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MIXING OF FRICTIONLESS, INCOMPRESSIBLE SUBSTANCES, I

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PHYSICS

ABSTRACT

The familiar system of equations for the mixing of two frictionless incompressible substances in one dimension is discussed. While it is known that external forces are necessary to initiate such mixing, the equations themselves admit solutions in the absence of such forces. By means of a contact transformation on the variables involved, the general solution of the system under this assumption is determined, and a large subclass discussed in considerable detail. All of these are found to be stable in the sense that the width of the zone in which appreciable mixing occurs tends to zero as time increases. For some such solutions, however, the surface between the two materials actually reestablishes itself, with a portion of one material being detached from its parent substance and sifting into the other. These, therefore, must be regarded as unstable for reasons which lie outside the one-dimensional theory.

MIXING OF FRICTIONLESS, INCOMPRESSIBLE SUBSTANCES, I

1. We assume that in a rectilinear cylinder of infinite cross-section, the region $x \leqslant x_o(t)$ ["t" denotes time] is occupied by an incompressible fluid S_1 with density unity, velocity $u(x,t)$, the region $x \geqslant x_1(t)$ [$x_1 > x_o$] by a second incompressible fluid S_2 with density β, velocity $v(x,t)$. In the region $x_o < x < x_1$, the substances are assumed to exist in a mixed state, with $\alpha(x,t)$ denoting the proportion of S_1 at position x at time t. If the pressure on both substances be taken the same, and frictional forces may be neglected, the equations governing the motion in the region $x_o \leqslant x \leqslant x_1$ are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \beta\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right), \tag{1}$$

$$\frac{\partial \alpha}{\partial t} + \frac{\partial(\alpha u)}{\partial x} = 0, \tag{2}$$

$$\frac{\partial(1-\alpha)}{\partial t} + \frac{\partial((1-\alpha)\ v)}{\partial x} = 0. \tag{3}$$

The equations (2), (3) added yield the first integral

$$\alpha u + (1-\alpha)\ v = F(t). \tag{4}$$

For $\alpha = 1$, this reduces to $u = F(t)$, so the motion of S_1 defines $F(t)$. Evidently, also, by definition of the points $x = x_o(t)$, $x = x_1(t)$, we have the following boundary conditions:

$$\text{At} \quad \alpha = 1, \quad dx = v\,dt, \tag{5}$$

and

$$\text{at} \quad \alpha = 0, \quad dx = u\,dt. \tag{6}$$

The above system has been studied by various people, and in particular, under the assumption $F(t) = t$, a one parameter family of solutions of similarity type have been found by Rosenbluth. We wish here to consider the system with $F(t) = \text{const.}$

2. Before specializing in this manner, we first observe that the transformation $x - \int_0^t F(t)\,dt \longrightarrow x$; $u - F(t) \longrightarrow u$, $v - F(t) \longrightarrow v$ has the following effects: equation (1) becomes

$$\frac{\partial u}{\partial t} + \frac{u \partial u}{\partial x} - \beta\left(\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x}\right) = (\beta - 1)\,F' \quad ; \tag{7}$$

equation (2) is unaffected,

$$\frac{\partial \alpha}{\partial t} + \frac{\partial(\alpha u)}{\partial x} = 0; \tag{2}$$

and equation (4) becomes

$$\alpha u + (1 - \alpha)\,v = 0. \tag{8}$$

The boundary conditions (5), (6) are unaffected.

Now let

$$w = \alpha u,$$

so that, by (8),

$$u = \frac{w}{\alpha}, \qquad v = \frac{-w}{1-\alpha}, \tag{9}$$

then (2) and (7) become

$$\left.\begin{aligned} &\frac{\partial\alpha}{\partial t} + \frac{\partial w}{\partial x} = 0 \\ &\left(\frac{1}{\alpha} + \frac{\beta}{1-\alpha}\right)\frac{\partial w}{\partial t} - 2w\left(\frac{1}{\alpha^2} - \frac{\beta}{(1-\alpha)^2}\right)\frac{\partial\alpha}{\partial t} \\ &\qquad - w^2\left(\frac{1}{\alpha^3} + \frac{\beta}{(1-\alpha)^3}\right)\frac{\partial\alpha}{\partial x} = (\beta - 1)\,F'. \end{aligned}\right\} \tag{10}$$

Our next step is to take α, w, instead of x, t as independent variables. Setting

$$J = \frac{\partial x}{\partial\alpha}\frac{\partial t}{\partial w} - \frac{\partial x}{\partial w}\frac{\partial t}{\partial\alpha}, \tag{11}$$

we have

$$\left.\begin{aligned} &\frac{\partial\alpha}{\partial x} = J^{-1}\frac{\partial t}{\partial w}, \qquad \frac{\partial w}{\partial x} = -J^{-1}\frac{\partial t}{\partial\alpha}, \\ &\frac{\partial\alpha}{\partial t} = -J^{-1}\frac{\partial x}{\partial w}, \qquad \frac{\partial w}{\partial t} = J^{-1}\frac{\partial x}{\partial\alpha}. \end{aligned}\right\} \tag{12}$$

Thus (10) becomes

$$\frac{\partial x}{\partial w} + \frac{\partial t}{\partial\alpha} = 0, \tag{13}$$

and

$$f(\alpha)\,\frac{\partial x}{\partial \alpha} + 2w\, f'(\alpha)\,\frac{\partial t}{\partial \alpha} - \frac{w^2}{2}\, f''(\alpha)\,\frac{\partial t}{\partial w} = (\beta - 1)\, F'\, J, \tag{14}$$

where

$$f(\alpha) = \frac{1}{\alpha} + \frac{\beta}{1-\alpha}\,. \tag{15}$$

Next, we note that (13) permits the introduction of $y(\alpha, w)$ with

$$x = y_\alpha\,, \qquad t = -y_w\,, \tag{16}$$

and that (14) then becomes

$$\begin{aligned} f\, y_{\alpha\alpha} - 2\, w\, f'\, y_{\alpha w} + \frac{w^2}{2}\, f''\, y_{ww} &= \\ &= (\beta - 1)\, G(y_w)(y_{\alpha w}^2 - y_{\alpha\alpha}\, y_{ww}), \end{aligned} \tag{17}$$

where

$$G(-t) = F'(t). \tag{18}$$

This is a second order equation of Monge-Ampere type; it is readily found to be elliptic.

In connection with the change of variable just made, it is of course to be remarked that care must be exercised in interpreting solutions of (17) as solutions of the original problem. This question revolves around the behavior of the Jacobian (11), and since that depends on the function F' or G, one cannot discuss it except with reference to a particular choice of G. For the case which we consider--namely, G = 0 -- it will be considered later.

3. Our first step is to remove the cross-derivative term from the equation; this we do by replacing w by the variable

$$\lambda = w\, f(\alpha) = u - \beta v \, . \tag{19}$$

We have

$$y_{\alpha\alpha} - 2\lambda f' f^{-2} y_{w\alpha} + \frac{1}{2}\lambda^2 f'' f^{-3} y_{ww} = 0 \, ,$$

$$y_{\alpha\alpha} = y_{\alpha\alpha} + 2\lambda f' f^{-1} y_{\lambda\alpha} + \lambda^2 f'^2 f^{-2} y_{\lambda\lambda} + \lambda f'' f^{-1} y_\lambda \, ,$$

$$y_{\alpha w} = \lambda f' y_{\lambda\lambda} + f\, y_{\lambda\alpha} + f' y_\lambda \, ,$$

$$y_{ww} = f^2 y_{\lambda\lambda} \, .$$

Thus, we obtain

$$y_{\alpha\alpha} + \left(\frac{1}{2}\frac{f''}{f} - \frac{f'^2}{f^2}\right)\left(\lambda^2 y_\lambda\right)_\lambda = 0$$

or

$$\alpha(1-\alpha)\bigl(1 + (\beta-1)\alpha\bigr)^2 y_{\alpha\alpha} + \beta(\lambda^2 y_\lambda)_\lambda = 0 \tag{20}$$

We now make the further change of variable

$$\alpha = \frac{1 - r}{(\beta-1)\, r + \beta + 1} \, , \qquad y = \bigl(1 + (\beta-1)\,\alpha\bigr)\, \psi, \tag{21}$$

and note

$$1 - \alpha = \frac{(1 + r)}{(\beta-1) r + \beta + 1} \, , \quad 1 + (\beta-1)\,\alpha = \frac{2\beta}{(\beta-1) r + \beta + 1} . \tag{22}$$

It is then readily found that (20) becomes

$$(1 - r^2)\, \psi_{rr} + (\lambda^2 \psi_\lambda)_\lambda = 0 \tag{23}$$

In addition, we shall need

$$\left.\begin{aligned} t &= \frac{-4\beta}{1-r^2}\,\psi_\lambda \\ x &= \frac{(1-r)^2-\beta(1+r)^2}{1-r^2}\,\lambda\psi_\lambda - \left((\beta-1)\,r+\beta+1\right)\psi_r+(\beta-1)\psi \end{aligned}\right\} \quad (24)$$

We have now to consider the boundary conditions in terms of our new variables. For $r = $ const., we have

$$dx = \frac{\beta(1+r)^2-(1-r)^2}{4\beta}\,\lambda\,dt$$

$$+\, d\lambda\left[\frac{(1-r)^2-\beta(1+r)^2}{1-r^2}\,\psi_\lambda + (\beta-1)\,\psi_\lambda - \left((\beta-1)\,r+\beta+1\right)\psi_{\lambda r}\right]$$

At $\alpha = 1,\ r = -1,\ u = 0,\ \lambda = u-\beta v = -\beta v$, so

$$dx = v\,dt + \lim_{r\to 1}\left[\frac{(1-r)^2-\beta(1+r)^2}{1-r^2}\,\psi_\lambda + (\beta-1)\,\psi_\lambda - \right.$$

$$\left. -\left((\beta-1)\,r+\beta+1\right)\psi_{\lambda r}\right] d\lambda$$

Except for $t = \infty$, $\psi_\lambda \rightarrow 0$ for $r \rightarrow -1$, by (24), so we have

$$\lim_{r\to -1}\left[\frac{\psi_\lambda}{1+r} - \psi_{\lambda r}\right] = 0$$

Thus simply the existence and continuity of $\psi_{\lambda r}$ is sufficient for the boundary condition to be satisfied.

