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THE MULTIPLICATION RATE FOR UNTAMPED CYLINDERS

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ABSTRACT

The determination of the critical dimensions of untamped cylinders, infinite and finite in length, and of the multiplication rate for non-critical sizes, is treated by an extrapolated end-point method and verifies for a few specific lengths and diameters by the variation method with a three-parameter trial function.

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THE MULTIPLICATION RATE FOR UNTAMPED CYLINDERS

The size, shape, composition, and multiplication rate of untamped masses of active material are related by the integral equation

$$n(\underline{x}) = \frac{1+f}{4\pi}\sigma \int d\underline{x}'\, n(\underline{x}')\, \frac{e^{-(\underline{x}-\underline{x}')(\sigma+\gamma/v)}}{(\underline{x}-\underline{x}')^2} \qquad (1)$$

where the integration is carried over the volume of the active material. A number of different methods of treating this equation have been developed for use with spheres and slabs. (Cf. LA Report 8 and a future report in this series). The two most useful and accurate of these, the variation and extrapolated end-point methods, are here applied to untamped cylinders.

For the variation method cylinders differ from slabs and spheres only in that the integrals involved are more cumbersome. The evaluation of the integrals is so laborious that the variation method has been applied only to a few special cases, the results being used to check on the accuracy of the more easily applied extrapolated end-point method.

The use of the extrapolated end-point method depends on the fact that the integral equation (1) can be solved exactly for the plane solution in a half infinite medium by the methods outlined by F. Smithies, London Math. Soc. 46, 409 (1939). The details and results of this treatment are given in LA Report 8. The solutions so obtained are usefully characterized by the phase and wavelength of the asymptotic sinusoidal or hyperbolic solution. The phase has usually been specified by giving the "extrapolated end-point", the distance beyond the boundary of the first root of the asymptotic solution. The slab has been treated by the approximate method of applying this boundary condition independently at the two boundaries. The accuracy of this approximation has been checked by the use of the variation method. The two results check to better than one percent in thickness throughout the useful range.

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The application of the extrapolated end-point method to the untamped sphere depends upon the fact that by the transformation $n(r) = ru(r)$ the integral equation (1) for [illegible] here is reduced to equation (1) for an odd solution, $u(x)$, in an untamped slab. The extrapolated end-point solution for the sphere can therefore be expected to be much more accurate than that for the slab as the two boundary conditions which are assumed independent are about twice as far apart. A comparison with the variation method solution shows that the error made is completely negligible for any useful radius and rises to about one percent for a sphere of zero radius.

We have not been able to find a corresponding identification of the integral equation for the infinite cylinder (i.e., infinite in length) with a slab problem with a displacement kernel, so that no equally justifiable extrapolated end-point method has been found for the cylinder. However, the success of the extrapolated end-point method for the slab and sphere and the identity of the end-point distance in the two cases suggests that the end-point distance for an infinite cylinder might well be approximately the same as the end-point distance for a slab or sphere of the same material and multiplication rate.

Making use of the hypothesis, we computed critical radii for infinite cylinders for various values of f in the following manner: The form of the solution in the interior was taken to be $J_0(kr)$ where k is determined by the condition $\tan^{-1}k/k = 1/(1+f)$. This condition is appropriate in the description of the solution far from any boundary. The radius was taken less than the first root of this $J_0(kr)$ by the extrapolated end-point distance determined previously for spheres and cylinders. The radius as a function of f so determined has no rigorous theoretical justification. However, when compared with the variation-method results obtained by Inglis with a parabolic trial function (Cf. LA Report 26) it proves accurate to less than one percent throughout the useful range.

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The same procedure has been used for cylinders of finite length. Here the interior solution is $J_0(k_1 r)\cos(k_2 z)$ where $k_1^2 + k_2^2 = k^2$ and as before $\tan^{-1}k/k = 1/(1+f)$. The radius is then taken less than the first root of $J_0(k_1 r)$ by the extrapolated end-point distance (a function of f alone) as before and the half-length less than the first root of $\cos(k_2 r)$ by the same amount. The results of these calculations are presented in Fig. 1. Here again a comparison with the variation method results shows no appreciable discrepancy.

