cos\phi = \tanh\beta \quad \text{where} \quad 0 < \phi < \pi/2$$

The recurrence-relation is

$$Q^1_{2s+1} = -\left(\frac{4s+1}{4s-3}\right)Q^1_{2s-3} - \left[(4s+1)\cot^2\phi + \frac{8s^2-4s-3}{4s-3}\right]\frac{(4s-1)\,Q^1_{2s-1}}{(2s-1)(2s)} \qquad (42')$$

The first four functions are

$$Q_1^1 = -\cos\phi + \frac{\phi}{\sin\phi}$$

$$Q_3^1 = \frac{15}{2}\left\{\left[\cot^2\phi + \frac{13}{15}\right]\cos\phi - \left[\cot^2\phi + \frac{1}{5}\right]\frac{\phi}{\sin\phi}\right\}$$

$$Q_5^1 = \frac{105}{8}\left\{-\left[3\cot^4\phi + 4\cot^2\phi + \frac{113}{105}\right]\cos\phi + \left[3\cot^4\phi + 2\cot^2\phi + \frac{1}{7}\right]\frac{\phi}{\sin\phi}\right\}$$

$$Q_7^1 = \frac{13}{6}\left\{\left[\frac{231}{8}\left(\cot^2\phi + \frac{19}{39}\right)\left(3\cot^4\phi + 4\cot^2\phi + \frac{113}{105}\right) - 5\cot^2\phi - \frac{13}{3}\right]\cos\phi - \left[\frac{231}{8}\left(\cot^2\phi + \frac{19}{39}\right)\left(3\cot^4\phi + 2\cot^2\phi + \frac{1}{7}\right) - 5\cot^2\phi - 1\right]\frac{\phi}{\sin\phi}\right\}$$

9. Numerical Evaluation of R_0 for one set of experimental data.

This section is a collection of data and the results of computation made by Whitman and his group.

$a = 5'' = 12.70$ cm $\qquad p/2\pi = .65(10)^6$ cycles/sec

Primary	Secondary
$N_1 = 60$ turns	$N_2 = 3$ turns
$x_1 = .5''$	$x_2 = 39.37''$
$\rho_1 = 1''$	$\rho_2 = 10''$
$\theta_1' = .17985$ (radians)	$\theta_2' = .22167$ (
$r_1' = 5.5900''$	$r_2' = 45,483''$
$\alpha_1 = .25441$ radians	$\alpha_2 = 3.10358$
$\beta_1 = .721170$	$\beta_2 = .7219$
$\mu_1 = \cos\alpha_1 = 96781$	$\mu_2 = -.99928$
$P_1' = .25168$	$P_1' = .03800$
$P_3' = 1.3905$	$P_3' = .22759$
$P_5' = 2.9780$	$P_5' = .5671$
$P_7' = 4.3719$	$P_7' = 1.0537$
$Q_1' = +.53317$	$Q_1' = +.53249$
$Q_3' = -.17778$	$Q_3' = -.17728$
$Q_5' = +.051468$	$Q_5' = +.05122$
$Q_7' = -01404$	$Q_7' = -0138$

Primary

$B_1^\infty = +.037954$

$B_2^\infty = -.011653$

$B_3^\infty = +.002890$

$B_4^\infty = -.0006199$

Secondary

$B_1^\infty = .0022243$

$B_2^\infty = -.0007392$

$B_3^\infty = +.0002129$

$B_4^\infty = -.00005717$

b_{ns}/π

	n = 1	n = 2	n = 3	n = 4
s=1	.153426	-.049064	-.001100	-.001987
s=2		.048404	-.014597	.000299
s=3			.020548	-.006924
s=4				.011182

For Primary

	C_1	C_2	C_3	C_4
using 3 equations	.09229	.04061	.07856	—
using 4 equations	.09890	.05769	11436	.06920

For Secondary

	C_1	C_2	C_3	C_4
using 3 equations	.005297	.002021	.005017	—
" 4 "	005633	.002893	006846	.003335

$U_{11}^{\infty} = .021479$

$U_{12}^{\infty} = .001283$

$U_{22}^{\infty} = 0000769$

$V_{11}\ .013477$

$V_{12} = 00079096$

$V_{21} = 00079099$

$V_{22} = 0000466$

$R_1'/R_0 = 38.814$ ohms

$R_2'/R_0 = 600335$ "

$R_{12}'/R_0 = .3602$ "

$L_1' = 1.234\,(10)^{-6}$ henries

$L_2' = .110\,(10)^{-9}$ "

$M_{12}' = 11.66\,(10)^{-9}$ "

Coil constants

Primary

$R_1 = 50$ ohms

$L_1 = 160(10)^{-6}$ henries

$C_1 = 375(10)^{-12}$ farads

Secondary

$R_2 = 35$ ohms

$L_2 = 11.15(10)^{-6}$ henries

$C_2 = 2300\ 10^{-12}$ farads

For plotting $|I_2|$ against R_0

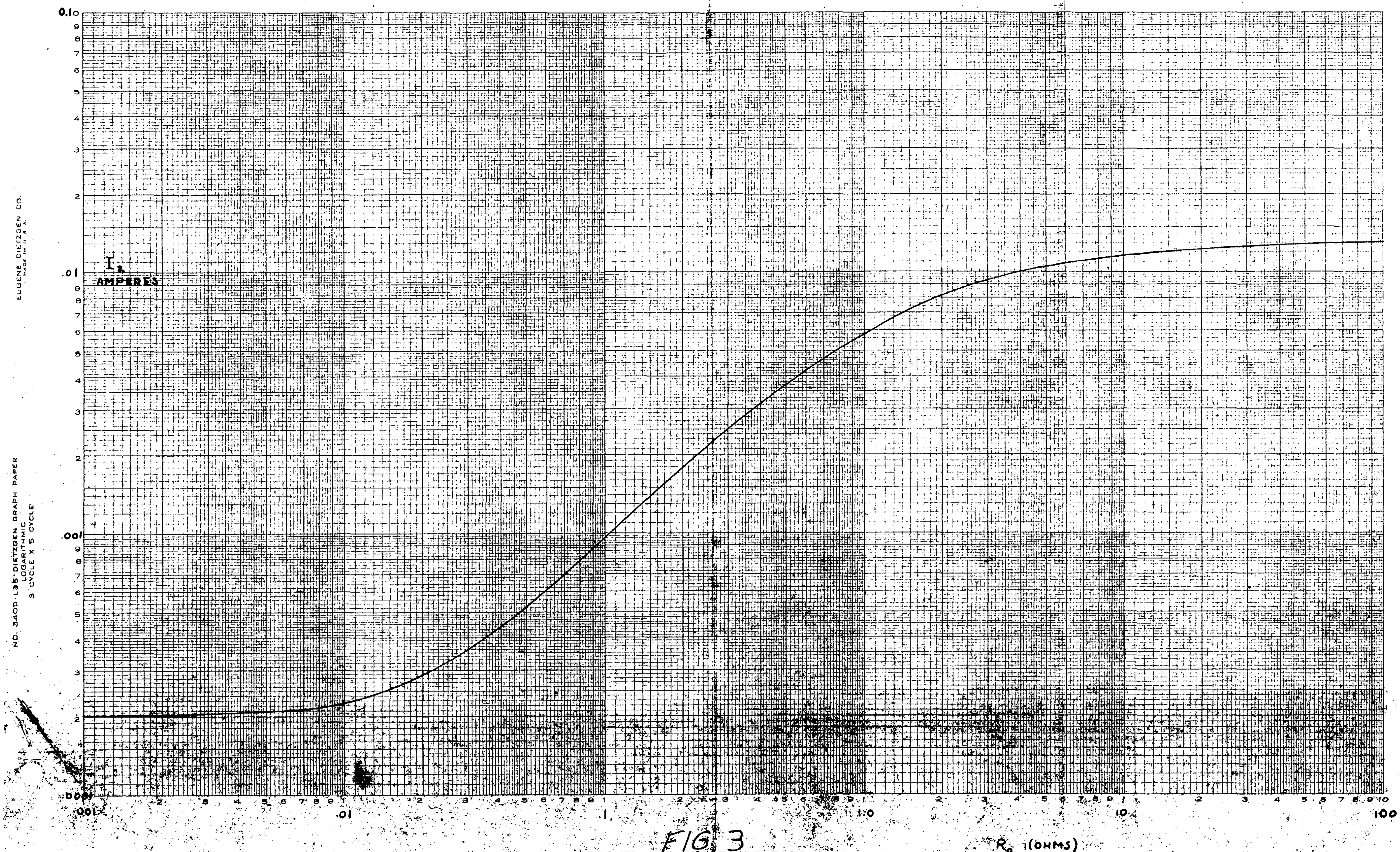
R_0 (ohms)	$\lvert I_2 \rvert$ amperes
10000.	.0132
1000.	.0132
100.	.0130
10.	0117
1.	.00577
.1000	.000966
.083	.000819
.080	.000792
010	.000224
001	000202

