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A DETERMINATION OF THE DISTRIBUTION OF NEUTRONS ESCAPING A SEMI-INFINITE MEDIUM OF DEUTERIUM, USING ELASTIC AND INELASTIC DIFFERENTIAL CROSS SECTIONS

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ABSTRACT

A source of 14.1 Mev neutrons is assumed to be uniformly distributed and isotropically directed in a semi-infinite medium of deuterium at uniform density, subject to the known laws of elastic and inelastic scattering. The distribution, in energy and angle normal to the surface, of those neutrons escaping the medium with energies above 6 Mev, is determined by Monte Carlo methods.

The dependence of the escape distribution on the depth of the source below the surface is also given, so that the results may be applied to a medium with a depth-dependent burning rate.

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1. The Neutron in Flight

The medium is considered to occupy the lower half of space. The geometric parameters characterizing a neutron in flight are taken to be the z-coordinate on an axis perpendicular to the surface and the angle η from the vertical to the line of flight. The square root v of the neutron energy measured in units of 100 Mev and the "weight" w complete the list of neutron parameters.

Source neutrons, having $w = 1$ and $v^2 = .141$, are uniformly distributed in a layer of width $\lambda = 24.596$ cm parallel to the surface. Escape distributions are obtained for such a source layer placed at various depths $(0, \lambda, 2\lambda, 3\lambda, 4\lambda)$ below the surface.

The source is assumed to be isotropic in direction; thus $\cos \eta = 0.5 - r$ is the Monte Carlo formula[1] for η, r representing here, and throughout the report, a random number equi-distributed on the interval $0 \leqq r \leqq 1$.

For the sake of definiteness, the deuterium medium is assumed to have normal numerical density $N_o = 5.08212 \times 10^{22}$ atoms per cubic centimeter. The total cross section $\tilde{\sigma}_t$ (in barns) is regarded[2] as being defined by the formula $\tilde{\sigma}_t = .3004/v$. Thus we have

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[1] Cf. LA-1583, LA-1592 for general remarks on the Monte Carlo method.

[2] All cross section data used in this report were supplied by J. L. Gammel. Cf. (1) R. S. Christian and J. L. Gammel, "Elastic Scattering of Protons and Neutrons by Deuterons," Phys. Rev. 91, 100 (1953). (2) Cf. also R. M. Frank and J. L. Gammel, "Inelastic Scattering of Protons and Neutrons by Deuterons," to appear in Phys. Rev.

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$y = (-\ln r)/10^{-24}\tilde{\sigma}_t N_o = -65.502v(\ln r)$ for the distance from the point of departure to the point of collision C, the extent of the medium permitting. Comparison of the z-coordinate of the point C with that of the surface distinguishes between escape and collision within the medium.

In the event of escape, the neutron is classified with respect to its energy group h and the angular zone j in which falls its normal angle of escape. Its weight w is then tallied in a counter which records the number of neutrons of type (h,j) escaping.

The polynomial

$$\tilde{\sigma}_i = -.08445331 + 2.80169574v^2 - 8.37914536v^4$$

is considered to define the total inelastic cross section in barns. Hence, in the event of collision, comparison of a new random number r with the ratio $x = \tilde{\sigma}_i/\tilde{\sigma}_t$ serves to determine the nature of the collision as inelastic or elastic according as $r < x$ or $r > x$, respectively.

2. Elastic Collision

An elastic collision suffered by a neutron with energy v^2 has the property that specification of the angle ψ' of scattering from the line of flight in the center of mass system uniquely determines the energy v'^2 of the scattered neutron according to the formula

$$v'^2/v^2 = \tfrac{1}{2}(1 + s) + \tfrac{1}{2}(1 - s)\cos\psi'$$

where $s = (A - 1)^2/(A + 1)^2$, $A = m_2/m_1$, $m_2 = 2.01473$, and $m_1 = 1.00893$.