For finite cylinders, variation-method radii were determined by the maximization of the expression

$$\frac{I}{N} = \frac{(1/4\pi)\int d\underline{x}\, d\underline{x}'\, n(\underline{x})\, n(\underline{x}')\, e^{-|\underline{x}-\underline{x}'|}/(\underline{x}-\underline{x}')^2}{\int n^2(x)\, dx}$$

where $n = 1 - \alpha r^2 - \beta z^2 + \epsilon r^2 z^2$

To perform the integration the following representation of the kernel was used:

$$\frac{e^{-|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|^2} = \frac{1}{8\pi^3}\int_0^\infty dy \int_1^\infty dz\, e^{-z^2 y} \int d\vec{k}\, e^{-k^2 y} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

With this representation, the integral I integrates to

$$\begin{aligned} I = 4\pi a^2 z_0 \int_0^\infty dy\, E_r(y) \Bigg\{ & K_0\left(\frac{y}{z_0}\right)\left[\frac{(1-\beta-\alpha+\epsilon)^2}{2} G_0\left(\frac{y}{a}\right) + 2(1-\beta-\alpha+\epsilon)(\alpha-\epsilon)G_1\left(\frac{y}{a}\right) \right. \\ & \left. + (\alpha-\epsilon)^2 G_2\left(\frac{y}{a}\right)\right] + 4K_1\left(\frac{y}{z_0}\right)\left[\frac{(1-\beta-\alpha+\epsilon)}{2}(\beta-\epsilon)G_0\left(\frac{y}{a}\right) + \right. \\ & \left. \left\{\epsilon(1-\beta-\alpha+\epsilon) + (\alpha-\epsilon)(\beta-\epsilon)\right\}G_1\left(\frac{y}{a}\right) + \epsilon(\alpha-\epsilon)G_2\left(\frac{y}{a}\right)\right] \\ & + \frac{8}{3}K_2\left(\frac{y}{z_0}\right)\cdot\left[\frac{(\beta-\epsilon)^2}{2} G_0\left(\frac{y}{a}\right) + 2\epsilon(\beta-\epsilon)G_1\left(\frac{y}{a}\right) + \epsilon\, G_2\left(\frac{y}{a}\right)\right]\Bigg\} \end{aligned}$$

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a = radius of cylinder

$2z_0$ = length of cylinder

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$$K_o(y) = -y(1 - e^{-1/y^2}) + 2\int_0^{1/y} dx\, e^{-x^2}$$

$$K_1(y) = \frac{2y^3}{3} + \frac{y}{3}(1 - 2y^2)\, e^{-1/y^2} + (\frac{2}{3} - y^2)\int_0^{1/y} dx\, e^{-x^2}$$

$$K_2(y) = y^3 - \frac{2y^5}{5} + \frac{y}{5}(2y^4 - 3y^2 + 1)e^{-1/y^2} + (\frac{2}{5} - y^2)\int_0^{1/y} dx\, e^{-x^2}$$

$$G_o(y) = 1 - {}_1F_1(\frac{1}{2}, 2, -\frac{1}{y^2})$$

$$G_1(y) = -y^2 + \frac{1}{4} + y^2\, {}_1F_1(\frac{1}{2}, 2, -\frac{1}{y^2})$$

$$G_2(y) = -y^2 + \frac{1}{6} + y^2\, {}_1F_1(\frac{1}{2}, 3, -\frac{1}{y^2})$$

$$E_r(y) = \frac{2}{\sqrt{\pi}}\int_y^{\infty} dx\, e^{-x^2}$$

${}_1F_1$ = confluent hypergeometric function

The integration over y was done numerically.

The maximum was determined subject to variation of all three parameters of the trial function n. The results compare with the extrapolated end-point method as follows:

length	diameter	f_{var}	$f_{end\text{-}point}$
4.0	2.0	.702	.70
4.0	1.60	.892	.89
4.0	1.33333	1.086	1.08
2.0	2.0	.857	.85
2.0	1.60	1.046	1.04
2.0	1.33333	1.239	1.23

The values in the last column are taken from Fig. 1.

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In Fig. 2 is presented the dependence of the critical mass of finite cylinders on their shape. The curve gives the critical mass, measured in units of the critical mass of an untamped sphere of the same material, as a function of the ratio of the length to diameter. The curve given was calculated for $f = .5$, however the dependence of the curve on the value of f is so slight that it seems unnecessary to give further curves.

The same approximation method of calculating critical sizes by the extrapolated end-point method has been applied to rectangular solids. The linear dimensions so determined check with those calculated by the variation method by Olum and Davis to within about one percent. The success of the end-point method in giving reasonably accurate results for problems to which it is not obviously applicable suggests that this method may prove of value in calculating critical sizes for still more irregular shapes.