The log-log graph of I_2 against R_0 from preceding column is shown in fig 3.
It would have been better, had the ordinates been $\log \frac{|I_2|}{|I_2^0|}$ where $|I_2^0|$ is the amplitude of the pickup current before the conducting shell makes its appearance. The ratio of amplitudes $|I_2|/I_0$ is obtained from one scope trace (At the particular instant when the mean radius of the shell is a, the pickup current is I_2)
For the case given by Rosen $|I_2|/I_0 = \frac{1}{2}$ when $a = 5'' = 12.7$cm
This by use of fig 3 gives $R_0 = .079$ ohms.

Since the relative configurations of the primary and secondary are not greatly dissimilar in the hemispherical shots and in implosions it is safe to draw one general conclusion from an inspection of these results. It is

L_1' is negligible compared to L_1
L_2' " " " to L_2
R_2' " " " " R_2
R_1' is small compared to R_1 (about 8% in this case.)

The fact that the coil-constants R_1, L_1 and C_1 appearing as factors of I_2 in eq (6) are all practically unchanged by creation of the shell means that a knowledge of these constants is unimportant since R_0 is determined from the ratio $|I_2|/|I_0|$.



0.10
.01
.001
.0001
I_2
AMPERES
.001
.01
.1
1.0
10
100
R_o (OHMS)
NO. 340D-L35 DIETZGEN GRAPH PAPER
LOGARITHMIC
3 CYCLE X 5 CYCLE
EUGENE DIETZGEN CO.
MADE IN U.S.A.

FIG 3

10. Form and Method for the General Solution.

If the occasion should arise which requires a solution when R_0/ap is not small, the system (25) must be solved. Setting

$$\left.\begin{array}{l} \lambda \equiv \pi a p / 2R_0 \text{, and } \bar{B}_n \equiv (4n-1)B_n \\ \text{the system is} \\ \left[b_{nn} - \frac{i}{(4n-1)\lambda}\right]\bar{B}_n + \sum_{s=1}^{\infty (n)} b_{ns}\bar{B}_s = \sum_{s=1}^{\infty} b_{ns}\bar{B}_s^{\infty} \end{array}\right\} \quad (43)$$

for $n = 1, 2, 3, \ldots \infty$, where $\sum_{s=1}^{\infty (n)}$ indicates ommission of the term $s = n$ from the summation.

If the points P and P_0 (or ultimately P_1 and P_2) are not close to the shell, four of these equations would give an approximate determination of $\bar{B}_1$ $\bar{B}_2$ $\bar{B}_3$ and $\bar{B}_4$. If terms $\bar{B}_n$ of order n higher than 4 are required, reference to the Table in section 8) shows that a first approximation would be given by neglecting all but diagonal terms in the array b_{ns} for $n, s > 4$ so that

$$\bar{B}_n \sim \frac{\sum_{s=1}^{\infty} b_{ns}\bar{B}_s^{\circ}}{b_{nn} - \frac{i}{(4n-1)\lambda}} \quad \text{for } n > 4.$$

A second approximation would be obtained by retaining the diagonal and the two adjacent lines, in which case there would be a recurrence relation beginning with $n = 5$, between three contiguous coefficients $\bar{B}_n$.

The manner in which the coefficients B_s, determined by the system (25), will depend upon α_1 and β_1 (which appear in the second members of (25)) is predicted by the reciprocal theorem. This shows that the solution (21) will take the symmetrical form

$$U(\alpha, \beta; \alpha_1, \beta_1) = 4\sqrt{\sin\alpha \cosh\beta \sin\alpha_1 \cosh\beta_1} \sum_{n=1}^{\infty} \sum_{s=1}^{\infty} \frac{Q'_{2s-1}(i\sinh\beta)\, P'_{2s-1}(\mu)}{(2s-1)\,2s} \frac{Q'_{2n-1}(i\sinh\beta_1)\, P'_{2n-1}(\mu_1)}{(2n-1)\,2n} \mathcal{C}_{n,s}(\lambda) \qquad (44)$$

where

$\mathcal{C}_{n,s}(\lambda) = \mathcal{C}_{s,n}(\lambda)$ (independent of the points P and P_1).

This means

$$(4s-1)B_s \equiv \overline{B}_s \equiv \sqrt{\sin\alpha_1 \cosh\beta_1} \sum_{t=1}^{\infty} \frac{Q'_{2t-1}(i\sinh\beta_1)\, P'_{2t-1}(\mu_1)}{(2t-1)\,2t} \mathcal{C}_{t,s}(\lambda) \qquad (45)$$

Instead of the singly-infinite set of coefficients B_s there is now a doubly-infinite set of coefficients c_{ns} to be found, which however have the advantage of being functions of λ only. Substituting this expression for B_s in the system (25) or (43) gives

$$\sum_{s=1}^{\infty} \frac{Q'_{2s-1}(i\sinh\beta_1)\, P'_{2s-1}(\mu_1)}{(2s-1)\,2s} \left[-\frac{i\, c_{n,s}}{(4n-1)\lambda} + \sum_{t=1}^{\infty} b_{nt}\, c_{ts} - (4s-1)\, b_{ns} \right] = 0$$

which must be true for every positive integral n. Moreover it must be true independently of the values of μ_1 and β_1.

