INDUCED CURRENTS IN A HEMISPHERICAL SHELL
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ABSTRACT

A mathematical study of the induced currents in a hemispherical shell has been made to serve as background and interpretive material for the experiments of Fitzhugh and Rosen (LA-521). Ed.
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Induced Currents in a Hemispherical Shell

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1 Object

This problem originates in the picture of a spherically symmetric wave-front of detonation giving the neighboring material a high electrical conductivity $\kappa$, which dies out soon after the wave passes. Considering the radius $R(t)$ of the spherical wave-front as fairly well known from detonation theory, the electromagnetic effect of this thin shell moving in an applied magnetic field may be calculated if $R_0$ is a known constant or known function of the time. The resistance of a cube of the conducting material whose edges have a length equal to the radius thickness $\Delta r$ of the shell is $R_0 = 1/k\Delta r$. Although the effective thickness $\Delta r$ might actually be an inch, the shell will be treated as infinitely thin as far as concerns the currents induced in it and their magnetic field. In fact the current distribution has been previously formulated in terms of integrals involving the unknown function of the time $R(t)$. Taking into account
also the mutual interaction between the currents in the shell and those induced in an inner (perfectly conducting) sphere. When \( R_0 \) is known this gives an explicit formula for the current in the pick-up coil before the metal sphere begins to move as well as later. With large scales, when the implosion phase (of the metal sphere) begins, the pick-up current is not the sum of the two effects that would be produced separately by sphere and shell. There is in addition a term of mutual action which may contribute 3 to 5\% to the total record when the shell is expanding. It may contribute 20 to 50\% if, as is now supposed, we have to consider a converging wave of detonation.

The photographic record of an alternating current signal received in the pick-up coil in Rosen's experiments are capable (in principle) of determining \( R_0 \). In these experiments a thick hemi-spherical shell of explosives is detonated on the outside as symmetrically as possible so that it is supposed to produce a converging wave front of detonation similar to that in implosion shots except that it is hemispherical. No magnetic field is applied, but instead an exciting current is caused to oscillate at 150 kilocycles. This circuit is coaxial with the pick-up coil and with the axis of symmetry of the thick shell of explosives. Both coils...
are electrically shielded, the shielding metals being cut so they could carry no currents circulating around the axis. It is assumed that the motion of the hypothetical, thin, conducting shell has no effect comparable with that of its mere position in the alternating off the primary exciting current.

From these views and experiments arises the mathematical problem of finding the distribution of current in a thin hemispherical shell (which is stationary) under the influence of the primary current but also in the presence of the secondary current in the pick-up coil. The latter current, being wholly induced, farther from the shell, and outside it, will have a smaller influence than the primary current upon the current distribution in the shell. However, there is no reason to ignore it, as the presence of the secondary does not complicate the problem in any essential manner. If the current distribution in the shell can be found in the presence of one alternating current only, the effect of two or more simultaneous currents oscillating with the same frequency but different phases is found by superposition - the effect of phases being fully taken into account by the use of complex vector potential.
2. The Two Circuit Equations

The instantaneous primary and secondary currents are the real parts of \( I_1 e^{ipt} \) and \( I_2 e^{ipt} \) where \( p \) is \( 2\pi \) times the frequency and the complex constants \( I_1 \) and \( I_2 \) are to be found. These circuits will be considered as having all their turns \( N_1 \) or \( N_2 \) concentrated at the mean turn in each case, these being circle coaxial with the \( x \)-axis in the planes \( x_1, x_2 \) with radii \( r_1, r_2 \). The traces of these circles in the meridian half-plane pass the points \( P_1 \) and \( P_2 \) in fig 1. It is convenient to refer to any point by its rectangular coordinates \( x, p \) or in some breath by its polar coordinates \( r, \theta \) with the same axis and origin.

The conducting shell has for its meridian section the thin quarter-circle \( A_0A \) of fig 1 of radius \( a \)

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Fig.1 The \( x, p \) half-plane and section of hemi-spherical shell.
The resistance, self-inductance, and capacitance in the
two circuits are $R_1$, $L_1$, $C_1$ and $R_2$, $L_2$, $C_2$. Their and their
mutual inductance $M_{12}$ are all known for the particular
frequency $\omega/2\pi$. An emf $V$ acting = real part of $V e^{i\omega t}$ is
applied in the primary, $V$ being a known positive constant.

The currents in the shell produce magnetic flux through
the primary and secondary circuits (in the positive $x$-direction)
whose instantaneous values may be denoted by the real
parts of $\phi_1 e^{i\omega t}$ and $\phi_2 e^{i\omega t}$.

The steady periodic state being established, the two
elementary circuit equations to determine $I_1$ and $I_2$ are

$$[R_1 + i(\omega L_1 - \frac{1}{\omega C_1})]I_1 + i \omega M_{12} I_2 + i \omega \phi_1 = V$$

$$i \omega M_{12} I_1 + [R_2 + i(\omega L_2 - \frac{1}{\omega C_2})]I_2 + i \omega \phi_2 = 0$$

The constants $\phi_1$ and $\phi_2$ are found (eq(35) below) to be of the
form

$$\phi_1 = -2\pi N_1 N_2 \sqrt{\frac{R_2}{L_2}} U_1 I_1 - 2\pi N_2 \sqrt{\frac{R_2}{L_2}} U_2 I_2$$

$$\phi_2 = -2\pi N_1 N_2 \sqrt{\frac{R_2}{L_2}} U_1 I_1 - 2\pi N_2 \sqrt{\frac{R_2}{L_2}} U_2 I_2$$

where the three constants $U_1$, $U_2$, and $U_2$ are special
values of a dimensionless, separatrix function of
two points $U(r_1, \theta; r_2, \theta) = U(r_1, \theta; r_2, \theta)$.  

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In equation (31.82) it is found to consist of 

\[ U = U^* + i \frac{2 \text{Re} V(r, \theta; r', \theta')}{\text{Re} \sigma} \]

(for small values of \( \text{Re} \sigma \))

and a method is obtained for computing the real functions 

\( U^* \) and \( V \). The case of a shell with infinite conductivity 

\( (R_0 = 0) \) quadrature \( U \) a real function, has designated by \( U^* \). 

Adiabatic formula for its evaluation is found; the computation 

of the real function \( V \) was more laborious. 

The meaning of the subscripts is 

\[ V_z = V(r, \theta; r_z, \theta_z), \text{ etc.} \]

With these relations the two circuit equations become

\[ \left[ R' + R_1 + i \left( \frac{\sigma}{p(L_i - L'_i)} - \frac{1}{PC} \right) \right] I_1 + \left[ R'_2 + i \sigma (M_2 - M'_2) \right] I_2 = V \]  

\[ \left[ R'_2 + i \sigma (M_2 - M'_2) \right] I_1 + \left[ R_2 + R'_2 + i \left( \frac{\sigma}{p(L_i - L'_i)} - \frac{1}{PC} \right) \right] I_2 = 0 \]

\[ R'_1 = 4 R_0 N_2 \frac{\alpha}{\omega} V_i \]

\[ L'_1 = 2 \pi N_2 \frac{\alpha}{\omega} U_i \]

\[ R'_2 = 4 R_0 N_2 \frac{\alpha}{\omega} \frac{\alpha}{\omega} V_2 \]

\[ L'_2 = 2 \pi N_2 \frac{\alpha}{\omega} U_2 \]

\[ M'_2 = 4 \pi N_2 \frac{\alpha}{\omega} \sqrt{P_2} \frac{\alpha}{\omega} U_2 \]
Electromagnetic rgs units are used in the mathematical equations, so these are the units which apply in the preceding equations. To use them with practical units, remembering that $U$ and $V$ are dimensionless, it is only necessary to multiply the definitions of $\frac{1}{2} \Omega \frac{1}{2} \Omega$ and $\frac{1}{2} \Omega$ by $10^3$ to convert them to theives. Then $R_0$ and all the initial coil constants may be taken in practical units.

The expression for $U$ is derived on the assumption that $\frac{1}{2} \Omega \frac{1}{2} \Omega$ is so small that terms in its square may be neglected in comparison with terms in $R_0$. From the large difference between electromagnetic effects observed with copper shells from those observed for shots, it is thought that this assumption is good.

The inductive, or geometrical, constants $L_1^' \frac{1}{2} \frac{1}{2} \frac{1}{2} L_1 ^'$ depend only upon the positions of the points $P_1$ and $P_2$ and the radius $a$ of the shell, but involve no material constants. The same is true of $V_1^', V_2^', V_2$ as the resistances $R_1 ^', R_2 ^', R_2$ are all proportional to the unknown material constant $R_0 = 1 \frac{1}{2} \frac{1}{2} R_0$.

Consequently as soon as the positions and coil-constants and the shell radius $a$ are known, the amplitude of the signal received in the pick-up coil (which is the absolute value $|I_2|$ of the complex constant $I_2$) could be substituted for several values of $R_0$ and a graph drawn plotting $|I_2|$ against $R_0$. The value of $R_0$ where these ordinates of
this curve reaches the observed value of $|I_1|$ is the roof of the shell.

It is to be expected that the functions $U$ and $V$ will turn out to be roughly of the same order of magnitude in the sense that $\rho_1 a$ and $a/R$ are of the same order. This is phrase is rather loose and will be applied when any $\rho_1 a \leq 1$ and $a/R \leq \frac{1}{10}$. (In one of Rosens their other hemisphere $a = 2.5''$, $a = 13/16''$ and $b = 9.5''$).
3. The Static Experiments.

The static experiments with various metallic hemisphere shells require for their theory only eq. (19) in which $|I_2| = 1$

The thing determined is the amplitude $|I_2|$ in the secondary when $|I_1| = 1$

under two conditions (without and with the shell).

In the first case $R_2', R_2', M_2'$, and $L_2'$ are zero, and if the amplitude $|I_2|$ had been measured in amperes, this would give a check upon the correctness of the initial soil-constants $R_2, L_2, M_2$, and $C_2$, but nothing more.

In the second case all four constants $R_2', R_2', L_2'$ and $M_2'$ are in eq. (19) but only $R_2'$ and $R_2''$ vary with the material constant $R_0$ which is known. Hence measurement of $|I_2|$ would be practicable for the determination of $R_0$ (if it were unknown), only if $R_0$ is so large that the two resistances $R_0'$ and $R_0''$ are of appreciable importance compared with the other terms in eq. (20). If they are not important this means that the curve of $|I_2|$ against $R_0$ is so flat as to be practically useless.

To say whether or not this is the case would require that computations be made of $V_2', V_2', U_2'$ and $U_2''$ for this particular configuration.

The thinnest copper hemisphere shell had a radial thickness $\Delta n = 0.15'' = 0.035\text{ cm}$. The resistivity of copper at $20^\circ\text{C}$ is about $1.72 \times 10^5 \text{ ohm} \cdot \text{cm}$ so that $R_0 = 0.000045 \text{ ohms}$. It might
appear at first sight that these indicate the shells to be perfect conductors as far as this method is concerned. However there is in eq(2) a large factor \( N_i \) \((=45\text{ mm})\) in the formula for \( R_2' \). The corresponding term in the formula for \( R_2' \) is \( N_i^2 \) when \( N_i=3 \). But this term appears added to \( R_2 \) which is also very small. Before saying that \( R_2' \) is of no importance it would be necessary to consider the relative magnitude of the term \( p_k^2-1 \cdot \frac{1}{p_k^2} \) in which it must be remembered that \( \frac{1}{p_k} \) and \( N_i^2 \) are to be multiplied by \( \sqrt{N_i} \).

Until such computations are made all we can say is that we share the general feeling that the shells are, for this purpose, practically perfect conductors, which means that the observed decrease of \( I_0 \) \((=7\text{ mm})\) is practically all to be accounted for by the appearance of the constants \( l_2^2 \) and \( M_2^2 \) in eq(15), which depend upon \( \frac{1}{p_k} \) and are independent of material constants. If so these experiments would serve only as a check upon the computations of \( l_2^2 \) and \( M_2^2 \).

If the experimental difficulties of obtaining shells of a material with sufficiently large \( R_0 \) could be overcome then the method ought to be a practicable method of substitution, by which the \( R_0 \) of a detonation wave-front could be found by comparison. This would avoid the need for any but elementary mathematics and computation.

The only alternative seems to be the mathematical
A problem followed by a serious job of computation which however is fairly straightforward.

The dynamical effects are so different from their static ones that it seems safe to conclude that $R_0$ for the explosive wave is large enough to make $|\Sigma|$ sensitive to its changes. This of course may be quite compatible with the assumption that $R_0/\text{ap}$ is small since $P = 400$. 
4. The Boundary Problem

In this section consider the shell in the presence of only one oscillating current, whose time may be any point P, not on the shell. Since the effect of capacity currents have been eliminated by electrical shielding, the only component of current in the shell which need be considered is that circulating around the axis. The volume density of this component may be called $J_\varphi$. Since the only component of the vector potential will then be $A_\varphi$, it is written $A$.

The law of conduction is

$$J_\varphi = -\kappa \frac{\partial A}{\partial r}$$

or

$$j_\varphi \equiv J_\varphi \Delta n = -(\kappa \Delta n) \frac{\partial A}{\partial r} \equiv -\frac{1}{R_0} \frac{\partial A}{\partial r}$$

where $j_\varphi$ is the equivalent surface density of current when we consider the shell infinitely thin. In that case the vector potential is continuous at the conducting arc $r = a$, $\theta > 0$. Its normal derivative have a discontinuity there which is a measure of $j_\varphi$ in accordance with the circuit law relation

$$\frac{H_\varphi(a+0, \theta) - H_\varphi(a-0, \theta)}{4\pi} = 4\pi j_\varphi$$

Since the magnetic field $H$ is the curl of $A$ the
The tangential component is given by
\[ H_0(r, \theta) = - \nabla \times A(r, \theta) - A(r, \theta) / r \]

The circuit relation is therefore
\[ \sum_{n=a_0}^{b_0} \left[ \frac{\nabla A(r, \theta)}{r} \right] = \frac{\mu_0}{2 \pi} \nabla A(a, \theta) \]

This with the law of conduction gives the boundary condition at the conducting wall.

\[ \sum_{n=a_0}^{b_0} \left[ \frac{\nabla A(r, \theta)}{r} \right] = \frac{\mu_0}{2 \pi} \nabla A(a, \theta) \]

These are real, instantaneous relations. In the case of a steady periodic state the instantaneous vector potential may be taken as the real part of \( A e^{i \omega t} \) so that \( \nabla A = i \omega A \).

With this understanding from here on, \( A \) will denote a complex function of position, independent of time.