Since we are only interested in neutrons escaping with energies above 6 Mev, we may regard any neutron which scatters at an angle ψ'

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greater than a certain critical angle ψ'_c (that for which the resulting energy $v'^2 = .06$) as effectively lost, or, as we shall say, elastically absorbed. At this point we use a semi-deterministic method. Instead of selecting ψ' at random from the proper distribution on the <u>full</u> range and losing the neutron in the event that $\psi' > \psi'_c$, we compute in advance the probability

$$e = \int_{\mu'_c}^{1} \tilde{\sigma}_e(v^2,\mu')d\mu' \Big/ \int_{-1}^{1} \tilde{\sigma}_e(v^2,\mu')d\mu'$$

of scattering at an angle $\psi' \leqq \psi'_c$, tally the weight $(1 - e)w$ in a counter for the number of neutrons lost to elastic absorption, and assign to the neutron the new weight $w' = ew$, this amount being regarded as elastically scattered at an angle less than critical.

In the formula for e, μ' denotes $\cos \psi'$, $\tilde{\sigma}_e(v^2,\mu')$ is the differential cross section (barns per steradian) for elastic scattering, and $\mu'_c = \cos \psi'_c = 4(-.31221 + .0337326v^{-2})$, the latter resulting from the expression for v'^2/v^2 when $v'^2 = .06$. The function e can be fitted by the polynomial

$$e = 2^{20}v(-A_o + A_1v^2 - A_2v^4 + A_3v^6 - A_4v^8)$$

where $A_o = .00003\ 20690$, $A_1 = .00118\ 61091$, $A_2 = .01562\ 88950$, $A_3 = .09165\ 31253$, and $A_4 = .20215\ 48272$.

The angle of scattering can now be determined by the formula

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$$r = \int_{\mu_c'}^{\mu'} \tilde{\sigma}_e(v^2, \mu')d\mu' \Big/ \int_{\mu_c'}^{1} \tilde{\sigma}_e(v^2, \mu')d\mu'$$

where r is random. For this purpose, we use a fit for $\tilde{\sigma}_e$, on the range $\mu_c' \leqq \mu' \leqq 1$, of the form $\tilde{\sigma}_e(v^2, \mu') = K(v^2)\left[a(v^2) + b(v^2)\mu' + c(v^2)\mu'^2\right]$, where a, b, c are suitable quadratic functions of v^2, and $K(v^2)$ is immaterial for the present application.

In this way we obtain the equation $A\nu + B\nu^2 + C\nu^3 - D = 0$ for the determination of $\nu = \mu'/2$, where

$$A = .00384\,60637 + .07959\,38700v^2 - .36363\,60000v^4$$

$$B = -\,.00267\,08480 + .25414\,20300v^2 - .86687\,70000v^4$$

$$C = -\,.00054\,62883 + .16717\,14660v^2 - .15952\,39996v^4$$

$$D = r(A/2 + B/4 + C/8) + (1 - r)\,(A\nu_c + B\nu_c^2 + C\nu_c^3)$$

$$\nu_c = \mu_c'/2.$$

Inspection of the above cubic in ν shows it to be monotone increasing and concave upward on the range $\nu_c \leqq \nu \leqq .5$ for all energies on the relevant range $.06 \leqq v^2 \leqq .141$. Newton's method is therefore appropriate for the evaluation of ν, a convenient initial approximation being $\nu_0 = .5$.

The formulas

$$v' = v(.55533 + .88935\nu)^{1/2}$$

$$\cos\psi = 2(.05 + .19969\nu) \,/\, (.049876 + .079876\nu)^{1/2}$$

[redacted]

are used for the computation of the energy v'^2 of the scattered neutron and the laboratory angle ψ of scattering from the original line of flight.

Consideration of Fig. 1 shows that the angular parameter η' for the scattered neutron may be obtained from the equation

$$\cos \eta' = \cos \psi \cos \eta - \sin \psi \sin \eta \cos \beta$$

where $\beta = \pi r$, and r is random.

Thus the weight w', energy v'^2, and angle η' of the elastically scattered neutron are determined. Its z-coordinate z' is, of course, that of the point C of collision.

3. Inelastic Collision

By an inelastic collision is meant a disruption of the deuterium nucleus by the impinging neutron, the end products being two neutrons and a proton. It is shown in Section 5 that, for an incident neutron having an energy E exceeding twice the binding energy b (2.228 Mev) of the deuteron, the particles resulting from the reaction may come off at any laboratory angle ψ with the original line of flight. Moreover, for a particular ψ, the possible energies range from zero to an upper bound $\overline{E} = \overline{E}(E, \psi)$, where $(\overline{E}E)^{1/2} \cos \psi = \frac{3}{2}\overline{E} - \frac{1}{2}E + b$, or, in our notation, $\overline{v}\, v \cos \psi = \frac{3}{2}\overline{v}^2 - \frac{1}{2}v^2 + .02228$.