This leads to the doubly-infinite system of equations

$$\frac{-i\, c_{ms}}{(4m-1)\lambda} + \sum_{t=1}^{\infty} b_{mt}\, c_{ts} = (4s-1)\, b_{m,s} \qquad (46)$$

for $m = 1, 2, 3, \cdots \infty$ and $s = 1, 2, 3, \cdots \infty$.

If we hold s constant and give m all positive integral values, the coefficients c_{1s} c_{2s} c_{3s} --- are those in the s^{th} row of the array c_{ms}. Holding s constant in eq (46), it merely serves as a label to indicate the row in question. Each selection of s therefore constitutes a selection of a singly infinite system of equations to determine all the singly infinite coefficients c_{ms} in the s^{th} row independently of those in any other row. The method of approximation is obvious. We could determine the 16 coefficients in which m and s range from 1 to 4. If computation were to be made for a number of pairs of points P_1 and P_2 it might be worth while to determine these 16 coefficients c_{ms} once for all.

When λ is not very large, and the solution U is obtained either by the system (43) or the system (46), its resolution into real and imaginary parts would conform with the previous notation, if taken in the form

$$U(\alpha,\beta;\alpha_1,\beta_1) = U^{r}(\alpha,\beta;\alpha_1,\beta_1;\lambda) + i\,\frac{2R_0}{\pi a p}\,V(\alpha,\beta;\alpha_1,\beta_1;\lambda) \qquad (47)$$

where the real functions U^{r} and V are symmetric functions of the two points $(\alpha_1 \beta_1)$, $(\alpha_2 \beta_2)$ and also of $\lambda \equiv \pi a p / 2 R_0$. This agrees with the notation adopted in section 2 and in eq (32a) (32b) for the asymptotic solution when λ is large and U^λ becomes U^∞ which (with V) becomes independent of λ.

It should be noticed that the asymptotic solution which has been applied in the preceding sections, assume that λ is large, and it is practicable only when β_1 and β_2 are not very small, that is when primary point P_1 and secondary P_2 are not close to the shell. In case several pairs of points (P_1, P_2) are to be considered it might save labor to take U in the symmetric form (44) in which the upper limits of n and s in the double series are in each case 4. Instead of determining the first sixteen e_{ns} as suggested above for general values of λ, their asymptotic values may be determined. To find these, it is readily verified by writing out the double system (46) that the solution e_{ns}^∞ of this complete system, for a perfectly conducting shell ($\lambda = \infty$) is

$$\left.\begin{aligned} e_{ns}^\infty &= (4n-1)\,\delta_{ns} \\ \text{where } \delta_{ns} &= 1 \text{ if } n=s, = \text{zero otherwise} \end{aligned}\right\} \qquad (48)$$

Hence, the asymptotic solution is

$$e_{ns} = e_{ns}^\infty + \frac{1}{\lambda} d_{ns} \qquad (49)$$

where the double array $d_{ns} = d_{sn}$ are determined by the doubly-infinite system

$$\sum_{t=1}^{\infty} b_{n,t} d_{ts} = \delta_{ns} = 1 \text{ if } n=s, = \text{zero otherwise} \qquad (50)$$

Here also, the coefficients d_{ns} in the s^{th} row are determined independently of those in any other row

II. Case of Closed Spherical Shell (at rest)

a. Periodic Currents

In this case polar coordinates r, θ, are most appropriate. The exact solution U in series form is obtainable. When the point $P(r,\theta)$ comes on the shell ($r \to a \pm 0$), and the source-point $P_1(r_1, \theta_1)$ is not on it, the expansion given in section 4, page 16, may be written in the condensed form

$$2 Q_{1/2}\left(\frac{\frac{a^2+r_1^2}{2ar_1} - \cos\theta\cos\theta_1}{\sin\theta\sin\theta_1}\right) = 2\pi\sqrt{\sin\theta\sin\theta_1}\sum_{n=1}^{\infty}\left(\frac{r_1}{a}\right)^{\pm(n+\frac{1}{2})}\frac{P_n^1(\cos\theta)P_n^1(\cos\theta_1)}{n(n+1)} \qquad (51)$$

where $\left(\frac{r_1}{a}\right)^{\pm(n+\frac{1}{2})} = \left(\frac{r_1}{a}\right)^{n+\frac{1}{2}}$ if $r_1 \leq a$ (Source P_1 inside)

$= \left(\frac{a}{r_1}\right)^{n+\frac{1}{2}}$ if $r_1 \geq a$ (Source P_1 outside)

Using the abreviation introduced in eq (43), namely $\lambda \equiv \frac{\pi a p}{2R_0}$, it is easilly verified that the solution of the boundary problem formulated in the system of equation (7) is

$$U(r,\theta;r_1,\theta_1) = 2\pi\sqrt{\sin\theta\sin\theta_1}\sum_{n=1}^{\infty}\frac{\left(\frac{r}{a}\right)^{\pm(n+\frac{1}{2})}\cdot\left(\frac{r_1}{a}\right)^{\pm(n+\frac{1}{2})}P_n^1(\cos\theta)P_n^1(\cos\theta_1)}{n(n+1)\left(1 - i\,\frac{2n+1}{8\lambda}\right)} \qquad (52)$$

(The two $\pm$ signs represent two independent alternatives)
Resolving this into real and imaginary parts as in (47)

$$U(r,\theta;r_1,\theta_1) = U^r(r,\theta;r_1,\theta_1;\lambda) + i\,\frac{2R_0}{\pi a p}V(r,\theta;r_1,\theta_1;\lambda) \qquad (53)$$

the formulas for computing U^r and V are

$$U^{\eta}(r,\theta;r_1,\theta_1;\lambda) = 2\pi\sqrt{\sin\theta\sin\theta_1}\sum_{n=1}^{\infty}\frac{\left(\frac{r}{a}\right)^{\pm(n+\frac{1}{2})}\left(\frac{r_1}{a}\right)^{\pm(n+\frac{1}{2})}P_n^1(\cos\theta)\,P_n^1(\cos\theta_1)}{n(n+1)\left[1+\left(\frac{2n+1}{8\lambda}\right)^2\right]} \quad (54_a)$$

$$V(r,\theta;r_1,\theta_1;\lambda) = \frac{\pi}{4}\sqrt{\sin\theta\sin\theta_1}\sum_{n=1}^{\infty}\frac{(2n+1)\left(\frac{r}{a}\right)^{\pm(n+\frac{1}{2})}\cdot\left(\frac{r_1}{a}\right)^{\pm(n+\frac{1}{2})}P_n^1(\cos\theta)\,P_n^1(\cos\theta_1)}{n(n+1)\left[1+\left(\frac{2n+1}{8\lambda}\right)^2\right]} \quad (54_b)$$

For the primary P_1, r_1 is less than a, and for secondary P_2, r_2 is $> a$. Hence the product of the two factors with $\pm$ exponents is

$\left(\frac{r_1}{a}\right)^{2n+1}$ in computing U_{11}^{η} and V_{11}

$\left(\frac{a}{r_2}\right)^{2n+1}$ in computing U_{22}^{η} and V_{22}

and $\left(\frac{r_1}{r_2}\right)^{n+\frac{1}{2}}$ in computing U_{12}^{η} and V_{12}

The asymptotic formulas come from (54_a) and (54_b) by replacing the factor $\left[1+\left(\frac{2n+1}{8\lambda}\right)^2\right]$ of the denominators by 1. Then U^{η} becomes U^{∞} and this (with V) becomes independent of λ.

