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PHENOMENOLOGICAL THEORY OF REACTIVE ASSEMBLIES

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CHAPTER 2

PHENOMENOLOGICAL THEORY OF REACTIVE ASSEMBLIES

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The complete theoretical treatment of reactive assemblies is exceedingly complex, and a large part of Volume IV of the Technical Series is dedicated to the problem of predicting the properties of a chain-reactive assembly from the nuclear properties of the materials in it and from the geometrical arrangement of these materials.

For the purpose of this volume we include a brief phenomenological theory of reactive assemblies which is based mainly on the following assumptions:

(1) Each neutron produces K daughter neutrons on an average.

(2) If a prompt neutron produces fission it does so, on an average, after the time τ_0.

(3) Delayed neutrons have a δ times greater chance of causing fission than prompt neutrons.

In this treatment we neglect the fact that the probability for a neutron to cause fission depends on the place where it is born and on the time that has elapsed since its birth. We shall not discuss the methods by which K, τ_0, and δ can be calculated from the nuclear and geometrical characteristics of the assembly. (Some of these methods are described briefly in Chapter 2.) We shall discuss, wherever needed, how K, τ_0, and δ are affected by those changes in configuration or composition of the assembly which are made in the course of experiment.

2.1 TIME BEHAVIOR OF REACTIVE ASSEMBLIES

The fission rate F(t) at the time t in a reactive assembly is made up of two kinds of fissions. First, there are the "source fissions"; that is, those caused by the presence of a (non-fission) neutron source, and the spontaneous fissions; in other words, all those fissions which are not caused by neutrons due to some previous fission. Secondly, there are the "daughter fissions"; that is,

all those which are caused by a neutron due to some previous fission. This would include, for instance, photoneutrons emitted from a beryllium tamper because of the gamma-radiation of some fission product.

The rate of source fissions, $S(t)$ may or may not depend explicitly on the time t, but it certainly does not depend on the previous fission history of the system. The daughter fissions, however, do. Some of them are caused by "prompt" neutrons, i.e., those which are emitted within an exceedingly short time ($< 10^{-12}$ seconds according to nuclear theory, $< 10^{-9}$ seconds according to experimental evidence) of the fission itself, and cause fission after an average time τ_0 which lies between about 10^{-8} for a Pu-239 or U-235 metal assembly, and about 10^{-3} seconds for a graphite pile. Others are caused by "delayed" neutrons which are emitted by certain fission fragments ("pregnant nuclei") after a time of the order of seconds. The decay curve for the pregnant nuclei is fairly well known both in the case of Pu-239 and U-235.

We introduce the function $K\,p(\tau)d\tau$ to indicate the probability that a given fission should produce a daughter fission at the time $\tau \pm 1/2\, d\tau$. The average number of daughter fissions produced by a single fission is found by integration as $K\int_0^\infty p(\tau)d\tau$. This number has previously been called K and our way of introducing $p(\tau)$ therefore normalizes it, so that

$$\int_0^\infty p(\tau)\,d\tau = 1 \qquad (1)$$

We can now write down the fission rate $F(t)$ at the time t as the sum of source fissions and daughter fissions and get

$$F(t) = S(t) + K\int_0^\infty F(t-\tau)p(\tau)d\tau \qquad (2)$$

This equation is a convenient starting point for all calculations in this chapter. It connects the fission rate at a given time t with the physical conditions of the system at this time and with the fission rates at all previous

times. It assumes that the configuration of the system, and hence both K and $p(\tau)$, do not change rapidly with time. We shall discuss later how the Equation (2) has to be amended if this condition is not fulfilled.

Let us now consider some simple cases.

(a) Everything constant:

$$F = S + KF\int_0^\infty p(\tau)d\tau = S + KF$$

$$\text{or } F = \frac{S}{1-K} \qquad (3)$$

This equation can also be obtained simply by adding up source fissions, primary, secondary, tertiary daughter fissions and so on:

$$F = S + KS + K^2S + K^3S + \ldots = S(1 + K + K^2 + K^3 + \ldots) = S/(1-K)$$

As K approaches unity, the equilibrium fission rate goes to infinity.

(b) S = constant, K = 1:

There is no stationary solution (other than $F = \infty$) but a solution where F increases linearly with time.

$$F(t) = at = S + \int_0^\infty a(t-\tau)p(\tau)d\tau$$

$$= S + at - a\bar{\tau}$$

$$a = \frac{S}{\bar{\tau}}$$

$$F = S\frac{t}{\bar{\tau}} \qquad (4)$$

Physically this is easy to see. If K = 1 each fission produces exactly one daughter fission (on the average) after the time $\bar{\tau} = \int_0^\infty \tau p(\tau)d\tau$. Hence every extra fission produced will increase the fission rate permanently by a rate of one fission every $\bar{\tau}$ seconds. The value of $\bar{\tau}$ is obtained by averaging over all neutrons, prompt and delayed, and is about .1 second.

(c) S and K constant but not F:

To solve our integral Equation (2) we must make some assumption about $p(\tau)$. The simplest assumption that agrees reasonably well with the facts in most cases is that $p(\tau)$ is a sum of exponentials:

$$p(\tau) = \sum_{k=1}^{n} \beta_k \lambda_k e^{-\lambda_k \tau}, \quad \lambda_k > 0 \qquad (5)$$

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The condition $\int_0^\infty p(\tau)d\tau = 1$ is then fulfilled if $\sum_{k=1}^{n} f_k = 1$. Under this assumption about $p(\tau)$, the general solution of our integral equation is of the form

$$F(t) = \frac{S}{1-K} + \sum a_j e^{\Lambda_j t} \qquad (6)$$

where the a_j are arbitrary constants which, in every special case, have to be computed from the boundary conditions of the problem.

To calculate the values of Λ_j we assume $a_1 = 1$, $a_2 = a_3 \ldots = 0$ and write Λ for Λ_1. Our integral equation then becomes

$$\frac{S}{1-K} + e^{\Lambda t} = S + K \int_0^\infty \left(\frac{S}{1-K} + e^{\Lambda(t-\tau)}\right) \sum f_k \lambda_k e^{-\lambda_k \tau} d\tau$$

$$e^{\Lambda t} = K \int_0^\infty e^{\Lambda(t-\tau)} \sum f_k \lambda_k e^{-\lambda_k \tau} d\tau \qquad (7)$$

$$= K e^{\Lambda t} \int_0^\infty \sum f_k \lambda_k e^{-(\Lambda+\lambda_k)\tau} d\tau$$

$$\frac{1}{K} = \sum f_k \lambda_k \int_0^\infty e^{-(\Lambda+\lambda_k)\tau} d\tau = \sum_{k=1}^{n} f_k \frac{\lambda_k}{\Lambda+\lambda_k}$$

Equation (7) is of the n^{th} degree in Λ and hence cannot be solved explicitly for $n > 4$. However, some simple conclusions can be drawn without solving the equation.

(1) Being of the n^{th} degree, it must have n solutions for Λ.

(2) All solutions are real. Plotting the right hand side of Equation (7) against Λ we get a curve which has a negative slope everywhere except at the points where $-\Lambda$ becomes equal to one of the λ_k. At these points the curve jumps from $-\infty$ to $+\infty$. A typical curve is shown in Figure 1. The intersections of this curve with a line drawn at height 1/K are the solutions and there must be n of them.