For a fixed initial energy v^2, the greatest $\overline{v}$ occurs at $\psi = 0$, and this $\overline{v} = \overline{v}(v,0)$ is a decreasing function of v. It is therefore clear that inelastic collisions induced by neutrons with energies less than

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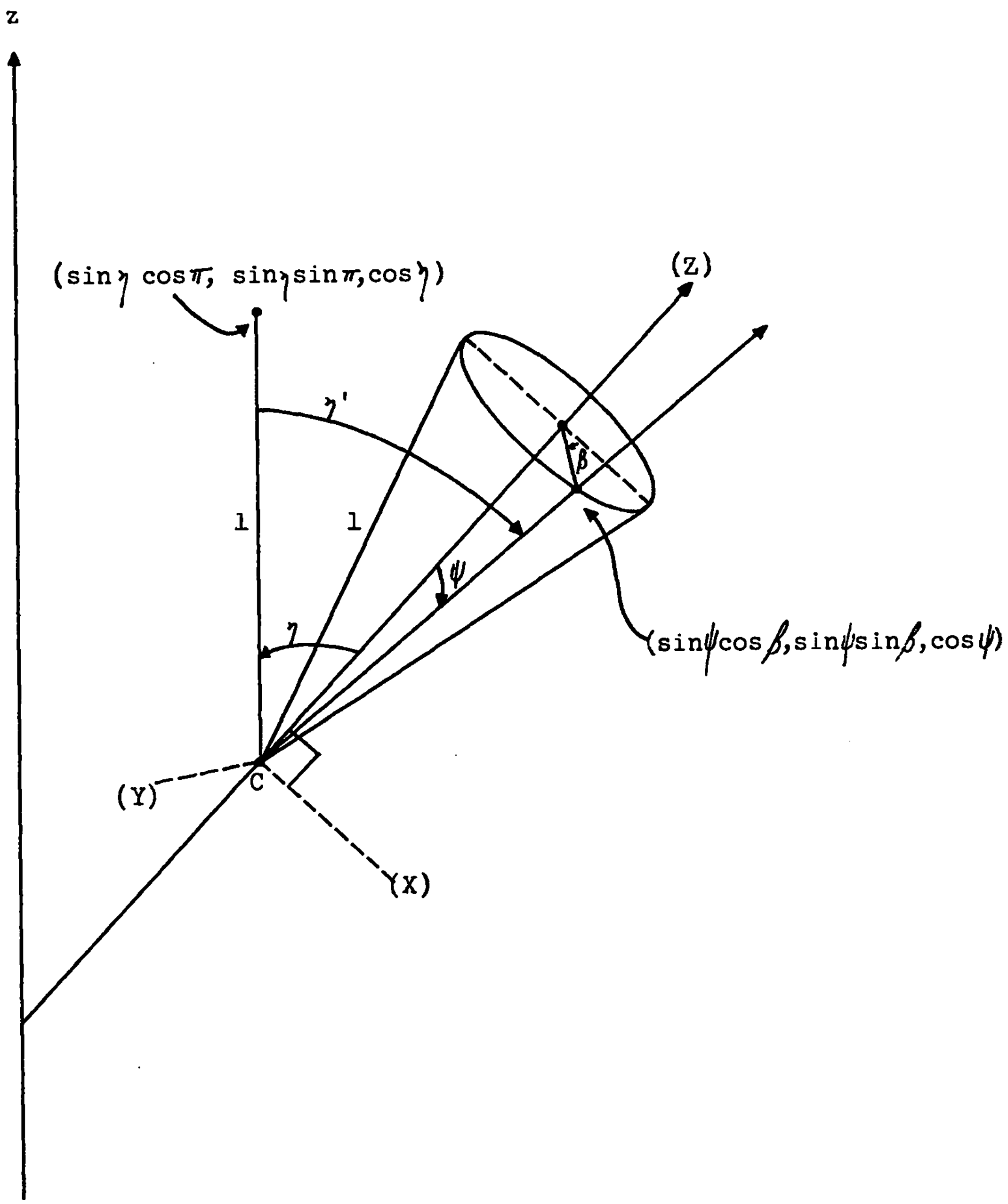

Fig. 1 Geometry of scattering.