In the case for which computations were made with cyclidic coordinates as described in section 9, the ratios r_1/a and a/r_2 were so small that the first two or three terms suffice in computing the series (54_a) and (54_b)

Equations (54a), (54e) are exact (for a stationary closed shell) for all values of λ, that is, for all values of R_0 and of the frequency for which the quasi-stationary equations of the electromagnetic field are applicable. This means all frequencies in the technological range as distinguished from the optical range, in other words all frequencies which it would be practicable to set up in the actual circuits.

The values of U_{11}^{r} U_{22}^{r} U_{12}^{r}, (which determine the apparent inductances L_1', L_2' and L_{12}' by eq 2), as well as the values of V_{11}, V_{22}, V_{12} (which determine the apparent resistances R_1', R_2', and R_{12}' by 2) are all functions of the frequency because of the dependence of U^r and V upon λ which appears in the denominators of (54a) and (54b)

However it appears from section (9), where a value R_0 of about .08 ohms was obtained, that 8λ is of the order of $4(10)^7$. Consequently in all the implosions, and for all frequencies of interest in them, the values of U^r and V given by (54a) (54b) will be practically their asymptotic values so all the apparent resistances and inductances will be constants (independent of the frequency)

Hence these constant apparent resistance and inductances are to be computed by eq(2) using the following

$$
\left.
\begin{aligned}
U_{11}^{*} = U_{11}^{\infty} &= 2\pi \sin\theta_1 \sum_{n=1}^{\infty} \left(\frac{r_1}{a}\right)^{2n+1} \frac{\left[P_n^1(\cos\theta_1)\right]^2}{n(n+1)} \\
U_{22}^{*} = U_{22}^{\infty} &= 2\pi \sin\theta_2 \sum_{1}^{\infty} \left(\frac{a}{r_2}\right)^{2n+1} \frac{\left[P_n^1(\cos\theta_2)\right]^2}{n(n+1)} \\
U_{12}^{*} = U_{12}^{\infty} &= 2\pi \sqrt{\sin\theta_1 \sin\theta_2} \sum_{1}^{\infty} \left(\frac{r_1}{r_2}\right)^{n+\frac{1}{2}} \frac{P_n^1(\cos\theta_1)\cdot P_n^1(\cos\theta_2)}{n(n+1)} \\
&= 2\, Q_{1/2}\left(\frac{\frac{r^2+r_1^2}{2rr_1} - \cos\theta\cos\theta_1}{\sin\theta\sin\theta_1}\right)
\end{aligned}
\right\} \quad (55_a)
$$

$$
\left.
\begin{aligned}
V_{11} &= \frac{\pi}{4}\sin\theta_1 \sum_{1}^{\infty} \frac{2n+1}{n(n+1)} \left(\frac{r_1}{a}\right)^{2n+1} \left[P_n^1(\cos\theta_1)\right]^2 \\
V_{22} &= \frac{\pi}{4}\sin\theta_2 \sum_{1}^{\infty} \frac{2n+1}{n(n+1)} \left(\frac{a}{r_2}\right)^{2n+1} \left[P_n^1(\cos\theta_2)\right]^2 \\
V_{12} &= \frac{\pi}{4}\sqrt{\sin\theta_1 \sin\theta_2} \sum_{1}^{\infty} \frac{2n+1}{n(n+1)} \left(\frac{r_1}{r_2}\right)^{n+\frac{1}{2}} P_n^1(\cos\theta_1)\, P_n^1(\cos\theta_2) \\
&= \frac{1}{4} r_1 D_{r_1} U_{12}^{\infty}
\end{aligned}
\right\} \quad (55_b)
$$

The constants U_{12}^{∞} and V_{12} are independent of a and could be evaluated in finite terms by use of tables of elliptic integrals, but for the present applications r_1/a and a/r_2 are both generally so small that it is easier to compute all six of these constants by the above series. Generally all terms after the second or third are negligible.

With a closed shell, these asymptotic constants show that the primary and secondary circuits will have an apparent "resistance coupling" through the mutual resistance constant R'_{12}, the direct electromagnetic coupling being annulled. This becomes evident on using the expression $2Q_{12}$ for V^{∞}_{12} given in (55a) to compute M'_{12} by the last of equations (2). Comparing the result with the expression for the mutual inductance M_{12} between primary & secondary in the absence of the shell (eq on page 51) it is seen that

$$M_{12} - M'_{12} = 0 \quad (\text{for a closed shell}).$$

Hence I_2 enters equation (1_a) only in the term $R'_{12} I_2$ and I_1 enters (1_b) only in the term $R'_{12} I_1$ which justifies the name "resistance-coupling"

This fact, together with the relations noted in section 9 enables us to rewrite the two fundamental circuit equations for primary and secondary separated by a closed stationary conducting shell in a much simpler form.
~~This form is a help to the ... of the effect of the shell in such a simple form that it is easy to extend the~~

~~results from the case of periodic a.c. currents to that of most general possible transient currents.~~

The relations referred to in section 9 amount in short to the statement that in all set-ups used in implosion the constants R_2, L_2, C_2 of the secondary and L_1, C_1 of the primary are unaffected by the introduction of the shell. The essential and important constants that are introduced by the closed shell are the mutual resistance R'_{12} and apparent inductance $M'_{12} = M_{12}$

Without the shell the two circuit equations are

$$\left[R_1 + i\left(pL_1 - \frac{1}{pC_1}\right)\right] I_1 \quad + \quad ipM_{12} I_2 \quad = V \qquad (56_a)$$

$$ipM_{12} I_1 + \left[R_2 + i\left(pL_2 - \frac{1}{pC_2}\right)\right] I_2 = 0 \qquad (56_b)$$

With the closed shell they are

$$\left[R_1 + R'_1 + i\left(pL_1 - \frac{1}{pC_1}\right)\right] I_1 \quad + \quad R'_{12} I_2 \quad = V \qquad (57_a)$$

$$R'_{12} I_1 \quad + \left[R_2 + i\left(pL_2 - \frac{1}{pC_2}\right)\right] I_2 = 0 \qquad (57_b)$$

All the constants appearing in these equations are independent of the frequency except possibly R_1 & R_2 (inductances change very little by skin-effect and proximity-effect). If the coils were wound with fine wire, not too closely packed the AC resistances R_1 and R_2 could be considered constants.