At all ordinary points it satisfies the partial differential equation
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) A = 0 = \left( \frac{\partial^2}{\partial \rho^2} + \frac{3}{\rho} \right) A \]
which is the same as
\[ r \nabla \left( \nabla \times (r A) \right) + \left( \nabla^2 - \frac{3}{\rho^2} \right) r A = 0 \]

Also \( A \to 0 \), like \( \rho \) when \( \rho \to 0 \)
and \( A \to 0 \), like \( \sin \rho r \) when \( r = \infty \).
The one remaining condition describes the nature of the singularity, or manner in which \( A \) becomes infinite when the variable point \( P(x, y) \) approaches the fixed source-point \( P(x, 0) \). This will appear after some preliminaries which should be brought in at this point. They result in the replacement of the dependent function \( A \) by a function \( U \).

Since the frequency is in the radio range (as distinguished from the optical range) the quasi-stationary equations of the electromagnetic field apply. This means that when a current changes we ignore the time taken for the change of field to be propagated to any point in space and therefore compute instantaneous fields at all distant points by the stationary formulas. (The phase relationships are properly accounted for by use of the complex potential).

It is known that the stationary vector potential at any point \( P(x, y) \) (or \( P(x, 0) \)) due to a unit steady current (circulating around the \( x \)-axis in the positive sense) in a circle of radius \( \rho \), coaxial with the \( x \)-axis in the plane \( x = 0 \), is

\[
2 \sqrt{\frac{\rho}{c}} \frac{Q}{(1 + \frac{D^2}{2\rho^2})} = 2 \sqrt{\frac{\rho}{c}} \frac{Q}{v} \left( \frac{\frac{r^2 + x^2}{2\pi n}}{\sin \theta \sin \phi} \right)
\]
where the distance, measured in the meridian plane of fig. 1, between the two points $P$ and $P'$ is $D$.

$$D^2 = (x-x')^2 + (\rho - \rho')^2.$$ 

$Q_{1/2}(z)$ denotes the accepted symbol for the second kind of Legendre function of $z$ with parameter $1/2$. This function of the argument indicates in always positive, except that it vanishes when one of the points moves to the $x$ axis or to infinity. It is always finite except when the points approach coincidence in which case it goes to $+\infty$, and in such a manner that \[Q_{1/2}(1 + \frac{D^2}{2\rho_2}) = \log \rho_2 + \text{terms which are finite when } D = 0.\]

The mutual inductance $M$ between two parallel circles whose traces are the points $P_1, P_2$ is given by

$$\frac{M}{4\pi \sqrt{\rho_1 \rho_2}} = Q_{1/2} \left(1 + \frac{D^2}{4\rho_2^2}\right) = 2(E-E)/K - k R$$

where the modulus of the complete elliptic integral $K$ of $E$ is given by

$$k^2 = \frac{4\rho_1 \rho_2}{D_0^2 + 4\rho_1 \rho_2} = \frac{4\rho_1 \rho_2}{(x_1 - x_2)^2 + (\rho_1 - \rho_2)^2}.$$ 

There are tables available for getting numerical values of $M$ directly without the use of elliptic integrals.

The mutual inductance $M_{12}$ appearing in the ($N_1 N_2 M_2$)

$$M_{12} = N_1 N_2 M_2 \frac{2Q_{1/2}(\frac{4\rho_1 \rho_2}{D_0^2 + 4\rho_1 \rho_2})}{2Q_{1/2}(\frac{4\rho_1 \rho_2}{D_0^2 + 4\rho_1 \rho_2})}.$$
The function \( Q_{\frac{1}{2}} \left( 1 + \frac{D^2}{4FP} \right) \) is a symmetric function of the two points which is the fundamental function for the solution of potential problems of this type in which the boundary conditions are given on surface of revolution. There are an infinite number of orthogonal coordinate systems \((x, \rho)\) which are "separable" coordinates for the potential equation.

This means that \(x\) and \(\rho\) are such conjugate functions of \(x\) and \(\rho\) that partial differential equation for the potential (of which that for \(A\) is a special case) may be reduced to ordinary differential equations.

In each of these systems \(Q_{\frac{1}{2}}\) has a so-called addition theorem, which is the canonical expansion in normal functions of \(Q_{\frac{1}{2}}\), this being the nucleus of the integral equation which is the key to the solution of such problems.

With polar coordinates this canonical expansion is

\[
Q_{\frac{1}{2}} \left( \frac{2 + r^2}{2r} \right) = \pi \sum_{n=1}^{\infty} \left( \frac{\alpha}{\beta} \right)^{n+1} \frac{P_n(\cos \alpha)P_n(\cos \beta)}{n(n+1)}
\]

where \(0 < r < r_1\), these being interchanged otherwise.

The set of functions \(P_n(\cos \theta)\) are associated Legendre functions, these being normal functions for a spherical surface but not for a semi-spherical one.

The boundary problem here of interest can be formulated...
so neatly in terms of polar coordinates as to tempt me to try for its solution with them. Results are disappointing for the expansion just written is not an expansion in normal functions for this hemispherical problem. One is driven to search for a coordinate system $(r, \theta)$ in which such an expansion for $Q_{mn}$ is obtainable.

The total complex vector potential produced at any point $P(r, \theta)$ by the complex current $I_i$ in the primary and the currents which it induces in the shield may be written

$$A(r, \theta) = N.I. \sqrt{\frac{r}{p}} G(r, \theta; r, \theta)$$  \hspace{1cm} (3)

and this (Green's) function $G$ may be taken in the form

$$G(r, \theta; r, \theta) = 2Q_{nn}(1 + \frac{D^2}{4r^2}) - U(r, \theta; r, \theta)$$  \hspace{1cm} (4)

Since we regard the source-point $P(r, \theta)$ as fixed anywhere but on the conducting arc we may suppose its coordinates and write $G(r, \theta)$ and $U(r, \theta)$ where the point $P(r, \theta)$ is assigned freely.

From the partial differential equations derived above for the vector potential $A$ it is evident
that $G(r, \theta)$ is determined by the following equations

\[
\begin{align*}
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{3}{4r^2} \right) G &= 0 \\
\end{align*}
\]

that is

\[
\begin{align*}
\partial_r^2 (r \partial_r G) + \partial_{\theta}^2 G - \frac{3G}{4 \sin^2 \theta} &= 0 \\
G &\to 0 \text{ like } \rho^{2n} \text{ when } \rho \to 0 \\
G &\to 0 \text{ like } (\sin \theta)^{-\frac{3n}{2}} \text{ when } \rho \to \infty
\end{align*}
\]

\[
G(a+0, \theta) = G(a-0, \theta) \text{ for } n \geq \theta > 0
\]

\[
\partial_n G(n, \theta) - \partial_n G(n, \theta)_{a-0} = 0 \text{ for } n \geq \theta > \frac{\pi}{2}
\]

\[
\partial_n G(n, \theta) - \partial_n G(n, \theta)_{a-0} = \frac{\mu n R}{R_0} G(a, \theta) \text{ for } n > \theta > 0
\]

$G \to \infty$ like $2 \log 1/D$ when $P \to P$.

Since the $Q$-functions in eq(4) prescribes the nature of the singularity for the Green's function, it is evident that $U(r, \theta)$ must be finite and continuous for all functions of the point $P$ including the case where it becomes identical with the fixed source point $P$. Hence the equations to determine $U$, are the same as the above with the omission of the last and the replacement of $\tilde{G}(0, \theta)$ by $U(a, \theta) - R_0 \left[ u_1 - u_2 \right] - \frac{\mu n R}{R_0} G(a, \theta) \text{ for } \frac{\pi}{2} \leq \theta > 0.$
This boundary condition is the only place through which the source \( P \) enters in the determination of \( V \) or \( \Omega \). Through it they will depend upon \((r, \theta)\) and the reciprocal theorem foretells their general independence.

Its proof does not depend upon the shape of the conducting arc which could be any curve. The particular values of \( R_0 \) do not enter the proof: it could be real or complex, zero or infinite (if infinite the boundary condition at \( r = \infty \) needs an obvious relaxation to permit the current from the source to escape to infinity).

To prove it, make a cut along the conducting arc and consider both sides of this cut as part of the external boundary of the \( x \)-\( p \) half-plane, the remainder of the external boundary being the entire \( x \)-axis and a semi-circle with center at the origin and radius which ultimately becomes infinite. Insert two internal boundaries in the shape of two circles each of radius \( \varepsilon \), one with center at the source \( P \), the other with center at any other point \( P_2 \) distinct from \( P \), neither being on the cut.

Green's integral transformation may be written

\[
\iint (U \nabla^2 U - U \nabla^2 U) \, dx \, dp = -\oint (U \frac{\partial^2 U}{\partial x^2} - U \frac{\partial^2 U}{\partial y^2}) \, ds
\]

where \( \nabla^2 = \frac{1}{x^2} + \frac{1}{y^2} \) and the surface integral is taken over
the half-plane with the external and internal boundaries specified. The line integral is around the complete boundary with the normal pointing inward. At points within the complete boundary \( \mathcal{C} \) and \( \mathcal{C}^* \) together with their first and second derivatives must be finite and continuous. These conditions are satisfied by

\[
U(x, \rho) = G(x, \rho; x_1, \rho_1) \quad \text{and} \quad U(x, \rho) = G(x, \rho; x_2, \rho_2)
\]

In the limit \( \varepsilon = 0 \), this transformation gives

\[
G(x_1, \rho_1; x_2, \rho_2) = G(x_2, \rho_2; x_1, \rho_1)
\]

every one of the six equations of the system (5) being necessary for the proof. Since the \( \mathcal{Q} \)-function in (4) is obviously symmetric, the significant form of the reciprocity theorem is that \( U \) is unaltered by interchange of source-point and receiving point, that is

\[
U(r, \theta; r, \theta) = U(r, \theta; r, \theta)
\]  \hspace{1cm} (6a)

Although the proof assumed that neither \( P \) nor \( P^* \) is on the cut, it is true when one but not both are on it, (from continuity).

A second relation, the inversion theorem, is quite independent of the reciprocity theorem. It also is independent of the particular value of \( R_0 \), but does depend upon the fact that the cut is some part of the arc of the semi-circle, \( r=a \). It need not be the quarter-circle of
Fig 1. But could be the arc \( \theta_2 > \theta > 0 \)

where \( \theta_2 \) is any positive angle less than \( \theta \).

It is easily verified that all the equations which determine \( U(r, \theta) \) including (5) retain the same form in the variables \( r', \theta \) when we make the substitution \( r = a^2/r' \). Two points \( P(r, \theta) \) and \( P'(r', \theta') \) are inverse points (or images of one of the other) by reflection in the semi-circular mirror \( r = a \), if

\[ rr' = a^2 \quad \text{and} \quad \theta' = \theta. \]

Consequently if \( P \) is any fixed point not on the cut

the inversion theorem states that it induces such a current distribution in the shell, that the value of \( U \) at every point \( P \) is the same as at the image \( P' \) of \( P \)

\[ U(r, \theta; r', \theta) = U(a^2/r, \theta; r', \theta). \tag{62} \]

Applying one theorem after the other shows that the value of \( U(r, \theta) \) at all points in the half-plane is unaltered by moving any source-point \( P \) to its image \( P' \).

These two theorems account for the preference in studying the symmetrical function \( U \) rather than the \( \phi \) vector potential \( A \). By their combined use it is evident that the boundary problem for \( U \) may be stated
in the following simpler form.

Let \( P(a, \theta) \) be any point not on the cut.

A function \( V(r, \theta) \) is required only inside the semi-circle \( r = a, \pi \leq \theta < 2\pi \) which is determined by

\[
\begin{align*}
(\partial^2_r + \partial^2_\theta - \frac{2}{r} \partial_r) V &= 0, \\
\partial_\theta (n \partial_\theta V) + \partial^2_r V - \frac{3V}{\sin^3 \theta} &= 0
\end{align*}
\]

(7a)

that is

\[
\partial_\theta (n \partial_\theta V) + \partial^2_r V - \frac{3V}{\sin^3 \theta} = 0
\]

(7b)

\[ V \to 0 \text{ like } \sin^3 \theta \text{ when } \sin \theta \to 0 \]

(7c)

\[ \partial_\theta U(n, 0) \to 0 \text{ when } r \to a - 0 \text{ if } \pi \leq \theta < 2\pi \]

(7d)

\[
U(a, 0, \theta) + \underbrace{Re}_{\text{anip}} \frac{1}{\partial_\theta} \frac{1}{n} \left. \partial_\theta U(n, 0) \right|_{n = a - 0} = 2Q \left( \frac{\frac{a^2 - n^2}{2a^2}}{\sin \theta \cos \theta} \right) \text{ for } \theta \to 0 \]

(7e)

The partial differential equation with rectangular coordinates \((x, y)\) has been retained because it will be found to be

the simpler form to transform into cylindrical \((r, \theta)\).
5. Cylindrical Coordinates

Let \( z = x + iy = re^{i\theta} \) be a complex variable, and make a cut in its half-plane along the circular arc, \( r = a \), from \( \theta = 0 \) to \( \theta = \theta_0 \), where the constant \( \theta_0 \) is any angle between zero and \( \pi \). There is no appreciable complication in this generalization of the quadrantals arc of Fig. 1. We may later place \( \theta_0 = \pi/2 \).

The half-plane these cut may be represented conformally upon a semi-infinite strip of the plane of the complex variable \( w = \alpha + i\beta \). We take this strip as \( 0 < \alpha < \pi \) and \( 0 < \beta < \infty \). The conformance is shown by similar lettering points on Figures 2, 2a, and 2b.

The equation of transformation is

\[
z = -a \frac{(\sin \omega - i \sinh \nu)}{i \sin \omega + i \sinh \nu} \tag{8}
\]

from which

\[
\frac{dz}{dw} = -i \frac{2a \sinh \nu \cos \omega}{(\cosh \omega + i \sinh \nu)^2}
\]

where

\[
\frac{1}{h} = \left| \frac{dz}{dw} \right| = \frac{2a \sinh \nu \sqrt{\cosh \beta - \sin \omega}}{[\cosh (\beta - \gamma) + \cosec \omega][\cosh (\beta - \gamma) - \cosec \omega]}
\]

that is

\[
\sqrt{dx^2 + dp^2} = \frac{\sqrt{d\alpha^2 + d\beta^2}}{h} = \frac{2a \sinh \nu \sqrt{\cosh \beta - \sin \omega}}{[\cosh (\beta - \gamma) + \cosec \omega][\cosh (\beta - \gamma) - \cosec \omega]} \frac{\sqrt{d\alpha^2 + d\beta^2}}{h} \tag{9}
\]

The conformance breaks down where \( \frac{dz}{dw} \) is zero or infinite.
It is zero when \( \beta = 0 \), \( \alpha = \frac{\pi}{2} \), which corresponds to the end \( A_0 \) of the cut. It is also zero when \( \beta \to \infty \) as the point \( C \) at the left end of the semi-circle is carried to infinity in the \( w \)-strip. It is infinite when \( \beta = \gamma \) and \( \alpha = \frac{\pi}{2} \) so that the infinite semi-circle on the \( z \)-half-plane \( \{ Re \, z \geq 0 \} \) is projected to the point \( B \) on the boundary of the \( w \)-strip.