Similarly, at $r = +1$, we have the condition

$$\lim_{r \to +1} \left[\frac{\psi_\lambda}{1-r} + \psi_{\lambda r} \right] = 0,$$

which is satisfied under the same circumstances.

We have of course to bear in mind that at $r = \pm 1$, $\psi_\lambda \sim 1 \mp r$, except for a λ which corresponds to $t = \pm \infty$. For $t = 0$, of course, ψ_λ vanishes to a higher order.

4. We return now to the question of the justifiability of our change of variable. To that end we compute $\frac{\partial(x,t)}{\partial(\alpha,\lambda)}$, and find readily

$$\frac{\partial(x,t)}{\partial(\alpha,\lambda)} = \frac{\alpha(1-\alpha)}{1+(\beta-1)\alpha}\left(\frac{\partial t}{\partial \alpha}\right)^2 + \frac{\lambda^2}{(1+(\beta-1)\alpha)^3}\left(\frac{\partial t}{\partial \lambda}\right)^2 \tag{25}$$

This therefore never changes sign, vanishes if and only if

$$\frac{\partial t}{\partial \alpha} = \frac{\partial t}{\partial \lambda} = 0, \quad \text{or} \quad \frac{\partial t}{\partial \alpha} = \lambda = 0,$$

for $0 < \alpha < 1$, and becomes infinite if and only if one or more of the quantities

$$\lambda, \; \frac{\partial t}{\partial \alpha}, \; \frac{\partial t}{\partial \lambda}$$

become infinite. Thus any solution of (23) must be examined with respect to these possibilities.

We further note that a solution of (23) to be physically satisfactory must have the property that it gives, for each fixed t, α and λ as single-valued functions of x, with α taking on values between 1 and 0 on an interval $x_0(t) < x < x_1(t)$, and thus r taking on values between -1 and 1 on that interval. Now, for t = const., we have

$$dx = \frac{\partial x}{\partial \alpha} d\alpha + \frac{\partial x}{\partial \lambda} d\lambda$$

$$0 = \frac{\partial t}{\partial \alpha} d\alpha + \frac{\partial t}{\partial \lambda} d\lambda ,$$

and so

$$\left. \begin{aligned} d\alpha &= \frac{\dfrac{\partial t}{\partial \lambda}}{\dfrac{\partial (x,t)}{\partial (\alpha,\lambda)}} dx. \\ d\lambda &= \frac{-\dfrac{\partial t}{\partial \alpha}}{\dfrac{\partial (x,t)}{\partial (\alpha,\lambda)}} dx. \end{aligned} \right\} \qquad (26)$$

Thus α will have extrema as a function of x only if $\frac{\partial t}{\partial \lambda}$ changes sign, and λ will have extrema only if $\frac{\partial t}{\partial \alpha}$ changes sign (so if and only if $\frac{\partial t}{\partial r}$ changes sign).

5. We now determine the general solution of the equation (23), finding for it both a series, and an integral representation.

First we seek solutions of the form

$$\psi(r,\lambda) = z(r)\, \varphi(\lambda). \qquad (27)$$

Substituting (27) in (23), we obtain

$$\left. \begin{aligned} (1 - r^2)\, z'' + \mu z &= 0, \\ (\lambda^2 \varphi')' - \mu z &= 0, \end{aligned} \right\} \qquad (28)$$

where μ is constant. Setting

$$z = (1 - r^2)\, P(r), \qquad (29)$$

we have

$$(1 - r^2)\, P'' - 4\, r\, P' + (\mu - 2)\, P = 0. \qquad (30)$$

With $\mu = (n+1)(n+2)$, (30) has as a solution a polynomial P_n of degree n for every integral n:

$$P_n = r^n + \sum_{k=1}^{n/2 \text{ or } n-1/2} (-1)^k \frac{n(n-1) \ldots (n-2k+1)\, r^{n-2k}}{2^k\, k!(2n+1)(2n-1)\ldots(2n-2k+3)} .$$

[Note: the P_n are a special case of Jacobi polynomials. The latter are defined as those functions P_n such that

$$\int_{-1}^{1} (1-r)^{\alpha} (1+r)^{\beta} P_n P_m \, dr = 0, \quad n \neq m, \quad \alpha, \beta > -1.$$

In the case before us $\alpha = \beta = 1$. See Courant-Hilbert, I, pp. 76-77.]

It will be convenient to set

$$Q_n = \left(\int_{-1}^{1} (1-r^2)\, P_n^2 \, dr \right)^{-1/2} P_n, \tag{31}$$

and

$$z_n = (1-r^2)\, Q_n. \tag{32}$$

Evidently then

$$\int_{-1}^{1} (1-r^2)^{-1} z_n z_m \, dr = \begin{matrix} 0, & n \neq m \\ 1, & n = m \end{matrix} \tag{33}$$

The corresponding solutions of the second of the equations (28) are readily seen to be

$$\varphi_n = \lambda^{n+1} \quad \text{or} \quad \varphi_n = \lambda^{-n-2}. \tag{34}$$

Thus the general solution of (23) may be written

$$\psi(r,\lambda) = \sum_{n=0}^{\infty} z_n(r)\left(a_n \lambda^{n+1} + b_n \lambda^{-n-2}\right). \qquad (35)$$

Consider, in particular, the solution

$$\psi_n(r,\lambda) = z_n(r)\left(a'_n \lambda^{n+1} + b'_n \lambda^{-n-2}\right).$$

For such a solution we have

$$t = -4\beta P_n(r)\left((n+1)\, a_n \lambda^{n} - (n+2)\, b_n \lambda^{-(n+3)}\right),$$

$$x = \left((1-r)^2 - \beta(1+r)^2\right) P_n \left(na_n \lambda^{n+1} - (n+3)\, b_n \lambda^{-(n+2)}\right)$$

$$-(1-r^2)\left(\beta + 1 + (\beta-1)\, r\right) P_n'(r)\left(a_n \lambda^{n+1} + b_n \lambda^{-(n+2)}\right).$$

For n = 0, this gives

$$\left.\begin{aligned} t &= 4\beta(2b\lambda^{-3} - a), \\ x &= 3\left(\beta(1+r)^2 - (1-r)^2\right) b \lambda^{-2}, \end{aligned}\right\}$$

or, with a convenient choice of constants

$$\left.\begin{aligned} t &= \lambda^{-3}, \\ x &= \tfrac{3}{8}\left((1+r)^2 - \beta^{-1}(1-r)^2\right) \lambda^{-2}. \end{aligned}\right\}$$

This yields

$$\left.\begin{aligned} x &= \frac{3}{2}\,\frac{(1-\alpha)^2-\beta\alpha^2}{(1+(\beta-1)\alpha)^2}\,t^{2/3},\\ u-\beta v &= t^{-1/3},\\ \alpha u+(1-\alpha)\,v &= 0, \end{aligned}\right\} \qquad (36)$$

a solution of "similarity" type.

For $n > 0$, on the other hand, none of the solutions ψ_n are satisfactory. For every P_n has zeros interior to $-1 < r < 1$, and hence at each such point, for $t \neq 0$, either

$$\lambda = 0 \quad \text{or} \quad \lambda = \pm\infty.$$

Hence, at such a point, since $P_n'(r) \neq 0$, we have $x = \pm\infty$.

6. We now obtain various integral representations of the solution of (23). To that end we note first that it is known that

$$F(r,\lambda) = \frac{1}{(1-2r\lambda+\lambda^2)^{1/2}\left(1-\lambda+(1-2r\lambda+\lambda^2)^{1/2}\right)\left(1+\lambda+(1-2r\lambda+\lambda^2)^{1/2}\right)}$$

is a generating function for the polynomials $P_n(r)$; more precisely

$$F(r,\lambda) = \sum_{n=0}^{\infty}\left(\frac{n+1}{2(2n+3)(n+2)}\right)^{1/2} P_n(r)\,\lambda^n.$$

(Cf. Pólya-Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. II, p. 93).

The function $F(r,\lambda)$ may be simplified:

$$F(r,\lambda) = \frac{1}{2(1-2r\lambda+\lambda^2)^{1/2}\left(1-r\lambda+(1-2r\lambda+\lambda^2)^{1/2}\right)}$$

$$= \frac{(1-2r\lambda+\lambda^2)^{1/2}-1+r\lambda}{2\lambda^2(1-r^2)(1-2r\lambda+\lambda^2)^{1/2}}$$

Thus, if we set

$$G(r,\lambda) = \frac{(1-2r\lambda+\lambda^2)^{1/2}-(1-r\lambda)}{\lambda(1-2r\lambda+\lambda^2)^{1/2}}\quad, \tag{37}$$

we have

$$G(r,\lambda) = \sum_{n=o}^{\infty}\left(\frac{2(n+1)}{(2n+3)(n+2)}\right)^{1/2} z_n(r)\lambda^{n+1}, \tag{38}$$

so that $G(r,\lambda)$ is a solution of (23), as may be verified directly.

Now, observe also, that we may write

$$2(1-r^2)\,F(r,\lambda) = \frac{1}{\lambda^2} + \frac{r\lambda-1}{\lambda^2(1-2r\lambda+\lambda^2)^{1/2}}$$

$$= \sum_{n=o}^{\infty}\left(\frac{2(n+1)}{(2n+3)(n+2)}\right)^{1/2} z_n(r)\lambda^n,$$

and integrating, obtain

$$K(r,\lambda) = \sum_{n=o}^{\infty}\left(\frac{2}{(2n+d)(n+2)(n+1)}\right)^{1/2} z_n(r)\,\lambda^{n+1}, \tag{39}$$

where

$$K(r,\lambda) = \frac{(1-2r\lambda + \lambda^2)^{1/2} - 1}{\lambda} + r, \qquad (40)$$

the constant of integration being determined by the condition that $K(r,0) = 0$. Further, if we multiply (39) by λ and differentiate, we get

$$H(r,\lambda) = \sum_{n=0}^{\infty} \left(\frac{2(n+2)}{(2n+3)(n+1)}\right)^{1/2} z_n(r)\lambda^{n+1} \qquad (41)$$

where

$$H(r,\lambda) = \frac{\lambda - r}{(1-2r\lambda + \lambda^2)^{1/2}} + r \, . \qquad (42)$$

We now base our further analysis on (41), (42), although one could equally well use (37) or (40) instead of (42).