b. Non-periodic Currents

To consider transient currents the use of imaginary quantities will be discontinued and from here on $I_1(t)$ and $I_2(t)$ will denote the real instantaneous primary and secondary currents. Their real vector potentials (ϕ-components only) at the general point $P(r,\theta)$ or $P(x,\rho)$ are $A_1(r,\theta,t)$ and $A_2(r,\theta,t)$ while $A_3(r,\theta,t)$ is that of the currents in the shell. The total vector potential at P is

$$A(r,\theta,t) = A_1 + A_2 + A_3$$

The shell currents produce magnetic fluxes $\phi_{13}(t)$ through the primary and $\phi_{23}(t)$ through the secondary where (lengths being expressed in centimeters)

$$\phi_{13}(t) = 2\pi\rho_1 N_1 A_3(r_1,\theta_1,t)10^{-9} \text{ and } \phi_{23}(t) = 2\pi\rho_2 N_2 A_3(r_2,\theta_2,t)10^{-9} \qquad (61)$$

these fluxes being in practical units

Consider the electrical constants R_1, L_1, C_1 of the primary and R_2, L_2 of the secondary as known constants (in practical units). It is assumed that there is no current anywhere at time $t \le 0$, but at time $t=0$ the initial charge $Q_1(0)=C_1V_1$, on the condenser charged to V_1 volts, begins to discharge through the series resistance R_1 and self inductance L_1. No other emf is applied anywhere. The discharge current $I_1(t) = -D_t Q_1 = -\dot{Q}_1(t)$ is governed by the equation

$$L_1\dot{I}_1(t) + R_1 I_1(t) - Q_1(t)/C_1 + M_{12}\dot{I}_2(t) + \dot{\phi}_{13}(t) = 0$$

Eliminating Q_1 by differentiating, we may take the fundamental equations in the form

$$I_1(t) = -\dot{Q}_1(t) \qquad (62_a)$$

$$L_1\ddot{I}_1(t) + R_1\dot{I}_1(t) + I_1(t)/C_1 + M_{12}\ddot{I}_2(t) + \ddot{\Phi}_{13}(t) = 0 \qquad (62_b)$$

$$L_2\dot{I}_2(t) + R_2 I_2(t) \qquad + M_{12}\dot{I}_1(t) + \dot{\Phi}_{23}(t) = 0 \qquad (62_c)$$

with the initial conditions

$$Q_1(0) = C_1 V_1 \;, \; I_1(0) = I_2(0) = \Phi_{13}(0) = \Phi_{23}(0) = 0 \qquad (62_d)$$

The trace of the primary circuit is the point $P_1(r_1, \theta_1)$ and that of the secondary is the point $P_2(r_2, \theta_2)$ where $r_1 < a$ and $r_2 > a$, the shell radius being $\underline{a}$ centimeters. Their ordinary mutual inductance M_{12} may be computed by

$$M_{12} = 4\pi N_1 N_2 \sqrt{\rho_1\rho_2}\; Q_{1/2}\left(\frac{\frac{r_1^2+r_2^2}{2r_1r_2} - \cos\theta_1\cos\theta_2}{\sin\theta_1\sin\theta_2}\right) 10^{-9}$$

$$= \frac{(2\pi\rho_1 N_1)(2\pi\rho_2 N_2)}{r_2}\sum_{n=1}^{\infty}\left(\frac{r_1}{r_2}\right)^n \frac{P_n^1(\cos\theta_1)P_n^1(\cos\theta_2)}{n(n+1)}\, 10^{-9} \text{ henries} \qquad (63)$$

The vector potential A_3 of the shell-currents must be of the form

$$\left.\begin{aligned} A_3(r,\theta,t) &= \sum_{n=1}^{\infty} J_n(t)\left(\frac{a}{r}\right)^{n+1} P_n^1(\cos\theta)/\sqrt{n(n+1)} \text{ where } a \le r \le \infty \\ &= \sum_{n=1}^{\infty} J_n(t)\left(\frac{r}{a}\right)^{n} P_n^1(\cos\theta)/\sqrt{n(n+1)} \text{ where } 0 \le r \le a \end{aligned}\right\} \qquad (64)$$

where the coefficients $J_n(t)$ are in amperes.

Since lengths are here measured in centimeters and the practical electromagnetic units are here implied, in which the unit of length is 10^{-9} cm, the boundary condition which was derived in section (4) ~~becomes~~

$$\mathcal{D}_t A(a,\theta,t) \equiv \dot{A}_1(a,\theta,t) + \dot{A}_2(a,\theta,t) + \dot{A}_3(a,\theta,t) =$$

$$= \frac{R_0}{4\pi \cdot 10^{-9}}\left[\mathcal{D}_r A_3(r,\theta,t)_{r=a+0} - \mathcal{D}_r A_3(r,\theta,t)_{r=a-0}\right]$$

or by (64)

$$\sum_{n=1}^{\infty}\left[a\,10^{-9}\dot{J}_n(t) + \frac{(2n+1)R_0}{4\pi}J_n(t)\right]\frac{P_n^1(\cos\theta)}{\sqrt{n(n+1)}} = -a\,10^{-9}\left[\dot{A}_1(a,\theta,t) + \dot{A}_2(a,\theta,t)\right] \quad (65)$$

Now A_1 and A_2 are given everywhere by

$$\left.\begin{aligned} A_1(r,\theta,t) &= 2N_1 I_1(t)\sqrt{\frac{\rho_1}{\rho}}\,Q_{1/2}\left(\frac{\frac{r_1^2+r^2}{2r_1 r} - \cos\theta_1\cos\theta}{\sin\theta_1\sin\theta}\right) \\ A_2(r,\theta,t) &= 2N_2 I_2(t)\sqrt{\frac{\rho_2}{\rho}}\,Q_{1/2}\left(\frac{\frac{r_2^2+r^2}{2r_2 r} - \cos\theta_2\cos\theta}{\sin\theta_2\sin\theta}\right)\end{aligned}\right\} \quad (66)$$

Since the primary is inside, and the secondary outside, the shell, (51) gives

$$\left.\begin{aligned} A_1(a,\theta,t) &= \frac{2\pi\rho_1 N_1 I_1(t)}{a}\sum_{n=1}^{\infty}\left(\frac{r_1}{a}\right)^n\frac{P_n^1(\cos\theta_1)P_n^1(\cos\theta)}{n(n+1)} \\ A_2(a,\theta,t) &= \frac{2\pi\rho_2 N_2 I_2(t)}{r_2}\sum_{n=1}^{\infty}\left(\frac{a}{r_2}\right)^n\frac{P_n^1(\cos\theta_2)P_n^1(\cos\theta)}{n(n+1)}\end{aligned}\right\} \quad (67)$$

