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SOME CONSIDERATIONS ON HELMHOLTZ INSTABILITY

PHYSICS

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ABSTRACT

Sections 2 - 6 of this report deal with the case of two incompressible fluids of equal densities. Dimensional arguments are given in Sections 2 and 3 regarding the early and late stages of Helmholtz instability. The second of these arguments is in support of a conjecture that the amplitude eventually grows at a constant rate, of the order of the velocity discontinuity 2v. This conjecture is in agreement with numerical calculations of Rosenhead and Carter, which indicate that the steady-state rate of growth is about 0.34U. It is pointed out in Section 4 that such a steady-state will give rise to a "wiping coefficient", as defined by Ingraham and Wheeler, of about 0.17. The character of the mixing in the late stages is discussed in Section 5, where it is concluded that the mixing will be quite fine and complete. The wavelength for maximum growth is derived in Section 6, by considering the transition from exponential to steady-state growth. Using a result of Carter's version of the Rosenhead calculation, this wavelength is found to be about 5.2π times the initial amplitude. In the last section, the initial rate of growth is found for the case of compressible fluids. It is concluded that compressibility will increase the instability, and will not determine a wavelength of maximum growth.

1. Introduction

Helmholtz was the first to remark[1] that a perturbation on an interface separating two portions of fluid having velocities relative to each other is unstable. A clear discussion, including the effects of a gravitational acceleration and surface tension, is given by Lamb,[1] while the effects of viscosity have been considered by Rayleigh.[2] Rosenhead[3] has performed a numerical calculation of the growth of the instability into the region where the non-linear inertia terms become important and has given analytical results to the third order.

It is the purpose of this report to consider the general nature of the instability and the effects of compressibility on the growth for early stages.

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(1) Lamb, Hydrodynamics, 6th Ed., Dover, N.Y., 1945.

(2) Rayleigh, Theory of Sound, 2nd Ed., Dover, N.Y., 1944.

(3) Rosenhead, Proc. Roy. Soc. A170, 1931.

2. The Small-amplitude Theory

Within the realm of the small-amplitude approximation we consider a sinusoidal perturbation on an interface separating two streams of velocity U and - U, density ρ, as in Fig. 1,

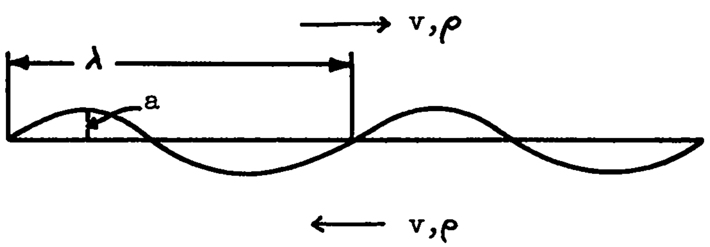


Fig. 1

and inquire as to the mechanism causing growth.

In the free stream, at a distance of the order of $\lambda\!\!\!^{-}(= \lambda/2\pi)$ away from the interface, the momentum per unit volume is of order ρU. If one considers a cylinder with an axis in the direction of the flowing stream, of unit cross sectional area, and of height $U\delta t$, the momentum in this volume is $\rho U^2 \delta t$ and the momentum transferred across unit area per second is $\sim \rho U^2$. This, then, is just the free stream pressure. A pressure gradient must therefore exist across the interface whose magnitude is of order $\rho U^2/\lambda\!\!\!^{-}$ and a pressure at the interface of order $\rho U^2 a/\lambda\!\!\!^{-}$. The resulting force on the interface is then of order $\rho U^2 a$ and the mass which must be accelerated is $\sim \lambda\!\!\!^{-2} \rho$.

The second law then leads to

$$\ddot{a} = \left(U^2/\lambda^2\right) a$$

and thus the expected exponential growth. It is immediately clear from the above discussion that the instability is governed entirely by pressure forces.

3. A Conjecture Concerning the Steady State

Rosenhead's[3] calculation of the later stages of the development of the instability shows the interface to have the form pictured in Fig. 2.