To make the point \( A_0 \) at the end of the cut (which is \( z = a e^{i \alpha} \)) correspond to the point \( \alpha = \frac{\pi}{2}, \beta = 0 \) the positive real constant \( \gamma \) must be so chosen as to bring this about. Placing \( \omega = \frac{\pi}{2} \) and \( z = a e^{i \alpha} \), it becomes

\[
 \alpha e^{i \beta} = -a \left( \frac{1 - i \tan \gamma}{1 + i \tan \gamma} \right) = a \left( \frac{\cosh \gamma - i \sinh \gamma}{\cosh \gamma} \right)
\]

so

\[
\begin{align*}
\cos \theta_0 &= \frac{\sinh \gamma - 1}{\cosh \gamma} \quad \text{and} \quad \sin \theta_0 &= \frac{2 \sinh \gamma}{\cosh \gamma} \\
\sinh \gamma &= \cot \theta_0 / 2, \quad \cosh \gamma &= 1/\sin \theta_0, \quad \text{and} \quad \tanh \gamma = \csc \theta_0.
\end{align*}
\]

We can take \( \theta_0 = \frac{\pi}{2} \) as \( \sinh \gamma = 1, \ \gamma = 0.881, \ \cosh \gamma = 2 \).

Resolving the second member of (10) into real and imaginary parts and equating the final to \( x \), the second to \( iy \) gives the two real equations expressing the rectangular co-ordinates in terms of \( \alpha \) and \( \beta \):

\[
\begin{align*}
x &= \frac{a (\sinh \gamma - \sinh \beta - \sin \alpha)}{[\cosh (\beta - \gamma) + \cos \alpha] [\cosh (\beta + \gamma) - \cos \alpha]} \\
p &= \frac{2 \sinh \gamma \cosh \beta \sin \alpha}{[\cosh (\beta + \gamma) + \cos \alpha] [\cosh (\beta - \gamma) - \cos \alpha]}
\end{align*}
\]
If amplitudes and angles are equated in eq (15), this gives the real equations expressing the polar coordinates in terms of $\alpha$ and $\beta$.

\[
R = \alpha \sqrt{\frac{[cosh(\alpha - \gamma) - cos\alpha]}{[cosh(\alpha + \gamma) + cos\alpha]} \cdot \frac{[cosh(\alpha + \gamma) + cos\alpha]}{[cosh(\alpha - \gamma) + cos\alpha]}}}
\]

\[
\cos \theta = \frac{\sinh \gamma \sinh \beta \sin \alpha}{\sqrt{[cosh(\alpha - \gamma) - cos\alpha] \cdot [cosh(\alpha + \gamma) - cos\alpha]}}
\]

\[
\sin \theta = \frac{2 \sinh \gamma \cosh \beta \sin \alpha}{\sqrt{[cosh(\alpha - \gamma) - cos\alpha] \cdot [cosh(\alpha + \gamma) - cos\alpha]}}
\]

These equations show that a pair of inverse points in the $z$-plane are carried into two points of the w-plane, which are equidistant from the base $\beta = 0$ and from the vertical bisector $\alpha = \pi/2$. If one of them is $P(\alpha, \beta)$ whose inverse image is $P'(\pi - \alpha, \beta)$, any function of $z = \cos \alpha + \sin \beta \sin \alpha$ must be an even function of $\alpha$, will have the same value at two points which are images of each other.

Any given point will actually be described by assigning numerical values to any of its polar coordinates. It is but a few minutes work to find the numerical values of its coordinates $\alpha$, $\beta$. The steps are
perhaps more simple if we first find the numerical values of its polar coordinates \( r', \theta' \) referred to the point \( C \) as origin, so

\[
\begin{align*}
    r' \cos \theta' &= r \cos \theta + a = x + a \\
    r' \sin \theta' &= r \sin \theta = \rho
\end{align*}
\]

(13)

It is then found that

\[
\begin{align*}
    \frac{r' \cos \alpha}{2a \sin \beta} &= \frac{\cos \theta' - r' \cos \theta}{\sin \beta} \\
    \frac{r' \sin \alpha}{2a \sin \beta} &= \frac{\sin \theta'}{\cosh \beta}
\end{align*}
\]

(14a)

which may also be written

\[
\begin{align*}
    \frac{r' \cosh \beta}{2a \sin \beta} &= \frac{\sin \theta'}{\sinh \alpha} \\
    \frac{r' \sinh \beta}{2a \sin \beta} &= \frac{\cosh \theta' - r' \cosh \theta}{\cosh \alpha}
\end{align*}
\]

(14b)

Between the two latter, \( \beta \) may be eliminated by use of the relation \( \cosh \beta \cdot \sinh \beta = 1 \), and this gives

\[
\frac{r' \sin \alpha}{4a^2} - \sin \alpha \left\{ \frac{r'^2}{4a^2} + \sinh \beta \left[ \sin^2 \theta' + (\cosh \theta' - \frac{r'}{2a})^2 \right] \right\} + \sinh \beta \cdot \sin^2 \alpha = 0 \quad (15)
\]

Solving for \( \sin \alpha \) gives

\[
2 \sin^2 \alpha = 1 + \left( \frac{2a \sin \beta}{r'} \right)^2 \left\{ \sin^2 \theta' + (\cosh \theta' - \frac{r'}{2a})^2 \right\}
\]

\[
- \sqrt{\left[ \left( \frac{r'}{2a \sin \beta} + \sin \theta' \right)^2 + (\cosh \theta' - \frac{r'}{2a})^2 \right] \left[ \left( \frac{r'}{2a \sin \beta} - \sin \theta' \right)^2 + (\cosh \theta' - \frac{r'}{2a})^2 \right]} \quad (16)
\]

This determines \( \alpha \) when \( \frac{r'}{2a}, \theta' \), and \( \gamma \) are numerically given, after
which \( \beta \) is found by the root of \((14.8)\). In this manner the coordinates \((\alpha_1, \beta_1)\) of the trace \(P\) of the primary circuit may be found, and similarly \((\alpha_2, \beta_2)\) of the secondary. Reference to figures \((2b)\) and \((5a)\) shows that \(\alpha\) and \(\alpha_2\) will both be less than \(\pi\) while \(\cos \alpha\), will be positive and \(\cos \alpha_2\) negative.

To trace the correspondence between the \(z\)-plane and the \(w\)-strip we may first find the equation of the locus of a curve which is a member of the family \(\alpha = \text{constant}\). From eq \((15)\), after placing \(\sin \psi = \cot \phi_2\) this is found to be

\[
r' = \frac{(\alpha_2 \cot \phi_2)}{(\cot \phi_2 - \cos \phi)} \cos \phi \left\{ 1 - \frac{\cos \phi}{\cot \phi_2} \sqrt{1 - (\cot \phi_2 \cos \phi) \tan^2 \phi} \right\}
\] (17)

This is the equation of the meridians curves of a family of confocal, cyclides of revolution. It inverts into a family of one-sheeted, confocal, hyperboloids of revolution. Each member is a curve beginning at the point \(C\) and ending perpendicularly on the cut. If the constant parameter \(\alpha\) of the curve is less than \(\pi / 2\), it ends on the inside of the cut, if greater than \(\pi / 2\) on the outside. If \(\alpha = \pi / 2\), it is the arc of the semi-circle \(r = \alpha\) which complements the cut. This one is the only member of the family drawn in fig \(2a\).
The equation of the family of curves $\beta = \text{constant}$, orthogonal to the first family, is obtained by squaring and adding together the two equations (14a) thus eliminating $\alpha$.

This gives

$$r' = \frac{2a \cot \alpha_2}{\cot \alpha_2 - \sin \beta} \left\{ \cos \beta' + \frac{\sin \beta}{\cot \alpha_2} \sqrt{1 - \frac{\sin^2 \beta'}{\cos^2 \beta \sin^2 \alpha_2}} \right\}$$

(17c)

This family of the meridians curves of confocal cyclids of revolution inserts into a family of meridian curves of confocal oblate spheroids. Every curve begins perpendicularly on the $x$-axis, and with one exception, ends perpendicularly on that axis. The exception is the open curve $\beta = \pi$ which begins at 0 and goes to $\infty$ with a vertical asymptote passing through $C$. All curves with $\beta > \pi$ lie to the left of this open curve, all to the right of $\beta < \pi$ to the right. (This is where primary and secondary are located.)

The locus in the $\pi$-plane of the equation $\beta = \pi$ is the infinite semi-circle $BBB$ together with the open curve $ODB$ whose equation is

$$r' = a \sqrt{1 - \sin^2 \beta' \sin^2 \beta}$$

(17c)

If $\beta$ is small, the locus $\beta = \text{constant}$ hugs the cut on both sides, and the locus $\beta = 0$ is both sides of the cut. As we go up the inside of the cut $\alpha$ increases from zero to the value $\pi/2$ when its edge $\beta_0$ is reached at $\beta = \pi/2$. Returning then to the $x$-axis on the outside.
of the cut, $\alpha$ goes on increasing to a limit $\pi$.
Two adjacent points on opposite sides of the cut in fig. 2 are coincident image points, which are transformed into two widely separated points on the base line of the $w$-strip, where $\alpha = 0$, and whose coordinates are $\alpha$ and $\pi - \alpha$.

Consequently any function of position on the $x$-plane which is single valued in a region which includes the cut (such as $U(x, \theta)$), will transform into a function of $\theta$ and $\beta$ which is an even function of $\Pi$ ($= \cos \theta$).
6. Boundary Problem with Cylindrical Coordinates

To transform the partial differential equation (7a) one finds

\[(D_x^2 + D_y^2)U = h^2(D_x^2 + D_y^2)U \]

so that (7) becomes

\[(D_x^2 + D_y^2)U - \frac{3}{4h^2}U = 0 \]

Reference to (9) and (11) shows that

\[\frac{1}{h^2} = \frac{1}{\sin^2 \alpha} - \frac{1}{\cos^2 \beta} \]

Hence \(U(\alpha, \beta)\) must satisfy the equation

\[\left\{D_x^2 + D_y^2 + \frac{3}{4\sin^2 \alpha} - \frac{1}{\cos^2 \beta}\right\}U = 0 \quad (18)\]

at all points in the strip.

This has solutions of the form

\[U = \psi(\alpha) \phi(\beta)\]

where

\[\frac{d^2}{d\alpha^2}U + \left[(m+\frac{1}{2})^2 - \frac{3}{4\sin^2 \alpha}\right]U = 0 \]

\[\frac{d^2}{d\beta^2}U - \left[(m+\frac{1}{2})^2 - \frac{3}{4\cos^2 \beta}\right]U = 0 \]

where \(m\) is an arbitrary constant.
In the first make the double substitution
\[ \mu = \cos \alpha \quad \text{and} \quad \nu(\alpha) = \tan \alpha \frac{y}{\sin \alpha} \]
It becomes
\[ \frac{d}{d \mu} \left[ (1-\mu^2) \frac{dy}{d\mu} \right] + \left[ \frac{n(n+1)}{\mu^2} - \frac{1}{1-\mu^2} \right] y = 0 \]
which is satisfied by the associated Legendre function \( P_n^m \).

The double substitution
\[ \nu = \sinh \beta \quad \text{and} \quad \nu = \cosh \beta \frac{y}{\cosh \beta} \]
carries the equation for \( \nu \) into the same form
\[ \frac{d}{d \nu} \left[ (1-\nu^2) \frac{dy}{d\nu} \right] + \left[ \frac{n(n+1)}{\nu^2} - \frac{1}{1-\nu^2} \right] y = 0 \]

For this problem the elements of the solution again are the particular solutions of the form
\[ U = Q'(i \sinh \beta) P_n^m(\sinh \beta) \]

For \( 0 < \mu < 1 \), the even functions of \( \mu, P_n^{2s}(\sin \theta) \), \( s=1,2,3,\ldots \) constitute a complete set of orthogonal functions,
\[ \int P_n^{2s}(\sin \theta) P_{n'}^{2s}(\sin \theta) \sin \theta \, d\theta = (2s-1) \frac{2s}{4s-1} \delta_{n,n'} \quad (19) \]
The functions \( Q'_n(i \sinh \beta) \) are real, their definiton being
\[ Q'_n(i \sinh \beta) = \pi^{\frac{1}{2}} \left( \frac{(2s)!}{(2s+1)!} \right) \frac{\sin \theta}{\Gamma(2s+1)} \frac{\Gamma\left(\frac{1}{2} - \frac{3}{4} + \frac{1}{2} \frac{2s}{1+e^{2\beta}}\right)} {\sqrt{1+e^{2\beta}}} \quad (20) \]
where \( F \) is the hypergeometric function.

From this, one finds

\[
\begin{aligned}
Q_{-1}^{2}(\alpha) &= \sqrt{\pi} \left(-\frac{\alpha}{(\alpha+1)} \right)^{1/2} \text{ and } B_{n} \left[ Q'_{-1}(\sinh \alpha) \right] = -2 \sqrt{\pi} \frac{1\!n_{1/2}}{1/2} \frac{\alpha^{n}}{(\alpha+1)^{1/2}} \\
\end{aligned}
\]  
(20)

From the preceding discussion it is evident that

\( U \) will satisfy all but the last boundary condition

if it is represented by the convergent series, valid everywhere,

\[
U(\alpha, \beta) = \sqrt{\sinh \alpha} \sum_{s=1}^{\infty} \frac{(4s-1)!}{(2s-1)2s} B_{s} Q'_{-1}(\sinh \alpha) P_{\mu}^{s} 
\]  
(21)

When \( \theta \to 0 \) while \( \theta > \Theta \geq 0 \), this means that

\( \beta \to 0 \) while \( \pi \beta > \alpha > 0 \), that is, the cut is

approached from within, so that by eq (9)

\[
\frac{d \alpha}{\sinh \beta} = -2 \alpha \frac{\sinh \beta}{\cosh \beta - \mu^{2}} d \beta
\]

By use of (40), the last boundary condition (17b) becomes

\[
\begin{aligned}
\frac{R_{0}}{2 \pi \epsilon_{p}} \sum_{s=1}^{\infty} \frac{(4s-1)!}{(2s-1)2s} B_{s} P_{\mu}^{s} + \frac{\sinh \beta}{\cosh \beta - \mu^{2}} \sum_{s=1}^{\infty} \frac{s+1}{(2s+1)!} B_{s} P_{\mu}^{s} & = \\
& = \frac{\sinh \beta}{(\cosh \beta - \mu^{2})} \sum_{s=1}^{\infty} \frac{s+1}{(2s+1)!} B_{s} P_{\mu}^{s} \\
& = -2 \alpha \frac{\sinh \beta}{\cosh \beta - \mu^{2}} \frac{1}{\sqrt{\pi \cot \alpha}} Q_{1/2}^{1}(\frac{\frac{2}{\alpha}}{2 \alpha \cot \alpha}) \\
& \quad \text{which must be satisfied for } 0 < \mu < 1.
\end{aligned}
\]  
(22)
The canonical expansion of the function \( Q_{\nu_2} \left( \frac{x + \frac{D^2}{2\eta} }{2\eta} \right) \) in these coordinates is derived in a publication of the National Bureau of Standards MT 15 (1942).

