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Informal Report

A Derivation of the Physical Equations Solved in the Inertial Confinement Stability Code DOC

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A DERIVATION OF THE PHYSICAL EQUATIONS SOLVED IN THE INERTIAL CONFINEMENT STABILITY CODE DOC

by

Anthony J. Scannapieco and Charles W. Cranfill

ABSTRACT

There now exists an inertial confinement stability code called DOC, which runs as a postprocessor. DOC (a code that has evolved from a previous code, PANSY) is a spherical harmonic linear stability code that integrates, in time, a set of Lagrangian perturbation equations. Effects due to real equations of state, asymmetric energy deposition, thermal conduction, shock propagation, and a time-dependent zeroth-order state are handled in the code. We present here a detailed derivation of the physical equations that are solved in the code.

I. INTRODUCTION

The viability of an inertial confinement fusion target design is dependent upon the fluid stability of that design. The problems faced in obtaining stability information about a target are copious and, for the most part, have shown themselves to be analytically intractable.

To consider, in a realistic fashion, the fluid stability of an imploding fusion pellet, one must deal with, at least, a time-dependent zeroth-order state, compressible fluid dynamics, real equations of state, thermal conduction, shocks, and a spherical geometry. Given the above items, which must be considered, even a linear analytic perturbation analysis is hopeless.

Given the hopelessness of ever obtaining an analytic result and the necessity of understanding the fluid stability of the fusion pellet, a computational procedure has been employed. The culmination of the computational procedure is the code DOC.

DOC is a linear stability code that integrates in time a set of perturbed fluid equations. The necessary information describing the time-dependent zeroth-order state is obtained from any one-dimensional Lagrangian hydro-code by dumping this information onto disk every hydro timestep. DOC then operates as a postprocessor on this data.

In this report we are specifically concerned with the physics contained in DOC. For another discussion of the techniques used, we refer the reader to the paper by McCrory, Morse, and Taggart, "Growth and Saturation of Instability of Spherical Implosions Driven by Laser or Charged Particle Beams."[1] DOC is an outgrowth of the code PANSY described in the first two sections of that paper. However, the major differences are that the zeroth-order state in DOC is obtained, as was stated above, from any one-dimensional Lagrangian hydro-code, and the thermodynamics in DOC are done for real equations of state rather than for ideal equations of state. DOC in its present form is a one-temperature code; however, it will be upgraded to a three-temperature code in the very near future. A physical viscosity will also be treated in the near future.

This report has been divided into five more sections. Section II describes the set of fluid moment equations that is being perturbed. The basic physics contained in DOC is presented in this section. Section III presents a discussion of the Eulerian and Lagrangian perturbation schemes and the connection between the two. Section IV presents the Eulerian perturbed moment equations. In Sec. V the Eulerian perturbation equations are transformed to a set of Lagrangian perturbed moment equations. Finally, Section VI specializes to a spherically symmetric zeroth-order state, and a spherical harmonic decomposition of the equations is performed to obtain the set of equations that are solved in DOC in its present form.

II. THE SET OF EQUATIONS DEFINING THE FLUID FLOW

The implosion of a fusion pellet can be described by a closed set of fluid moment equations. The particular set of equations chosen dictates what physics will be treated in the analysis of the perturbed fluid equations. Since we are interested in the fluid stability of the pellet, the set of moment equations that follow characterize the fluid aspects of the pellet implosion.

The set of moment equations from which the perturbed equations are derived is obtained as follows. Start with the equation for the conservation of mass

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0, \tag{1}$$

where ρ is the density, and $\underline{v}$ the local average fluid velocity. The local average position $\underline{R}(t)$ of a given element of fluid can then be defined in terms of the local average fluid velocity by

$$\frac{d\underline{R}}{dt} = \underline{v}(\underline{R},t) \;. \tag{2}$$

Next, the equation for the conservation of momentum is employed to calculate the velocity