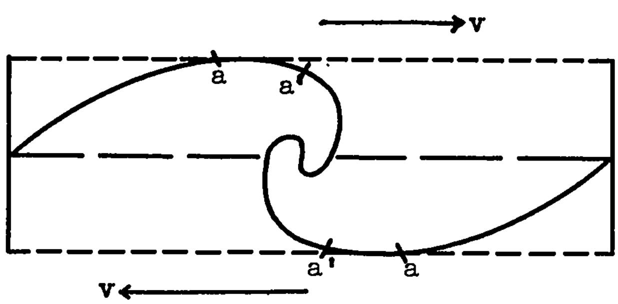
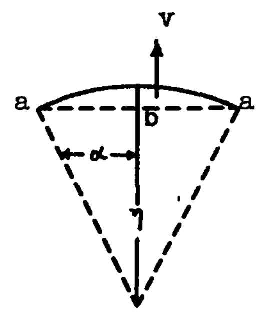


Fig. 2

It is natural to think that the only steady motion that can occur is one of constant velocity in the neighborhood of a-a' in the vertical direction. The only parameters available to determine a steady state, for a fluid which is infinite in extent in all directions, are U and λ.

Therefore, if a steady state exists, the only value that the velocity can attain in the vertical direction is of the order of magnitude of U.

A physical argument which supports this contention can be given. Consider the region a-a of Fig. 2.



Suppose η is the radius of curvature and α is an angle of the first order of small quantities. If we further suppose that the velocity component beneath the line aba , and parallel to it, is small, then the pressure drop across the mass of fluid aa'ba is of order ρU^2 by the argument given previously. The area this works on is $2\alpha\eta(a-a')$. The mass transferred into the volume across the face aba', in a time δt, is of order $\rho v\delta t(2\alpha\eta)$. Therefore the momentum transfer per unit time is $\sim\rho v^2 2\alpha\eta$. Application of the second law leads to $v\sim U$.

This argument is supported by the results of Rosenhead's calculations as given in Table II of his paper. There one can see that the velocity of the area considered does become approximately constant.

A test of this point has been made in a numerical calculation by D. Carter. In the Rosenhead paper, the vortex layer on the interface was approximated by 12 vortices distributed over a wavelength, while in the recalculation this number has been increased to 24 in addition to the use of a more accurate integration scheme. The results again indicate the existence of the steady state, but an even more accurate scheme must be used to carry the integration further.

4. The "Wiping" Coefficient

In LA Report 1593, R. Ingraham and J. Wheeler have introduced a quantitative measure, characteristic of the instability, known as the wiping coefficient. This number is defined as the ratio of the depth of mixing to the relative displacement or "slide" along the interface. The slipstream occurring behind a plane shock incident on a wedge provides an excellent experimental source of Helmholtz instability. The values found for the wiping coefficient from this source range from ~ 0 to 0.3.

The existence of the steady state, postulated above, leads to the conclusion that there does exist a theoretical value for the wiping coefficient. Following Ingraham and Wheeler, we define $a(x)$ to be the maximum displacement measured from the interface in a direction normal to it, and $\Delta(x)$ to be the relative displacement of the two fluids for a particle which has moved a distance x in the lower fluid (see Fig. 3).

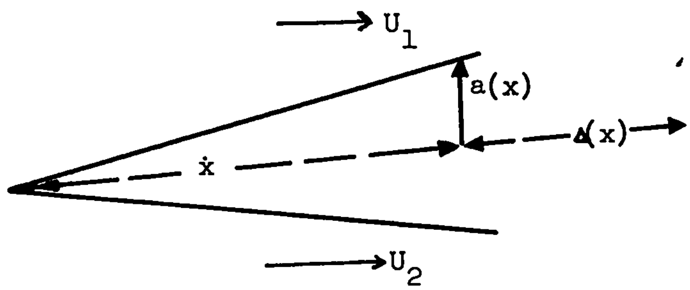


Fig. 3

Then I, the wiping coefficient is $a(x)/\Delta(x)$.

For calculations of Rosenhead and Carter, we can consider the vertical displacement to have taken place with constant velocity v from a virtual time origin t_o. Then I is $V(t - t_o)/(U_1 - U_2)(t - t_o)$, and in the reference frame in which both velocities are equal and opposite this is just $V/2U$.

Rosenhead's calculation leads to a value of 0.2 for I, while the more accurate Carter calculation results in 0.17, both of which lie in the range of the experimental values.