We require in (22) only the case where \( P \) is on the cut, \( r = a \).

Thus (23), (page 252 eq 21),

\[
Q_{\nu_2} \left( \frac{x + \frac{D^2}{2\eta} }{2\eta} \right) = \sqrt{\frac{\pi}{\eta}} \sum_{s=1}^{\infty} \frac{(4s-1)!}{(2s-1)s!} \frac{Q\left(\text{cosh}^2 \theta \right)}{2s-1} \frac{P_{2s}^1}{2s-1} \frac{P_{2s}^1}{2s-1} \tag{23}
\]

which is valid for \( P \), any point in the complex plane.

Hence the boundary condition (22) takes the form

\[
\sum_{s=1}^{\infty} \frac{(4s-1)!}{(2s-1)s!} \frac{B_s P_{2s}^1}{2s-1} P_{2s}^1 \left( \frac{x + \frac{D^2}{2\eta} }{2\eta} \right) = \frac{\sinh \gamma \mu}{\cosh \gamma \mu - \mu^2} \sqrt{\frac{\pi}{\eta}} \sum_{s=1}^{\infty} \frac{(4s-1)!}{(2s-1)s!} \frac{Q\left(\text{cosh}^2 \theta \right)}{2s-1} \frac{P_{2s}^1}{2s-1} \frac{P_{2s}^1}{2s-1} \tag{24}
\]

for \( 0 < \mu \leq 1 \)

If this equation is multiplied by \( \frac{P_{2s}^1}{2s-1} \, d\mu \) and integrated from \( \mu = 0 \) to \( \mu = 1 \), taking account of the integral formula (19), the result is an infinite system of linear equations to determine the unknown coefficients \( B_s \).

This set of equations may be written.
\[ \frac{2R_0}{\pi \rho a} B_0 + \sum_{s=1}^{\infty} \left( \frac{45-1}{s} \right) \frac{1}{\rho_0} B_s^0 = \sum_{s=1}^{\infty} \left( \frac{45-1}{s} \right) \frac{1}{\rho_0} B_s^0 \]  

where \( \rho \) takes in succession the values \( n=1, 2, 3, \ldots, \infty \). The coefficients for a perfectly conducting shell (\( R_0=0 \)) are the real constants

\[ B_s^0 = \sqrt{\sin \theta, \cos \theta, \frac{Q(\sin \theta)}{45-0.25}} \]  

The real coefficients are pure numbers, defined by

\[ b_{n,s} = b_{s,n} = (-1)^{n+s} \frac{\Gamma(n+1)}{n! \Gamma(s+1)} \sinh \left( \frac{\mu}{s} \right) \left( \frac{P_n^s(\mu)}{\sinh \gamma} \right) \]  

The only place where the angular extent \( \theta_0 \) of the conducting shell enters these equations is in the determination of these coefficients.

For the \( \theta_0 = \pi/2 \), which generates a hemispherical shell we take

\[ \sinh \gamma = 1 \quad \text{and} \quad \cosh \gamma = 2 \]

Such an application to a sphere it is thought that the constant \( 2R_0 \rho_0 a \) is probably small since \( \rho = 4(16/3) \). Hence terms with the square of this constant as a factor may be neglected so that writing

\[ B_s = B_s^0 + \frac{i(2R_0)}{\pi \rho a} C_s \]

the set of equations to determine \( B_s^0 \)
real constants $C_s$,

$$\sum_{s=1}^{\infty} l_{m,s} C_s = B_s^\infty \quad (n=1,2,3,\ldots,\infty) \quad (29)$$

On writing out a few of these equations:

$$b_1 C_1 + b_{12} C_{12} + b_{13} C_3 + b_{14} C_4 + \cdots = B_1^\infty$$

$$b_2 C_1 + b_{22} C_{12} + b_{23} C_3 + b_{24} C_4 + \cdots = B_2^\infty$$

$$b_3 C_1 + b_{32} C_{12} + b_{33} C_3 + b_{34} C_4 + \cdots = B_3^\infty$$

$$b_4 C_1 + b_{42} C_{12} + b_{43} C_3 + b_{44} C_4 + \cdots = B_4^\infty$$

The numerical evaluation of the coefficient $C_s$ gives

the asymptotic solution of the problem for large values of depth $s$.

A few of the reasons may be indicated here for believing

that this solution is quite feasible and that an asymptotic

solution for the applications in view may be attained

with a finite number of these coefficients $C_s$ possibly

the first three, perhaps four. It is better to

postpone a discussion of numerical methods to a later section but

one or two relations may be given here.

With large integers $s$, the functions $P_n(s)$ becomes infinite

like $e^s$. For if $s$ is large

$$\text{value of } P_n(s) \sim (e^s)$$

$$\text{value of } Q_n(s) \sim (e^s)$$

$$\text{value of } Q_{n+1}(s) \sim (e^s)$$

$$\text{value of } Q_{n+2}(s) \sim (e^s)$$
hence for large $s$

$$B_s^\infty \sim \frac{e^{-\beta_s s}}{2s} \cos \left( \frac{\beta_s + \frac{1}{2}s}{2} \right)$$  \hspace{1cm} (301)

These terms are the second members of the equations (29).

The symmetrical coefficients $b_{ss}$ vanish like $\frac{1}{n}$ when $n \to \infty$ (s constant).

If $\beta$ were large the second members of (29) would rapidly approach zero with increasing order $s$.

The coordinates $\beta_1, \alpha_2$ for primary and secondary circuits cannot exceed $\beta$ which is $80^\circ$ for the hemispherical problem. The nearer these circuits are to the shell the smaller $\beta_1, \alpha_2$. Effectively as $\beta_1$ is near $80^\circ$, and $\beta_2$ is near this value $100^\circ$, the larger is $\beta_2$. As a representative example, take

$$\beta_1 = 75^\circ$$  \hspace{1cm} (302)  \hspace{1cm} approximately,

$$B_4^\infty \sim \frac{e^{-6.375}}{8} \cos \left( 7.50^\circ, -\frac{\pi}{2} \right) = .0002 \cos \left( 7.50^\circ, -\frac{\pi}{2} \right)$$

These are the reasons for expecting a reasonable approximation may be obtained by neglecting all but the first terms on four coefficients $b_{ss}$ and $c_{ss}$.

In the later section a method is derived for computing the numerical coefficients $b_{ss}$. It seems both to invest a summary at this point.
7. Summary and Meaning of the Solution

\[ U(\alpha, \beta; \alpha', \beta') = U(\alpha, \beta; \alpha', \beta') + i \frac{\gamma}{\pi a} V(\alpha, \beta; \alpha', \beta') \] (31)

where \( U^* \) and \( V^* \) are real functions of \( \Pi \) functions independent of \( R_0/a \), and may be computed by

\[ U(\alpha, \beta; \alpha', \beta') = 4 \sqrt{\sin \alpha \cos \beta \sin \alpha' \cos \beta'} \sum_{\nu=1}^{\infty} \left[ \frac{Q^1_{\nu}(i \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \nu \n

where \( \nu = m \alpha \), and the coefficients \( C_\nu \) are functions of \( \Pi, \alpha, \beta \) found as solutions of the set of equations (29).

As indicated by (30), three or four terms may be sufficient for computing a sufficient approximation to these series. By (32a) the real function \( U^* \) for a perfectly conducting shell automatically satisfies the reciprocity theorem and theory of inversion theorems. Since \( V \) is a first approximation (with high frequency) it is not expected to satisfy the reciprocity theorem exactly, although it may be expected to be a good approximation in the conducting core.

Up to this point the currents in the shell have been considered as a periodic current distribution existing on the...
presence of one periodic (complex) current, effectively N. I., in
a circle whose trace is the source-point P. The phases of the
induced currents with respect to the phase of the primary current
are taken into account by the use of the complex potential
of the form given in equations (3) and (4).

Consequently when both complex current sources are present
the complex vector potential is not $\sigma I \cdot$, This is to be
replaced by

$$A_r + \sigma \cdot G(r, \theta; r, \theta_0) \cdot G(r, \theta; r, \theta_0)$$

where the Green's function in each case is defined similarly
to that given in (4). The evaluation of the $\sigma I \cdot$ function given above
applies in either case. The part of each Green's function which
is represented by $\sigma I \cdot$ contributes to the self and mutual
inductances of the coils as if there were no shell present.

The initial circuit constants of primary and secondary had
lumped inductances and capacitances which were not effective
in producing an alternating field at the shell, and these
constants are assumed to have been determined experimentally
for the particular frequency.

The vector potential $A_3(r, \theta)$ which is produced by all
currents in the shell is then for $\sigma I \cdot$ (33) and (4)

$$A_3(r, \theta) = \frac{N. I.}{B} U(r, \theta; r, \theta) - \frac{N. I.}{B} U(r, \theta; r, \theta)$$

They produce a flux through primary and secondary
which is $\phi_3 = \int \Phi A_3(r, \theta)$ and $\phi_3 = \int \Phi A_3(r, \theta)$, that
\[
\begin{align*}
\Phi_3 &= -2\pi N_1^2 U_1 I_1 - 2\pi N_1 N_2 \sqrt{A} U_2 I_2 \\
\Phi_{23} &= -2\pi N_2 N_2 \sqrt{A} U_2 I_2 - 2\pi N_1^2 P_1 U_2 I_2
\end{align*}
\]

(35)

These and eq (31) have been used in the discussion in section 2 so that the job which confronts the computer should now be clear.

The points \( P_1 (x, y, 0), P_2 (x, 0) \) (traces of primary and secondary) being numerically given, as well as the radius \( a \) of the shell, it is first necessary to find the corresponding cylindrical coordinates \( \alpha, \beta, \) and \( \alpha', \beta'. \) This is a small portion of work as explained in equations (13) to (16).

Then \( U_1, U_2, \) and \( U_2' \) are to be computed by the series (32).

Before \( V_1, V_2, V_3 \) by the series (32), it is necessary to compute a number of the coefficients \( C_1, C_2, C_3, C_4 \), and for this the coefficients \( \beta_1, \) must be computed.

In view of the experimental application it would seem that an error of at least 10% could be tolerated in evaluating \( V_1, V_2, V_3, V_4, V_5, V_6, \) and \( V_7. \)
8. Methods for computing $\mathcal{E}_{n,s}$

Since $\mathcal{E}_{n,s} = \mathcal{E}_{m,n}$, the first sixteen of these coefficients
are determined by the equation $m = s, 1, 2, 3, 4$ if we compute the ten
in the triangle which includes the diagonal, namely
$1 \leq s \leq n = 1, 2, 3, 4$. In that case the highest value of
$n + s - 2$ is 6.

To compute them in finite terms begin with the
polynomial
\[
\frac{\Gamma(m-\frac{1}{2})}{m!} \sum_{t=0}^{m-1} \frac{(-1)^t (1+\mu)^t}{t! (t+1)! (m-1-t)!} \int \frac{dt}{(m-t+1)}
\]  

(36)

Write the same equation with $m$ replaced by $s$ and multiply
the two. Rearranging the product of the two polynomials
on the right in ascending powers of $(1+\mu)$ we get
\[
\frac{\Gamma(m+\frac{1}{2}) \Gamma(s+\frac{1}{2})}{m! s!} P_{m}(\mu) P_{s}(\mu) = 4 \frac{\Gamma(m+\frac{1}{2}) \Gamma(s+\frac{1}{2})}{(m-1)! (s-1)!} \sum_{k=0}^{M+S-2} (-1)^k D_k^{m,s} (1+\mu)^{k+1}
\]  

(37)

where
\[
D_k^{m,s} = D^{s,m}_k = \sum_{t=0}^{k} \frac{1}{t! (t+1)! (k-t)! (k-t+1)!} \left[ \frac{(m-1)!}{(m-1-t)!} \frac{(s-1)!}{(s-1-k)!} \right]
\]  

(38)