By use of these expansions the boundary condition (65) becomes

$$\sum_{n=1}^{\infty} \frac{P_n^1(\cos\theta)}{\sqrt{n(n+1)}}\left\{a(10)^{-9}\dot{J}_n(t) + \frac{(2n+1)R_0}{4\pi}J_n(t) + \frac{2\pi(10)^{-9}}{\sqrt{n(n+1)}}\left[\rho_1 N_1\left(\frac{r_1}{a}\right)^n P_n^1(\cos\theta_1)\,\dot{I}_1(t) + \rho_2 N_2\left(\frac{a}{r_2}\right)^{n+1} P_n^1(\cos\theta_2)\,\dot{I}_2(t)\right]\right\} = 0$$

Since this boundary condition must be fulfilled at every point of the shell ($0 < \theta < \pi$) the coefficient of each function $P_n^1(\cos\theta)$ must be zero at every instant. This gives an infinite system of equations which may be written

$$\left.\begin{aligned} &L_3\dot{J}_n(t) + R^{(n)}J_n(t) + m_{n1}\dot{I}_1(t) + m_{n2}\dot{I}_2(t) = 0 \\ &\text{for} \\ &\quad n = 1, 2, 3, \text{---}\infty \end{aligned}\right\} \quad (68)$$

where

$$\left.\begin{aligned} L_3 &\equiv a(10)^{-9} \text{ henries} \quad (a \text{ in cm}) \\ R^{(n)} &\equiv (2n+1)R_0/4\pi \text{ ohms} \quad (R_0 \text{ in ohms}) \\ m_{n1} &\equiv 2\pi\rho_1 N_1\left(\frac{r_1}{a}\right)^n P_n^1(\cos\theta_1)(10)^{-9}/\sqrt{n(n+1)} \text{ henries} \quad (\rho_1 \text{ in cm}) \\ m_{n2} &\equiv 2\pi\rho_2 N_2\left(\frac{a}{r_2}\right)^{n+1} P_n^1(\cos\theta_2)(10)^{-9}/\sqrt{n(n+1)} \text{ henries} \quad (\rho_2 \text{ in cm}) \end{aligned}\right\} \quad (69)$$

With this notation eqn (64) used in (61) gives

$$\Phi_{13}(t) = \sum_{n=1}^{\infty} m_{n1}J_n(t) \quad \text{and} \quad \Phi_{23}(t) = \sum_{n=1}^{\infty} m_{n2}J_n(t) \qquad (70)$$

As far as the influence of shell currents is concerned these are equivalent to an infinite set of fictitious linear currents J_n in linear circuits. The self inductance of each is the constant L_3, the resistance of the n^{th} fictitious circuit is $R^{(n)}$. Its mutual inductance with the primary is m_{n1} and with the secondary is m_{n2} as shown by eq (70). This interpretation is admitted also by the system of equations (68) which imply that there is no mutual inductance between members of the system of currents J_n. This special relation originates in the fact that the vector potential A_3 in eq (64) consists of the sum of normal solutions.

There is also a fundamental relation of importance which characterizes the closed shell. The sum of all the products of mutual inductance of form $m_{n1} \cdot m_{n2}$ is proportional to the mutual inductance M_{12} between primary and secondary circuits, that is by (63) and (69)

$$\sum_{n=1}^{\infty} m_{n1} \cdot m_{n2} = L_3 M_{12} \qquad (71)$$

All currents vanish at time $t = 0$

The differential equations for the entire system of currents becomes

$$I_1(t) = -\dot{Q}_1(t) \qquad (72a)$$

$$L_1 \ddot{I}_1(t) + R_1 \dot{I}_1(t) + I_1(t)/C_1 + \frac{\ddot{I}_2(t)}{L_3}\sum_{n=1}^{\infty} m_{n1} m_{n2} + \sum_{n=1}^{\infty} m_{n1} \ddot{J}_n(t) = 0 \qquad (72b)$$

$$L_2 \dot{I}_2(t) + R_2 I_2(t) + \frac{\dot{I}_1(t)}{L_3}\sum_{n=1}^{\infty} m_{n1} m_{n2} + \sum_{n=1}^{\infty} m_{n2} \dot{J}_n(t) = 0 \qquad (72c)$$

and

$$L_3 \dot{J}_n(t) + R^{(n)} J_n(t) + m_{n1} \dot{I}_1(t) + m_{n2} \dot{I}_2(t) = 0, \text{ for } n = 1, 2, 3 \ldots \infty \qquad (72d)$$

The initial conditions are

$$\left.\begin{aligned} &Q_1(0) = C_1 V_1 \\ &I_1(0) = I_2(0) = J_n(0) = 0 \quad (n = 1, 2, 3 \ldots \infty) \end{aligned}\right\} \qquad (73)$$

The case of [illegible] conducting [illegible] is $R_n \to \infty$ or $R^{(n)} \to \infty$

Dividing each equation of (72d) by R^n gives every $J_n \equiv 0$. Hence replacing $\frac{1}{L_3}\sum m_{n1} m_{n2}$ by its equivalent M_{12} the system reduces to

$$\left.\begin{aligned} &I_1(t) = -\dot{Q}_1(t) \\ &L_1 \ddot{I}_1(t) + R_1 \dot{I}_1(t) + I_1(t)/C_1 + M_{12} \ddot{I}_2(t) = 0 \\ &L_2 \dot{I}_2(t) + R_2 I_2(t) + M_{12} \dot{I}_1(t) = 0 \end{aligned}\right\} \qquad (74a)$$

In case of a perfectly conducting shell $R^{(n)} \to 0$

so by (72d) $J_n(t) = -(m_{n1} I_1 + m_{n2} I_2)/L_3$ which gives

$$\left.\begin{aligned} &\left(L_1 - \frac{\sum m_{n1}^2}{L_3}\right)\ddot{I}_1(t) + R_1 \dot{I}_1(t) + I_1(t)/C_1 = 0 \\ &\text{and} \\ &\left(L_2 - \frac{\sum m_{n2}^2}{L_3}\right)\dot{I}_2 + R_2 I_2 = 0 \end{aligned}\right\} \qquad (74b)$$

In this case the primary and secondary current satisfy independent equations, their coupling being annulled by introduction of the perfectly conducting, closed, shell between them. Their self-inductances are both decreased.

The system (72) has particular solutions of the form

$$\left.\begin{aligned} Q_1(t) &= A e^{-\beta t} \text{ and } I_1(t) = \beta A e^{-\beta t} \\ I_2(t) &= \beta B e^{-\beta t} \text{ and } J_n(t) = \beta^2 C^{(n)} e^{-\beta t} / (L_3 \beta - R^{(n)}) \end{aligned}\right\} \quad (75)$$

where β has certain characteristic values to be found. One of these is $\beta = 0$ which corresponds to $Q_1(t) = A =$ constant and $I_1(t) \equiv I_2(t) \equiv J_n(t) \equiv 0$. As this is of no present interest we ignore it in cancelling a factor β from the equations which result on using these form in the set of equations 72). The system (72d) require

$$C^{(n)} = -m_{n1} A - m_{n2} B \quad \text{for } n = 1, 2, 3, \cdots \infty \qquad (76)$$