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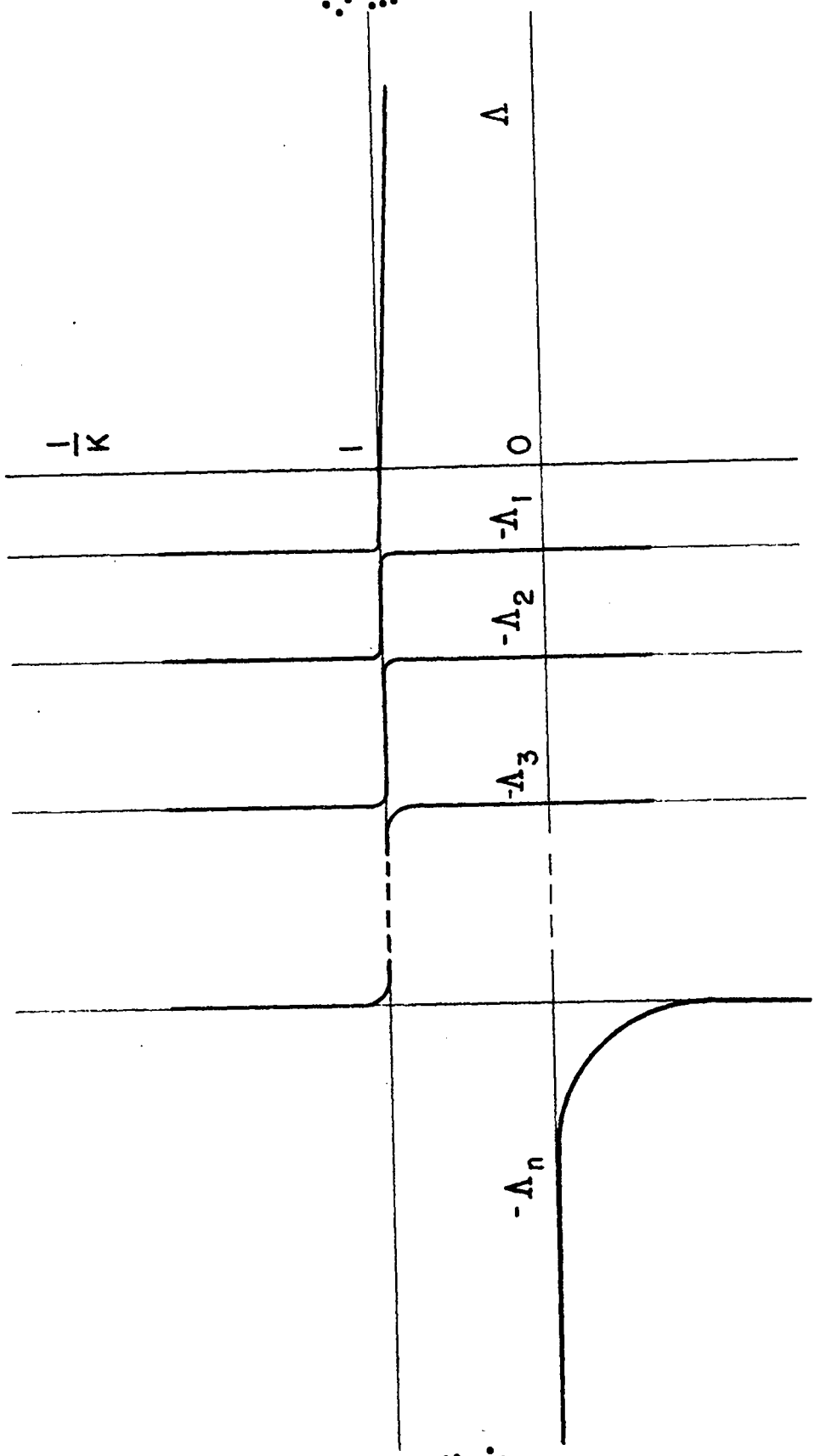

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(3) As long as $K < 1$, all solutions are negative. (Since $\sum f_k = 1$, a positive value of Λ would make the right hand side of Equation (7) < 1 or $K > 1$.) Therefore, as long as $K < 1$ all the time dependent solutions of our integral equationare sums of exponentials which decrease with time, leaving eventually only the time-independent solution $S/(1 - K)$ which is,therefore,indeed the equilibrium fission rate.

(4) If $K > 1$, one solution for Λ becomes positive. It represents an exponential which grows with time and which eventually exceeds all other terms. The other solutions, however, are all still negative and not very different from the corresponding values of λ_k (with the exception of the smallest λ_k).

It is seen that the time behavior of the assembly is fairly complicated and not in general to be described by a simple exponential. If one suddenly alters the conditions of the system (for instance, by removing or introducing a neutron source or by altering the arrangement of tamping material) the subsequent fission rate is a superposition of all the solutions just mentioned, with coefficients a_i which depend on the character of the alteration. After a while the solutions with the greater (negative) values of Λ die out and only the one with the least negative Λ is left. Hence, after changing the reactivity of an assembly, for instance in an attempt to make it just critical, one always has to wait about a minute or two for the transitory effects to die out before one can judge the effect of the change on the reactivity.

We said before that a certain fraction, which we shall call f, of all the neutrons is emitted not in the fission act itself but as a result of the β-decay of certain fission fragments ("pregnant nuclei"). The decay curve of these delayed neutrons has been measured repeatedly, both for U-235 and Pu-239, and several attempts have been made to represent the curve as a sum of exponentials (which it ought to be, if there are several types of pregnant nuclei,

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each with its characteristic decay period). The two longest periods are quite accurately known since it has been possible to isolate the corresponding radioactive isotopes (of bromine and iodine) by chemical methods. Concerning the shorter periods, different observers don't agree too well. Actually, the disagreement hardly effects the shape of the decay curve or the calculated time behavior of an assembly.

Since the delayed neutrons have lower energies than the prompt neutrons, their chance of producing a fission is different (usually larger) by a factor which we call γ. (An approximate calculation of γ was attempted by F. DeHoffmann in LA-471). Strictly speaking we should introduce a different γ for each of the different of pregnant nuclei, but not enough is yet known about the respective neutron energies to justify such niceties.

The prompt neutrons are emitted within a negligible time of the instant when the neutron which caused the fission hit the nucleus. However, they have to travel some distance and in some cases have to be slowed down, before they can cause fission themselves. The average time lost in this way we call τ_o. Of course some neutrons may take shorter times and some (for instance those which return after having suffered several collisions in the tamper) may take much longer:

Taking all this into account we can write

$$p(\tau) = (1-\gamma f)\, p_o(\tau) + \gamma f \sum q_i \lambda_i e^{-\lambda_i \tau} \qquad (8)$$

where the first term refers to the prompt neutrons and f is the fraction of neutrons which are delayed. If we approximate this term by an exponential $p_o(\tau) = (1/\tau_o)\, e^{-\tau/\tau_o}$ we have expressed $p(\tau)$ entirely as a sum of exponentials and can now express the time behavior of the system by Equation (6), with the values of Λ determined by Equation (7).

Equation (7) can also be used to determine K if the period $T = 1/\Lambda$ of the system has been measured. We rewrite Equation (7) by subtracting both

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sides from 1 and get

$$1 - \frac{1}{K} = \frac{K-1}{K} = \sum f_k - \sum f_k \frac{\lambda_k}{\Lambda + \lambda_k} = \sum f_k \frac{\Lambda}{\Lambda + \lambda_k} = \sum f_k \frac{\tau_k}{T + \tau_k}$$

$$\left(\tau_k = \frac{1}{\lambda_k} \right)$$

If we furthermore make use of Equations (5), (8) we get

$$\frac{K-1}{K} = (1 - \gamma f) \frac{\tau_o}{T + \tau_o} + \gamma f \sum q_i \frac{\tau_i}{T + \tau_i}$$

This equation can be further simplified if one considers that $\gamma f \ll 1$ (about .005 to .01), that τ_o is, at the most, a few milliseconds while practical values of T are usually a second or more, and that in the interesting cases, K is close to 1. Hence we can write approximately

$$\delta K = K - 1 = \frac{\tau_o}{T} + \gamma f \sum \frac{q_i \tau_i}{T + \tau_i} \qquad (9)$$

This reactivity equation is often called the "inhour equation", for reasons that will be apparent in the next section.

2.2 MEASUREMENT OF K

The behavior of a system for which K = 1 is so characteristic that in this case the measurement of K presents no problem. In systems which are slightly sub-critical or super-critical, the quantity we want to measure is really K - 1 or δK which is often called the excess reactivity. This is a non-dimensional quantity and does not, as such, require a special unit. However, since the excess reactivity is usually a small fraction of 1, it is convenient to use a suitable unit smaller than 1, for instance the microre (μRe) or micro-reactivity unit. The reactivity in microres is equal to $\delta K \times 10^6$.

The absolute measurement of the excess reactivity is quite difficult. One method which has been used with some success will be described presently. On the other hand, relative measurements are fairly easy. That is, it is fairly

easy to compare the excess reactivities of one reactor in two different states. Several arbitrary units have been used, such as the "inhour" and the "cent". Their value on the absolute scale differs from one reactor to the other depending on the value of β (the relative efficiency of delayed and prompt neutrons), and on the material (plutonium or U-235) used.