The $k^{th}$ coefficient $D_k^{m,s}$ consists of

(39)
and rational functions of \( n \) and \( s \)

\[
D_0^{ns} = 1
\]

\[
D_1^{ns} = \frac{1}{16} \left[ (n-2)(m+\frac{1}{2}) + (s-2)(s+\frac{1}{2}) \right]
\]

\[
D_2^{ns} = \frac{1}{12} \left[ (n-2)(n-3)(m+\frac{1}{2})(m+\frac{3}{2}) + 3(n-1)(m+\frac{1}{2})(s-1)(s+\frac{1}{2}) + (s-2)(s-3)(s+\frac{1}{2})(s+\frac{3}{2}) \right]
\]

\[
D_3^{ns} = \frac{1}{3!4!} \left\{ (n-2)(n-3)(n+\frac{1}{2})(n+\frac{3}{2}) \left[ (n-3)(n+\frac{5}{2}) + 6(s-1)(s+\frac{1}{2}) \right]
\right. \\
+ (s-1)(s-2)(3+\frac{1}{2})(s+\frac{3}{2}) \left[ (s-2)(s+\frac{5}{2}) + 6(n-1)(n+\frac{1}{2}) \right] \bigg\}
\]

\[
D_4^{ns} = \frac{1}{4!5!} \left\{ (n-1)(n-2)(n-3)(n+\frac{1}{2})(n+\frac{3}{2})(n+\frac{5}{2})(n+\frac{7}{2}) \right. \\
+ \text{some term with } n \text{ replaced by } s \\
+ 10(n-1)(n-2)(n-3)(n+\frac{1}{2})(n+\frac{3}{2})(n+\frac{5}{2})(n+\frac{7}{2})(s-1)(s+\frac{1}{2}) \\
+ \text{some term with } n \text{ and } s \text{ interchanged} \\
+ 20(n-1)(n-2)(n+\frac{1}{2})(n+\frac{3}{2})(s-1)(s-2)(s+\frac{1}{2})(s+\frac{3}{2}) \bigg\}
\]

\[
D_5^{ns} = \frac{1}{5!6!} \left\{ (n-1)(n-2)(n-3)(n-4)(n-5)(n+\frac{1}{2})(n+\frac{3}{2})(n+\frac{5}{2})(n+\frac{7}{2})(n+\frac{9}{2}) \right. \\
+ \text{some function of } s \\
+ 15(n-1)(n-2)(n-3)(n-4)(n+\frac{1}{2})(n+\frac{3}{2})(n+\frac{5}{2})(n+\frac{7}{2})(s-1)(s+\frac{1}{2}) \\
+ \text{some term with } n \text{ and } s \text{ interchanged} \\
+ 50(n-1)(n-2)(n-3)(n+\frac{1}{2})(n+\frac{3}{2})(n+\frac{5}{2})(s-1)(s-2)(s+\frac{1}{2})(s+\frac{3}{2}) \\
+ \text{some term with } n \text{ and } s \text{ interchanged} \bigg\}
\]
\[ D_{n,s}^m = \]
\[ = \frac{1}{6! \cdot 7!} \left\{ (n-1)(n-2)(n-3)(n+\frac{1}{2})(n+\frac{3}{2}) - \cdots - (n + \frac{11}{2}) \right\} + \text{some function of } s \]
\[ + 21(n-1)(n-2)(n-5)(n+\frac{1}{2})(n+\frac{3}{2}) - \cdots - (n + \frac{15}{2}) (5-\sqrt{5}+\frac{1}{2}) \]
\[ + \text{some with } n \text{ and } s \text{ interchanged} \]
\[ + 105(n-1)(n-2)(n-4)(n+\frac{1}{2})(n+\frac{3}{2}) - (n + \frac{23}{2})(5-2)(5-\sqrt{5}+\frac{1}{2}) + \text{some with } n \text{ and } s \text{ interchanged} \]
\[ + 175(n-1)(n-2)(n-3)(n+\frac{1}{2})(n+\frac{3}{2})(5-\sqrt{5}+\frac{1}{2})(5-\sqrt{5}-\frac{1}{2}) (5+\frac{1}{2})(5+\frac{3}{2})(5+\frac{5}{2}) \]
\[ \text{etc.} (\text{using } \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)} = \frac{1, 3, 5, \ldots, (2n-1)}{2, 4, 6, \ldots, 2n} \text{ where } \Gamma(n)^2 = n \]
\[ \text{and } 2 \Gamma(2) = \Gamma(2+1) \]

If eq (37) is multiplied by \((-1)^{n+s} \mu \, d\mu / (2-\mu^2)\) and integrated from zero to one, the first member becomes \(D_{n,s}^m\) as defined in (27) for the hemispherical shell \(\sin \theta = 1\). The result is

\[ D_{n,s}^m = (-1)^{n+s} \frac{\Gamma(n+\frac{1}{2}) \Gamma(s+\frac{1}{2})}{(n-1)! (s-1)!} \sum_{k=0}^{n-s-2} (-1)^k D_{k}^{m,s} S_k \]

where

\[ S_k = 4 \int_0^1 \frac{(1-\mu^2)^{k+1}}{2-\mu^2} \, d\mu = 2 \int_0^1 \frac{z^{k+1}}{1+z} \, dz = \psi \left( \frac{k+2}{2} + \frac{1}{2} \right) - \psi \left( \frac{k+2}{2} \right) \]

where \(\psi(z) = \frac{d}{dz} \log \Gamma(z)\)
The $S_k$ may be computed by

\[
S_0 = 2 - \log_4 4
\]

\[
S_{2n} = \frac{2}{2n+1} + \sum_{t=1}^{n} \frac{1}{t(2t-1)} - \log 4 \]

\[
S_{2n-1} = \log_4 4 - \sum_{t=1}^{n} \frac{1}{t(2t-1)}
\]

On (since \( \log 2 = \sum_{t=1}^{\infty} \frac{(-1)^{t+1}}{t} \)),

\[
S_k = 2 \left[ \sum_{t=1}^{K+1} \frac{(-1)^{t+1}}{t} - \log 2 \right] = -2 \sum_{t=K+2}^{\infty} \frac{(-1)^{t+1}}{t}
\]

By use of the recurrence relation

\[
S_{k+1} = \frac{2}{K+2} - S_k,
\]

the eq (39) may be written

\[
\frac{4}{n+5} \frac{P_{m,s}}{D_{n}^{s}} = (-1)^{s+5} \left[ \frac{1.3.5\cdots(2n-1)}{2^{n+2} \cdot (m-1)!(s-1)!} \right].
\]

\[
\left\{ 2 + D_1^s + \frac{5}{3} D_2^s + \frac{7}{6} D_3^s + \frac{17}{30} D_4^s + \frac{37}{30} D_5^s + \frac{819}{210} D_6^s + \cdots - \left[ 1 + D_1^s + D_2^s + D_3^s + D_4^s + D_5^s + D_6^s + \cdots \right] \log 4 \right\} \tag{41}
\]

where \( \log_4 4 = 1.386294361 \) and \( D_k^s = 0 \) if \( n+s-2 > k \)

This eliminates $S_k$ for computing $D_k$ up to $k=6$.
In the next section is described the numerical application of the asymptotic solution to experimental data. This was made by the computing group under P. Whitman. It applied to two given points P and P₂ (primary and secondary), and to a given shell-radius a. The latter was selected at a particular instant on the scope-time recording pickup current I₂ on the basis of a known velocity of the front of the converging detonation wave. Since the coefficients rₙₙ are independent of these data, and appear in other connections, the first sixteen thus found are tabulated here, or rather 4 \( \ell_{1,8} \)

\[
\frac{4 \ell_{1,8}}{\pi}
\]

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<th>n = 2</th>
<th>n = 3</th>
<th>n = 4</th>
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<td>-0.0440</td>
<td>-0.0794</td>
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<td>0.00119</td>
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<td>-0.02769</td>
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<td>0.00119</td>
<td>-0.02769</td>
<td>0.04473</td>
</tr>
</tbody>
</table>
For computing $P'_{25-1}(\mu)$ where $\mu = \cos \alpha$

$$P'_1(\mu) = \sin \alpha$$
$$P'_3(\mu) = \frac{3}{2} \sin \alpha (5 \mu^2 - 1)$$
$$P'_5(\mu) = \frac{195}{8} \sin \alpha (3 \mu^4 - 2 \mu^2 + \frac{5}{2})$$

$$P'_7(\mu) = \frac{231}{16} \sin \alpha (13 \mu^6 - 15 \mu^4 + \frac{45}{11} \mu^2 - \frac{5}{23})$$

The recurrence relation is

$$P'_{25+1}(\mu) = -\left(\frac{45+1}{45-3}\right) P'_{25}(\mu) + \left[(45+1)\mu^2 - \frac{85-45-3}{45-3}\right] \frac{(45-1)}{\left(25-1\right)\left(25\right)}$$ (42)

The functions $Q'_{25-1}(\imath \sin \beta)$ satisfy the same relation provided that $\mu$ is replaced by $\imath \sin \beta$. To compute them in finite terms let the positive acute angle $\phi$ (in radians) be determined by

$$\cos \phi = \tanh \beta \text{ where } 0 < \phi < \frac{\pi}{4}$$

The recurrence relation is

$$Q'_{25+1} = -\left(\frac{45+1}{45-3}\right) Q'_{25} - \left[(45+1) \cot^2 \phi + \frac{85-45-3}{45-3}\right] \frac{(45-1)}{\left(25-1\right)\left(25\right)}$$ (42')
The first four functions are

\[ Q_1 = -\cos \phi + \frac{\phi}{\sin \phi} \]

\[ Q_3 = \frac{15}{2} \left\{ \left[ \cot^2 \phi + \frac{13}{15} \right] \cos \phi - \left[ \cot^2 \phi + \frac{1}{5} \right] \frac{\phi}{\sin \phi} \right\} \]

\[ Q_5 = \frac{105}{8} \left\{ -\left[ 3 \cot^4 \phi + 4 \cot^2 \phi + \frac{113}{105} \right] \cos \phi + \left[ 3 \cot^2 \phi + 2 \cot \phi + \frac{1}{7} \right] \frac{\phi}{\sin \phi} \right\} \]

\[ Q_7 = \frac{13}{6} \left\{ \left[ \frac{231}{8} (\cot^2 \phi + \frac{19}{39}) (3 \cot^2 \phi + 4 \cot \phi + \frac{113}{105}) - 5 \cot^2 \phi - \frac{13}{3} \right] \cos \phi \right. \]

\[ \left. - \left[ \frac{231}{8} (\cot^2 \phi + \frac{19}{39}) (3 \cot^2 \phi + 2 \cot \phi + \frac{1}{7}) - 5 \cot^2 \phi - 1 \right] \frac{\phi}{\sin \phi} \right\} \]
9. Numerical Evaluation of \( R_0 \) for one set of experimental data.

This section is a collection of data and the results of computation made by Whitman and his group.

\( a = 5'' = 12.70 \text{ cm} \) \hspace{1cm} \( P/2\pi = .65(10) \text{ cycle/sec} \)

**Primary**

\( N_1 = 60 \text{ turns} \)

\( r_1 = .5'' \)

\( \rho_1 = 1'' \)

\( \theta_1 = 179.85 \text{ (radians)} \)

\( r_1' = 5.5900'' \)

\( \alpha_1 = .25441 \text{ radians} \)

\( \beta_1 = .721170 \)

\( \chi_1 = .96781 \)

\( P_1 = .29168 \)

\( P_1' = 1.3905 \)

\( P_1'' = 2.9780 \)

\( P_1''' = 4.3719 \)

\( \theta_1 = .53317 \)

\( \theta_1' = -.17778 \)

\( \theta_1'' = .051468 \)

\( \theta_1''' = -0.1404 \)

**Secondary**

\( N_2 = 3 \text{ turns} \)

\( r_2 = 39.37'' \)

\( \rho_2 = 10'' \)

\( \theta_2 = 221.67 \text{ (radians)} \)

\( r_2' = 45.483'' \)

\( \alpha_2 = 3.10358 \)

\( \beta_2 = .7219 \)

\( \chi_2 = -.99928 \)

\( P_2 = .03800 \)

\( P_2' = 1.22759 \)

\( P_2'' = 5.671 \)

\( P_2''' = 1.0537 \)

\( \theta_2 = .53249 \)

\( \theta_2' = .17728 \)

\( \theta_2'' = .05122 \)

\( \theta_2''' = -0.138 \)
Primary

\[ B_1 = +0.037954 \]
\[ B_2 = -0.011653 \]
\[ B_3 = +0.007576 \]
\[ B_4 = -0.006199 \]

Secondary

\[ B_1 = 0.0022243 \]
\[ B_2 = -0.0007392 \]
\[ B_3 = +0.0002129 \]
\[ B_4 = -0.00005717 \]

\[ \frac{L_{ms}}{\pi} \]

<table>
<thead>
<tr>
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<th>( n = 2 )</th>
<th>( n = 3 )</th>
<th>( n = 4 )</th>
</tr>
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<td>-0.001003</td>
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<tr>
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<td>( s = 3 )</td>
<td>0.020548</td>
<td>-0.006924</td>
<td>0.011824</td>
<td></td>
</tr>
</tbody>
</table>

From Primary

\[ C_1 \]
\[ C_2 \]
\[ C_3 \]
\[ C_4 \]

Using 3 equations

\[ 0.09229 \]
\[ 0.04061 \]
\[ 0.07856 \]

Using 4 equations

\[ 0.09890 \]
\[ 0.05769 \]
\[ 0.11436 \]
\[ 0.06920 \]

For Secondary

\[ C_1 \]
\[ C_2 \]
\[ C_3 \]
\[ C_4 \]

Using 3 equations

\[ -0.005297 \]
\[ 0.002621 \]
\[ 0.005017 \]

Using 4 equations

\[ 0.003633 \]
\[ 0.002893 \]
\[ 0.006846 \]
\[ 0.003335 \]
\[
\begin{align*}
U_n^0 &= 0.21479 \\
U_{n2}^0 &= 0.001283 \\
U_{o2}^0 &= 0.000769 \\
V_n^0 &= 0.013477 \\
V_{o2} &= 0.00079096 \\
(V_{o1} &= 0.00079099 \\
V_{o2} &= 0.000466 \\
R_1' / R_o &= 368.824 \text{ ohms} \\
R_2' / R_o &= 600335 \quad \text{ohms} \\
R_3' / R_o &= 3602 \quad \text{ohms} \\
L_1' &= 1234 \times 10^{-6} \text{ henries} \\
L_2' &= 110 \times 10^{-9} \quad \text{ohms} \\
M_o' &= 11.66 \times 10^{-9} \quad \text{ohms}
\end{align*}
\]

Coil constants

Primary

\begin{align*}
R_1 &= 50 \text{ ohms} \\
L_1 &= 160(10^{-6}) \text{ henries} \\
C_1 &= 375(10^{-12}) \text{ farads}
\end{align*}

Secondary

\begin{align*}
R_2 &= 35 \text{ ohms} \\
L_2 &= 110.15(10^{-6}) \text{ henries} \\
C_2 &= 2300 \times 10^{-12} \text{ farads}
\end{align*}

For plotting \( I_2 \) against \( R_0 \)

\begin{align*}
R_0 \text{ (ohms)} & \quad \left| I_2 \text{ (amperes)} \right| \\
10000 & \quad 0.0132 \\
1000 & \quad 0.0132 \\
100 & \quad 0.0130 \\
10 & \quad 0.0117 \\
1 & \quad 0.00577 \\
0.000 & \quad 0.0000966 \\
0.0083 & \quad 0.000819 \\
0.080 & \quad 0.000792 \\
0.010 & \quad 0.000224 \\
0.001 & \quad 0.0002202
\end{align*}
The log-log graph of $I_2$ against $R_0$ from preceding column is shown in fig 3.

It would have been better, had the ordinates been $\log \left| \frac{I_2}{I_0} \right|$ where $|I_2|$ is the amplitude of the pickup current before the conducting shell makes its appearance. The ratio of amplitudes $|I_2|/I_0$ is obtained from one scope trace (at the particular instant when the median radius of the shell is $a$, the pickup current is $I_0$).