Purely formally, if one replaces λ in (41) by $\lambda\tau^{-1}$, multiplies by $a(\tau)$ and integrates from 0 to ∞ with respect to λ, one obtains

$$\int_0^{\infty} \left[\frac{\lambda - r\tau}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{1/2}} + r\right] a(\tau)\, d\tau \qquad (43)$$

$$= \sum_{n=0}^{\infty} a_n z_n(r)\lambda^{n+1} ,$$

where

$$a_n = \left[\frac{2(n+2)}{(2n+3)(n+1)}\right]^{1/2} \int_0^{\infty} \tau^{-(n+1)} a(\tau)\, d\tau \, .$$

Further, irrespective of the validity of this procedure, or the existence of the above moments, one may verify that the left member of (43) is a solution of (23) provided only that the requisite differentiation is justified. Next observe that if we replace λ by λ^{-1} in (39), and differentiate, we obtain

$$\operatorname{sgn}\lambda \frac{(\lambda - r)}{(1 - 2r\lambda + \lambda^2)^{1/2}} - 1 = - \sum_{n=0}^{\infty} \left[\frac{2(n+1)}{(2n+3)(n+2)}\right]^{1/2} z_n(r)\lambda^{-(n+2)}$$

Thus, if

$$b_n = - \left[\frac{2(n+1)}{(2n+3)(n+2)}\right]^{1/2} \int_0^{\infty} \tau^{n+2}\, b(\tau)\, d\tau,$$

$$\int_0^{\infty} \left[\frac{\operatorname{sgn}\lambda\,(\lambda - r\tau)}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{1/2}} - 1\right] b(\tau)\, d\tau = \sum_{n=0}^{\infty} b_n\, z_n(r)\, \lambda^{-(n+2)}, \qquad (44)$$

and again, of course, one may verify directly that (44) is a solution of (23).

Next we observe that the effect of the term "r" under the integral sign in (43) and of the "-1" under the integral sign in (44), insofar as the physically pertinent quantities x, t are concerned, is simply to add to x a constant -- the time remains unaffected. So, since any linear function of r is a solution of (23), we may as well write the solution arising out of (43), (44) in the form

$$\psi(r,\lambda) = -\int_0^\infty \frac{\lambda - r\tau}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{1/2}} (a + \operatorname{sgn}\lambda\, b)\, d\tau, \tag{45}$$

where we have replaced a,b by -a, -b.

From this we obtain

$$\psi_\lambda = -(1-r^2) \int_0^\infty \frac{\tau^2(a + \operatorname{sgn}\lambda\, b)}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{3/2}} d\tau$$

$$\psi_r = \int_0^\infty \frac{(\tau - \lambda r)\tau^2}{(\tau^2 - 2r\lambda\tau + \lambda^2)} (a + \operatorname{sgn}\lambda\, b)\, d\tau$$

Thus

$$t = 4\beta \int_0^\infty \frac{\tau^2(a + \operatorname{sgn}\lambda\, b)}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{3/2}} d\tau \tag{46}$$

$$x = -\int_0^\infty \frac{(\beta-1)\lambda^3 - 3(\beta-1)r\tau\lambda^2 - 3(\beta+1)r\tau^2\lambda + (\beta+1)\tau^3}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{3/2}} (a + \operatorname{sgn}\lambda\, b)\, d\tau \tag{47}$$

Setting $a + b = f(\tau)$, $a - b = g(\tau)$, we have

$$t = 4\beta \int_0^\infty \frac{\tau^2 f(\tau)}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{3/2}} d\tau \quad (48)$$

$$x = -\int_0^\infty \frac{\left((\beta-1)\lambda^3 - 3(\beta-1) r\tau\lambda^2 - 3(\beta+1) r\tau^2\lambda + (\beta+1)\tau^3\right)}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{3/2}} f(\tau)\, d\tau \quad (49)$$

for $\lambda > 0$, and the same expressions with f replaced by g for $\lambda < 0$. Evidently $\int_0^\infty f(\tau)\, d\tau$ must be convergent, and similarly for $g(\tau)$ if it occurs, since otherwise x would always be infinite.

$$\left.\begin{aligned} \frac{\partial t}{\partial \lambda} &= -12\beta \int_0^\infty \frac{(\lambda - r\tau)\tau^2 f(\tau)\, d\tau}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{5/2}}, \\ \frac{\partial t}{\partial r} &= 12\beta\lambda \int_0^\infty \frac{\tau^3 f(\tau)}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{5/2}} d\tau; \end{aligned}\right\} \quad (50)$$

$$\left.\begin{aligned} \frac{\partial x}{\partial \lambda} &= 3\lambda \int_0^\infty \frac{(1-r)^2(\lambda+\tau) - \beta(1+r)^2(\lambda-\tau)}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{5/2}} \tau^2 f(\tau)\, d\tau, \\ \frac{\partial x}{\partial r} &= 3\lambda^2 \int_0^\infty \frac{(\beta+1)(\lambda+r\tau) + (\beta-1)(r\lambda+\tau)}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{5/2}} \tau^2 f(\tau)\, d\tau. \end{aligned}\right\} \quad (51)$$

The first of these, (50), is of particular use in the sequel.

7. We want now to examine various properties of the solution (48), (49). We note first that $\lambda = 0$ gives $x = -(\beta+1)\int_0^\infty f(\tau)\, d\tau$, $t = 4\beta \int_0^\infty \tau^{-1} f(\tau)\, d\tau$. The latter integral may or may not be divergent; in the former case $\lambda = 0$ corresponds to $t = \pm\infty$. At first glance one is tempted to conclude from this that λ cannot change sign, since $\lambda = 0$ gives x and t constant. This argument is not rigorous, however, because of possible singular behavior of the solution in the neighborhood of $\lambda = 0$. We shall therefore defer consideration of this matter until we have studied further the properties of (48), (49).

Next consider the lines $r = \pm 1$. We have

for $r = -1$, $$t = 4\beta\int_0^\infty \frac{\tau^2 f(\tau)\, d\tau}{|\tau+\lambda|^3}$$

for $r = 1$, $$t = 4\beta\int_0^\infty \frac{\tau^2 f(\tau)}{|\tau-\lambda|^3}\, d\tau.$$

It follows that the only values of λ possible at $r = 1$ are positive, or such that $f(-\lambda) = 0$ and $\int_0^\infty \frac{\tau^2 f(\tau)}{|\tau+\lambda|} d\tau$ is convergent. Similarly, at $r = 1$, the only possible values of λ are negative or such that $f(\lambda) = 0$ and $\int_0^\infty \frac{\tau^2 f(\tau)\, d\tau}{|\tau-\lambda|^3}$ is convergent. Since, at each end, λ is a constant multiple of the velocity of the boundary, however, we must expect that the values of λ assumed at each point constitute an interval -- or else that λ is constant, a possibility that we must evidently take account of. In view of this rather complicated situation, it is worthwhile for the purposes of orientation to specialize somewhat.

We shall therefore hereafter in this report restrict our attention to the case that $f(\tau)$ is everywhere positive or zero. The more general case will be considered in a later discussion.

To begin, we assume that $f(\tau) = 0$ for $\tau > m$, and prove that in this case, for small t, the formulae (48), (49) always provide a solution of the problem before us. Indeed we shall see that for this case, for t small, x, λ are always monotone-increasing functions of r on the interval $-1 \leqq r \leqq 1$.

We note first from the formulae (48), (49), that for $\lambda \longrightarrow \infty$, $t \rightarrow 0$, $x \rightarrow -(\beta-1)\int_0^m f(\tau)\, d\tau$, the latter integral being necessarily finite. It is accordingly convenient to think of the motion as starting at $t = 0$, with $\lambda = \infty$. Next, from (51) we see that $\frac{\partial t}{\partial r}$ is always positive, while $\partial t/\partial\lambda$ is certainly negative for all $r < 0$, and, if $r > 0$ for $\lambda > rm$. Hence on any curve $t = \text{const.}$ in the (r, λ) plane which lies above the curve $\lambda = 0$ for $r < 0$, $\lambda = rm$ for $r > 0$, $d\lambda/dr > 0$. Further, since $\partial t/\partial\lambda < 0$, $dx/dr > 0$ also (cf. § 4 above, and recall $d\alpha/dr < 0$).

It remains then only to show that there are curves $t = \text{const.}$ extending from $r = -1$ to $r = 1$ in the region under consideration. To that end, consider any $\lambda = \lambda_0 > m$, and the corresponding value of t at $r = -1$; $t_0 = 4\beta\int_0^m \frac{\tau^2 f(\tau)\, d}{(\lambda_0 + \tau)^3}$. Now obviously

$t_1 = 4\beta \int_0^m \frac{\tau^2 f(\tau)\, d\tau}{(\lambda_0 - \tau)^3}$ at $r = 1$ is a larger value of t, and on $\lambda = \lambda_o$, t increases monotonically from $t = t_o$ at $r = -1$ to $t = t_1$ at $r = 1$. Hence on every segment $r = r_o$, $\lambda > \lambda_o$, t must take on the value t_o once and only once for all $-1 < r_o \leqslant 1$. This establishes our result.

We have now to examine the behavior of this solution as $t \to \infty$. To that end, it will be convenient to assume first that $f(\tau)$ vanishes for $\tau < \epsilon$ and that $\int_\epsilon^m \frac{\tau^2 f(\tau)\, d\tau}{|\lambda - \tau|^3}$ is divergent for $\lambda = m$ and $\lambda = \epsilon$.