2.2-1 Absolute Measurements (The "Boron Bubble")

Ideally, the multiplication constant of a reactor could be measured by introducing a calibrated neutron source and then measuring how many additional neutrons are produced through the fission chains. In practice such a procedure would be very difficult. The source would have to have the same energy distribution as the fission neutrons, and its distribution in space would have to be the same. This might be achieved by suitable averaging of a number of measurements carried out with the source placed at different points. However, there would still be the difficulty of an absolute measurement of the total number of neutrons produced in the system.

An ingenious way of avoiding these difficulties is the so-called "boron bubble" method. A small portion of the reactor is replaced by a boron-containing material which has the same absorption and scattering of neutrons, but contains no fissile material. This "boron bubble" is moved to different places and a suitable average of the results is taken. The introduction of this boron bubble obviously decreases the reactivity of the assembly. If it represents a fraction v/V of the total volume of the fissile material, its introduction is, on an average, equivalent to a reduction of K by that same fraction. Such a change, known in absolute units, can be used to calibrate any of the relative methods which are to be described in the next paragraph. The disadvantage of the method is that the energy dependence of the fission cross section of U-235 and Pu-239 cannot be simulated by any non-fissile absorber. In order to calculate the right amount of boron, one has to know beforehand the energy spectrum of the neutrons

in the system, which, furthermore, usually varies from point to point. Also, it is not accurately true that all the boron bubble does is to absorb as many neutrons as the fissile material which it replaces, because this statement is true only on an average and not for each individual neutron energy. Experiments which have been carried out by this method are described in Chapters 4 (Waterboiler) and 5 (Hydrides) where the difficulties of this method are discussed in more detail.

2.2-2 Relative Measurements

For subcritical systems, a scale linear in K can be established by the simple procedure of measuring the neutron intensity in several different states of the system; for instance for several different positions of the control rod. The reciprocal of the neutron intensity is a measure of 1 - K, the negative "excess reactivity". However, the units are completely arbitrary and depend, furthermore, on the constancy of the neutron detector used.

Several methods for measuring the excess reactivity depend on the time behavior of the system. If a reactor is slightly above critical, the neutron flux will grow exponentially with time (in the absence of a neutron source), and this rate of growth is connected with the excess reactivity by Equation (9). In particular, for a system exceedingly close to critical, the rate of growth of the chain reaction is a linear function of K - 1 and the number of times the flux e-folds in one hour is directly proportional to K - 1. This reasoning leads to the unit called one "inhour" which we may define as the change in reactivity which will turn a just critical system into one with an e-folding time of one hour. If T is one hour, it is much larger than any of the τ_i and Equation (9) becomes $K - 1 = \gamma \bar{\tau}/T$, where $\bar{\tau}$ is the average time between fissions, as defined in the first section. If we insert $\gamma = 1$, $\bar{\tau} = .09$ seconds, $T = 3600$ seconds, we find that one inhour is a change of K by 2.5×10^{-5}. However, if $\gamma > 1$, as in most of the assemblies discussed in this volume, then the value of the inhour is greater than 2.5×10^{-5} by the same factor.

With an appropriately rapid control mechanism, T can be made as small

as about one second; on the other hand, Equation (9) can also be used for negative values of T, but the measurement of K becomes very inaccurate for K smaller than, say, .98, as can be seen from Figure 1. From Equation (9) or from a graphical representation of its relevant portion (the "inhour curve") the excess reactivity of a reactor in a given state can be immediately obtained in units of one inhour, from a measurement of T.

The values of K_1 and K_2 of a reactor in two different states can also be compared, provided K_1 and K_2 are < 1, by making a sudden transition from one state to the other, for instance by rapidly shifting the control rod. The immediate effect of this transition depends exclusively on the change in K_p, the prompt-multiplication constant; the delayed neutrons take some time to adjust themselves and may, therefore, be regarded as part of the source, in the first instant. Since the prompt-multiplication is due to the fraction $1 - \gamma f$ of all the neutrons emitted as a result of fission (see Equation (8)), we have

$$K_p = K(1 - \gamma f) \qquad (10)$$

Immediately after the transition we find the fission rate changed by a factor

$$A = \frac{1-K_{p2}}{1-K_{p1}} = \frac{1-K_2\ (1-\gamma f)}{1-K_1(1-\gamma f)} \qquad (11)$$

After several minutes the fission rate will have adjusted itself to the new state of the system and will have changed altogether by a factor

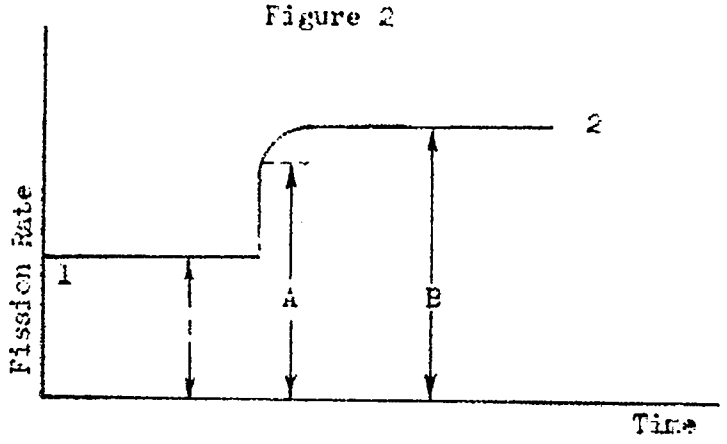
$$B = \frac{1 - K_2}{1 - K_1} \qquad (12)$$

If both A and B are measured we can solve the two linear Equations (11) and (12) and get the two excess reactivities:

$$1 - K_1 = \gamma f \frac{1-B}{A-B} \quad ; \qquad 1 - K_2 = \gamma f \frac{(1-B)B}{A-B}$$

The difficulty lies in measuring A; it is necessary both to effect the transition and to record the change in fission rate within a time short compared to the fastest delayed-neutron period, that is, within a fraction of a second. The change of fission rate with time is illustrated in the Figure 2 below. The transition can sometimes be performed with sufficient speed, but pen

Figure 2



recorders as were used in most of our experiments are not fast enough to permit an accurate determination of A. (See Chapter 5, Section 5.2)

The excess reactivity by this method is obtained in units of β f. The fraction f of delayed neutrons is .006 for U-235 and .003 for Pu-239; β varies from one system to another. It has been suggested to use β f as a practical unit of excess reactivity and to call one hundredth of β f "one cent". In other words, 100 cents is the change in K from a just critical system to the point where the prompt neutrons alone can support the reaction.

A variant of the "sudden change" method is that in which K is left constant but the source is suddenly changed. If, for instance, a source is suddenly introduced into a subcritical system, the multiplication is at first only $1/(1 - Kp)$ but grows within a few minutes to its equilibrium value $1/(1 - K)$. A comparison between the fission rate immediately after the insertion of the source, and that some time later, gives the value of $(1 - K - K\beta f)/(1 - K) = C$ from where we find $K = \beta f/(C - 1)$ plus higher

terms in γf which are negligible. The difficulty of measuring C is the same as that of measuring A in the "change-of-K" method. Of course, both the immediate and the ultimate change in fission rate is the same (apart from the sign), whether a given source is suddenly inserted or suddenly removed, and the latter is often easier to do.

2.3 MEASUREMENT OF γ

As it was pointed out in the last section, the value of the inhour for a U-235 system with $\gamma = 1$ is 2.5×10^{-5}; that is, if a just critical system is changed so that K increases by 2.5×10^{-5}, the fission rate will then e-fold in one hour. This is very nearly the case for a graphite pile where the main loss of neutrons is in the system itself (for instance in U-238) rather than by leakage; the competition between fission and absorption depends very little on the speed of the neutrons.

However, in small reactors such as those which are described in this volume, leakage is the main process competing with fission, and the slow delayed neutrons (with mean energies of around 0.5 million electron volts) leak out much less than the faster prompt neutrons (mean energy 1 to 2 million electron volts). Hence, the factor γ which indicates the relative efficiency of delayed and prompt neutrons is > 1. The value of the inhour is then greater than 2.5×10^{-5} by a factor γ. Hence, an absolute calibration of the K scale, for instance by the boron bubble method, amounts to a measurement of γ. In this way values of 1.3 - 1.6 for γ were observed in some cases. An experimental measurement of the quantity γf is given in Chapter 4, Section 4.2.