5. Character of Mixing Arising from Helmholtz Instability

The implications of a steady state, independent of wave number, go beyond the existence of a wiping coefficient. If any particular wavelength, of the possible Fourier components of an original disturbance, is chosen for consideration, it will certainly be subject to parasitic wavelengths growing unstably on it. Contrary to the case of Taylor instability, where the velocity of rise of the "bubble" is a function of the wave number, these parasitic disturbances will not be "washed downstream". Each of them will spend a certain time growing exponentially (the time is governed by the wavelength) until the amplitude is $0.3\,\lambda$. After this time, all wavelengths grow with the same constant velocity.

One is then led to the conclusion that the solitary wave shown in Fig. 2, with its feature of ever-narrowing but undisturbed swirls or tongues is not representative of a true interface, but that the region $d - d'$ is one of quite fine and complete mixing.

6. Extent of the Linear Phase and Wavelength for Maximum Growth

The Carter calculation, starting as it does from a quite small amplitude, can be used to determine the point at which the small amplitude approximation breaks down. If one plots the logarithm of the velocity against time one finds a curve such as that shown in Fig. 4.

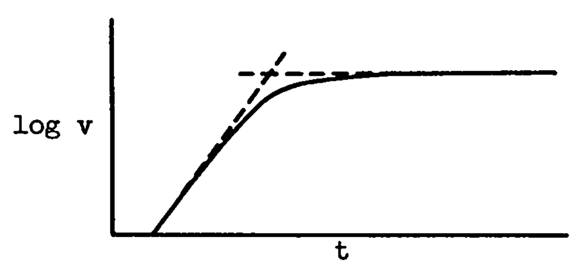


Fig. 4

By extending both linear portions as indicated it is found that the exponential phase can be considered to occur until the amplitude is $0.3\mathcal{X}$, after which the constant velocity phase takes place. The error introduced by this procedure is quite small.

If η_o is the original amplitude of the disturbance, we can define a time τ by

$$0.3\mathcal{X} = \eta_o \, e^{U/\lambda \, \tau}$$

After time τ, the amplitude is given by

$$\eta = v(t-\tau) + 0.3\mathcal{X}$$

and one finds that this expression is maximized by

$$\mathcal{X} = \frac{\eta_o}{0.3} \, e^{0.3 \frac{U}{v} - 1}$$

$$\approx 2.6 \; \eta_o$$

7. The Influence of Compressibility on the Growth of Helmholtz Instability

We take the equations of hydrodynamics in Eulerian form, where the symbols are as used by Lamb[1]

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{\partial p}{\partial y} \tag{1}$$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

We assume an isothermal equation of state for simplicity

$$p = c^2 \rho \tag{2}$$

The method of solution is a perturbation around the steady state specified by the constant stream velocity (in the x-direction) u_o.

Then let

$$u = u_o + u_1$$

$$v = \quad v_1 \tag{3}$$

$$\rho = \rho_o + \rho_1$$

where the quantities with subscript "(1)" are small. It is a trivial result that the zero-th order equations are satisfied by ρ_o = constant.

The only terms remaining in the first order equations are

$$\rho_o\left(\frac{\partial u_1}{\partial t} + u_o \frac{\partial u_1}{\partial x}\right) = - c^2 \frac{\partial \rho_1}{\partial x} \tag{4}$$

$$\rho_o\left(\frac{\partial v_1}{\partial t} + u_o \frac{\partial v_1}{\partial x}\right) = - c^2 \frac{\partial \rho_1}{\partial y} \tag{5}$$

$$\frac{\partial \rho_1}{\partial t} + u_o \frac{\partial \rho_1}{\partial x} + \rho_o\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) = 0 \tag{6}$$

By analogy with the theory for the incompressible case, we assume solutions of the form

$$u_1 = U(y)e^{i(\sigma t - kx)} \tag{7}$$

$$v_1 = V(y)e^{i(\sigma t - kx)} \tag{8}$$

$$\rho_1 = \rho(y)e^{i(\sigma t - kx)} \tag{9}$$

and further assume that the interface is given by

$$\eta = ae^{i(\sigma t - kx)} \tag{10}$$

From (x) we get

$$\rho = \rho_o U/kc^2 \left[\sigma - kU_o\right] \tag{11}$$

while (5) leads to

$$V = \frac{ic^2}{\rho_o[\sigma - kU_o]} \quad \frac{\partial \rho}{\partial y} \tag{12}$$