As an extreme but simple instance of this situation, let us take the case that $f(\tau) = \delta(1)$, when we have

$$\left.\begin{aligned} \frac{t}{4\beta} &= \frac{1}{(1 - 2r\lambda + \lambda^2)^{3/2}} \\ x &= -\frac{(\beta-1)\lambda^3 - 3(\beta-1) r\lambda^2 - 3(\beta+1) r\lambda + \beta + 1}{(1 - 2r\lambda + \lambda^2)^{3/2}}. \end{aligned}\right\} \quad (52)$$

This yields

$$\lambda^2 - 2r\lambda - (\sigma^{-2/3} - 1) = 0, \quad (53)$$

where

$$\sigma = \frac{t}{4\beta},$$

or

$$\lambda = r + \sqrt{r^2 + (\sigma^{-2/3} - 1)} \quad , \tag{54}$$

so long as $\sigma \lesseqgtr 1$. Figure 1 shows the curves t = const. in the (r,λ)-plane for various $t \lesseqgtr 4\beta$. As is evident from (54), $t = 4\beta$ yields the broken line $\lambda = 0$ for $r < 0$, $\lambda = 2r$ for $r > 0$. To interpret this let us look at Figs. 2a, 2b. The first shows x as a function of α for various $t < 4\beta$. The second shows x for $t = 4\beta$ and for one greater value (the latter curve is not completed to r = 1 at the top since the value of λ there is extremely large). Note that for all $r < 0$, so for all $\alpha < (\beta+1)^{-1}$, we have $x = -(\beta+1)$. That is to say at $t = 4\beta$ the value of α at the left-hand end of the mixing zone drops discontinuously from $\alpha = 1$ to $\alpha = (\beta+1)^{-1}$. Now observe that one gets $t = 4\beta$ also for $r > 0$ and $\lambda = 0$, and thus for values of $r > 0$ and λ small but positive, values of t larger, but as little larger as one pleases, than 4β. Thus for $t > 4\rho$, α drops to the value zero at the left-hand end of the mixing zone. The boundary between the two materials is thus reestablished, but only at the expense of a certain mass of the material to the left having become detached from its parent substance and mixed into the material to the right. The subsequent motion of this piece is indicated by the curves $t = 256\beta/27$ in Figs. 1, 2b.

In more complicated situations, this piece may break up still further. Consider for example the case

$$\frac{t}{4\beta} = \frac{1}{(1 - 2r\lambda + \lambda^2)^{3/2}} + \frac{1}{(4 - 4r\lambda + \lambda^2)^{3/2}} \,. \tag{55}$$

Figs. 3, 4 show the curves $t = \text{const.}$ in the (r,λ)- and (r,x)-planes, respectively. Here, at $t = \frac{9}{2}\beta$ we have exactly the same phenomenon as above, and then at $t = 64\beta$ the detached mixed material separates into two pieces.

There is, furthermore, no a priori reason why a "break-off" of the kind just indicated cannot occur before one of the sort discussed in the first example. That is, at three successive values of t, $t = t_1, t_2, t_3$ the curves $t = \text{const.}$ in the (r,λ)-plane may look as in Fig. 5. There may of course also be more than one instance of this kind.

With these facts before us, let us return to the general discussion, still with $f(\tau) \geqslant 0$, $f(\tau) = 0$, for $\tau > m$ and $\tau < \epsilon$, and $\int_\epsilon^m \frac{\tau^2 f(\tau)}{|\lambda - \tau|^3}\, d\tau$ divergent for $\lambda = m$ or ϵ. Then, for some $\lambda_0 > m$, we have clearly $\int_0^m \frac{\tau^2 f(\tau)\, d\tau}{|\lambda - \tau|^3} = \int_0^m \frac{f(\tau)}{\tau}\, d\tau$.

Now let us suppose that $f(\tau) > 0$ throughout the interval $\epsilon \leqq \tau \leqq m$. Then for every $\lambda < \lambda_0$, t considered as a function of r increases monotonically from a value less than $t_0 = 4\beta \int_\epsilon^m \tau^{-1} f(\tau)\, d\tau$ at $r = -1$ to a greater value at $r = 1$. Thus at exactly one point it takes on the value t_0. Further, since at $r = 0$ it takes on this value only at $\lambda = 0$, $t = t_0$, for $r > 0$ is a curve extending from

$r = 0, \lambda = 0$ to $r = 1, \lambda = \lambda_0$. On it, moreover, r is a single-valued function of λ, though not necessarily vice versa. It remains then to consider the case that $f(\tau)$ vanishes, either at isolated points $\tau = \tau_0$, or on intervals. Its zeros at isolated points, moreover, can only distort the above picture if $\int_\epsilon^m |\tau - \tau_0|^{-3} f(\tau)\, d\tau$ converges. In the latter case, the point $\lambda = \tau_0$, $r = 1$ is a singular point of the differential equation $\frac{\partial t}{\partial \lambda} d\lambda + \frac{\partial t}{\partial r} dr = 0$ and every curve $t \geqslant 4\beta\int_\epsilon^m |\tau - \tau_0|^{-3} f(\tau)\, d\tau$ goes through it. The configuration there is that shown in Fig. 6. If $f(\tau)$ vanishes on an interval, the situation is similar, except that all of the curves $t =$ const. except one are broken like the curve $t = t_3$ in Fig. 5. Evidently, therefore the zeros of $f(\tau)$ simply correspond to the appearance in the mixing zone of intervals at both ends of which $r = 1$ ($\alpha = 0$). If these zeros are isolated, the intervals do not break off, whereas when $f(\tau)$ vanishes on an interval, they do.

In consequence, we can conclude, even in the presence of zeros of $f(\tau)$, that for $r > 0$, $t = t_0$ is a curve extending from $r = 0$, $\lambda = 0$ to $r = 1$, $\lambda = \lambda_0$, except for breaks of the kind just described. Thus the situation is always qualitatively like that in the simple examples for which graphs have been given.

We now note briefly the situation which occurs if $\int_\epsilon^m \tau^2 |\tau - m|^{-3} f(\tau)\, d\tau$ is convergent. We assume that $f(\tau) > 0$ for τ arbitrarily near m; otherwise we could change the value of m. It is then readily seen that in the right angle $\lambda < m$, $r < 1$, all values of t greater than $t^* = 4\beta\int_{\epsilon}^{m} \tau^2 \left|\tau - m\right|^{-3} f(\tau)\, d\tau$ are assumed. Thus all curves $t = t_1$, $t_1 \geqslant t^*$ go through $r = 1$, $\lambda = m$, and the right-hand end of the mixing zone having achieved the velocity $u = m$ at $t = t^*$ maintains it constant thereafter. Except for this the situation is not different from that just discussed. Evidently, there is a similar possibility at $r = 1$, $\lambda = \epsilon$, if $\int_{\epsilon}^{m} \tau^2 \left|\tau - m\right|^{-3} f(\tau)\, d\tau$ is convergent.

Next we consider the case that $f(\tau)$ is different from zero in every neighborhood of $\tau = \infty$. The solution in this case may clearly be approximated by considering solutions of the sort just discussed, and allowing m to tend to infinity; the evident conclusion is that all curves $t =$ const. go through the point $r = 1$, $\lambda = \infty$, and thus the right-hand end-point of the mixing zone lies always at $x = \infty$. But of course for values of t sufficiently large, if $f(\tau)$ has an interval of zeros which is also bounded away from the origin, the mixing zone will have a break in it, and the solution exclusive of this break describes a mixing regime, by itself. So solutions of this sort are not to be neglected.

Next let us drop the condition that $f(\tau)$ vanish in some neighborhood of the origin and consider the consequences which can arise, supposing first that $\int_0^m \tau^{-1} f(\tau)\, d\tau$ is convergent, but of course that $f(\tau)$ is different from zero in every neighborhood of the origin. Observe first that t is a uniformly continuous function on the half-strip $-1 < r < 1 - \epsilon$, $\lambda > 0$. Thus it is bounded there, say

$t \leqslant T$. Hence all larger values of t must correspond to curves lying in the region $r > 1 - \epsilon$. Moreover, at $t = 4\beta \int_0^m \tau^{-1} f(\tau)\, d\tau$, the value of r at the left-hand end-point of the mixing zone $x = x_o$, changes discontinuously from -1 to +1, and so α drops discontinuously from 1 to 0 at that time and place, $x = x_o(t)$ remaining constant and equal to $-(\beta + 1) \int_0^\infty f(\tau)\, d\tau$, thereafter. The structure of the solution as t increases, depends of course on the zero of $f(\tau)$.

Now let us assume that $\int_0^\infty \tau^{-1} f(\tau)\, d\tau$ is divergent at $\tau = 0$, and consider any value of $t > 0$. Corresponding to this value of t we find a unique $\lambda = \lambda_o$ on $r = -1$, $t = 4\beta \int_0^\infty \frac{\tau^2 f(\tau)}{(\tau + \lambda)^3}\, d\tau$, and by a type of argument already used, for each r on $-1 < r \leqslant 0$, one and only one value of $\lambda > \lambda_o$ corresponding to this value of t, and thus a continuous arc from $r = -1$ to $r = 0$ corresponding to this value of t. Further, it crosses $r = 0$ with $d\lambda/dr > 0$.

The continuation of this curve cannot cross the line $r = 0$ again for $0 \leqslant \lambda \leqslant \infty$ except possibly at $\lambda = 0$, and cannot intersect $r = 1$ except at a point λ where $\int_0^\infty \frac{\tau^2 f(\tau)}{(\lambda - \tau)^3}\, d\tau$ is convergent, cannot intersect $\lambda = 0$ except at $r = 0$, nor $\lambda = \infty$ except at $r = 1$. Moreover, at every point with $0 < r < 1$, $0 < \lambda < \infty$, r is an analytic function of λ. Finally the possibility that it crosses $r = 0$ at $\lambda = 0$ may be ruled out, because this would imply the existence of a point at which $\frac{d\lambda}{dr} = 0$, so $\frac{\partial t}{\partial r} = 0$, and this cannot occur. Thus negative values of λ do not occur for any of these solutions, and the curves $t =$ const. all cross $r = 1$, although this intersection may lie at $\lambda = \infty$. It cannot, however, in the case before us lie at $\lambda = 0$, since this corresponds to $t = \infty$. There may of course be additional arcs belonging to the value of t under consideration; these will have both end points on $r = 1$, and correspond to the motion of portions of the mixture which have broken loose in the manner already indicated above.