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that for which $\bar{v}^2(v,0) = .06$ must result in the production of neutrons below 6 Mev. Hence, a neutron entering an inelastic collision with energy below 8.323 Mev is effectively lost, and we tally its weight w in a counter reserved for total "inelastic absorption."

If a neutron with energy $v^2 > .08323$ induces an inelastic collision, we are assured that $\bar{v}^2(v,0)$ exceeds .06. However, as ψ increases, $\bar{v}(v,\psi)$ decreases until it attains the value $(.06)^{1/2}$. For a given v, that angle ψ for which this occurs is called the critical angle $\psi_c = \psi_c(v)$. Neutrons resulting from inelastic collision and travelling at angles $\psi > \psi_c$ necessarily have energies below 6 Mev. These remarks are illustrated in Fig. 2.

Setting $\bar{v}^2 = .06$ in the maximum energy formula yields $\cos \psi_c = 4v^{-1}(.114595 - .51031v^2)$ for the critical angle corresponding to the incident energy v^2.

For inelastic collision, a differential cross section $\tilde{\tau}(E,E',\psi)$ (barns per Mev per steradian) is defined in such a way that $10^{-24}\tilde{\tau}(E,E',\psi)\, N_o\, \Delta x\, S(E)$ is the expected number of neutrons of energy E' and angle ψ', per Mev per steradian, resulting from inelastic collisions of a source of S(E) neutrons of energy E in traversing a medium of numerical density N_o and thickness Δx. Since <u>two</u> neutrons result per inelastic collision, manifestly, for $E > 2b$

$$2\pi \int_{-1}^{1} \int_{o}^{\bar{E}(E,\psi)} \tilde{\tau}(E,E',\psi)\, dE'\, d(\cos\psi) = 2\,\tilde{\sigma}_i(E)$$

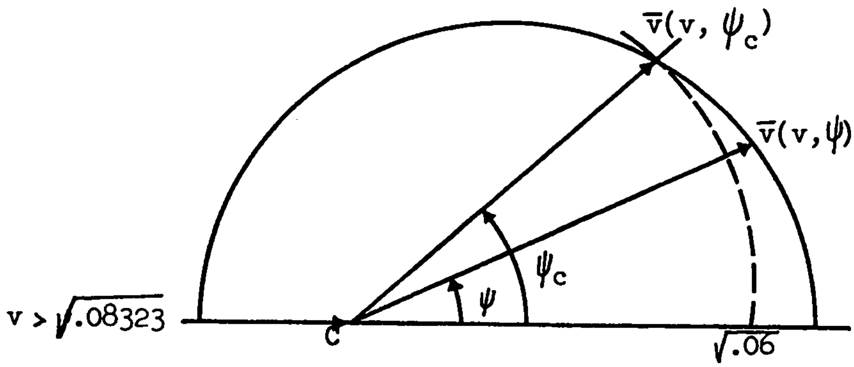

Fig. 2 The critical angle relations.

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where $\tilde{\sigma}_i(E)$ is the ordinary total inelastic cross section.

For E = 14.1, and E = 9.66, graphs were prepared showing the values of

$$\tilde{\tau}(E,\psi) \equiv \int_6^{\bar{E}(E,\psi)} \tilde{\tau}(E,E',\psi)dE'$$

plotted against $\mu = \cos\psi$ on the range $\mu_c = \cos\psi_c \leq \mu \leq 1$. An excellent fit for these two curves is provided by the formula

$$\tilde{\tau}(E,\psi) = 10^{-3}\left[23.82 + \frac{E - 9.66}{14.1 - 9.66}(3.58)\right](\mu - \mu_c)\Big/(1.391 - \mu)^2$$

in case E = 14.1 or 9.66, and the formula is considered to define the function $\tilde{\tau}(E,\psi)$ for $8.323 \leq E \leq 14.1$ on the corresponding range $\mu_c(E) \leq \mu \leq 1$.