For the case given by Brain $|I_2|/I_0 = \frac{1}{2}$ when $a = 5'' = 12.7\text{em}$.

This by use of fig 3 gives $R_0 = 0.079\text{ohms}$.

Since the relative configurations of the primary and secondary are just greatly dissimilar in the hemispherical slits and in implosions it is safe to draw one general conclusion from an inspection of these results. It is

$L'$ is negligible compared to $L$, $L'$ is negligible compared to $L_0$, $R_0'$ is small compared to $R_0$ (about 8% in this case.).

The fact that the coil-constants $R$, $L$, and $C$ appearing as factors of $I_2$ in eq (16) are all practically unchanged
By creation of the shell means that a knowledge of these constants is unimportant since \( R_0 \) is determined from the ratio \( I_2/I_0 \).
10. Form and Method for the General Solution

If the occasion should arise which requires a solution when \( R_0 \) is not small, the system (25) must be solved. Setting
\[
\lambda \equiv \frac{n \alpha p}{2 R_0}, \text{ and } \overline{B}_n \equiv (4n-1)B_n,
\]
the system is
\[
\left[ -\frac{i}{(4n-1)\lambda} \overline{B}_n + \sum_{s=1}^{\infty} (4s-1) b_{ns} \overline{B}_s \right] \overline{B}_n = \sum_{s=1}^{\infty} \frac{b_{ns}}{\lambda} \overline{B}_s.
\]
for \( n = 1, 2, 3, \ldots \infty \), where \( \sum_{s=1}^{\infty} \) indicates omission of the terms \( s = n \) from the summation.

If the points \( P \) and \( P' \) (or ultimately \( P \) and \( P_a \)) are not close to the shell, four of these equations would give an approximate determination of \( \overline{B}_1, \overline{B}_2, \overline{B}_3, \) and \( \overline{B}_4 \). If terms \( \overline{B}_n \) of order \( n \) higher than 4 are required, reference to the table in section 8 shows that a first approximation would be given by neglecting all but diagonal terms in the array \( b_{ns} \) for \( n, s > 4 \) so that
\[
\overline{B}_n \approx \frac{\sum_{s=1}^{\infty} b_{ns} \overline{B}_s}{b_{nn} - \frac{1}{(4n-1)^2}} \quad \text{for } n > 4.
\]

A second approximation would be obtained by retaining the diagonal and the two adjacent lines, in which case there would be a recurrence relation beginning with \( n = 5 \), between three contiguous coefficients \( \overline{B}_n \).
The manner in which the coefficients $B_\xi$, determined by the system (25), will depend upon $\alpha$, $\omega$, $\beta$, (which appear in the second member of (25)) is predicted by the reciprocal theorem. This shows that the solution (21) will take the symmetrical form

$$U(\alpha, \beta; \alpha, \beta) = 4 \sqrt{\sin \alpha \cos \beta \sin \alpha \cos \beta} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \frac{Q'(i \sinh \eta) P'_{(m)}}{(2s-1) 2s \lambda} \frac{Q'(i \sinh \eta) P'_{(m)}}{(2m-1) 2m \lambda} \eta_m \lambda_s$$

(44)

where

$$L_{nm} = \frac{L_n (\lambda)}{5^m}$$ (independent of the points $P_0$ and $P_0$).

This means

$$(4s-1) B_s \equiv B_s = \sqrt{\sin \alpha \cos \beta \sin \alpha \cos \beta} \sum_{m=1}^{\infty} \frac{Q'(i \sinh \eta) P'_{(m)}}{(2s-1) 2s \lambda} \eta_m \lambda_s$$

(45)

Instead of the singly-infinite set of coefficients $B_s$ there is now a doubly-infinite set of coefficients $\xi_{nm}$ to be found, which however have the advantage of being functions of $\lambda$ only. Substituting this expression for $B_s$ in the system (25) or (43) gives

$$\sum_{s=1}^{\infty} \sum_{r=1}^{\infty} \frac{Q'(i \sinh \eta) P'_{(m)}}{(2s-1) 2s \lambda} \left[ - \frac{i \xi_{nm}}{(4n-1) \lambda} + \sum_{t=1}^{\infty} \frac{L_t \xi_s}{\eta_s} - (4s-1) \eta_m \right] = 0$$

which must be true for every positive integral $n$. Moreover, it must be true independently of the values of $\mu$ and $\beta$.

This leads to the doubly-infinite system

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of equations

\[
\frac{i}{4n-1}\lambda c_{ns} + \sum_{t=1}^{\infty} \frac{c_{mt} c_{ts}}{s} = (4s-1) c_{ms} \tag{46}
\]

for \( n = 1, 2, 3, \ldots \infty \) and \( s = 1, 2, 3, \ldots \infty \).

If we hold \( s \) constant and give \( n \) all positive integral values, the coefficients \( c_{1s}, c_{2s}, c_{3s}, \ldots \) are those in the \( s \)th row of the array \( c_{ns} \). Holding \( s \) constant in eq. (46), it merely serves as a label to indicate the row in question. Each selection of \( s \) therefore constitutes a selection of a singly infinite system of equations to determine all the singly infinite coefficients \( c_{ns} \) in the \( s \)th row independently of those in any other row. The method of approximation is obvious. We could determine the 16 coefficients in which \( n \) and \( s \) range from 1 to 4. If computations were to be made for a number of pairs of points \( P_1 \) \& \( P_2 \), it might be worth while to determine these 16 coefficients \( c_{ns} \) once for all.

When \( n \) is not very large, the solution \( V \) obtained either by the system (45) or the system (46), the resolution into real and imaginary parts would conform with the previous notation, if taken in the form

\[
U(a, b; a, b) = U_a(a, b; a, b; \lambda) + \frac{i R_0}{\pi \lambda} V(a, b; a, b; \lambda) \tag{47}
\]

where the real functions \( U \) and \( V \) are symmetric functions.
of the two points \((x, p), (x, z)\) and also of \(\lambda \equiv \pi D/2R_0\).

This agrees with the notation adopted in section 2 and in eqs. (32a), (32b) for the asymptotic solution when \(\lambda\) is large and \(V^*\) becomes \(U^*\) which (with \(V\)) becomes independent of \(\lambda\).

It should be noticed that the asymptotic solution which has been applied in the preceding sections, assumes that \(\lambda\) is large, and it is practicable only when \(\beta_1\) and \(\beta_2\) are not very small, i.e. when primary point \(P_1\), and secondary \(P_2\) are not close to the shell. In case several pairs of points \((P_i, P_j)\) are to be considered it might save labor to take \(U\), in the asymptotic form (44) in which the upper limits of \(m\) and \(s\) in the double series are in each case 4.

Instead of determining the first sixteen \(c_{ns}\) as suggested above for general values of \(\lambda\), their asymptotic values may be determined. To find these, it is readily verified by writing out the double system (46) that the solution \(\delta_{ns}^o\) of this complete system, for a perfectly conducting shell \((\lambda = \infty)\) is

\[
\delta_{ns}^o = (4n-1) \delta_{ns}
\]

where \(\delta_{ns} = 1\) if \(m = s\), = zero otherwise.

Hence, the asymptotic solution is

\[
\delta_{ns} = \delta_{ns}^o + \frac{i}{\lambda} d_{ns}
\]

\[
\delta_{ns}^o = (4n-1) \delta_{ns}
\]
Where the double array \( d_{n,s} = d_{s,n} \) are determined by the double-infinite system

\[
\sum_{t=1}^{\infty} \sum_{n} \epsilon_{n,t} d_{n,s} = \delta_{n,s} = 1 \text{ if } n=s = 0 \text{ otherwise}
\]

Thus also, the coefficients \( d_{n,s} \) in the \( s \)th row are determined independently of those in any other row.
11. Case of Closed Spherical Shell (strictly)

a. Periodic Current

In this case polar coordinates, \( r, \theta \), are most appropriate. The exact solution \( U \) in series form is obtainable. When the point \( P(r, \theta) \) lies on the shell \( (r = a \pm 0) \), and the source point \( P_i(r, \theta) \) is not on it, the expansion given in section 4, page 16, may be written in the continued form

\[
2Q_{12} \left( \frac{a^2 + r^2}{2a} \frac{\sin \theta \cos \theta}{\sin \theta \cos \theta} \right) = 2\pi \sqrt{\sin \theta \cos \theta} \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^{- \left( m + \frac{1}{2} \right)} \frac{P_n^{(1)}(\cos \theta) P_n^{(1)}(\cos \phi)}{\sin \theta \sin \phi} \quad (51)
\]

where \( \left( \frac{r}{a} \right)^{- \left( m + \frac{1}{2} \right)} = \left( \frac{r}{a} \right)^{m + \frac{1}{2}} \) if \( r \leq a \) (source \( P_i \), inside)

\[= \left( \frac{r}{a} \right)^{m + \frac{1}{2}} \] if \( r > a \) (source \( P_i \), outside)

Using the abbreviation introduced in eq.(43), namely \( \lambda = \frac{n \alpha \pi}{a} \), it is readily verified that the solution of the boundary problem formulated in the system of equations (7) is

\[
U(r, \theta; r, \theta) = 2\pi \sqrt{\sin \theta \cos \theta} \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^{- \left( m + \frac{1}{2} \right)} \left( \frac{r_0}{a} \right)^{m + \frac{1}{2}} \frac{P_n^{(1)}(\cos \theta) P_n^{(1)}(\cos \phi)}{\sin \theta \sin \phi} \quad (52)
\]

(The two \( \pm \) signs represent two independent alternatives)

Resolving this into real and imaginary parts as in (47)

\[
\overline{U}(r, \theta; r, \theta) = U^* + i \frac{2 \pi \alpha}{\lambda \alpha \beta} V(r, \theta; r, \theta) \quad (53)
\]

the formulas for computing \( U^* \) and \( V \) are
\[ U(r, \theta; R, \theta; \lambda) = 2\pi \sin\theta \sin\lambda \sum_{n=1}^{\infty} \left( \frac{R}{a} \right)^{\pm(m+\frac{1}{2})} \frac{\left( \frac{R}{a} \right)^{\pm(n+\frac{1}{2})} P^\prime_n(\cos\theta) P^\prime_n(\cos\lambda)}{n (n+1) \left[ 1 + \left( \frac{2n+1}{a^2} \right)^2 \right]} \] 

\[ V(r, \theta; R, \theta; \lambda) = \frac{\pi}{4} \sin\theta \sin\lambda \sum_{n=1}^{\infty} \left( \frac{2n+1}{a^2} \right)^{\pm(n+\frac{1}{2})} \frac{\left( \frac{2n+1}{a^2} \right)^{\pm(n+\frac{1}{2})} P^\prime_n(\cos\theta) P^\prime_n(\cos\lambda)}{n (n+1) \left[ 1 + \left( \frac{2n+1}{a^2} \right)^2 \right]} \] 

In the primary, if \( r_1 \) is less than \( a \), and for secondary \( P_2 \), \( r_2 > a \) 

Hence the product of the two factors with \( \pm \) exponents is

\[ \left( \frac{R}{a} \right)^{2m+1} \] in computing \( U^m \) and \( V^m \)

\[ \left( \frac{R}{a} \right)^{n+1} \] in computing \( U^n \) and \( V^n \)

and \[ \left( \frac{R}{a} \right)^{n+\frac{1}{2}} \] in computing \( U^1 \) and \( V^1 \)

The asymptotic formulas come from (54a) and (54b) 

by replacing the factor \( \left[ 1 + \left( \frac{2n+1}{a^2} \right)^2 \right] \) of the denominator by \( 1 \). Then \( U^m \) becomes \( U^m \) and this (with \( V^m \)) becomes independent of \( r \).

In the case for which computations were made with cyclides coordinates as described in section 9, the ratios \( R/a \) and \( a/r \) were so small that the first two or three terms of \( \lambda \) sufficed in computing the series (54a) and (54b)
Equations (54a), (54b) are exact (for a stationary closed shell) for all values of \( R_0 \), that is, for all values of \( R_0 \) and of the frequency for which the quasi-stationary equations of the electromagnetic field are applicable. This means all frequencies in the technological range as distinguished from the optical range, in other words, all frequencies which it would be practicable to set up in the actual circuits.

The values of \( U''', U''', U'''' \) (which determine the apparent inductances \( L_1', L_2', L_3' \), by eq 2), as well as the values of \( V_{1''}, V_{1''}, V_{2''} \) (which determine the apparent resistance \( R_1', R_2', R_3' \), by eq 2) are all functions of the frequency because of the dependence of \( U'' \) and \( V'' \) upon \( \beta \), which appears in the denominators of (54a) and (54b).

However, it appears from section (9), where a value \( R_0 \) of about 0.08 ohm was obtained, that \( \beta \) is of the order of 4(10^3). Consequently, in all the implications, and for all frequencies of interest in them, the values of \( U'' \) and \( V'' \) given by (54a) and (54b) will be practically their asymptotic values at all the apparent resistances and inductances will be constants (independent of the frequency).
Hence these constant apparent resistances and inductances are to be computed by eq.(2) using the following:

\[ U''_n = U''_n = 2\pi \sin \theta, \sum_{m=1}^{\infty} \left( \frac{R_1}{r_2} \right)^{m+1} \left[ \frac{P_m^2(\cos \theta_1)}{m(m+1)} \right] \]

\[ U'_n = U'_n = 2\pi \sin \theta, \sum_{m=1}^{\infty} \left( \frac{R_2}{r_2} \right)^{m+1} \left[ \frac{P_m(\cos \theta_2)}{m(m+1)} \right]^2 \]

\[ U' = U' = 2\pi \sin \theta \sin \theta \sum_{m=1}^{\infty} \left( \frac{R_2}{r_2} \right)^{m+1} \frac{P_m^2(\cos \theta_2)}{m(m+1)} \]

\[ = 2Q \left( \frac{\sin^2 \theta_2}{\sin \theta_2} \right) \]

\[ V''_n = \frac{\pi}{4} \sin \theta, \sum_{m=1}^{\infty} \frac{2m+1}{m(m+1)} \left( \frac{R_1}{r_2} \right)^{m+1} \left[ \frac{P_m^2(\cos \theta_1)}{m(m+1)} \right] \]

\[ V'_n = \frac{\pi}{4} \sin \theta, \sum_{m=1}^{\infty} \frac{2m+1}{m(m+1)} \left( \frac{R_2}{r_2} \right)^{m+1} \left[ \frac{P_m^2(\cos \theta_2)}{m(m+1)} \right]^2 \]

\[ V' = \frac{\pi}{4} \sqrt{\sin \theta \sin \theta} \sum_{m=1}^{\infty} \frac{2m+1}{m(m+1)} \left( \frac{R_2}{r_2} \right)^{m+1} \frac{P_m(\cos \theta_2)}{m} \frac{P_m(\cos \theta_2)}{m} \]

\[ = \frac{\pi}{4} \delta, r_1, U' \]

The constants \( U''_n \) and \( V''_n \) are independent of \( \alpha \) and could be evaluated in finite terms by use of tables of elliptic integrals, but for the present applications \( r_1 \), \( \alpha \) and \( \alpha / r_2 \) are both generally so small that it is easier to compute all six of these constants.
by the above series. Generally all terms after the second or third are negligible.