It might be possible to measure γ directly by comparing the fission rates obtained with two neutron sources simulating the energy spectra of delayed and prompt neutrons, respectively. No such measurements were attempted.

Of course γ can never be greater than ν; the delayed neutrons cannot be utilized better than 100 per cent, and of the prompt ones, at least the fraction

$1/\nu$ must be utilized to get the system critical.

F. de Hoffmann has shown how to calculate γ under certain simple assumptions. (IA-471).

2.4 MEASUREMENTS OF τ_o

To measure τ_o is difficult because of its smallness. In the water-boiler (See Chapter 4) τ_o is about 10^{-4} seconds, in a hydride (See Chapter 5) about 10^{-6} seconds, in metal about 10^{-8} seconds.

On the other hand, the very smallness of τ_o is useful since it helps to separate it from the much longer periods of the delayed neutrons. If any of the parameters which influence the fission rate (such as K or the source strength S), is suddenly changed, the fission rate will adjust itself rapidly to the new condition as far as the prompt neutrons are concerned while the rate of emission of delayed neutrons can be regarded as essentially constant. Our fundamental Equation (2) can then be written

$$F(t) = S + S' + K_p \int_o^\infty F(t-\tau)\ e^{-\tau/\tau_o}\ \frac{d\tau}{\tau_o} \qquad (13)$$

where S' represents the delayed neutrons and is treated as time-independent. The time dependent part of the solution of this equation is found to be proportional to

$$e^{\frac{K_p-1}{\tau_o}} = e^{\alpha t}, \quad \alpha = \frac{K_p-1}{\tau_o}$$

This follows directly from Equation (7) if one replaces K by K_p, the sum by a single term, λ by α, f_k by 1 and λ_k by $1/\tau_o$.

If $K_p > 1$, the system is supercritical for prompt neutrons and will blow up in a very short time unless steps are taken to prevent that. In Chapter 9 an experiment is described (the "dragon experiment") where the condition $K_p > 1$ was indeed realized, for a time short enough to prevent a disaster.

If $K_p < 1$, the system is not explosive, but the fission rate will grow in exponential fashion because of the delayed neutrons as long as $K > 1$. If we want to do measurements on a system in which the fission rate does not grow exponentially, we have to keep $K < 1$, and hence $K_p < 1 - \beta f$ (See Equation (10)). In such a system α is, of course, negative, and $|\alpha| > \beta f / \tau_o$. Since βf is of the order of 10^{-2}, we see that the "time constant" $1/|\alpha|$ of a waterboiler, a hydride reactor and a metal reactor are of the order of 10^{-2} seconds, 10^{-4} seconds and 10^{-6} seconds, respectively.

It is possible to produce neutron bursts of very short duration, for instance by modulating the beam of a cyclotron (See Volume I of the Technical Series). If such a burst of neutrons is made to enter a subcritical reactor, the fission rate immediately after the pulse will decrease exponentially with the time constant $|\alpha| = (1-K_p)/t_o$. The decrease can be observed by the use of counting "channels" each of which is operating only for a definite short-time interval, a definite time after the neutron burst. Equipment of this type has been used previously for measuring neutron velocities from their time of flight, and experiments in which time constants of fissile reactors are measured by means of a modulated neutron source are described in Chapter 8 of this volume.

The same chapter also describes an ingenious variant of the modulated source method which was suggested by Bruno Rossi. This variant depends on the fact that statistical fluctuations of any neutron source represent a kind of modulation, and that it is, therefore, not necessary to apply external modulation. A brief theory of this method is given in this chapter, in the section on fluctuations.

A third method which has been successfully tried depends on a moderately rapid, periodic variation of K. To calculate the effects of such a variation on the fission rate, we use again Equation (2). We assume that both K_p and the

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fission rate F are the sum of a constant and a small purely harmonic term

$$K_p = K_o + Ke^{iwt} \quad ; \quad F = F_o + fe^{iwt}$$

Equation (2) then becomes

$$F_o + fe^{iwt} = S + (K_o + Ke^{iwt}) \int \left(F_o + fe^{iw(t-\tau)}\right) e^{-\tau/\tau_o} \frac{d\tau}{\tau_o}$$

$$\begin{aligned} F_o &= S + K_o F_o \qquad F_o = S/(1-K_o) \\ f &= K F_o + K_o f \frac{1}{\tau_o} \int e^{-iwt-\tau/\tau_o} d\tau \\ &= K F_o + K_o f \frac{1}{1+iw\tau_o} \qquad (14) \\ f &= K F_o \frac{1}{1-\frac{K_o}{1+iw\tau_o}} = K F_o \frac{1-K_o+w^2\tau_o^2-K_o\, iw\tau_o}{(1-K_o)^2+w^2\tau_o^2)} \end{aligned}$$

(In this calculation the term with Kf has been neglected). The occurrence of an imaginary term indicates that the variation in fission rate is shifted in phase with respect to the variation. This phase shift is

$$\tan \phi = \frac{K_o\, w\, \tau_o}{1-K_o+w^2\tau_o^2} \qquad \text{or for } \phi \ll 1 \qquad \phi = \frac{K_o\, w\, \tau_o}{1-K_o} \qquad (15)$$

Now $(1 - K_o)/\tau_o$ is nothing else but the average value of $|\alpha|$, while K_o might be replaced by 1, for the sort of accuracy with which the phase shift can be measured. The phase shift then becomes simply $w|\alpha|$, the usual value for a system of time constant $|\alpha|$ (for instance, a resistance R and capacity C, such that $1/CR = |\alpha|$) disturbed by a force (for instance, an electromotive force) of frequency $w/2\pi$. Hence, by measuring the phase shift one determines α. An experiment of this type is described in LA-183 and in Chapter 4 of this Volume.

It is a common feature of all these methods of measuring τ_o that all one really measures is $\alpha = (K_p-1)/\tau_o$. Hence to get τ_o one has to know the (prompt) excess reactivity Kp - 1. The difficulties of measuring excess reactivities have been explained in a previous section.

2.5 FLUCTUATIONS IN REACTIVE ASSEMBLIES

A nuclear chain reaction consists of individual processes which happen at random. This has the effect that if the same experiment is repeated several times, the results fluctuate. Such fluctuation occurs always when random events are involved, as, for instance, in ordinary radioactive decay.; in nuclear chain reactions the fluctuations are accentuated by the multiplication mechanism and can assume formidable proportions.

R. Feynman in Volume VI of the Technical Series has described methods for calculating the fluctuations in certain cases which were of practical interest to the Project, carrying the calculation to a considerable degree of refinement and complexity where this was required. In this chapter a simplified treatment is introduced, of a less ambitious scope, sufficient for the interpretation of the experiments with which this volume deals. The methods used are largely based on Feynman's report, but some of his equations are derived in a different way which was first suggested by Ulam and Hawkins.

2.5-1 The Method of "Generating Functions".

A few general remarks will make what follows easier. We are concerned with experiments in which the result is a whole number, e.g. the number of pulses recorded during a given period. If one such experiment is repeated many times the result will in general fluctuate and we shall call p_k the probability that the result of one experiment is the number k. One then defines the generating function $P = \sum_0^\infty p_k x^k$ which is a very suitable vehicle for carrying all the statistical information about the experiment, and one from which useful information can be easily extracted. Furthermore, the generating function of a combined experiment is, in some cases, a simple algebraic combination of the generating functions of the individual experiments. It is convenient to associate generating functions with fictitious experiments as well; in other words, with any question starting with "How many?". This will be done, even if no method

has been devised to give an exact answer, or if the answer is not known. We shall use bold capitals to indicate generating functions; for instance we shall associate the function $\Phi = \sum_0^\infty \phi_k \; x^k$ with the question "How many neutrons are generated by an individual fission?" where ϕ_k is the probability that just k are generated. Equally, we shall use $\mathbf{T}$ in connection with the question "How many fissions occur in a single chain in a given (subcritical) system?" although we know no way of finding out.