Using this the equation (72b) and (72c) become

$$\frac{A}{B} = \frac{\beta^2 S_{12}(\beta)}{\left(L_1 - \sum_{n=1} \frac{m_{n1}^2}{L_3}\right)\beta^2 - R_1 \beta + \frac{1}{C_1} - \beta^2 S_1(\beta)} = \frac{\left(L_2 - \sum_{n=1}^{\infty} \frac{m_{n1}^2}{L_3}\right)\beta - R_2 - \beta S_2(\beta)}{\beta S_{12}(\beta)} \qquad (77)$$

The three S-functions may be computed as functions of β by the series

$$\left.\begin{aligned} S_1(\beta) \equiv \frac{1}{L_3}\sum_{n=1}^{\infty}\frac{m_{n1}^2 R^{(n)}}{L_3\beta - R^{(n)}} \quad \text{and} \quad S_2(\beta) \equiv \frac{1}{L_3}\sum_{n=1}^{\infty}\frac{m_{n2}^2 R^{(n)}}{L_3\beta - R^{(n)}} \\ S_{12}(\beta) \equiv \frac{1}{L_3}\sum_{n=1}^{\infty}\frac{m_{n1}\, m_{n2} R^{(n)}}{L_3\beta - R^{(n)}} \end{aligned}\right\} \quad (78)$$

The time-constant β must be some root of the second of equations (77). For each root β_s there is one arbitrary coefficient say B_s in terms of which the constant A is given by ~~either~~ of the two equations (77) and then the constants $C^{(n)}$ are determined by (76). By superposing all possible particular solutions one obtains the number of arbitrary constants B_s which are necessary and sufficient to be fitted to the initial conditions

The definitions of m_{n1} and m_{n2} given in eq (69) show that the S-series will converge very rapidly when the primary and secondary circles are not close to the shell, that is when r_1/a and a_2/r are both fairly small. In fact when these ratios are less than 1/4 and when the primary and secondary are also in the plane $x = 0$, $(\cos\theta_1 = \cos\theta_2 = 0)$, it is found that the first term of each S-series differs from the entire sum by much less than one percent. The basis of this approximation is that the term $m_{n_1}^2$ is proportional to

$$\left[\left(\frac{r_1}{a}\right)^n \frac{P_n'(\cos\theta_1)}{n(n+1)}\right]^2$$

Also $m_{n_2}^2$ is proportional to $\left[\left(\frac{a}{r_2}\right)^{n+1} \frac{P_n'(\cos\theta_2)}{n(n+1)}\right]^2$

For the experimental arrangement, $\cos\theta_1 = \cos\theta_2 = 0$. Since $P_n'(0)$ vanishes when n is even, every alternate term of the series drops out so that in passing from one term to the next there is a convergence ratio $(r_1/a)^4$ or $(a/r_2)^4$.

Hence we may take

$$\left.\begin{aligned}
m_{11} &= \pi\sqrt{2}\, N_1 \frac{\rho_1^2}{a}(10^{-9}) \text{ henries} \\
m_{12} &= \pi\sqrt{2}\, N_2 \frac{a^2}{\rho_2}(10^{-9}) \text{ henries} \\
M_{12} &= m_{11}\, m_{12}/L_3 \text{ henries} = 2\pi^2 N_1 N_2 \frac{\rho_1^2}{\rho_2}(10^{-9}) \text{ henries} \\
S_1(\beta) &= \frac{m_{11}^2 R_3}{L_3(L_3\beta - R_3)}, \quad S_2(\beta) = \frac{m_{12}^2 R_3}{L_3(L_3\beta - R_3)} \\
\text{so} \quad S_{12}(\beta) &= \frac{m_{11}\, m_{12} R_3}{L_3(L_3\beta - R_3)} = \sqrt{S_1(\beta)\, S_2(\beta)} \\
\text{where} \quad R_3 &\equiv R^{(1)} \equiv 3R_0/4\pi \text{ and } L_3 = a(10^{-9})
\end{aligned}\right\} \quad (79)$$

From this geometrical arrangement, the effect of currents in the shell is represented by the first current J_1 only.

For brevity let the four time-constants be defined by

$$\left.\begin{aligned} \alpha_1 &\equiv \left(R_1 + \sqrt{R_1^2 - 4L_1/C_1}\right)/2L_1 \\ \alpha_1' &\equiv \left(R_1 - \sqrt{R_1^2 - 4L_1/C_1}\right)/2L_1 \\ \alpha_2 &\equiv R_2/L_2 \quad \text{and} \quad \alpha_3 \equiv R_3/L_3 \end{aligned}\right\} \tag{80}$$

The characteristic time-constants β_s ($s = 1,2,3,4$) are the four roots of eq (77) which may be written

$$(\beta-\alpha_1)(\beta-\alpha_1')(\beta-\alpha_2)(\beta-\alpha_3) =$$

$$= \beta^2\left[\frac{m_{12}^2}{L_2L_3}(\beta-\alpha_1)(\beta-\alpha_1') + \frac{m_{11}^2}{L_1L_3}\beta(\beta-\alpha_2) - \frac{M_{12}^2}{L_1L_2}\beta(\beta+\alpha_3)\right] \tag{81}$$

In the case of no conducting shell this becomes on dividing by α_3 and then letting $\alpha_3 \to \infty$,

$$(\beta-\alpha_1)(\beta-\alpha_1')(\beta-\alpha_2) = \beta^3\frac{M_{12}^2}{L_1L_2} \tag{81'}$$

The system of equations (72) reduce to four

$$I_1(t) = -Q_1(t) \tag{82a}$$

$$(D_t+\alpha_1)(D_t+\alpha_1')I_1 = -(M_{12}\ddot{I}_2 + m_{11}\ddot{J}_1)/L_1 \tag{82b}$$

$$(D_t+\alpha_2)I_2 = -(M_{12}\dot{I}_1 + m_{12}\dot{J}_1)/L_2 \tag{82c}$$

$$\frac{m_{12}}{L_3}D_tI_2 = -\frac{m_{11}}{L_3}\dot{I}_1 - \dot{J}_1 - \alpha_3 J_1 \tag{82d}$$

where

$$Q_1(0) = C_1V_1 \quad \text{and} \quad I_1(0) = I_2(0) = J_1(0) = 0 \tag{83}$$

The solution is

$$
\left.\begin{aligned}
I_2(t) &= \sum_{s=1}^{4} \beta_s B_s e^{-\beta_s t} \\
Q_1(t) &= \frac{L_2}{M_{12}} \sum_{s=1}^{4} \left[\left(1-\frac{m_{12}^2}{L_2L_3}\right)\frac{\beta_s}{\alpha_3} - \left(1+\frac{\alpha_2}{\alpha_3}\right) + \frac{\alpha_2}{\beta_s}\right] B_s e^{-\beta_s t} \\
J_1(t) &= \frac{-L_2}{M_{12}} \sum_{s=1}^{4} \left[\left(1-\frac{m_{12}^2}{L_2L_3}\right)\frac{\beta_s^2}{\alpha_3} - \frac{\alpha_2}{\alpha_3}\beta_s\right] B_s e^{-\beta_s t}
\end{aligned}\right\} \quad (84)
$$

The four constants B_s are determined by the four linear equations which express the four initial conditions (83). (On writing these it is found that they are equivalent to the following simpler set

$$
\left.\begin{aligned}
\sum_{s=1}^{4} B_s/\beta_s &= C_1V_1M_{12}/R_2 \\
\sum_{1}^{4} B_s &= 0 \\
\sum_{1}^{4} \beta_s B_s &= 0 \\
\sum_{1}^{4} \beta_s^2 B_s &= 0
\end{aligned}\right\} \quad (85)
$$

A simpler and fairly accurate approximation to the solution (84) is obtainable by further restricting the values of the self inductances.