The curves of Fig. 1 can be used for non-critical configurations by the use of the similarity transformation (cf. L.A. 8). If the multiplication rate is γ then the quantity $1 + f$ is to be replaced by $(1 + f)/(1 + \gamma)$ and the linear dimensions are measured not in units of the mean free path but the mean free path divided by $1 + \gamma$. The multiplication rate γ is so defined that the number of neutrons increases (or decreases if γ is negative) in one mean free time by a factor e^{γ}.

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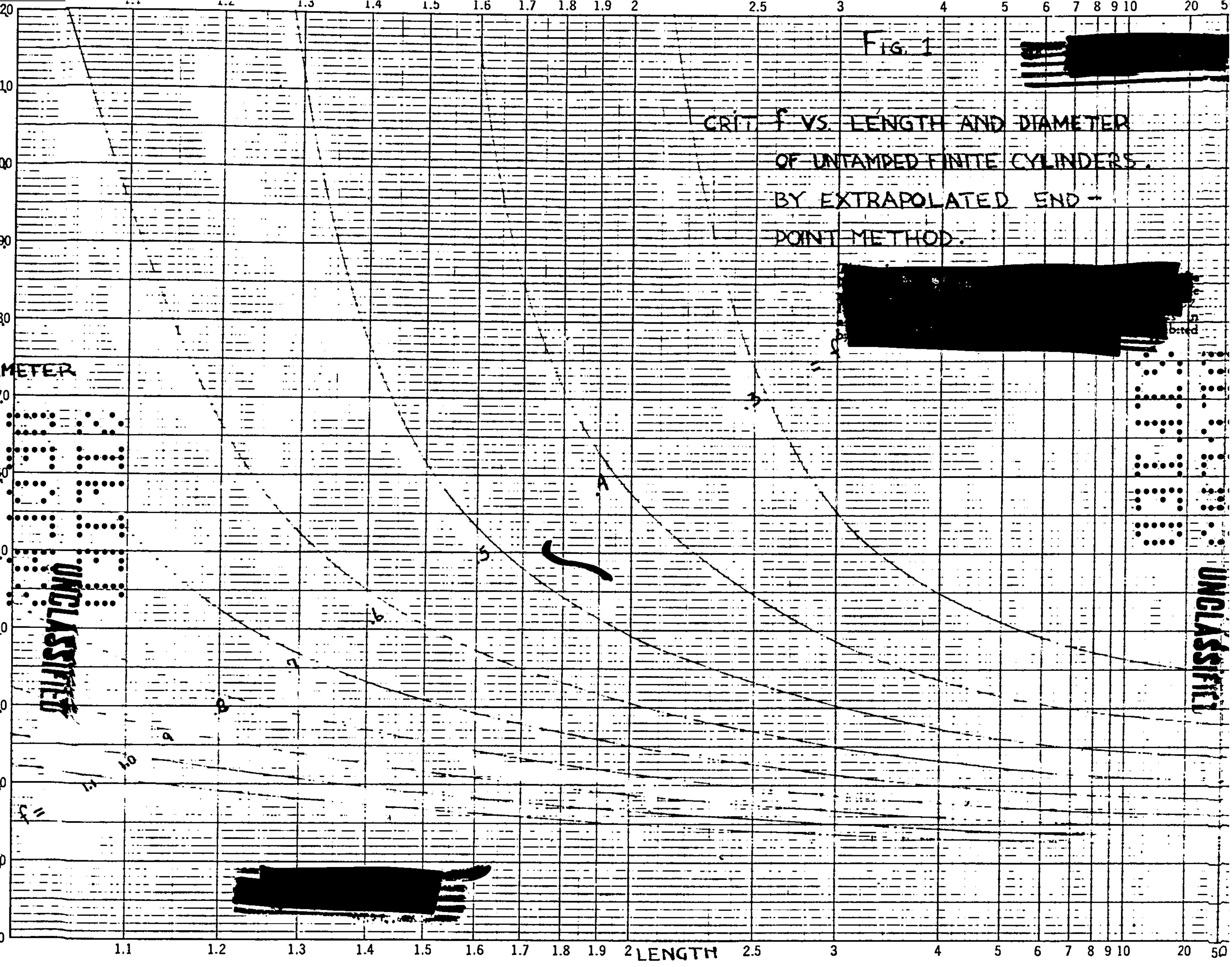