2.5-2 Combinations of Experiments

If two (similar or different) experiments with the respective generating function $\mathbf{F}$ and $\mathbf{G}$ are made and their results are added, the generating function of this combined experiment is the product of the individual generating functions, $\mathbf{F} \cdot \mathbf{G}$, provided the outcome of one experiment does not influence that of the other.

For instance, the probability of getting the result "4" is the probability of getting "0" in the first experiment and "4" in the second, plus that of getting "1" in the first and "3" in the second, etc. or $f_0g_4 + f_1g_3 + f_2g_2 + f_3g_1 + f_4g_0$ This, however, is just the coefficient of x^4 in the product $(f_0 + f_1x + f_2x^2 + \ldots\ldots)\cdot(g_0 + g_1x + g_2x^2 + \ldots\ldots)$. In particular, if an experiment is done n times, the generating function relating to the sum of the m results is the n^{th} power of the generating function $\mathbf{G}$ of the experiment.

In some applications the number n itself is determined by the outcome of a previous experiment with the generating function $\mathbf{F} = f_0 + f_1x + f_2x^2 + \ldots\ldots$. The generating function of the combined experiment is $f_0 + f_1\mathbf{G} + f_2\mathbf{G}^2 + \ldots\ldots = \mathbf{F}(\mathbf{G}_{(x)})$.

We may state this as follows:

If two experiments are performed in "cascade" (the outcome of the first decides how many times the second one is to be performed) the combined generating function is obtained by making that of the first operate on that of the second.

A few examples will make these things clearer. Let us consider a long-lived radioactive sample in which N atoms disintegrate per second on an average. What is the generating function P associated with the question "How many atoms disintegrate during one second?"

Let us first solve this problem for a very short interval dt . The probability that two or more atoms should disintegrate in dt is negligible; the probability that one should disintegrate is Ndt and the remainder, 1- Ndt , is the probability for no disintegration. The generating function is, therefore,

$$(1-Ndt) + (Ndt)x = 1 + Ndt\,(x-1)$$

A time interval of one second can be regarded as the sum of 1/dt intervals of the length dt. We therefore get the generating function referring to the interval t by taking the (t/dt)th power of the generating function which refers to dt. In this way we get

$$P = \left[1 + dt(x-1)N\right]^{1/dt} = e^{N(x-1)} = e^{-N} + e^{-N}Nx + e^{-N}\frac{N^2}{2!}x^2 + \ldots\ldots \qquad (16)$$

The coefficients give the probability that in a given second just 0, 1, 2 etc. atoms should disintegrate and we see that our method permits a fairly simple derivation of the well known Poisson distribution.

A second example refers to a particle counter (G.M. counter, boron chamber, etc.) We assume that the counter is used to record certain primary events, e.g. disintegration in a source, fission in a reactor etc. We define its efficiency E as the probability for one individual primary event to be recorded. (We thus include in E such factors as solid angle or absorption due to intervening material). The generating function of the counter is therefore

$$C = 1-E+Ex = 1+E\,(x-1) \qquad (17)$$

If n primary events occur, what the counter does is to repeat n times the experiment described by the generating function C , and the outcome is

[redacted]

therefore described by the generating function C^n.

If the number of primary events is itself subject to fluctuations and associated with a generating function P, then we have to let P operate on C in order to get the generating function of the counting experiment. If, in particular, $P = e^{N(x-1)}$ as before, (that is, if the number of primary events has a Poisson distribution) we get

$$P(C) = e^{N(C-1)} = e^{N(1+E(x-1)-1)} = e^{NE(x-1)}$$

Hence the number of pulses per second again follows a Poisson distribution; this is obvious since the pulses are independent events just as much as the primary events (the disintegrations in the source) only their average rate per second is not N but EN. However, the formalism can be used in cases where the primary events do not follow a Poisson distribution. We shall very soon encounter such cases.

2.5-3 Moments of the Distribution

The problems where one can actually determine the generating function relating to a certain experiment are not too numerous. However, it is often possible to calculate the first few moments of the distribution and in many cases this is all that is needed.

By differentiating the generating function once we get $\frac{\partial F}{\partial x} = f_1 + 2f_2x + 3f_3x^2 + \ldots$ If we now set $x = 1$, we obtain $\sum_0^\infty n \cdot f_n = \bar{n}$, the mean value of the result. Similarly, by differentiating twice and setting $x = 1$, we get

$$\left(\frac{\partial^2 F}{\partial x^2}\right)_{x=1} = \overline{n(n-1)} = \overline{n^2} - \bar{n}$$

and generally

$$\left(\frac{\partial^k F}{\partial x^k}\right)_{x=1} = \overline{n(n-1)(n-2)\ldots(n-k+1)}$$

Of course $F_{x=1}$ is always 1 since it is the sum of the probabilities of all possible results.

The first derivative in general merely offers a check since the mean value of the result can be calculated from a simpler type of theory which does not involve generating functions. The second derivative, however, gives a measure of the fluctuations. Higher derivatives can be calculated, usually with not much greater effort, to get more detailed information, but we shall confine our attention to the second derivative in what follows.

For the sake of readability, we shall use the letters a and b :

$$a = F'_{(x=1)} = \bar{n} ; \quad b = F''_{(x=1)} = \overline{n(n-1)} = \overline{n^2} - \bar{n}$$

Quantities which are often wanted are the mean square $\overline{n^2} = a + b$ and the mean square deviation $m = \overline{(n-\bar{n})^2} = \overline{n^2} - \bar{n}^2 = b+a-a^2$. The latter has the important property of being additive, if the results of two independent experiments are added. This can be easily verified. We use subscripts 1 and 2 to refer to the two experiments and the subscript 12 to refer to their sum, and find

$$a_{12} = \left[\frac{d}{dx}(F_1 \cdot F_2)\right]_{x=1} = \left[F_1' F_2 + F_2' F_1\right]_{x=1} = \left[F_1' + F_2'\right]_{x=1} = a_1 + a_2$$

$$b_{12} - a_{12}^2 = \left[\frac{d^2}{dx^2}(F_1 F_2) - \left\{\frac{d}{dx}(F_1 F_2)\right\}^2\right]_{x=1} = b_1 + 2a_1 a_2 + b_2 - (a_1+a_2)^2 = (b_1 - a_1^2) + (b_2 - a_2^2)$$

The first of these two equations merely indicates that the average is additive, which is obvious. By adding the two equations we get $m_{12} = m_1 + m_2$.

If two experiments are combined in cascade, we have $F_{12} = F_1(F_2)$.

One can then similarly verify the following relations:

$$\frac{d}{dx}\left[F_1(F_2)\right] = F_1'(F_2) \cdot F_2' \qquad \text{and for } x = 1: \; a_{1(2)} = a_1 \cdot a_2$$

$$\frac{d^2}{dx^2}\left[F_1(F_2)\right] = F_1''(F_2) \cdot (F_2')^2 + F_1'(F_2) \cdot F_2'' , \qquad \text{and for } x = 1: \; b_{1(2)} = b_1 a_2^2 + a_1 b_2$$

The mean square deviation m is of course no longer additive but follows a relation similar to that for b :

$$m_{1(2)} = m_1 a_2^2 + a_1 m_2$$

Let us again consider some examples. The generating functions for

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independent events if N is their average number, we found to be $P = e^{N(x-1)}$

By differentiating we get

$a = P'_{(x=1)} = Ne^{N(1-1)} = N$ (as it has to be)

$b = P''_{(x=1)} = N^2 e^{N(1-1)} = N^2$; $m=b+a-a^2 = N$ (as is well known)

Another example refers to the counting of non-Poisson events with a counter of efficiency E. The generating function of the primary events be F, that of the counter, $C = 1 + E(x-1)$. If a, b and m refer to the number of primary events, a_R, b_R, and m_R to the number of recorded events, we find

$$a_R = a \cdot E, \quad b_R = b \cdot E^2, \quad m_R = a_R\left(1+E\frac{m-a}{a}\right) = a_R\left(1+E\frac{b-a^2}{a}\right) \qquad (18)$$

Hence, if the deviation from a Poisson distribution is such that $m > a$, then $m_R > a_R$ as well, varying less strongly with $\frac{m}{a}$ as E decreases and not at all when E = 0, as must clearly be the case.

2.5-4 Distribution in Length of Individual Fission Chains.