It is evident from the above discussion that for any given ϵ, we can choose T so large that all curves $t =$ const. $\geqslant T$ lie in the right-angled strip bounded by $\lambda = 0$, $\lambda = \epsilon$, $r = 1 - \epsilon$, $r = 1$. Further, we can also show that for $r < 1 - \epsilon$, $x - x_o$, where $x_o = -(\beta + 1) \int_0^\infty f(\tau)\, d\tau$, tends to zero for $\lambda \longrightarrow 0$: Thus the width in space of the zone in which $r < 1 - \epsilon$, where $\alpha > \epsilon/2\beta$, tends to zero as $t \longrightarrow \infty$. To prove this, we have simply to show the uniform continuity of x as a function λ at $\lambda = 0$, for $r < 1 - \epsilon$. From the formula (49), we see that it is sufficient to show that each of the integrals $\int_0^\infty \frac{\lambda^k \tau^{3-k} f(\tau)}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{3/2}} d\tau$, $k = 1, 2, 3$ tends to zero uniformly in r for $r < 1 - \epsilon$, as $\lambda \longrightarrow 0$. One readily establishes the inequality

$$\frac{1}{(\tau^2 - 2r\lambda\tau + \lambda^2)^{3/2}} \leqslant \left(\frac{2}{1 - r}\right)^{3/2} \frac{1}{(\tau + \lambda)^3} ,$$

so that the problem reduces to the consideration of the integrals

$$\int_0^{\infty} \frac{\lambda^k \tau^{3-k}}{(\tau+\lambda)^3} f(\tau)\, d\tau .$$

For $k = 1$, we have

$$\int_0^{\infty} \frac{\lambda\tau^2}{(\tau+\lambda)^3} f(\tau)\, d\tau = \int_0^{\infty} \frac{\tau^2}{(\lambda+\tau)^2} f(\tau)\, d\tau - \int_0^{\infty} \frac{\tau^3}{(\lambda+\tau)^3} f(\tau)\, d\tau .$$

Since both integrals on the right have the limit x_o, as $\lambda \longrightarrow 0$, the one on the left must vanish. Now in the same way, we can show that $\int \frac{\lambda\tau}{(\tau+\lambda)^2} f(\tau)\, d\tau$ has the limit zero, whence, with the identity

$$\frac{\lambda^2\tau}{(\tau+\lambda)^3} + \frac{\lambda\tau^2}{(\tau+\lambda)^3} = \frac{\lambda\tau}{(\tau+\lambda)^2} ,$$

we conclude that $\int_0^{\infty} \frac{\lambda^2\tau}{(\tau+\lambda)^3} f(\tau)\, d\tau$

has the limit zero as $\lambda \longrightarrow 0$. Finally

$$\frac{\lambda^3}{(\lambda+\tau)^3} = \frac{\lambda}{\lambda+\tau} - \frac{2\lambda^2\tau}{(\lambda+\tau)^3} + \frac{\lambda\tau^2}{(\lambda+\tau)^3} , \qquad \text{and}$$

$$\int_0^{\infty} \frac{\lambda}{\lambda+\tau} f(\tau)\, d\tau = \int_0^{\infty} f(\tau)\, d\tau - \int_0^{\infty} \frac{\tau f(\tau)}{(\lambda+\tau)}\, d\tau ,$$

so $$\lim_{\lambda \to 0} \int_0^{\infty} \frac{\lambda^3}{(\lambda+\tau)^3} f(\tau)\, d\tau = 0.$$

We thus have the qualitative result that as $t \longrightarrow \infty$, the width of the region where there is appreciable mixing tends to zero, although small amounts of the material originally to the left may penetrate arbitrarily far into the material to the right. In this connection it is of some interest to calculate $\int_{x_0}^{x_1} \alpha\, dx$ as a function of time. From equation (2), we observe that $\int_{(x_0,t_0)}^{(x,t)} (\alpha dx - w dt)$ is independent of its path. Integrating by parts, we obtain for this integral $wt + \alpha x - x_0 \int_{(1,0)}^{(\alpha,w)} (x\, d\alpha - t\, dw) = x_0 - y(\alpha,w) + y(1,0)$, where y is the function defined by equation (16). Recalling $y = \left(1 + (\beta-1)\alpha\right)\psi$, we have

$$\int_{x_0}^{x_1} \alpha\, dx = -x_0 + \beta\, \psi(-1, \lambda_0) - \psi(1, \lambda_1),$$

where λ_0 and λ_1 are the values of λ at $r = -1$, $r = 1$ corresponding to the value of t in question. The above has been derived under the implicit assumption of no breaks in the solution, but it is easy to verify that it is correct even in the presence of such breaks if λ_1 is the maximum λ on $r = 1$ corresponding to the value of t in question. In consequence of this, one gets

$$\int_{x_0}^{x_1} \alpha\, dx = - x_0 - (\beta - 1) \int_0^{\infty} f(\tau)\, d\tau .$$

Since in the limit $t \longrightarrow \infty$, $-x_0 \longrightarrow (\beta + 1) \int_0^{\infty} f(\tau)\, d\tau$, we have

$$\lim_{t \to \infty} \int_{x_0}^{x_1} \alpha\, dx = 2 \int_0^{\infty} f(\tau)\, d\tau .$$

This completes our discussion of the case $f(\tau) \geqslant 0$. It will be observed that $f(\tau) \leqslant 0$ clearly describes a regime of "unmixing" since it amounts simply to reversing the sense of time and interchanging left and right in the solutions we have discussed. Since the general case can be considered as a superposition in the (λ, r)-space of a mixing and an unmixing, there do not appear to be any simple criteria for distinguishing those $f(\tau)$ which change sign, which provide mixing regimes over some interval of time. In a subsequent report we shall take this subject up further; here we conclude with two remarks. First, the assertion made above concerning the "stability" of the solutions with $f(\tau) \geqslant 0$ remains valid provided simply that $f(\tau)$ is absolutely integrable at the origin. Second, a completely new phenomenon appears when we come to $f(\tau)$ which change sign -- namely, solutions in which α, u, v change discontinuously at one of the boundaries, not just at one instant as in the examples above, but throughout an interval of time. That is, at the left-hand boundary, say, α may drop from the value 1 to a value α which is a function of time, u increasing at the same instant from zero to a value which also varies in time. This is, therefore, a sort of shock mixing. It does not, however, require any special equations to describe it; because the equation governing the motion is elliptic, the nature of discontinuities which can develop in a solution is controlled by the equation itself. This phenomenon appears whenever $t(r, \lambda)$ has a relative maximum on $r = -1$ or $r = 1$, and will be discussed in more detail in Part II of this report.

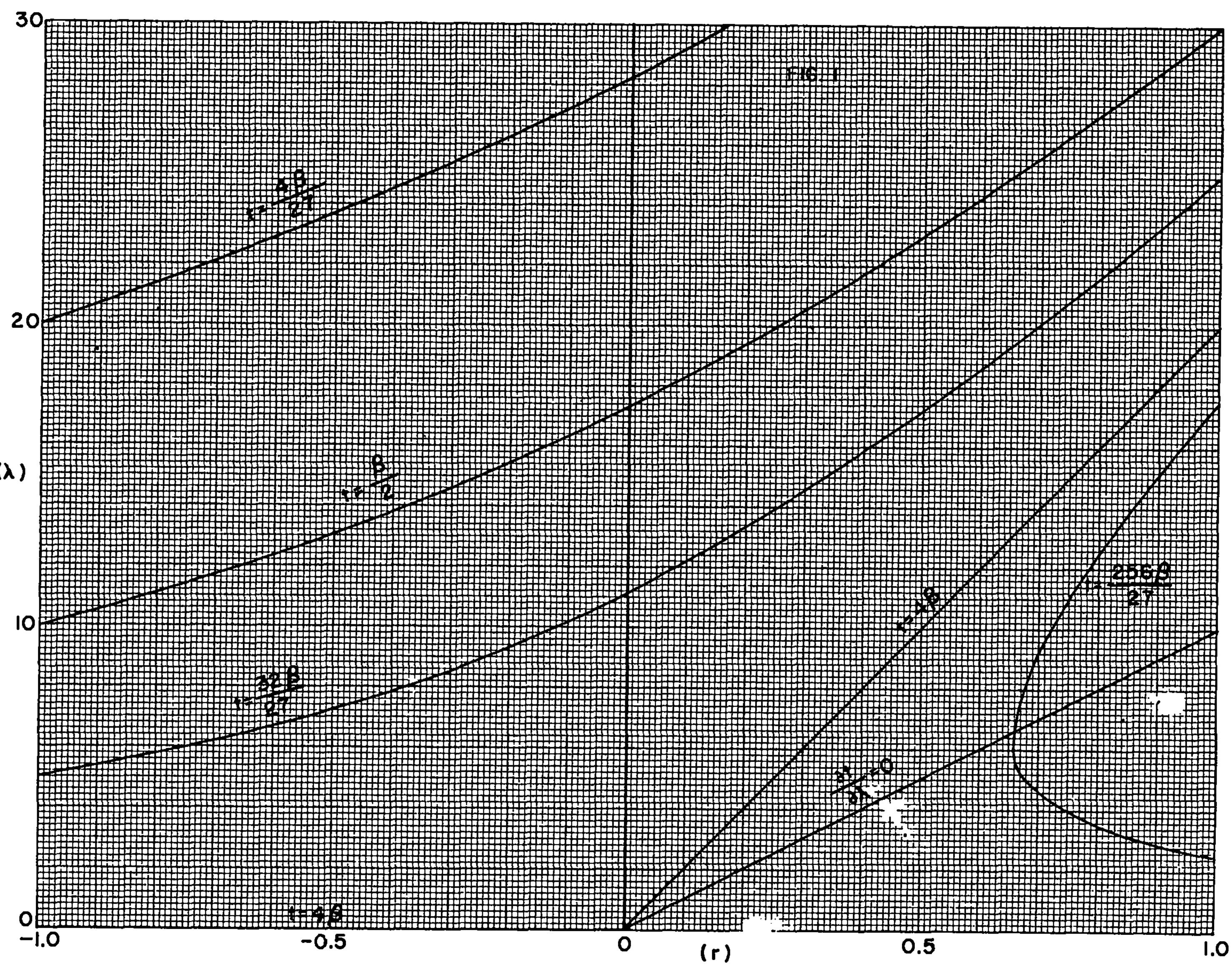
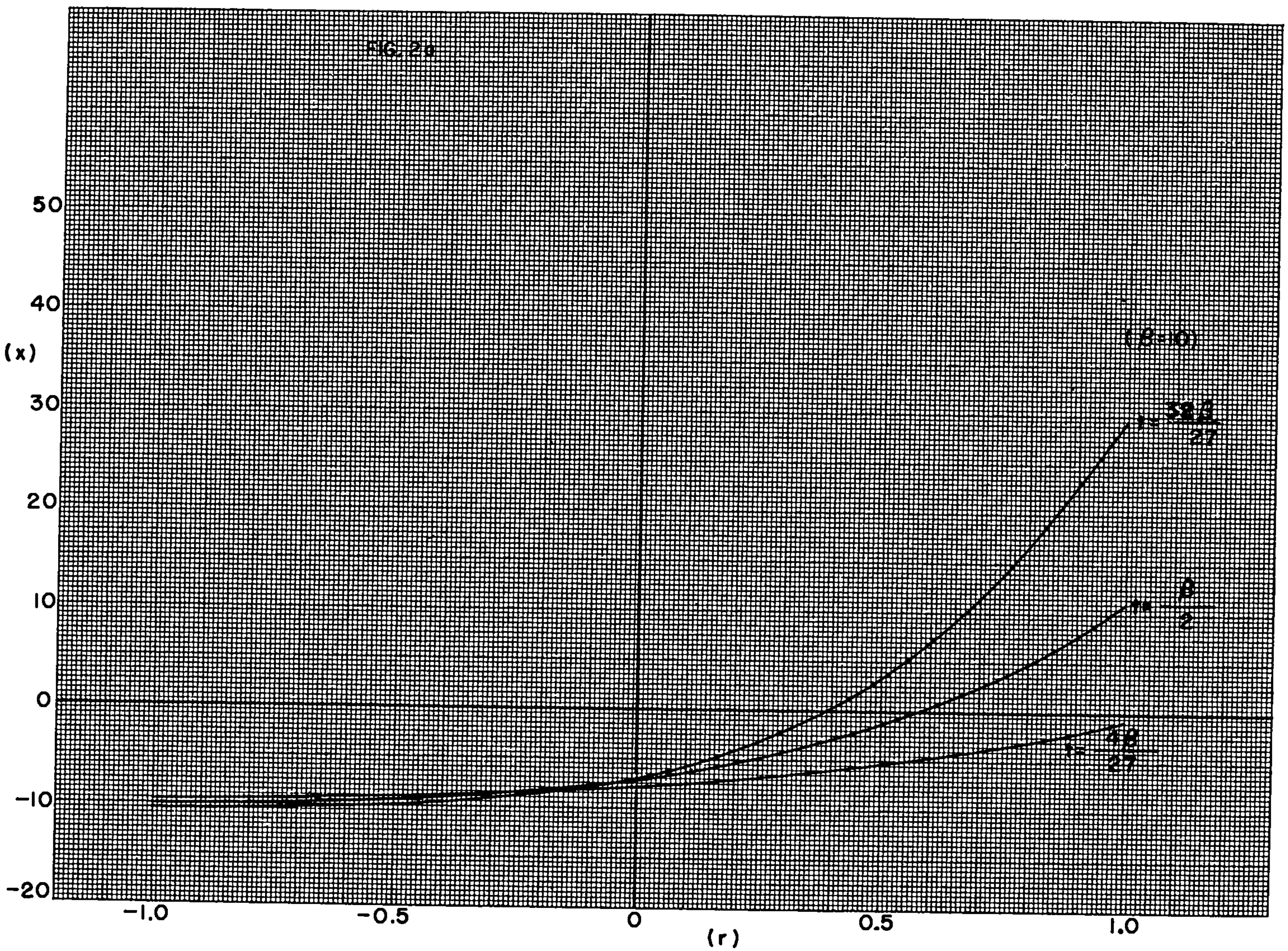