Fig. 1
[redacted]
CRIT f VS. LENGTH AND DIAMETER
OF UNTAMPED FINITE CYLINDERS.
BY EXTRAPOLATED END-
POINT METHOD.
[redacted]
DIAMETER
LENGTH
f =
[redacted]

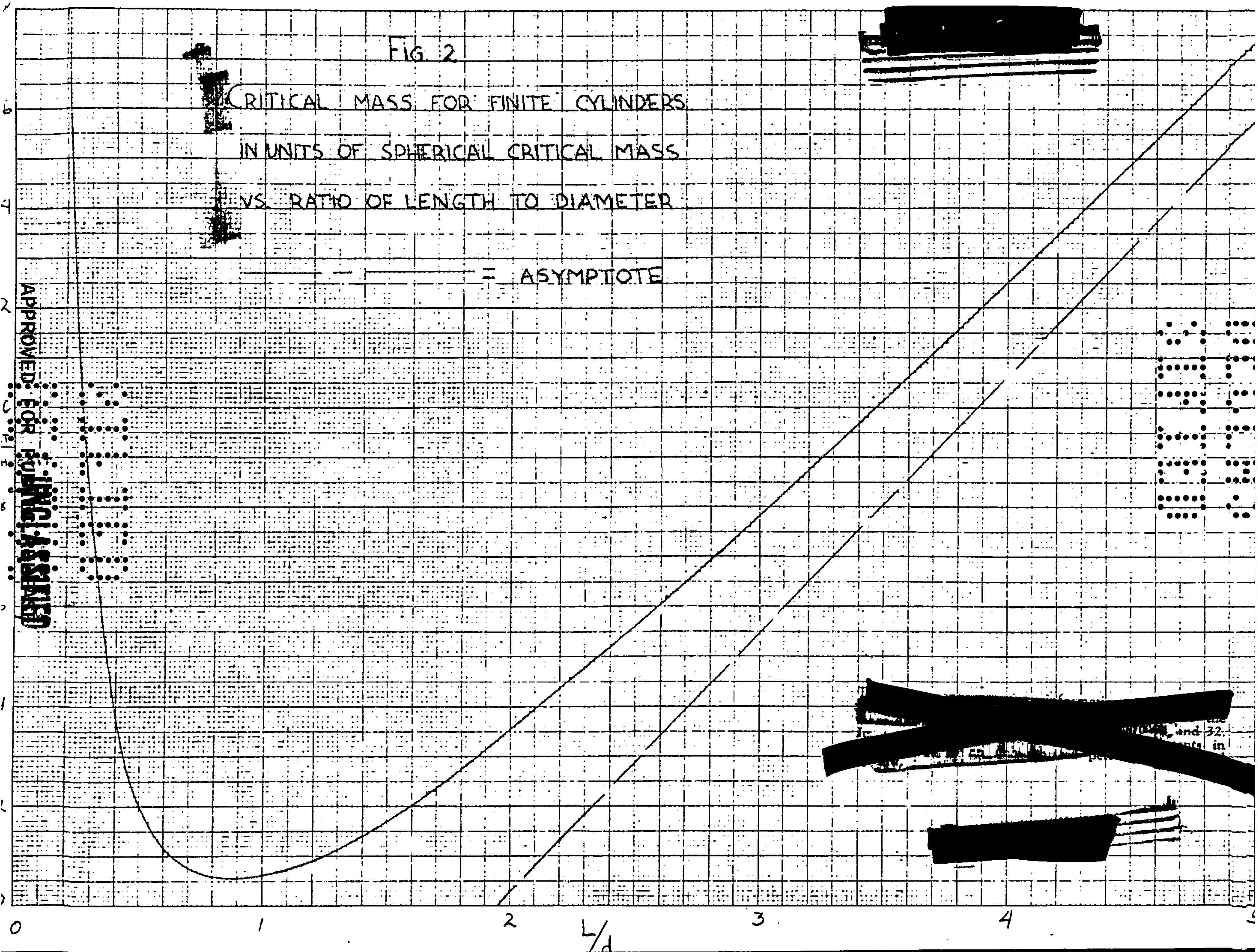

FIG. 2
CRITICAL MASS FOR FINITE CYLINDERS
IN UNITS OF SPHERICAL CRITICAL MASS
VS. RATIO OF LENGTH TO DIAMETER
— — = ASYMPTOTE
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0
1
2
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