As a first application to chain reactions, we ask what is the length of an individual fission chain, that is, the total number of fissions resulting from a single parent fission in a subcritical system (including the parent fission). This length will vary from one chain to another and we would like to know how big the variations are. We can obtain an equation by looking at this problem in two different ways and equating what we see. On the one hand, we can regard the question as one whole and associate with it the generating function $T(x) = \sum t_n x^n$ where t_n is the probability that the chain should consist of just n fissions. On the other hand, we can split the process into the following three stages: the parent fission generates some neutrons; some of these cause secondary fissions; each of these is itself the parent of a chain.

The first stage has the generating function $\Phi = \sum \phi_n x^n$ where ϕ_n is the probability of the liberation of just n neutrons. Each of these neutrons independently has a probability K/ν of causing fission and hence the

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number of fissions caused by one of them is governed by the generating function $F = 1 + (K/\nu)(x-1)$. The generating function for the total number of secondary fissions, M, is, therefore, obtained by letting Φ operate on F; $M = \Phi(F)$. The generating function for the number of fissions in each one of the chains which takes its origin from a secondary fission is again T, as defined above. Since M governs the number of these chains, the total number of fissions in all of them is governed by $M(T)$. In this number the original parent fission is not included; if we want to include it we must multiply $M(T)$ with x, the generating function for an experiment for which we know the answer is 1. The resulting expression $x \cdot M(T)$ now refers to the total number of neutrons in the entire chain and must, therefore, be the same as T.

Hence, we get this equation:

$$T = x \cdot M(T) \tag{19}$$

If we assume M to be known it would be possible in principle, but rather difficult[1] to calculate T. However, we can again calculate moments by

(1) Under the special assumption that the number of neutrons emitted in one fission follows a Poisson distribution, T can be worked out explicitly and the result is

$$T = \sum \frac{(Kn)^{n-1} e^{-nK}}{n!} \cdot x^n$$

taking derivatives and setting x=1. (In the formulae which follow, x is always equal 1 even though it is not indicated). Thus we get

$$T = a = 1 + M' \cdot T' \qquad a = 1/(1-M') \tag{20}$$

$$T'' = 2M'T' + M''(T')^2 + M'T''; \quad T'' = (2M'T' + M''(T')^2)/(1-M') = \frac{2M'(1-M') + M''}{(1-M')^3} = b$$

We can calculate the values of M' and M'' since $M = \Phi(F)$; $M' = \Phi' F'$ and $M'' = \Phi''(F')^2 + \Phi' F''$. If we remember that $F = 1 + (x-1)(K/\nu)$ we find (always for x=1) that $F' = k/\nu$ and $F'' = 0$. On the other hand, Φ is simply ν, the mean value of the number n of neutrons emitted in one fission, and Φ' the mean value of $n(n-1)$ which we shall call X_2. Hence $M' = K$, $M'' = K^2(X_2/\nu^2)$. (21)

Introducing this in equations (20) and (21) we get

$$a = \frac{1}{1-K} \quad ; \quad b = \frac{K^2(X_2/\nu^2) + 2K(1-K)}{(1-K)^3} \quad ; \quad m = b+a-a^2 = \frac{K^2(X_2/\nu^2)+K(1-K)}{(1-K)^3}$$

The first of these equations is again obvious since the average length of a chain is nothing else but the factor $1/(1-K)$ by which the number of original fissions is multiplied by the chain reaction. The secondequation shows that the fluctuations become very large if K approaches unity, and indicates that measurements of these fluctuations may be used to determine X_2.

In practical reactors one never deals with a single chain (except in bombs). Usually there are many "source fissions", that is, fissions which are not caused by a fission neutron, and usually they come at random. Therefore, if there are S on an average during the time t, the generating function for their number in t is $S = e^{S(x-1)}$ (See Equation (16)). To get the generating function for the total number of fissions in all the chains which originate in the time interval t, we have to let S operate on T which gives us

$$U = e^{S(T-1)}$$

Differentiating (and setting x = 1) we find

$$U' = sT' = S/(1-K); \quad U'' = s^2(T')^2 + sT'' = \frac{S^2}{(1-K)^2} + S\,\frac{K^2(X_2/\nu^2)+2K(1-K)}{(1-K)^3}$$

$$m = U'' + U' - (U')^2 = S\,\frac{K^2(X_2/\nu^2)+1-K^2}{(1-K)^3}$$

So far we have spoken about the number of fissions happening in a reactor although we know no way of actually determining the accurate number. If a counter of efficiency E is used to record the fissions then the fluctuations become accessible to direct measurement. By using equation (18) we find for the mean square deviation m_R of the recorded pulses

$$m_R = a_R\left[1 + E\,\frac{K^2(X_2/\nu^2)+2K(1-K)}{(1-K)^2}\right] \qquad (22)$$

where a_R is the mean recorded number.

With decreasing E the fluctuations of the recorded pulses become smaller and they approach a Poisson distribution. If E is made much smaller than $(1-K)^2$ then the fluctuations are "normal"; that is, no greater than if random events were being counted. At $E \approx (1-K)^2$ it takes $1/(1-K)$ chains to give a single pulse each chain giving $1/(1-K)$ events; yet the fluctuations are greater than "normal" hence the pulses are not independent. Since different chains are independent of each other, this must mean that there is an appreciable probability of obtaining two or more pulses from one chain, although the average number of pulses produced by one chain is only 1-K. This gives a vivid indication of the enormous disparity in the lengths of individual chains for close-to-critical systems (where 1-K is small).

2.5-5 Time-Dependent Generating Functions

If the properties of a system change with time, the probability P_k that a certain experiment will give the result k will in general also depend on time, and we have to use a time-dependent generating function $P(t,x) = \sum p_k(t)\, x^k$ to describe the experiment. In some cases of interest to us a differential equation for $P(t,x)$ can be set up, and from it the moments $\left(\frac{\partial P}{\partial x}\right)_{x=1}, \left(\frac{\partial^2 P}{\partial x^2}\right)_{x=1}$ etc. can be found as explicit functions of time.

Let us first consider a chain reactor without a neutron source. We shall use the symbol $P(t,x) \equiv \sum p_k(t)\, x^k$ for the generating function corresponding to the number of neutrons present in the system; that means, $p_k(t)$ is the probability that there should be k neutrons in the system at the time t. To establish a differential equation for $P(t,x)$ we proceed as follows. If at the beginning of the small time interval dt, one neutron is present, what is the generating function describing the number of neutrons at the end of this interval? The following things may happen during dt:

(1) The same neutron may still be there, (probability $1-dt/\tau$) if τ is the mean life of an individual neutron in the system. Since, in this case,

[redacted]

we find one neutron at the end of our interval dt, this case contributes a term $(1-dt/\tau)x$ to our generating function.

(2) The neutron may have caused fission, probability $(K/\nu)(dt/\tau)$. In this case m neutrons are produced with the probability ϕ_m, so our term is $(K/\nu)(dt/\tau)\sum \phi_m x^m$ or $(K/\nu)(dt/\tau)\Phi(x)$.

(3) The neutron may have leaked out or may have been absorbed, probability $(1-K/\nu)(dt/\tau)$; result: no neutron at the end of dt, hence the term in our generating function is just $(1-K/\nu)(dt/\tau)$.

Hence, the complete generating function which describes the number of neutrons at the end of the interval dt is

$$(1-\frac{dt}{\tau})\,x + (1-\frac{K}{\nu})\frac{dt}{\tau} + \frac{K}{\nu}\frac{dt}{\tau}\,\Phi(x) \equiv x + g\frac{dt}{\tau}$$

$$g = 1 - \frac{K}{\nu} - x + \frac{K}{\nu}\Phi(x) = 1 - \frac{K}{\nu}(\Phi(x) - 1) - x$$

If now the generating function at the time t is $P(t,x)$, the generating functions at time t + dt is obtained by letting P operate on the generating function $x + g\,dt/\tau$ which describes what happens in the interval dt.

Thus we get

$$P(t+dt,x) = P(t,\, x+g\frac{dt}{\tau})$$

$$P(t,x) + dt\frac{\partial}{\partial t}P(t,x) = P(t,x) + g\frac{dt}{\tau}\frac{\partial}{\partial x}P(t,x) \qquad (23)$$

hence

$$\tau\frac{\partial P}{\partial t} = g\frac{\partial P}{\partial x} \qquad (24)$$

The same equation is derived by R. Feynman in Volume VI of the Technical Series, by a slightly different procedure and his notation is slightly different.