Using the notation $\nu = \frac{1}{2}\cos\psi$, $\nu_c = \frac{1}{2}\cos\psi_c$, we obtain

$$\tilde{\tau}(E,\psi) = .05(.16031 + .80631\nu^2)(\nu - \nu_c)/(.6955 - \nu)^2$$

It is clear from the definition of $\tilde{\tau}$ that the number i of neutrons above 6 Mev produced per inelastic collision is

$$i = 2\pi\int_{\mu_c}^{1}\tilde{\tau}(E,\psi)d(\cos\psi)\Big/\tilde{\sigma}_i(E)$$

Using the $\tilde{\tau}$ and $\tilde{\sigma}_i$ formulas, we obtain

$$i = (A_5 + A_6 v^2)(L + U)/(-A_7 + A_8 v^2 - A_9 v^4)$$

where $A_5 = .050363$, $A_6 = .25331$, $A_7 = .0084453$, $A_8 = .28017$, $A_9 = .83791$, and

[redacted]

[redacted]

$$L = .2 \ \ln \left(\frac{.1955}{.6955 - \nu_c} \right)$$

$$U = 2(.25575 - .51151 \, \nu_c)$$

For the energy range within which we operate, i is less than unity. When a neutron of weight w and energy v^2 above .08323 collides inelastically, (1 - i)w is added to the total "inelastic absorption" counter, and w' = iw is the weight assigned to the inelastically scattered neutron. Moreover, the appropriate random number formula for assigning the laboratory angle ψ of scattering for the neutron of weight iw is

$$r = \int_{\mu_c}^{\mu} \tilde{\tau}(E,\psi) d(\cos\psi) \Big/ \int_{\mu_c}^{1} \tilde{\tau}(E,\psi) d(\cos\psi)$$

which leads to the equation

$$.2 \ \ln\left(\frac{a - \nu}{a - \nu_c} \right) + .2\left(\frac{\nu - \nu_c}{a - \nu} \right) - r(L + U) = 0$$

where $\nu = \mu/2$ and a = .6955.

Newton's method is applicable here on the same grounds as before, and is used to evaluate $\nu = \frac{1}{2} \cos \psi$. The value of $\cos \eta'$ for the scattered neutron is now obtained exactly as in the elastic case.

The function $\tau(E,E',\psi)$, considered as a function of E', is rather flat on the interval $6 \leq E' \leq \bar{E}(E,\psi)$, and we use the simple formula $v'^2 = .06 + r(\bar{v}^2 - .06)$ to determine the energy v'^2 of the scattered neutron. Solving the maximal energy equation gives

$$\bar{v} = \frac{2}{3} \left\{ \nu v + \left[(\nu v)^2 + (.75 v^2 - .03342) \right]^{1/2} \right\}$$

[redacted]

4. Results

The following tables show, for various sources, the number S of neutrons per source neutron escaping the surface, and the distribution in energy and angle of neutrons escaping. The lower bounds of the four energy groups h = 1, 2, 3, 4 are taken to be 14, 11 1/3, 8 2/3, and 6 Mev, respectively. The lower bounds of the angular zones j = 1, 2, ... , 9, 10 are defined to be cos η = .9, .8, ... , .1, 0 in that order. In the escape distribution tables, a_{jh} is the fraction of escaping neutrons in energy group h and angular zone j, while $e_h = \sum_j a_{jh}$.

Tables 1, 2, 3, 4, 5 give the results for a neutron source in a layer of thickness λ parallel to the surface and at depths 0, λ, ... , 4λ below it. Here λ(= 24.596 cm) is the total free path for 14.1 Mev neutrons in deuterium.

Table 6 shows the escape distribution for a source uniformly distributed in a layer extending from the surface to a depth of 5λ . The escape distribution for this case may be regarded as essentially that for the semi-infinite source.