FIG. 1
t = 4β/27
t = β/2
t = 32β/27
t = 4β
t = 256β/27
t = 4β
(λ)
(r)
30
20
10
0
−1.0
−0.5
0
0.5
1.0

FIG. 2a



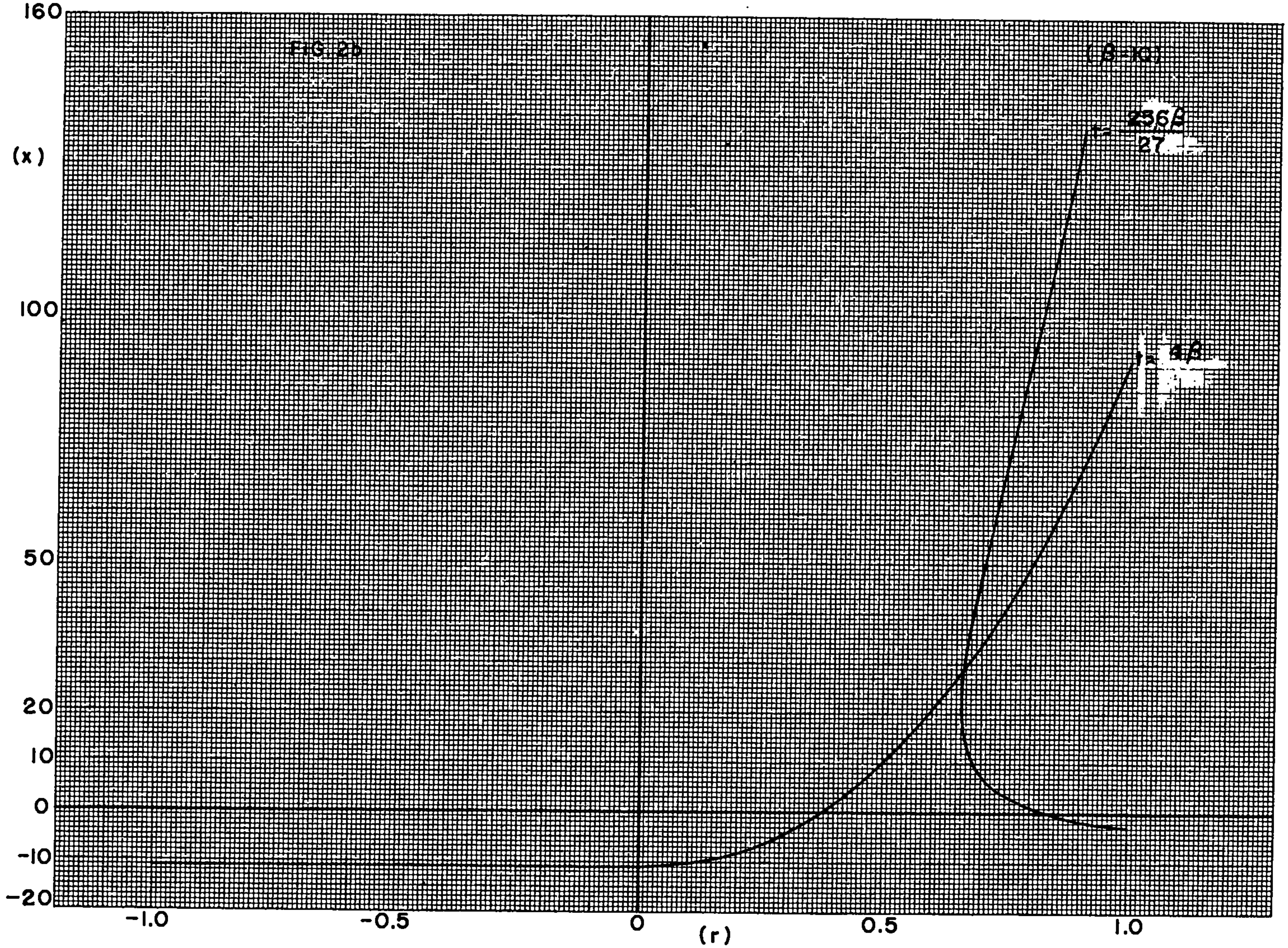


FIG. 2b
(β=10)
r = 256β/27
r = 4β
(x)
160
100
50
20
10
0
-10
-20
-1.0
-0.5
0
(r)
0.5
1.0

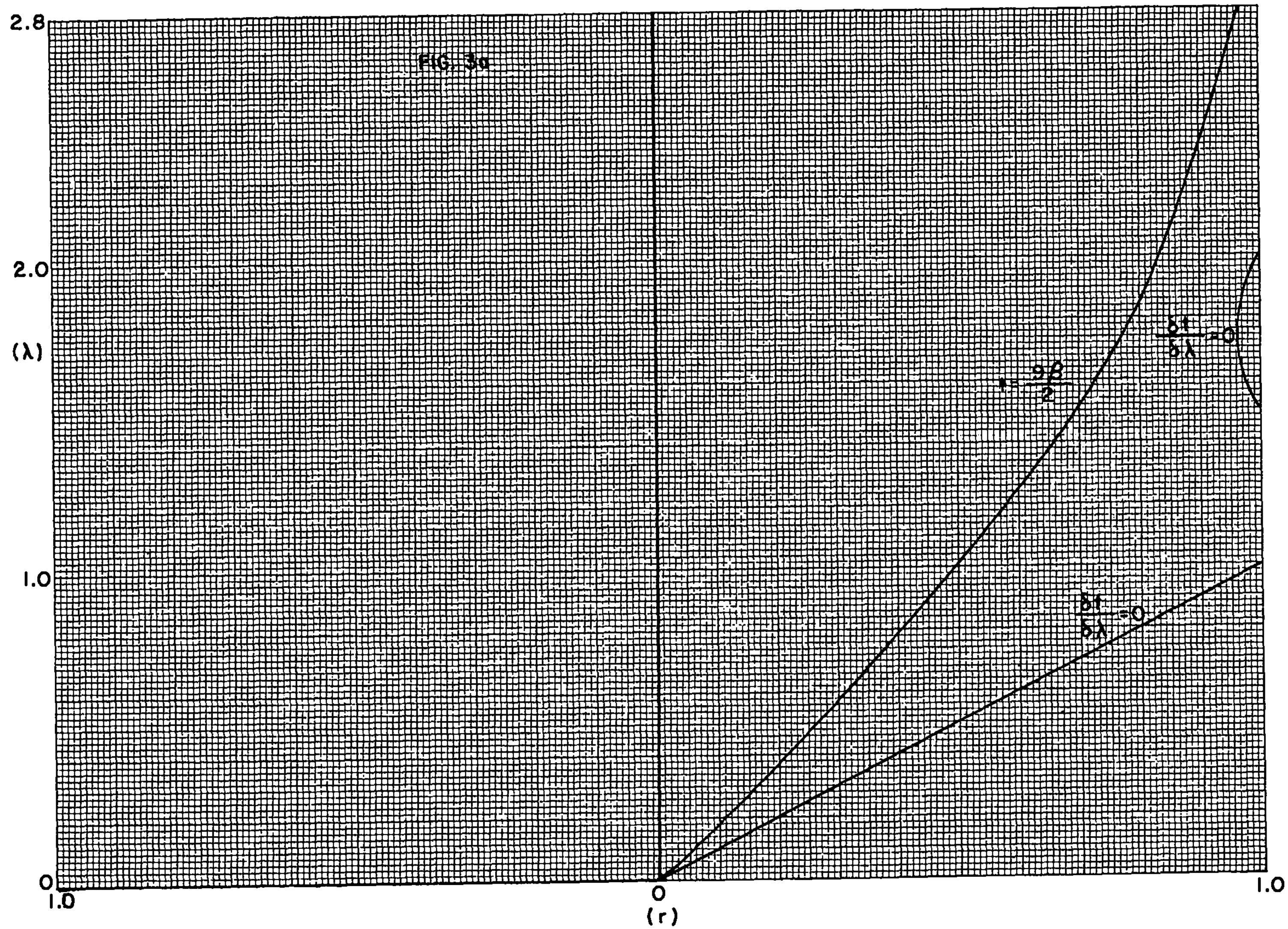

2.8
2.0
(λ)
1.0
0
1.0
0
(r)
1.0
$f = \frac{9\beta}{2}$
$\frac{\delta f}{\delta \lambda} = 0$
$\frac{\delta f}{\delta \lambda} = 0$