With a closed shell, these asymptotic contours show that the primary and secondary circuits will have an apparent "resistance coupling" through the mutual resistance constant $R_{12}$, the direct electromagnetic coupling being annulled. This becomes evident on using the expression $2Q_{12}$ for $V_{12}$ given in (55) to compute $M'_{12}$ by the last of equation (2). Comparing the result with the expression for the mutual inductance $M_{12}$ between primary & secondary in the absence of the shell (eg on page 51) it is seen that

$$M_{12} - M'_{12} = 0 \text{ (for a closed shell)}.$$ 

Hence $I_{2}$ enters equation (1a) only in the term $R_{12}I_{2}$ and $I_{1}$ enters (1a) only in the term $R_{12}I_{1}$ which justifies the name "resistance-coupling".

This fact, together with the relations noted in section 9 enables us to rewrite the two fundamental circuit equations for primary and secondary separated by a closed stationary conducting shell in a much simpler form.
result from the same grounds, because in these cases the most general form of transient currents.

The relations referred to in section 9 amount in short to the statement that in all setups used in practice the constants $R_2, L_2, C_2$ of the secondary and $L_1, C_1$ of the primary are unaffected by the introduction of the shell. The essential and important constants that are introduced by the closed shell are the mutual resistance $R_{12}'$ and apparent inductance $M_{12}' = M_{12}$.

Without the shell the two circuit equations are

$$[R_i + i(pL_1 - \frac{1}{pC})]I_i + ipM_{12}I_2 = V \quad (56)$$

$$ipM_{12}I_i + [R_i + i(pL_1 - \frac{1}{pC})]I_2 = 0 \quad (56a)$$

With the closed shell they are

$$[R_i + R_{12}' + i(pL_1 - \frac{1}{pC})]I_i + R_{12}'I_2 = V \quad (57a)$$

$$R_{12}'I_i + [R_i + i(pL_1 - \frac{1}{pC})]I_2 = 0 \quad (57a)$$

All the constants appearing in these equations are independent of the frequency except possibly $R_i$, $R_{12}'$ (inductances change very little by skin-effect and proximity-effect). If the coils are wound with fine wire, not too closely packed the AC resistance $R_i$ and $R_{12}'$ could be constant constants.
8. Non-periodic Currents

To consider transient currents the use of imaginary quantities will be discontinued and from here on \( I_1(t) \) and \( I_2(t) \) will denote the real instantaneous primary and secondary currents. Their real vector potentials (\( \phi \)-components only) at the general point \( P_x, P_y \) are \( A_1(r, \theta, t) \) and \( A_2(r, \theta, t) \) while \( A_3(r, \theta, t) \) is that of the currents in the shell. The total vector potential at \( P \)

\[
A(r, \theta, t) = A_1 + A_2 + A_3
\]

The shell currents produce magnetic fluxes \( \phi_3(t) \) through the primary and \( \phi_3(t) \) through the secondary coils.

\[
\phi_3(t) = \frac{2\pi r N_x A_3(r, \theta, t) d\theta}{\mu_0} (61)
\]

Thus the fluxes being in practical units.

Consider the electrical constants \( R, L, C \) of the primary and \( R_2, L_2 \) of the secondary as known constants (in practical units). It is assumed that there is no current anywhere at time \( t < 0 \), but at time \( t=0 \) the initial charge \( Q_0 = C \cdot V \), on the condenser charged to \( V \) volts, begins to discharge through the series resistance \( R_1 \) and self-inductance \( L_1 \). No other current is applied anywhere. The discharge current \( I_1(t) = Q(t) = -\frac{dQ}{dt} \) is governed by the equation

\[
L_1 \frac{d^2 I_1}{dt^2} + R_1 I_1(t) - \frac{Q(t)}{C} + M_1 \frac{dI_2}{dt} + \frac{d\phi_3(t)}{dt} = 0
\]

Eliminating \( Q_n \) by differentiating, we may take the
The fundamental equations in the form
\[ I_1(t) = -\dot{Q}_1(t), \]
\[ L_1 \ddot{I}_1(t) + R_1 I_1(t) + I_1(t)/C_1 + N_{11} \ddot{I}_2(t) + \dot{\Phi}_{i1}(t) = 0, \]
\[ L_2 \ddot{I}_2(t) + R_2 I_2(t) + N_{12} \ddot{I}_1(t) + \dot{\Phi}_{i2}(t) = 0, \]
with the initial conditions
\[ Q_1(0) = C_1 V_1, \quad I_1(0) = I_2(0) = \Phi_{i1}(0) = \Phi_{i2}(0) = 0. \]

The traces of the primary current is the point \( P_1(r_1, \theta_1) \)
and that of the secondary is the point \( P_2(r_2, \theta_2) \)
where \( r_1 < a \) and \( r_2 > a \), the shell radius being
a centimeters. Their ordinary mutual inductance \( M_{12} \) may
be computed by
\[ M_{12} = 4\pi N_1 N_2 \sqrt{P_1 P_2} \Omega \left( \frac{r_1^2 + r_2^2 - a^2 + 2a \cos \theta_1 - 2a \cos \theta_2}{\sin \theta_1 \sin \theta_2} \right)^{1/2}. \]

The vector potential \( A_3 \) of the shell-currents must be of the form
\[ A_3(r, \theta, t) = \sum_{n=1}^{\infty} J_n(t) \left( \frac{a}{r} \right)^{n+1} P_n^1(\cos \theta) \sqrt{\sin \theta}, \text{ where } 0 < r < a \]
\[ = \sum_{n=1}^{\infty} J_n(t) \left( \frac{a}{r} \right)^n P_n^1(\cos \theta) \sin \theta, \text{ where } 0 \leq r \leq a, \]
where the coefficients \( J_n(t) \) are in amperes.
Since lengths are here measured in centimeters and the practical electromagnetic units are here implied, in which the unit of length is 0.01 cm, the boundary condition which was derived in section (4) becomes

\[ \Theta_e A(a, \theta, t) = \dot{A}_1(a, \theta, t) + \dot{A}_2(a, \theta, t) + \dot{A}_3(a, \theta, t) = \]

\[ = \frac{R_0}{4\pi}\left[ \Theta A(\infty, \theta, t) - \Theta A(a, \theta, t) \right] \]

or by (64)

\[ \sum_{n=1}^{\infty} \left[ a(n+1) \int_{\alpha}^{\theta} \int_{\theta}^{\theta_n} P_{n}^{1}(\cos \theta, \cos \theta) \right] = -a(n+1) \int_{\alpha}^{\theta} \int_{\theta}^{\theta_n} P_{n}^{1}(\cos \theta, \cos \theta) \]

(65)

Now \( A_1 \) and \( A_2 \) are given everywhere by

\[ A_1(\infty, \theta, t) = 2N_1 I_1(\epsilon) \sqrt{\epsilon} Q \left[ \frac{1}{2n} \frac{\lambda}{\alpha} \frac{\lambda}{\alpha} - \cos \theta \cos \theta \right] \]

(66)

\[ A_2(\infty, \theta, t) = 2N_2 I_2(\epsilon) \sqrt{\epsilon} Q \left[ \frac{1}{2n} \frac{\lambda}{\alpha} \frac{\lambda}{\alpha} - \cos \theta \cos \theta \right] \]

Since the primary is inside and the secondary outside the shell, (61) gives

\[ A_1(a, \theta, t) = 2\pi N_1 I_1(\epsilon) \sum_{m=1}^{\infty} \left( \frac{\alpha}{\lambda} \right)^{m} \frac{P_{m}^{1}(\cos \theta, \cos \theta)}{m(m+1)} \]

(67)

\[ A_2(a, \theta, t) = 2\pi N_2 I_2(\epsilon) \sum_{m=1}^{\infty} \left( \frac{\alpha}{\lambda} \right)^{m} \frac{P_{m}^{1}(\cos \theta, \cos \theta)}{m(m+1)} \]
By use of these expansions the boundary condition (65) becomes

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ a_n \left( \frac{\partial^{(n)} J_m}{\partial \tau} \right) + \frac{(a+1)R_0}{m} \frac{\partial J_m}{\partial \tau} + \frac{2m(10)^9}{\sqrt{n(n+1)}} \left( \rho_1 N_1 \left( \frac{\alpha}{a_1} \right)^n P^l \left( \omega_1 e^{(0)} \right) \right) \frac{\partial I_1}{\partial \tau} + \rho_2 N_2 \left( \frac{\alpha}{a_2} \right)^{n+1} P^l \left( \omega_2 e^{(0)} \right) \frac{\partial I_2}{\partial \tau} \right] = 0
\]

Since this boundary condition must be fulfilled at every point of the shell \((0 < \theta < \pi)\), the coefficient of some function \(P^l (\cos \theta)\) must be zero at every instant. This gives an infinite system of equations which may be written

\[
L_3 \frac{\partial J_m}{\partial \tau} + R^{(n)} \frac{\partial J_m}{\partial \tau} + m_1 \frac{\partial I_1}{\partial \tau} + m_2 \frac{\partial I_2}{\partial \tau} = 0
\]

for

\[
m = 1, 2, 3, \ldots \infty
\]

where

\[
L_3 \equiv a_1 (10)^{-9} \text{ henries (} a \text{ in can)}
\]

\[
R^{(n)} \equiv (a+1)R_0/a \text{ ohms (} R_0 \text{ in ohms)}
\]

\[
m_1 \equiv 2\pi \rho_1 N_1 \left( \frac{\alpha}{a_1} \right)^n P^l \left( \omega_1 e^{(0)} (10)^9 / \sqrt{n(n+1)} \right) \text{ henries (} \rho \text{ in can)}
\]

\[
m_2 \equiv 2\pi \rho_2 N_2 \left( \frac{\alpha}{a_2} \right)^{n+1} P^l \left( \omega_2 e^{(0)} (10)^9 / \sqrt{n(n+1)} \right) \text{ henries (} \rho \text{ in can)}
\]

With this notation eq. (64) used in (68) gives

\[
\phi_1 (\theta) = \sum_{m=1}^{\infty} m_1 J_m (\theta) \quad \text{and} \quad \phi_2 (\theta) = \sum_{m=2}^{\infty} m_2 J_m (\theta)
\]
As far as the influence of shell currents is concerned there are equivalent to an infinite set of fictitious linear currents \( I \) in linear circuits. The self-inductance of each is the constant \( L_1 \), the resistance of the \( n \)th fictitious circuit is \( R^{(n)} \). Its mutual inductance with the primary is \( M \), and with the secondary is \( M_n \), as shown by eq.(70).

This interpretation is admitted also by the system of equations (68) which imply that there is no mutual inductance between members of the system of currents \( I_n \). This special relation originates in the fact that the vector potential \( A_3 \) in eq.(64) consists of the sum of normal solutions.

There is also a fundamental relation of importance which characterizes the closed shell. The sum of all the products of mutual inductance of pairs \( m_1, m_2 \) is proportional to the mutual inductance \( M_{1,2} \) between primary and secondary circuits, that is by (63) and (69)

\[
\sum_{n=1}^{\infty} m_1 m_2 = L_2 M_{1,2} \tag{71}
\]

All currents vanish at time \( t = 0 \).
The differential equations for the entire system of

reinlets becomes

\[ I_1(t) = -\dot{Q}_1(t) \] (72a)

\[ L_1 \ddot{I}_1(t) + R_1 \dot{I}_1(t) + I_1(t)/C, + \frac{\ddot{I}_2(t)}{L_2} \sum_{n=3}^{\infty} m_{2n} I_n(t) + \sum_{n=3}^{\infty} m_{2n} \dot{J}_n(t) = 0 \] (72b)

\[ L_2 \ddot{I}_2(t) + R_2 I_2(t) + \frac{\ddot{I}_1(t)}{L_2} \sum_{n=3}^{\infty} m_{2n} I_n(t) + \sum_{n=3}^{\infty} m_{2n} \dot{J}_n(t) = 0 \] (72c)

and

\[ L_3 \ddot{J}_3(t) + R_3 \dot{J}_3(t) + m_{2n} \ddot{I}_1(t) + m_{2n} \dot{I}_2(t) = 0, \text{ for } n=1,2,3,\ldots \] (72d)

The initial conditions are

\[ Q_1(0) = C, V, \]

\[ I_1(0) = I_2(0) = J_3(0) = 0 \text{ (n = 1,2,3...)} \] (73)

The case of the perfect conductor as \( R_1 \to 0 \) as \( R \to 0 \) is

Dividing each equation of (72a) by \( R \) gives every \( J_n \equiv 0 \).

Hence replacing \( \frac{\ddot{I}_1(t)}{L_2} \sum_{n=3}^{\infty} m_{2n} I_n(t) \) by its equivalent \( M_{21} \), the

equation reduces to

\[ I_1(t) = -\dot{Q}_1(t) \]

\[ L_1 \ddot{I}_1(t) + R_1 \dot{I}_1(t) + I_1(t)/C, + M_{21} \dot{I}_1(t) = 0 \]

\[ L_2 \ddot{I}_2(t) + R_2 I_2(t) + M_{21} \dot{I}_1(t) = 0 \] (74a)

In case of a perfectly conducting shell \( R \to 0 \)

\[ \dot{J}_3(t) = -\frac{(m_{21} I_1 + m_{22} I_2)}{L_3}, \text{ which gives} \]

\[ (L_3 - \frac{\sum m_{3n}^2}{L_3}) \ddot{I}_2(t) + R_1 \dot{I}_1(t) + I_1(t)/C, = 0 \]

and

\[ (L_2 - \frac{\sum m_{2n}^2}{L_2}) \ddot{I}_2(t) + R_2 I_2(t) = 0 \] (74b)
In this case the primary and secondary current satisfy independent equations, their coupling being annulled by introduction of the perfectly conducting, closed, shell between them. Their self-inductances are both decreased.