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TABLE 1

Source Depth = 0 Width = λ

S = .288050

$$\left[a_{jh}\right]$$

j \ h	1	2	3	4
1	.114547	.014998	.014476	.013720
2	.099546	.014419	.013964	.015218
3	.093745	.015745	.012824	.014920
4	.093413	.016275	.015010	.014196
5	.074232	.016488	.013711	.011353
6	.065553	.016462	.010451	.010048
7	.060824	.013769	.009590	.007724
8	.040081	.011009	.007194	.005771
9	.025573	.005232	.003037	.003463
10	.008086	.001534	.001183	.000616
e_h	.675600	.125931	.101440	.097029

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TABLE 2

Source Depth = λ Width = λ

S = .103708

$\left[a_{jh}\right]$

j \ h	1	2	3	4
1	.099428	.048919	.044051	.046724
2	.089995	.048995	.041347	.036991
3	.065086	.042743	.032848	.032317
4	.054916	.031002	.027818	.027363
5	.043873	.025339	.018724	.018825
6	.024946	.013683	.015214	.012100
7	.007994	.007725	.009235	.008087
8	.002893	.004438	.005221	.005442
9	.000163	.000727	.001482	.001630
10	.000000	.000447	.000968	.000301
e_h	.389294	.224018	.196908	.189780

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TABLE 3

Source Depth = 2λ Width = λ

$S = .040575$

$$\left[a_{jh}\right]$$

j \ h	1	2	3	4
1	.100214	.071224	.061659	.062459
2	.066743	.064204	.059892	.047582
3	.054936	.034003	.036815	.039269
4	.030291	.036081	.031980	.026172
5	.021148	.031112	.018999	.020874
6	.010058	.009583	.014583	.013024
7	.002465	.006104	.006192	.005910
8	0	.002123	.004235	.004833
9	0	.000872	.000893	.002163
10	0	.000439	.000716	.000150
e_h	.285855	.255745	.235964	.222436

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TABLE 4

Source Depth = 3λ Width = λ

$S = .016337$

$$\left[a_{jh}\right]$$

j \ h	1	2	3	4
1	.090532	.100203	.088870	.064570
2	.055088	.072234	.069825	.052711
3	.050746	.048436	.041034	.048913
4	.010960	.036078	.030550	.031220
5	0	.014599	.020039	.015206
6	.003060	.005324	.006176	.011666
7	0	.002769	.007408	.008189
8	0	0	.005588	.005027
9	0	0	0	.002152
10	0	0	0	.000821
e_h	.210386	.279643	.269490	.240481

[redacted]

TABLE 5

Source Depth = 4λ Width = λ

S = .006867

$\left[a_{jh}\right]$

j \ h	1	2	3	4
1	.109216	.087511	.102160	.098415
2	.058248	.040457	.075649	.055957
3	.050967	.037510	.042473	.059885
4	0	.045513	.017127	.018776
5	0	.011015	.019401	.020552
6	0	.014009	.005867	.004257
7	0	.004049	.008704	.007386
8	0	0	0	.003926
9	0	0	0	.000758
10	0	0	0	.000212
e_h	.218431	.240064	.271381	.270124

[redacted]

TABLE 6

Source Depth = 0 Width = 5λ

$$\left[a_{jh}\right]$$

j \ h	1	2	3	4
1	.108887	.031877	.029401	.028675
2	.092233	.029191	.027222	.025016
3	.081577	.025018	.020978	.022946
4	.074661	.022543	.020027	.018940
5	.058811	.019655	.015636	.014179
6	.048134	.014780	.011681	.010751
7	.040500	.011169	.009115	.007657
8	.026003	.008161	.006315	.005558
9	.016208	.003552	.002337	.002842
10	.005115	.001111	.001033	.000505
e_h	.552129	.167057	.143745	.137069

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It may be of interest to note the effect of the inelastic scattering upon the escape distribution. The following Table 7 shows the escape distribution which obtains for the essentially infinite medium under the assumption $\tilde{\sigma}_i(v) \equiv 0$, as compared with Table 6.

TABLE 7

$$\left[a_{jh}\right]$$

j \ h	1	2	3	4
1	.116670	.036243	.030673	.020804
2	.090508	.031691	.027612	.018231
3	.073196	.030968	.024158	.017628
4	.074131	.026365	.021749	.014373
5	.061407	.021450	.016211	.011409
6	.044183	.018015	.012835	.009053
7	.041452	.014823	.008664	.005205
8	.027272	.010203	.006828	.004753
9	.013928	.005565	.002911	.002310
10	.003770	.001219	.001219	.000315
e_h	.546517	.196542	.152860	.104081

[redacted]

[redacted]

5. Elementary Properties of Inelastic Collision

The preliminary statements made in Section 3 may be derived in a very elementary way from energy and momentum considerations.