By means of (11) this becomes

$$V = \frac{i}{k} \frac{\partial U}{\partial y} \tag{13}$$

Putting (11) and (13) in (6) we arrive at the differential equation which determines U(y)

$$U'' - \left[k^2 - \frac{(\sigma - kU_o)^2}{c^2}\right] U = 0 \tag{14}$$

The solutions are then (we use H and L to denote Heavy and Light, heavy fluid on top)

$$U_H = A_H e^{\beta_H y} \qquad \beta_H = -\left[k^2 - \left(\frac{\sigma - kU_{OH}}{c_H}\right)^2\right]^{1/2}$$

$$U_L = A_L e^{\beta_L y} \qquad \beta_L = +\left[k^2 - \left(\frac{\sigma - kU_{OL}}{c_L}\right)^2\right]^{1/2} \tag{15}$$

and these lead to

$$V_H = \frac{i}{k} \beta_H \; A_H \; e^{\beta_H y} \tag{16}$$

$$V_L = \frac{i}{k} \beta_L \; A_L \; e^{\beta_L y}$$

and

$$\rho_H = \frac{\rho_{OH}}{kc_H^{\,2}} \left[\sigma - kU_{OH}\right] A_H e^{\beta_H y}$$

$$\rho_L = \frac{\rho_{OL}}{kc_L^{\,2}} \left[\sigma - kU_{OL}\right] A_L \; e^{\beta_L y} \tag{17}$$

These solutions must satisfy the kinematical condition at the interface for each fluid

$$\frac{dy}{dt} - \frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial x} \frac{dx}{dt} = 0 \tag{18}$$

This condition leads to

$$A_H = \frac{ka}{\beta_H}(\sigma - kU_{OH})$$

$$A_L = \frac{ka}{\beta_L}(\sigma - kU_{OL}) \tag{19}$$

The remaining condition to be satisfied is that of pressure continuity across the interface. The pressure is given by

$$P = c^2(\rho_o + \rho_1)$$

$$= c^2\left[\rho_o + \frac{\rho_o}{kc^2}\left[\sigma - kU\right] Ae^{\beta y}\, e^{i(\sigma t - kx)}\right] \tag{20}$$

Evaluation of all terms at the interface, $y = \eta(x,t)$, leads to the algebraic equation which determines the growth factor

$$\rho_{OL}\frac{\left[\sigma - kU_{OL}\right]^2}{\beta_L} = \rho_{OH}\frac{\left[\sigma - kU_{OH}\right]^2}{\beta_H} \tag{21}$$

In the limit as $c \longrightarrow \infty$, $\beta_L = \beta_H = k$, and this then reduces to the result of the incompressible theory to the same order of accuracy.

$$\sigma_{inc} = k\frac{\rho_H U_H + \rho_L U_L}{\rho_H + \rho_L} + ik\frac{\sqrt{\rho_H \rho_L}}{(\rho_H + \rho_L)}(U_H - U_L)$$

$$k = \tau_{inc} \tag{22}$$

The last equation defines τ_{inc}.

One can solve (21) in the case where terms of higher order than $(U/c)^2$ are neglected. After much algebra one finds

$$\sigma = k\tau_{inc} + k \frac{\rho_H \rho_L}{(\rho_H + \rho_L)^2} (U_H - U_L) \left[\frac{(\tau_{inc} - U_{OH})^2}{2c_H^{\,2}} + \frac{(\tau_{inc} - U_{OL})^2}{2c_L^{\,2}} \right]$$

$$+ ik \frac{\sqrt{\rho_H \rho_L}}{(\rho_H + \rho_L)} (U_H - U_L) \left\{ 1 + \frac{\rho_L \dfrac{(\tau_{inc} - U_{OH})^2}{2c_H^{\,2}} + \rho_H \dfrac{(\tau_{inc} - U_{OL})^2}{2c_L^{\,2}}}{\rho_H + \rho_L} - \right.$$

$$\left. - \frac{1}{2} \left[\frac{(\tau_{inc} - U_{OH})^2}{2c_H^{\,2}} + \frac{(\tau_{inc} - U_{OL})^2}{2c_L^{\,2}} \right] \right\}$$

This result indicates that the influence of compressibility is to increase the growth factor σ over the incompressible case, a perhaps not unexpected result. It also shows that compressibility alone does not determine a wavelength of most rapid growth.