The system (72) has particular solutions of the form

\[
\begin{align*}
Q_{1}(t) &= \beta A e^{-\beta t} \quad \text{and} \quad I_{1}(t) = \beta A e^{-\beta t} \\
I_{2}(t) &= \beta B e^{-\beta t} \quad \text{and} \quad J_{1}(t) = \beta C^{(m)} e^{-\beta t} \left( L_{2} - R_{2} - \beta S_{1}(t) \right)
\end{align*}
\]  

\[
175
\]

where \( \beta \) has certain characteristic values to be found. One of these is \( \beta = 0 \) which corresponds to \( Q_{1}(t) = A = \text{constant} \) and \( I_{1}(t) = I_{2}(t) = J_{1}(t) = 0 \). As this is of no present interest we ignore it in cancelling a factor \( \beta \) from the equations which result on using this form in the set of equations (72). The system (72a) requires
\[
C^{(m)} = -m_{1} A - m_{2} B \quad \text{for} \quad m = 1, 2, 3, \ldots \infty \quad \text{(76)}
\]

Using this the equations (72b) and (72c) become

\[
\frac{A}{B} = \frac{\beta^{2} S_{2}(t)}{(L_{2} - \sum_{m=1}^{\infty} \frac{m_{2}^{2}}{L_{2}})^{2} - R_{2}^{2} + \frac{1}{\beta} - \beta^{2} S_{1}(t)} = \frac{(L_{2} - \sum_{m=1}^{\infty} \frac{m_{2}^{2}}{L_{2}}) - R_{2} - \beta S_{1}(t)}{\beta S_{2}(t)} \quad \text{177}
\]

The three \( S \)-functions may be computed as
functions of $\mathbf{p}$ by the series

\[ S_1(\mathbf{p}) = \frac{1}{L_3} \sum_{n=0}^{\infty} \frac{m_n^2 R^{(m)}_n}{L_3 \beta - R^{(m)}_n} \quad \text{and} \quad S_2(\mathbf{p}) = \frac{1}{L_3} \sum_{n=0}^{\infty} \frac{m_{n_2}^2 R^{(m)}_n}{L_3 \beta - R^{(m)}_n} \]

\[ S_3(\mathbf{p}) = \frac{1}{L_3} \sum_{n=0}^{\infty} \frac{m_n m_{n_2} R^{(m)}_n}{L_3 \beta - R^{(m)}_n} \]

The time-constant $\beta$ must be some root of the second of equations (77). For each root $\beta$, there is one arbitrary coefficient, say $B_\lambda$, in terms of which the constant $A$ is given by either of the two equations (77) and then the constants $C^{(m)}$ are determined by (77). By superposing all possible particular solutions one obtains the number of arbitrary constants $B_\lambda$ which are necessary and sufficient to be fitted to the initial condition.

The definitions of $m_n$ and $m_{n_2}$ given in eq. (69) show that the $S$-series will converge very rapidly when the primary and secondary circles are not close to the shell, that is when $\frac{r_2}{r_1}$ and $\frac{r_2}{r_4}$ are both fairly small. In fact when these ratios are less than 1/4 and when the primary and secondary are also in the plane $x = 0$, ($\cos \theta_1 = \cos \theta_2 = 0$), it is found that the first term of each $S$-series differs.
from the entire sum by much less than one percent. The basis of this approximation is that the term \( m_n^2 \) is proportional to

\[
\left( \frac{\alpha^2}{\mu} \right)^{n+1} \frac{R_0^2}{m (n+1)}
\]

Also \( m_n^2 \) is proportional to

\[
\left( \frac{\alpha^2}{\mu} \right)^{n+1} \frac{R_0^2}{m (n+1)}
\]

For the experimental arrangement, \( \cos \theta = \cos \theta_2 = 0 \).

Since \( P_n(0) \) vanishes when \( n \) is even, every alternate term of the series drops out so that in passing from one term to the next there is a convergence ratio \( (r/a)^4 \) or \( (a/r)^4 \).

Hence we may take

\[
m_1 = \pi \frac{a^2 N_1 P_1^2}{a} (10^9) \text{ henries}
\]

\[
m_2 = \pi \frac{a^2 N_2}{P_2} (10^9) \text{ henries}
\]

\[
M_1 = m_1 m_2 / L_3 \text{ henries} = 2 \pi N_1 N_2 P_1^2 P_2 (10^9) \text{ henries}
\]

\[
S_1 (\beta) = \frac{m_1^2 R_3}{L_3 (L_3 \beta - R_3)} \quad \Rightarrow \quad S_2 (\beta) = \frac{m_2^2 R_3}{L_3 (L_3 \beta - R_3)}
\]

\[
S_3 (\beta) = \frac{m_1 m_2 R_3}{L_3 (L_3 \beta - R_3)} = \sqrt{S_1 (\beta) S_2 (\beta)}
\]

where

\[
R_3 = R^{(i)} = 3 R_0 / 4 r_0 \quad \text{and} \quad a = 10^{10}
\]

From this geometrical circumstance, the effect of currents in the shell is represented by the first current \( J \) only.
For brevity let the four time-constants be defined by
\[
\alpha_i \equiv \frac{(R_i + \sqrt{R_i^2 - 4L_iC_i})}{2L_i}
\]
\[
\alpha'_i \equiv \frac{(R_i - \sqrt{R_i^2 - 4L_iC_i})}{2L_i}
\]
\[
\alpha_2 \equiv \frac{R_2}{L_2} \quad \text{and} \quad \alpha_3 \equiv \frac{R_3}{L_3}
\]

The characteristic time-constants \( \beta_s \) \((s = 1, 2, 3, 4)\) are the four roots of eq. (77) which may be written
\[
(\beta - \alpha_i)(\beta - \alpha'_i)(\beta - \alpha_2)(\beta - \alpha_3) = 0
\]
\[
= \beta^2 \left[ \frac{m_{12}^2}{L_1 L_2} (\beta - \alpha_i)(\beta - \alpha'_i) + \frac{m_{13}^2}{L_1 L_3} (\beta - \alpha_2) - \frac{m_{12}^2}{L_1 L_2} (\beta + \alpha_3) \right]
\]

In the case of no conducting shell this becomes on dividing by \( \alpha_3 \) and then letting \( \alpha_3 \to \infty \),
\[
(\beta - \alpha_i)(\beta - \alpha'_i)(\beta - \alpha_2) = \beta^3 \frac{M_{12}^2}{L_1 L_2}
\]

The system of equations (72) reduces to four
\[
I_i(0) = -Q_i(0)
\]
\[
(\partial_t + \alpha_i)(\partial_t + \alpha'_i) I_i = -\left( M_{12} I_2 + m_{13} J \right)/L_i
\]
\[
(\partial_t + \alpha_2) I_2 = -\left( M_{12} I_1 + m_{13} J \right)/L_2
\]
where
\[
\partial_t(0) = C_i V_i \quad \text{and} \quad I_{in}(0) = I_{in}(0) = J_{in}(0) = 0
\]
The solution is:

\[ I_s(t) = \sum_{s=1}^{4} \beta_s B_s e^{-\beta_s t} \]

\[ Q_s(t) = \frac{L_2}{M_1} \sum_{s=1}^{4} \left[ (1 - \frac{m^2}{L^2}) \frac{B_s}{\alpha^3} - (1 + \frac{\alpha_s}{\alpha^3}) + \frac{\alpha_s}{\alpha^3} \right] B_s e^{-\beta_s t} \]

\[ J_s(t) = -\frac{L_2}{M_2} \sum_{s=1}^{4} \left[ (1 - \frac{m^2}{L^2}) B_s^2 - \frac{\alpha_s}{\alpha^3} B_s \right] B_s e^{-\beta_s t} \]

The four constants \( B_s \) are determined by the four linear equations which express the four initial conditions (83). On writing these it is found that they are equivalent to the following simpler set:

\[ \sum_{s=1}^{4} B_s / \beta_s = C, V, M_1, / R_2 \]

\[ \sum_{s=1}^{4} B_s = 0 \]

\[ \sum_{s=1}^{4} \beta_s B_s = 0 \]

\[ \sum_{s=1}^{4} \beta_s^2 B_s = 0 \]

A simpler and fairly accurate approximation to the solution (84) is obtainable by further restricting the values of the self-inductances.

In the applications in mind, the self-inductances \( L_1 \) and \( L_2 \) of primary and secondary are of the same order of magnitude and this is at least one hundred times the value of the largest of the four positive constants \( \beta_s, \alpha, M_1, M_2, R_2 \), these
four being also of the same order of magnitude. To get the approximate equation, just eliminate $I_1$ between $(82x)$ and $(82y)$. The result is

$$\left\{ \left( 1 - \frac{m_2^2}{L_2^2} \right) \partial_t^2 + (\alpha_1 + \alpha_2) \partial_t + \alpha_1 \alpha_2 \right\} I_2 = -\frac{M_2 \alpha_3}{L_2} \dot{I}.$$ 

Since $\frac{m_2}{L_2}$ is of the order of magnitude of unity and $M_2 \alpha_2$ is of the order of 0.1, this equation may be replaced by

$$(\partial_t^2 + \alpha_2) (\partial_t + \alpha_2) I_2(t) = -\frac{M_2 \alpha_3}{L_2} \dot{I}(t) \quad (86)$$

Since $I_1$, $I_2$, $I_3$ vanish when $t = 0$, this is equivalent to the initial vanishing of $I_1$, $I_2$, and $I_3$, so $(86)$ gives

$$I_2(t) = -\frac{M_2 \alpha_3}{L_2} \int_0^t \left[ \frac{\alpha_2 e^{\alpha_2(t-t')}}{\alpha_2 - \alpha_3} - \frac{\alpha_3 e^{\alpha_3(t-t')}}{\alpha_2 - \alpha_3} \right] dt' \quad (87)$$

Similarly the second member of $(82x)$ contains factors $M_2 \alpha_3$ and $M_1 \alpha_1$, each less than 0.1. Neglecting it, the primary current is unaffected by secondary and shell currents so that since $\partial_t I_1 = 0$, the integration of $(82x)$ gives

$$...$$
\[ I_1(t) = \frac{V}{L} \left( e^{-\alpha_1 t} - e^{-\alpha_2 t} \right) \]  

Using this in (87) gives

\[ I_2(t) = \]

\[ = \frac{-V_1 M_2 \alpha_3}{L, L_2} \left\{ \frac{1}{\alpha_1 - \alpha_2} \left[ \frac{\alpha_1 e^{-\alpha_1 t}}{\alpha_2 - \alpha_1} - \frac{\alpha_1' e^{-\alpha_1' t}}{\alpha_2 - \alpha_1} \right] \right. \]

\[ + \frac{1}{\alpha_2 - \alpha_3} \left[ \frac{\alpha_2 e^{-\alpha_2 t}}{\alpha_3 - \alpha_2} - \frac{\alpha_2' e^{-\alpha_2' t}}{\alpha_3 - \alpha_2} \right] \left\} \]  

(88)

For comparison, the same primary current gives the following secondary when there is no shell (this is the limit \( \alpha_3 \to \infty \) of the preceding equation)

\[ I_2(t) = \]

\[ = \frac{-V_1 M_2}{L, L_2} \left\{ \frac{1}{\alpha_1 - \alpha_2} \left[ \frac{\alpha_1 e^{-\alpha_1 t}}{\alpha_2 - \alpha_1} - \frac{\alpha_1' e^{-\alpha_1' t}}{\alpha_2 - \alpha_1} \right] - \frac{\alpha_2 e^{-\alpha_2 t}}{\alpha_2 - \alpha_1} \right\} \]  

(88)

From the nature of the application, this solution is doubtless a sufficiently close approximation to the exact solution (84). To find the error, it would be necessary to compute the solution (84) for comparison.
Case of double roots.

In the example below the roots \( \alpha \) and \( \alpha' \) are identical. Evaluating the 000 sequence in the three preceding equations gives

\[
I_1(t) = \frac{V}{L} t e^{-\alpha t}
\]

The corresponding current \( I_2(t) \) which this primary current induces in the secondary when the shell is between them

\[
I_2(t) = \frac{-VM_2 \alpha_3}{L_1 L_2} \left[ \frac{e^{-\alpha_1 t}}{(-\alpha_1)(\alpha_3-\alpha_1)} \left[ -\alpha_1 t + \frac{\alpha_2 \alpha_3 - \alpha_2^2}{(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)} \right] \right]
\]

\[
+ \frac{1}{\alpha_2-\alpha_3} \left[ \frac{\alpha_2 e^{-\alpha_2 t}}{(\alpha_2-\alpha_1)^2} - \frac{\alpha_3 e^{-\alpha_3 t}}{(\alpha_3-\alpha_1)^2} \right]
\]

With no shell this reduces to

\[
I_2(t) = \frac{-VM_2}{L_1 L_2 (\alpha_2-\alpha_1)} \left\{ \frac{\alpha_2}{\alpha_1-\alpha_2} \left( e^{-\alpha_2 t} - \frac{\alpha_2}{\alpha_1-\alpha_2} \right) - \alpha_2 t e^{-\alpha_2 t} \right\}
\]

where

\[
\alpha_1 = R_1/2L_1 \quad \alpha_2 = R_2/2L_2 \quad \text{and} \quad \alpha_3 = R_3/2L_3
\]
The following is a typical set of data.

**Geometric Data**

\[ r_1 = r_2 = 3.0 \text{ cm}, \cos \theta = 0, \quad N_1 = 6 \text{ turns} \]

\[ r_2 = r_3 = 63.5 \text{ cm}, \cos \theta = 0, \quad N_1 = 1 \text{ turn} \]

\[ a = 12.7 \text{ cm} \]

These give

\[ m_1 = 0.189 (10)^{-6} \text{ henries} \]

\[ m_2 = 0.113 (10)^{-6} \quad " \]

\[ M_1 = 0.167 (10)^{-6} \quad " \]

This shows that \( J \) is the most important part of the shell—
current.

**Electrical Constants**

\[ L_1 = 3.0 (10)^{-6} \text{ henries} \quad R_1 = 1.733 \text{ ohms} \]

\[ L_2 = 6.6 (10)^{-6} \quad R_2 = 70. \quad " \]

\[ L_3 = 0.127 (10)^{-6} \quad R_3 = 0.0191 \quad (R_0 = 0.89) \quad " \]

\[ V_1 = 135 \text{ volts} \]

\[ C_1 = 4 (10)^{-6} \text{ farads} = 4k_1R_1^2 \quad \text{so} \quad \alpha' = \alpha_1 \]

These give

\[ \alpha_1 = \alpha'_1 = 0.289 (10)^{6} \quad \text{sec}^{-1} \]

\[ \alpha_2 = 10.6 \quad " \]

\[ \alpha_3 = 1.5 \quad " \]

With these numerical values the three currents
may be computed as functions of the time
by equations \((8.9)\), \((8.9a)\), and \((8.9c)\).