In the applications in view, the self-inductances L_1 and L_2 of primary and secondary are of the same order of magnitude and this is at least one hundred times the value of the largest of the four positive constants L_3, m_{11}, m_{12}, M_{12}, these four being also of the same order of magnitude.

To get this approximate solution first eliminate J_1 between (82c) and (82d). The result is

$$\left\{\left(1-\frac{m_{12}^2}{L_2L_3}\right)D_t^2+(\alpha_2+\alpha_3)D_t+\alpha_2\alpha_3\right\}I_2=-\frac{M_{12}\alpha_3}{L_2}\dot{I}_1$$

Since $\frac{m_{12}}{L_3}$ is of the order of magnitude of unity and m_{12}/L_2 of the order of .01, this equation may be replaced by

$$(D_t+\alpha_2)(D_t+\alpha_3)I_2(t)=-\frac{M_{12}\alpha_3}{L_2}\dot{I}_1(t) \qquad (86)$$

Since I_1 I_2 J_1 vanish when $t=0$ this is equivalent to the initial vanishing of I_2, $\dot{I}_2$ and I_1, so (86) gives

$$I_2(t)=-\frac{M_{12}\alpha_3}{L_2}\int_0^t I_1(t')\left[\frac{\alpha_2 e^{-\alpha_2(t-t')}-\alpha_3 e^{-\alpha_3(t-t')}}{\alpha_2-\alpha_3}\right]dt' \qquad (87)$$

Similarly the second member of (82a) contains factors M_{12}/L_1 and M_{11}/L_1 each less than .01. Neglecting it, the primary current is unaffected by secondary and shell currents so that since $Q_1(0)=C_1V_1$ and $I_1(0)=0$ the integration of (82a) gives

$$I_1(t) = \frac{V_1}{L_1}\left(\frac{e^{-\alpha_1' t} - e^{-\alpha_1 t}}{\alpha_1 - \alpha_1'}\right) \tag{88}$$

Using this in (87) gives

$$I_2(t) =$$

$$= \frac{-V_1 M_{12}\alpha_3}{L_1 L_2}\left\{\frac{1}{\alpha_1-\alpha_1'}\left[\frac{\alpha_1 e^{-\alpha_1 t}}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)} - \frac{\alpha_1' e^{-\alpha_1' t}}{(\alpha_2-\alpha_1')(\alpha_3-\alpha_1')}\right]\right.$$

$$\left. + \frac{1}{\alpha_2-\alpha_3}\left[\frac{\alpha_2 e^{-\alpha_2 t}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_1')} - \frac{\alpha_3 e^{-\alpha_3 t}}{(\alpha_3-\alpha_1)(\alpha_3-\alpha_1')}\right]\right\} \tag{88a}$$

For comparison, the same primary current gives the following secondary when there is no shell (This is the limit $\alpha_3 \to \infty$ of the preceding equation)

$$I_2(t) =$$

$$= \frac{-V_1 M_{12}}{L_1 L_2}\left\{\frac{1}{\alpha_1-\alpha_1'}\left[\frac{\alpha_1 e^{-\alpha_1 t}}{\alpha_2-\alpha_1} - \frac{\alpha_1' e^{-\alpha_1' t}}{\alpha_2-\alpha_1'}\right] - \frac{\alpha_2 e^{-\alpha_2 t}}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_1')}\right\} \tag{88b}$$

From the nature of the application this solution is doubtless a sufficiently close approximation to the exact solution (84). To find the error it would be necessary to compute the solution (84) for comparison.

Case of double-roots.

In the example below the roots α_1 and α_1' are identical. Evaluating the 0/0 occurring in the three preceding equations gives

$$I_1(t) = \frac{V_1}{L_1} t e^{-\alpha_1 t} \qquad (89)$$

The corresponding current $I_2(t)$ which this primary current induces in the secondary when the shell is between them is

$$I_2(t) = \frac{-V_1 M_{12} \alpha_3}{L_1 L_2} \left\{ \frac{e^{-\alpha_1 t}}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)} \left[-\alpha_1 t + \frac{\alpha_2\alpha_3 - \alpha_1^2}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)} \right] \right.$$

$$\left. + \frac{1}{\alpha_2-\alpha_3} \left[\frac{\alpha_2 e^{-\alpha_2 t}}{(\alpha_2-\alpha_1)^2} - \frac{\alpha_3 e^{-\alpha_3 t}}{(\alpha_3-\alpha_1)^2} \right] \right\} \qquad (89_a)$$

With no shell this reduces to

$$I_2(t) = \frac{-V_1 M_{12}}{L_1 L_2 (\alpha_2-\alpha_1)} \left\{ \frac{\alpha_2}{\alpha_2-\alpha_1} \left(e^{-\alpha_1 t} - e^{-\alpha_2 t} \right) - \alpha_1 t e^{-\alpha_1 t} \right\} \qquad (89_b)$$

where

$\alpha_1 = R_1/2L_1$, $\alpha_2 = R_2/L_2$ and $\alpha_3 = R_3/L_3$

The following is a typical set of data.

Geometrical Data

$r_1 = \rho_1 = 3.0$ cm, $(\cos\theta_1 = 0)$, $N_1 = 6$ turns

$r_2 = \rho_2 = 63.5$ cm, $(\cos\theta_2 = 0)$, $N_1 = 1$ turn

$a = 12.7$ cm

These give

$m_{11} = .0189(10)^{-6}$ henries

$m_{12} = .0113(10)^{-6}$ "

$M_{12} = .0168(10)^{-6}$ "

This shows that J_1 is the most important part of the shell-current.

Electrical Constants

$L_1 = 3.0(10)^{-6}$ henries $\qquad R_1 = 1.733$ ohms

$L_2 = 6.6(10)^{-6}$ " $\qquad R_2 = 70.$ "

$L_3 = .0127(10)^{-6}$ " $\qquad R_3 = .0191$ "

$V_1 = 135$ volts $\qquad (R_0 = 08$ " $)$

$C_1 = 4(10)^{-6}$ farads $= 4L_1/R_1^2$ (so $\alpha_1' = \alpha_1$)

These give

$\alpha_1 = a_1' = .289(10)^6$ sec^{-1}

$\alpha_2 = 10.6$ "

$\alpha_3 = 1.5$ "

With these numerical values the three currents may be computed as functions of the time by equations (89), (89a) and (89b)

Chester Snow, Nov 9, 1941