FIG. 3a

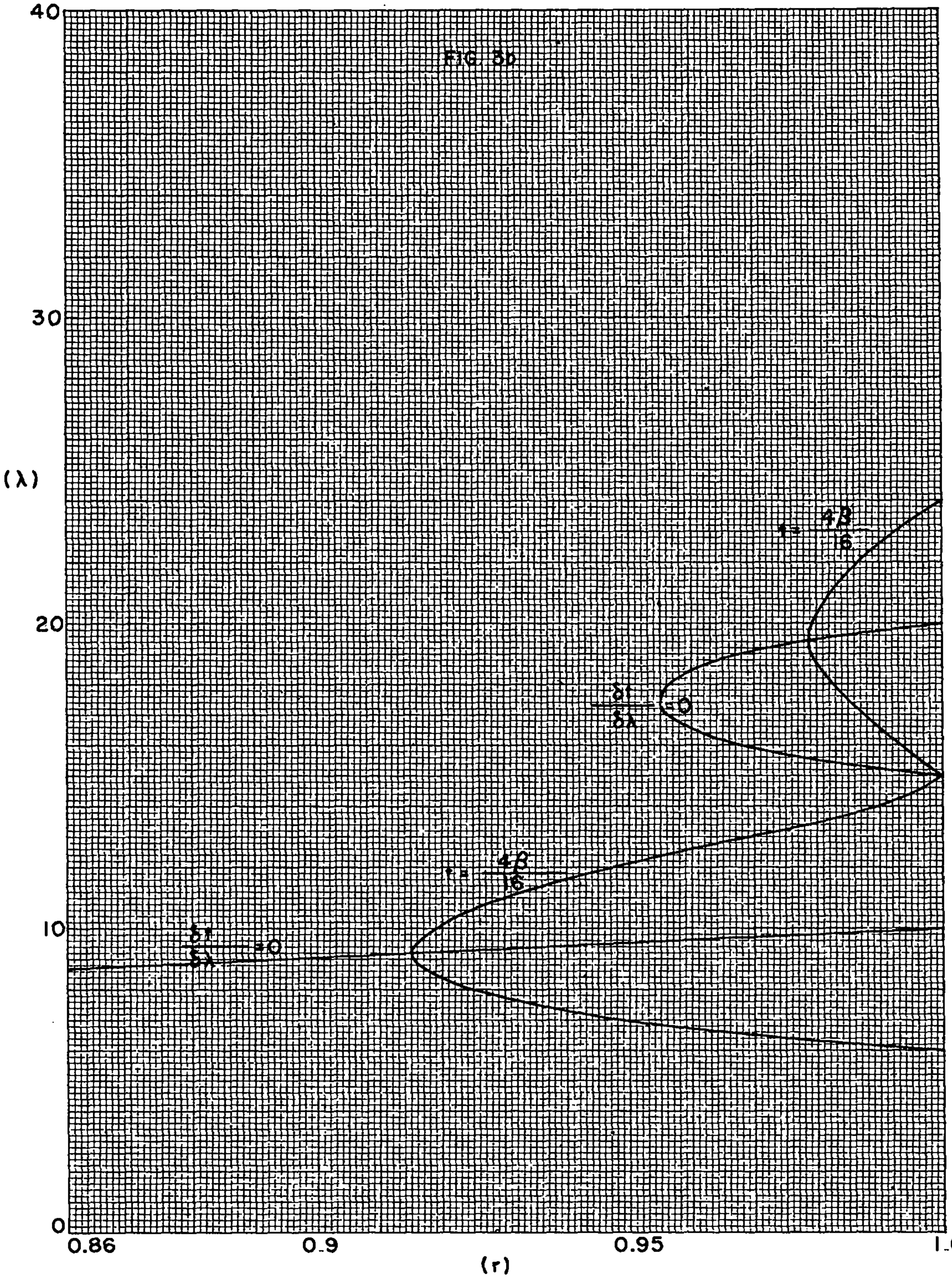


FIG. 3b
40
30
20
10
0
(λ)
0.86
0.9
0.95
1.0
(r)