We can again obtain the moments of the distribution by differentiating Equation (23) with respect to x, and then putting x = 1. By differentiating $P(x,t)$ once we get, since $\left(\frac{\partial P}{\partial x}\right)_{x=1} = \bar{n} \equiv a(t)$, the average number of neutrons present

[redacted]

at the time t, and remembering that $\left(\frac{\partial \Psi}{\partial x}\right)_{x=1} = \nu$, $g(x=1) = 0$, and differentiate (24) with respect to x,

$$\tau \frac{da}{dt} = a \frac{dg}{dx} = a(K-1), \quad \text{or} \quad a = c_1 e^{\frac{K-1}{\tau} t} = c_1 e^{\alpha t} \tag{25}$$

the familiar exponential increase (or decrease if $K<1$) of the number of neutrons with time in a chain-reactor.

By differentiating Equation (23) twice we find how $b \equiv \overline{n(n-1)} = \left(\frac{\partial^2 P}{\partial x^2}\right)_{x=1}$ with time:

$$\tau \frac{db}{dt} = 2b \frac{dg}{dx} + a \frac{d^2 g}{dx^2} = 2b(K-1) + aK(X_2/\nu)$$

This equation can be solved by making use of Equation (25) and the result is:

$$b = e^{2\alpha t}\left(-c_1 \frac{\beta}{\alpha} e^{-\alpha t} + c_2\right)$$

where $\alpha = \frac{1}{\tau}\left(\frac{dg}{dx}\right)_{x=1} = \frac{K-1}{\tau}$; $\beta = \frac{1}{\tau}\left(\frac{d^2 g}{dx^2}\right)_{x=1} = \frac{KX_2}{\tau\nu}$

If we start the reaction by inserting one neutron at the time $t = 0$, our boundary conditions are that

$$a_{t=0} = 1 ; \quad b_{t=0} = 0$$

These conditions serve to determine the integration constants c_1 and c_2, and our solutions become

$$a = e^{\alpha t} \quad \text{and} \quad b = \left(\frac{\beta}{\alpha}\right)\left(e^{2\alpha t} - e^{\alpha t}\right)$$

For a divergent chain ($\alpha > 0$) both a and b become very large for large t and the second term in b becomes quickly negligible so that b grows proportional to a^2. This is plausible since the fluctuations in a divergent chain arise at the beginning, when the numbers involved are still small.

In most problems of practical interest there is a source of neutrons in the system, liberating S neutrons per second, on an average. If we want to include such a source we have to change our Equation (23) into

$$P(t+dt, x) = P\left(t, x + g\frac{dt}{\tau}\right)\left(1 + S\,dt\right)^{[x-1]} \tag{26}$$

since $1 - S\,dt + S\,dt \cdot x$ is the generating function representing the arrival of a source neutron in the interval dt, the probability of which is independent of the number of neutrons already present. Our Equation (24) then changes into

$$\tau \frac{\partial P}{\partial t} = g \frac{\partial P}{\partial x} + P \cdot S\tau(x-1) \qquad (27)$$

This equation, too, is derived in Feynman's chapter in Volume VI (If our system contains a source of F fissions per second, rather than S neutrons, we merely have to replace the term $S\tau(x-1)$ in Equation (27) by $F\tau(\Phi - 1)$.

By forming the first and second derivatives of Equation 27 we get

$$\tau \frac{da}{dt} = a(K-1) + S\tau \qquad \text{or} \qquad \frac{da}{dt} = \alpha a + S$$

$$\text{and} \qquad \frac{db}{dt} = 2\alpha b + \beta a + 2Sa$$

If $\alpha < 0$ (i.e., in a subcritical system) a static solution will establish itself after awhile, when $da/dt = db/dt = 0$. In this case we have

$$a = \frac{S}{-\alpha} = \frac{S}{|\alpha|} = \frac{S\tau}{1-K}$$

$$b = (2S+\beta)\frac{a}{2|\alpha|} = a^2\left(\frac{1}{S\tau} \cdot \frac{KX_2}{2\nu} + 1\right) \qquad (28)$$

2.5-6 Fluctuations of a Dragon Pulse

Equation (28) has been derived for a subcritical system. However, if a system, after having run at a K slightly below 1 for some time, is slowly made slightly supercritical, one may expect Equation (28) still to hold. Both a and b will, of course, go up in an exponential fashion, but the ratio b/a^2 may

be expected to stay nearly constant at about $1+X_2/2\nu s\tau$, since K remains close to 1. In a dragon experiment (see Chapter 9) K is first kept slightly below 1 for a comparatively long time; it is then slowly raised above 1 (slowly if time is measured in terms of τ) and finally lowered again below 1. As soon as K exceeds 1 the neutrons begin to multiply very rapidly and soon their number is so large that b/a^2 may be regarded as accurately constant (the number is so large that no further increase of the fluctuations can take place). This means that the number of neutrons present, at a certain time after the system has become supercritical, fluctuates from one experiment to another, but the "shape" of the pulse (neutron level plotted against time) is always the same. Hence, the relative fluctuation of the (integrated) pulse size should be the same as the relative fluctuation of the neutron level, as indicated by Equation (28).

A more suitable measure than b/a^2 for the relative fluctuation is the relative mean square deviation $\overline{(n-\bar{n})^2}/\bar{n}^2 = (b+a-a^2)/a^2 = (b/a^2)+(1/a)-1$, where the term $1/a$ can be neglected since we are only interested in cases where a is very large.

So we get for the relative mean square deviation of a dragon pulse

$$\overline{(J-\bar{J})^2}/\bar{J}^2 = \frac{1}{st}\,\frac{X_2}{2\nu} \qquad (29)$$

where J, the intensity of the pulse, may be measured in arbitrary units. Since from dragon measurements one should be able to get a fairly accurate figure for τ in the dragon assembly, it might be possible to use a dragon, combined with a calibrated mock fission source to determine X_2 from observations of how the pulse size varies from one drop to another. Some observations of that sort are reported in Chapter 9, but no definite conclusion can be drawn from them.

2.5-7 Fluctuations in a Stationary Reactor

Equation (28) tells us how the instantaneous number of neutrons in a reactor fluctuates. To measure these fluctuations one would have to use a detector

which is made sensitive for periods of time $T \ll \tau$ so that the neutron level does not change during any one of these "gates". Such an experiment would be very hard to design in such a way that it gives relevant results.

If the gate time T is made longer the experiment becomes easier to perform, but more difficult to interpret. The mean value A for the number of fissions N happening during T can be formed by simply integrating the fission rate over the time T, but this is not so for B, the mean value of N(N - 1) because of the "memory" of the system. In general, this memory has very complicated effects because of the presence of delayed neutrons of several different periods. However, if T is small compared to these periods one can treat the delayed neutrons as a (slowly varying) source of random neutrons and include them in the value of S. In this case only the mean life τ_o of the prompt neutrons enters into the calculation. Measurements of this kind have been carried out (see Chapter 5) and can be used to get the value of τ_o and of x_2.