Consider particles of mass m_i, velocities V_i, total mass $m = \sum m_i$, center of mass velocity V and total momentum P defined by $mV \equiv \sum m_i\ V_i \equiv P$, relative velocities $V_i{}' \equiv V_i - V$ (whence $\sum m_i V_i{}' = 0$), and kinetic energy $k = \sum \frac{1}{2} m_i\ V_i{}^2$ ($= \sum \frac{1}{2} m_i\ V_i{}'^2 + \frac{1}{2} mV^2$).

Let $i = 1, 2$, and assume a momentum-conserving collision resulting in fragmentation of m_2 with conversion of its binding energy b into mass. Let the particles resulting from the collision have masses n_j, total mass $m = \sum n_j$ (ignoring mass increase), velocities W_j, center of mass velocity W, and momentum Q, where $mW \equiv \sum n_j\ W_j \equiv Q$, relative velocities $W_j{}' \equiv W_j - W$, and kinetic energy $\ell = \sum \frac{1}{2} n_j\ W_j{}^2$ ($= \sum \frac{1}{2} n_j\ W_j{}'^2 + \frac{1}{2} mW^2$).

We postulate the conservation equations

(a) $P = Q$

(b) $k = b + \ell$

Conservation of momentum (a) implies $V = W$, and (b) therefore reads $k = b + \frac{1}{2} mV^2 + \sum \frac{1}{2} n_j\ W_j{}'^2$. For such a reaction it is therefore necessary that $k - \frac{1}{2} mV^2 \geqq b$. In the special case $V_2 = 0$, $mV = m_1V_1$, $\frac{1}{2} mV^2 = (m_1/m)k$, and this necessary condition becomes $k \geqq (m/m_2)b$, or $k \geqq \frac{3}{2} b$ for a neutron impinging on deuterium. This serves to explain the fact that $\tilde{\sigma}_i$ drops to zero at 3.342 Mev (cf. Section 1).

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[redacted]

In the limiting case, $k - \frac{1}{2} mV^2 = b$, the resulting relative energy $\sum \frac{1}{2} n_j W_j'^2 = 0$ and all $W_j = V$ necessarily. Conversely, $W_j \equiv V$, $j = 1, 2, \ldots$ is a solution of the equations (a), (b) in this case.

Consider the case $k - \frac{1}{2} mV^2 > b$, and let J denote any <u>one</u> of the resulting particles j, $\sum'$ signifying summation over all $j \neq J$. Define $n' = \sum' n_j$, and let W_R be the velocity of the center of mass of the residual set R, i.e., $n'W_R \equiv \sum' n_j W_j$. Then $0 = \sum n_j W_j' = n_J W_J' + \sum' n_j (W_j - W)$ $= n_J W_J' + n'(W_R - W)$, and

$$\text{(c)} \quad n_J W_J' + n' W_R' = 0$$

where W_R' is the velocity of the center of mass of R relative to that of the whole system. Now define $W_j'' \equiv W_j - W_R = W_j' - W_R'$ for $j \neq J$. From these equations and (c) it follows that $\sum' n_j W_j'' = 0$ and $k - \frac{1}{2} mV^2 - b$ $= (m/n')\left(\frac{1}{2} n_J W_J'^2\right) + \sum' \frac{1}{2} n_j W_j''^2$.

This indicates that the <u>maximal</u> relative energy for the J^{th} particle is $(n'/m)(k - \frac{1}{2} mV^2 - b)$. If $\frac{1}{2} n_J W_J'^2$ equals this maximum, we necessarily have for all $j \neq J$ the equations $W_j'' = 0$ and $W_j = V - (n_J/n')W_J'$. Conversely, if W_J' is a vector of arbitrary direction with $\frac{1}{2} n_J W_J'^2$ $= (n'/m)(k - \frac{1}{2} mV^2 - b)$, then the definitions $W_J \equiv V + W_J'$ and $W_j \equiv V - (n_J/n')W_J'$ for $j \neq J$ provide a solution of the equations (a), (b) and the J^{th} particle attains the maximal relative energy.