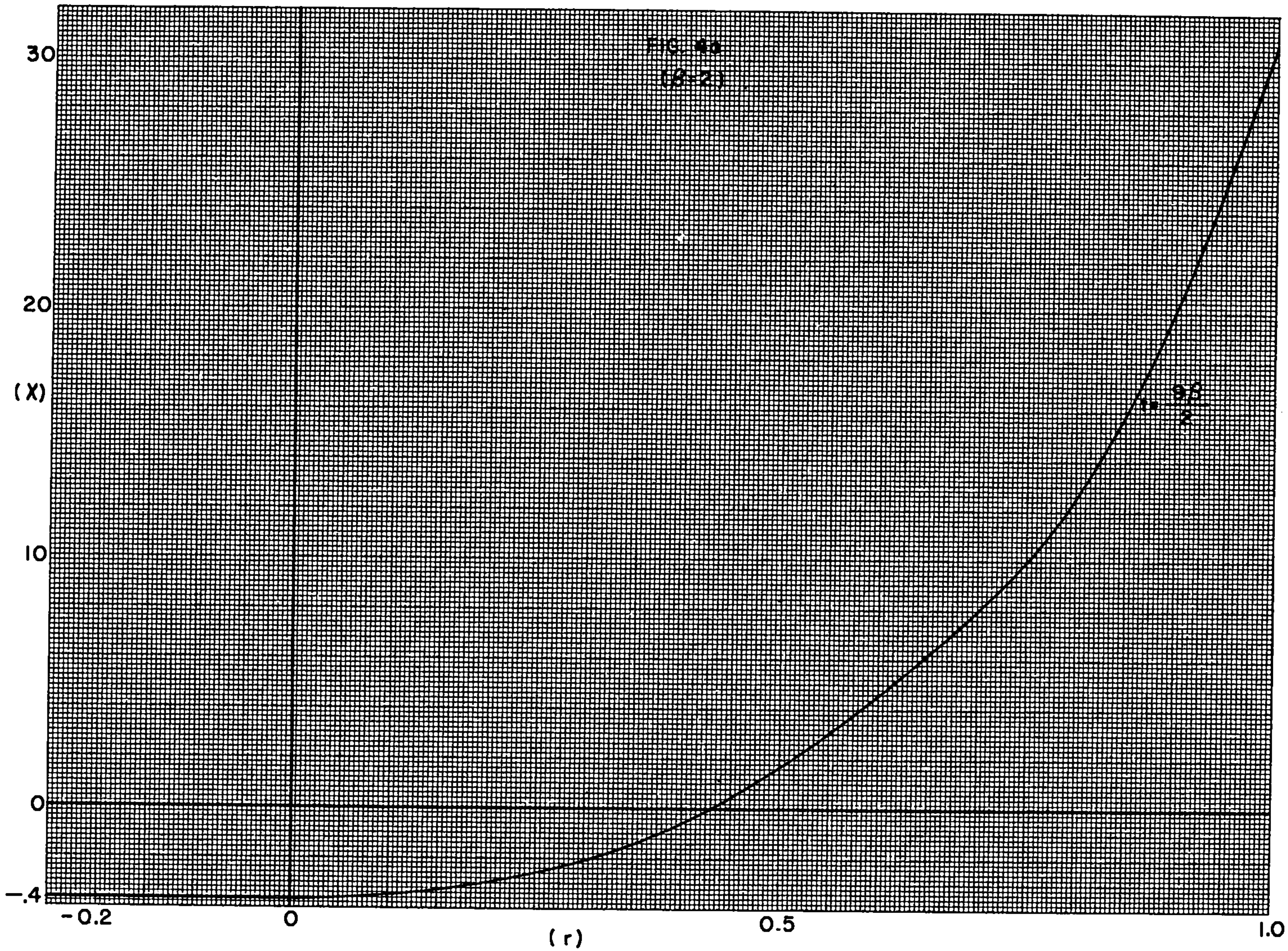


FIG. 4a
(β=2)
30
20
(X)
10
0
-.4
-0.2
0
0.5
1.0
(r)
9β/2

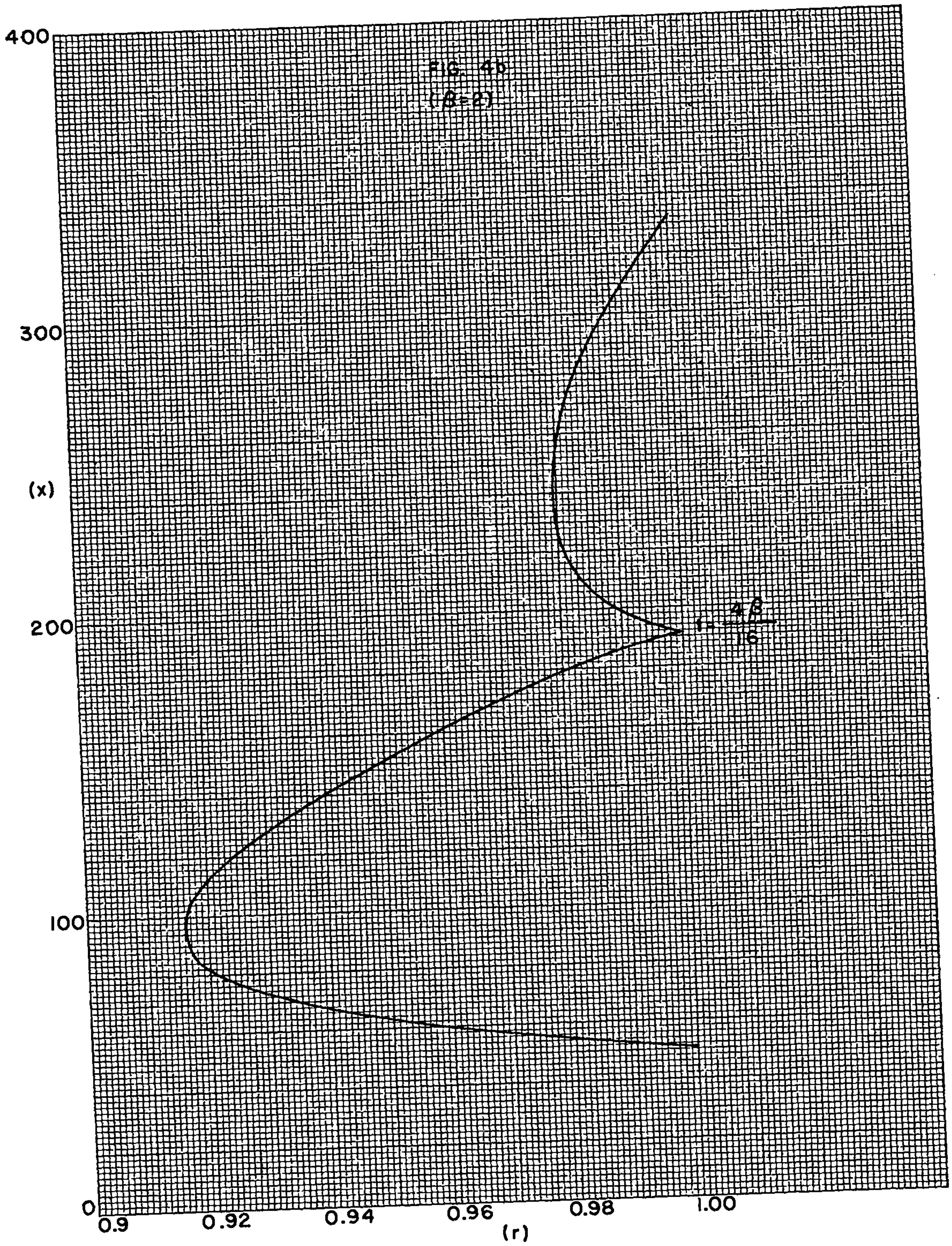


FIG. 4b
(β=2)
400
300
(x)
200
100
0
r = 4β/16
0.9
0.92
0.94
0.96
0.98
1.00
(r)

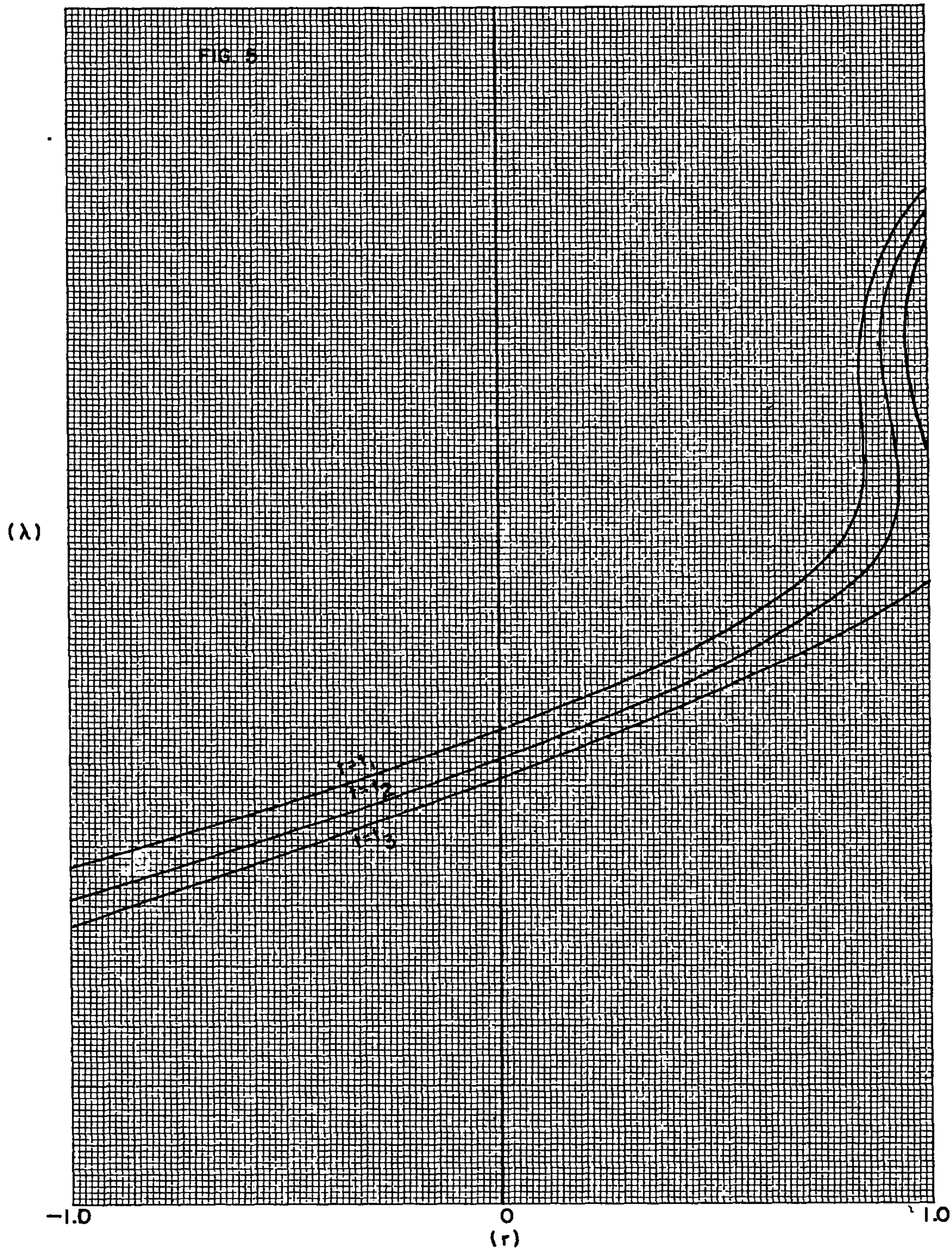


FIG. 5

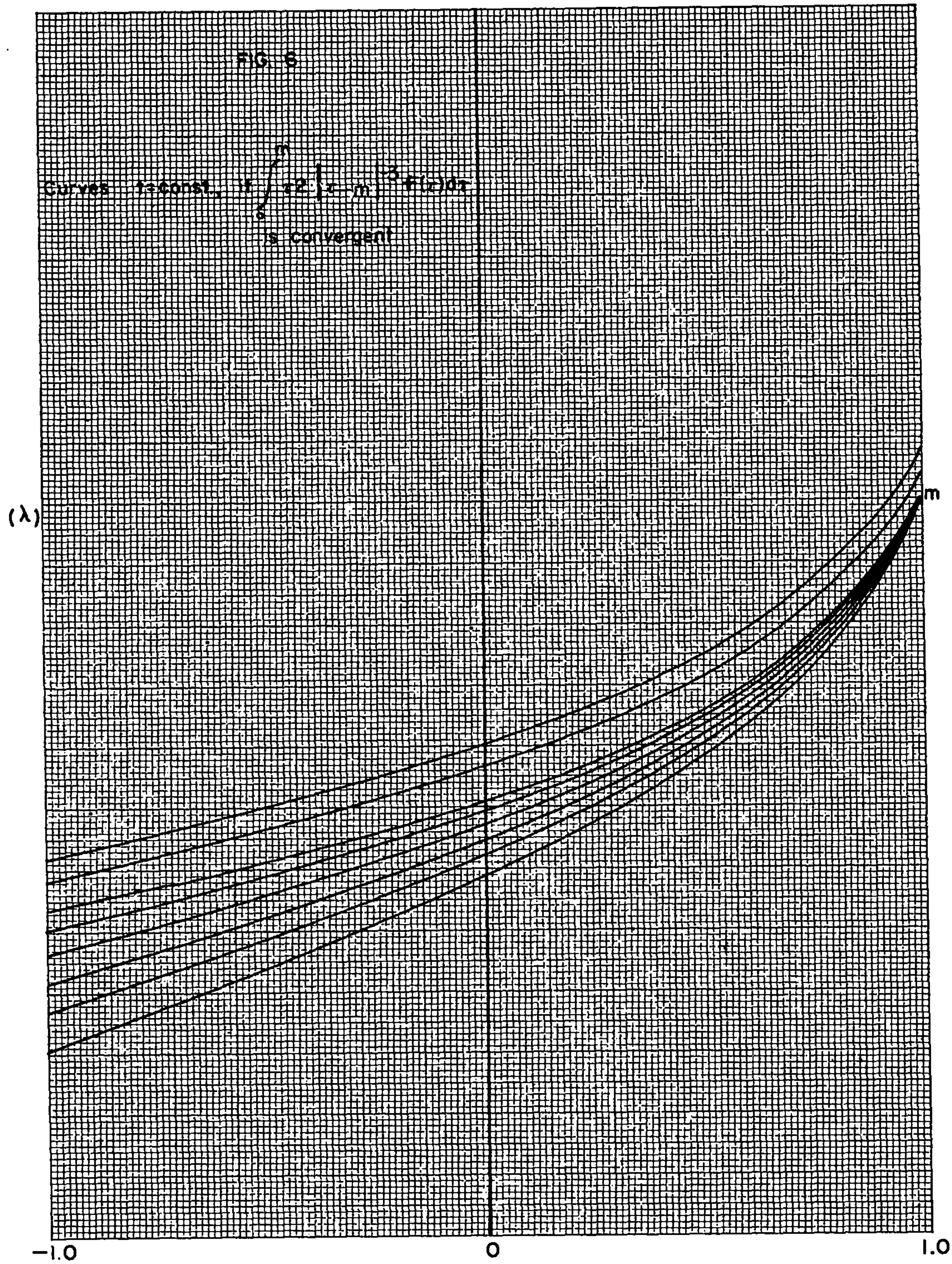


FIG. 6
Curves
is convergent
(λ)
m
-1.0
0
1.0
(r)