To calculate B, the mean value of N(N - 1), where N is the number of fissions occurring in the time T, we use a generating function of two variables x and y, where x refers (as before) to the number n of neutrons present at the time t while y refers to the "fission score", s at the time t; that is, to the number of fissions which have taken place between time zero and time t. In other words, if our generating function is $P(t, x, y) = \sum p_{ik}(t)\, x^i\, y^k$, then $p_{ik}(t)$ is the probability that at the time t there should be just i neutrons present and that, simultaneously, the fission score should be k. The calculation of mean values goes very much as in the case of one variable.

$$\left(\frac{\partial P}{\partial x}\right)_{x=y=1} = \bar{n} \equiv a \; ; \; \left(\frac{\partial^2 P}{\partial x^2}\right)_{x=y=1} = \overline{n(n-1)} \equiv b$$

$$\left(\frac{\partial P}{\partial y}\right)_{x=y=1} = \bar{s} \equiv A \; ; \; \left(\frac{\partial^2 P}{\partial y^2}\right)_{x=y=1} = \overline{s(s-1)} \equiv B$$

$$\left(\frac{\partial^2 P}{\partial x \partial y}\right)_{x=y=1} = \overline{n \cdot s} \equiv Z$$

The differential equation for P can again be written

$$\tau \frac{\partial P}{\partial t} = g \frac{\partial P}{\partial x} + P \cdot s\tau(x-1) \tag{30}$$

but g is now a function of both x and y

$$g = 1 - \frac{K}{\nu} - x + \frac{K}{\nu} y \Phi(x) \tag{31}$$

By forming the first and second derivatives of Equation (30) (including the mixed derivative) and setting x=y=1 one gets 5 equations from which A and B (in which we are mainly interested) can be calculated. Since we are considering a stationary system we assume $da/dt = db/dt = 0$. The fission score s and the quantities A, B, and Z relating to it keep growing with time and are obtained by integration, the integration constants are fixed by the condition that A, B, and Z must be zero at the time zero. The final results are

$$A = \frac{K S}{\nu(1-K)} t \; ; \quad B = \frac{K S}{\nu} \cdot \frac{X_2(K^2/\nu^2) + 2K(1-K)}{(1-K)^3} \cdot \left(t - \frac{1-e^{-|\alpha| t}}{|\alpha|} \right) \tag{32}$$

where $\alpha = \frac{K-1}{\tau} < 0$, hence $|\alpha| = \frac{1-K}{\tau}$

From Equation (32) we can calculate the fluctuations of the number of fissions recorded by a detector of efficiency E in periods of the duration T each. If a_R denotes the mean number of fissions counted in a period T, and m_R the mean square deviation of that number, we get, with the help of Equation (18)

$$m_R = a_R \left(1 + E \frac{B(T) - A^2(T)}{A(T)} \right) = a_R \left(1 + E \frac{X_2(K^2/\nu^2) + 2K(1-K)}{(1-K)^2} \left[1 - \frac{1-e^{-|\alpha| T}}{|\alpha| T} \right] \right) \tag{33}$$

The term in square brackets is for small values of T proportional to T. Hence, the deviation from a random error (for which $m_R = a_R$) rises linear with the length T of the "gate", until T becomes of the order $1/|\alpha|$. At greater values of T the square bracket approaches 1, and Equation (33) in the limit of large T becomes identical with Equation (22). This must be so since for large values of T the contribution of those chains which do not entirely fall within T becomes

negligible, and Equation (22) was derived under the (tacit) assumption that all chains which start in a given interval also end in it.

Thus, by measuring the dependance of the fluctuations on the gate width T, one can get information on $|\alpha|$, and from the asymptotic value (for large T) one can draw conclusions regarding X_2. Measurements of that kind are described in Chapters 4 and 8.

Strictly speaking, measurements of this kind should be interpreted on the basis of calculations which take the existence of several delayed neutron periods into account. Such calculations were attempted, but the resulting formulae are so clumsy that no attempt was made to evaluate them. In an approximate way the delayed neutrons may be regarded as part of the source. They differ from a true source in that they show greater than normal fluctuations, but these are comparatively slow. In the measurements described in Chapter 8, the fluctuations due to the varying length of prompt chains were computed from a number of short runs during which the delayed neutron intensity did not vary much. In this case Equation (33) may be used; K then means the multiplication factor for prompt neutrons alone (which we previously called K_p) and $|\alpha|$ is the time constant with which the prompt chains decay.

2.5-8 The Rossi Experiment

B. Rossi proposed a method for measuring α which depends on the existence of fluctuations and is, therefore, treated in this section[2]. The method of

(2) Since in this section α (as defined before) is always negative we shall use the letter α to denote $|\alpha|$, in contrast to the previous sections.

generating functions does not lend itself naturally to the treatment of this problem and I shall give an approximate calculation which was given by F. DeHoffman in LA-101. A more accurate calculation is given by R. Feynman in his chapter on fluctuations in Volume VI of the Technical Series. The

experimental method is described in Chapter 8 of this Volume. It determines the probability $P(t)$ that a certain time t after a pulse, recorded by counter No. 1 a pulse should be recorded by counter No. 2; the time t can be varied and the variation of this probability on t determined.

In order to calculate the probability $P(t)$ we consider first "related pairs", where the two pulses recorded by counters No. 1 and No. 2 are due to neutrons belonging to the same fission chain. In this case the two pulses have a "nearest common ancestor"; that is, a fission such that two different neutrons emitted in it caused the two sub-chains which, respectively, gave rise to the two pulses. We denote by t_1 and t_2 the times at which the two pulses occur, and by t_o the time of the "nearest common ancestor". The probability for this ancestor to happen in the time interval $t_o \pm \frac{1}{2} dt_o$ is $N\, dt_o$ if N is the fission rate in the system. The probability that just m neutrons are emitted in this fission is ϕ_m. The expected number of neutrons present in the system at the time t_1 is $m \cdot e^{-\alpha(t_1 - t_o)}$, and the probability that one of them should cause a pulse in counter No. 1 in the interval dt_1 is $E_1\, dt_1/\tau\nu$

The expected number of neutrons present at time t_2 is $(m-1)e^{-\alpha(t_2 - t)}$; here we write (m-1) instead of m because we know that the second pulse descends from one of the other m-1 neutrons emitted in the ancestor fission and not from the same neutron from which the first pulse descends; otherwise our fission would not be the "nearest common ancestor." The probability that one of these $(m-1)e^{-\alpha(t_2 - t)}$ neutrons should produce a pulse in counter No. 2 in the interval dt_2, is $E_2 dt_2/\tau\nu$. E_1 and E_2 are the efficiencies of the two counters, measured in counts per fission.

By multiplying all these factors we get

$$N dt_o \phi_m m e^{-\alpha(t_1 - t)} \frac{E_1 dt_1}{\tau\nu} (m-1) e^{-\alpha(t_2 - t)} \frac{E_2 dt_2}{\tau\nu}$$

$$= N \frac{E_1 E_2}{\tau^2 \nu^2} \phi_m m(m-1) e^{-\alpha(t_1 + t_2)} dt_1 dt_2 e^{2\alpha t_o} dt_o$$

This expression has to be integrated over all possible values of t_o ; that is, from $-\infty$ to t_1 if $t_1 < t_2$; , it must also be summed over all values of m. This gives

$$dt_1 dt_2 N \frac{E_1 E_2}{\tau^2 \gamma^2} X_2 e^{-\alpha(t_1+t_2)} \int_{-\infty}^{t_1} e^{2\alpha t_o} dt_o = N \frac{E_1 E_2}{2\alpha\tau^2} \frac{X_2}{\gamma^2} e^{-\alpha(t_2-t_1)} dt_1 dt_2$$

To this has to be added the number of "accidental" pairs; that is, of those where the two pulses recorded are caused by neutrons originating from different fission chains. Different chains are statistically independent and hence the number of these pairs is simply

$$N E_1 dt_1 \; N E_2 dt_2 = N^2 E_1 E_2 dt_1 dt_2$$

The total number of pairs, related and accidental, has to be divided by $N E_1 dt_1$, the probability of a pulse in counter No. 1 during dt , in order to get p(t) , where $t = t_2 - t_1$.

$$p(t)dt = \frac{N^2 E_1 E_2 dt_1 dt + N E_1 E_2 \frac{X_2}{2\alpha\tau^2\gamma^2} e^{-\alpha t} dt_1 dt}{N E_1 dt} = \left(N E_2 + E_2 \frac{X_2}{2\alpha\tau^2\gamma^2} e^{-\alpha t}\right) dt$$

Now the term NE_2 is simply the counting rate C in the second counter, and if we replace α by $(1-K_p)/\tau$ we get.

$$P(t)dt = C\,dt + \frac{E_2 X_2}{2\gamma^2\tau(1-K_p)} e^{-\frac{1-K_p}{\tau}t} dt$$

This is essentially identical with Equation (3) in Chapter 8 , where further discussion of these equations can be found together with a description of the experiments to which they refer.

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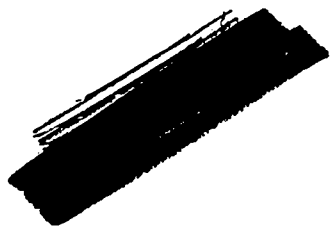

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