[redacted]

More generally, if e is any energy on the range $0 \leqq e \leqq (n'/m)(k - \frac{1}{2} mV^2 - b)$, there exists a solution of the system (a), (b) which assigns to the J^{th} particle the relative energy e. For, let $W_J{}'$ be any vector such that $\frac{1}{2} n_J W_J{}'^2 = e$, take $W_j{}''$ $(j \neq J)$ to be any solution of the system

$$\sum{}' \frac{1}{2} n_j W_j{}''^2 = k - \frac{1}{2} mV^2 - b - (m/n')e$$

$$\sum{}' n_j W_j{}'' = 0$$

and define $W_J = V + W_J{}'$, and $W_j = V - (n_J/n')W_J{}' + W_j{}''$ for $j \neq J$.

Because of the range of relative energy e we have $0 \leqq |W_J{}'| \leqq \rho_J = \left[(2n'/mn_J)(k - \frac{1}{2} mV^2 - b)\right]^{1/2}$ and the range of velocities W_J is therefore the closed sphere of radius ρ_J indicated in Fig. 3.

Three cases arise, depending on the incident kinetic energy.

<u>Case 1</u>. $\rho_J < |V|$. No velocities W_J can result for which $\psi_J >$ arc sin $\rho_J/|V|$. For a given ψ_J less than this bound, the range of possible speeds $|W_J|$ is bounded from zero.

<u>Case 2</u>. $\rho_J = |V|$. The range of possible ψ_J is $0 \leqq \psi_J \leqq \pi/2$. For such ψ_J, the range of $|W_J|$ extends from zero to its upper bound.

<u>Case 3</u>. $\rho_J > |V|$. In this case, ψ_J has range $0 \leqq \psi_J \leqq \pi$, and for a given ψ_J, $|W_J|$ ranges from 0 to its upper bound.

In all cases, the <u>maximum</u> speed $w_J \equiv \max |W_J|$ associated with a possible ψ_J is given by the greater root of the quadratic $\rho_J{}^2 = w_J{}^2 + V^2 - 2w_J|V| \cos \psi_J$. Substituting for ρ_J, and writing e_J for $\frac{1}{2} n_J w_J{}^2 = \max \frac{1}{2} n_J W_J{}^2$

[redacted]

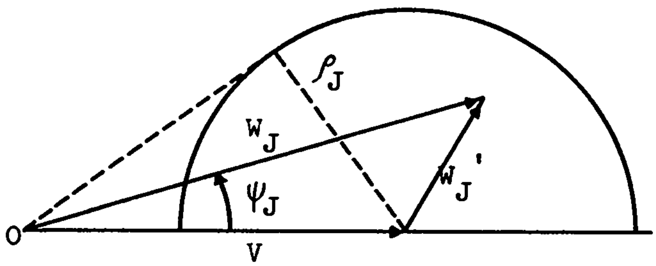

Fig. 3 The range of resultant velocities.

associated with ψ_J, we have

$$\frac{2(mn_J)^{1/2}}{n'} \cdot e_J^{1/2} \left(\tfrac{1}{2} mV^2\right)^{1/2} \cos \psi_J = \frac{m}{n'} \left(e_J + \tfrac{1}{2} mV^2\right) - k + b$$

From the definition of ρ_J it follows that $\rho_J \lesseqqgtr |V|$ according as $k - (m/n') (\tfrac{1}{2} mV^2) \lesseqqgtr b$, and, in the case $V_2 = 0$, according as $(n' - m_1)k/n' \lesseqqgtr b$. Moreover, in case $V_2 = 0$, the above equation takes the simpler form

$$\frac{2(m_1 n_J)^{1/2}}{n'} e_J^{1/2} k^{1/2} \cos \psi_J = \frac{m}{n'} e_J - \frac{(n' - m_1)k}{n'} + b$$

Specifically, in the case of a deuteron shattering the deuterium nucleus, cases 1, 2, and 3 obtain according as k is less than, equal to, or greater than, 2b, respectively, and the maximal energy e_J for direction ψ_J is given by the equation

$$e_J^{1/2} k^{1/2} \cos \psi_J = \tfrac{3}{2} e_J - \tfrac{1}{2} k + b$$

[redacted]