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla v}\right) = -\underline{\nabla} P \;, \tag{3}$$

where P is the scalar fluid pressure. The pressure is obtained from a density- and temperature-based equation of state

$$P = P(\rho,T). \tag{4}$$

To close the system, an equation for the temperature T is required. This relation comes from the general equation of heat transfer

$$T\frac{ds}{dt} = \frac{1}{\rho}\,\underline{\nabla}\cdot[\kappa(\rho,T)\underline{\nabla}T] + \frac{\dot{q}(\rho,T)}{\rho}, \tag{5}$$

where s is the entropy per unit mass, $\kappa(\rho,T)$ is the thermal conductivity, and $\dot{\underline{q}}(\rho,T)$ is any external sources of energy. Note that it has been assumed that κ and $\dot{\underline{q}}$ are known functions of density and temperature. In our calculations κ is chosen to be the Spitzer electron thermal conductivity, and $\dot{\underline{q}}$ represents energy deposition by laser or electron beam sources.

From the first law of thermodynamics

$$\frac{d\varepsilon}{dt} = \frac{Tds}{dt} + \frac{P}{\rho^2}\frac{d\rho}{dt} \; , \tag{6}$$

Therefore,

$$\frac{Tds}{dt} = \left(\frac{\partial\varepsilon}{\partial T}\right)_\rho \frac{dT}{dt} + \left[\left(\frac{\partial\varepsilon}{\partial\rho}\right)_T - \frac{P}{\rho^2}\right]\frac{d\rho}{dt} \; . \tag{7}$$

Substituting Eq. (7) into Eq. (5) yields

$$\left(\frac{\partial\varepsilon}{\partial T}\right)_\rho \frac{dT}{dt} + \left[\left(\frac{\partial\varepsilon}{\partial\rho}\right)_T - \frac{P}{\rho^2}\right]\frac{d\rho}{dt} = \frac{1}{\rho}\,\nabla\cdot(\kappa\nabla T) + \frac{\dot{q}}{\rho} \; . \tag{8}$$

To close the system of equations the specific internal energy ε is obtained from an equation of state

$$\varepsilon = \varepsilon(\rho,T) \; . \tag{9}$$

TABLE I BASIC FLUID EQUATIONS	
$$\frac{d\underline{R}}{dt} = \underline{v}(\underline{R},\ t)$$	(10)
$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{v}) = 0$$	(11)
$$\rho\left(\frac{\partial\underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\underline{\nabla}P$$	(12)
$$\left(\frac{\partial\varepsilon}{\partial T}\right)_\rho \frac{dT}{dt} = \left[\frac{P}{\rho^2} - \left(\frac{\partial\varepsilon}{\partial\rho}\right)_T\right]\frac{d\rho}{dt} + \frac{1}{\rho}\underline{\nabla}\cdot\kappa\underline{\nabla}T + \frac{\dot{q}}{\rho}$$	(13)
$$P = P(\rho,T)$$	(14)
$$\varepsilon = \varepsilon(\rho,T)$$	(15)
$$\kappa = \kappa(\rho,T)$$	(16)
$$\dot{q} = \dot{q}(\rho,T)$$	(17)

Our resulting set of equations is displayed in Table I.

III. EULERIAN AND LAGRANGIAN PERTURBATION SCHEMES

In an Eulerian perturbation scheme, the fluid variables are perturbed about a given position, say $\underline{r}_o$. Thus, the perturbed fluid quantity is given as

$$\phi_1^E(\underline{r}_o,t) = \phi(\underline{r}_o,t) - \phi_o(\underline{r}_o,t). \tag{18}$$

However, in a Lagrangian perturbation scheme the fluid variables are perturbed about a given fluid element. After the perturbation the fluid element may have moved to a new position, say $\underline{R}$, where

$$\underline{R} = \underline{r}_o + \underline{\xi}(\underline{R},t). \tag{19}$$

We define the quantity $\underline{\xi}$ as the "perturbed displacement vector". The perturbed fluid quantity, in the Lagrangian scheme, is

$$\phi_1^L(\underline{R},t) = \phi(\underline{R},t) - \phi_o(\underline{r}_o,t). \tag{20}$$

The connection between the Eulerian and the Lagrangian perturbed fluid quantities is then easily obtained by a Taylor-series expansion of $\phi(\underline{R},t)$ about the position $\underline{r}_o$. Thus,

$$\phi(\underline{R},t) \cong \phi(\underline{r}_o,t) + \underline{\xi}\cdot\underline{\nabla}_o\phi_o(\underline{r}_o,t), \tag{21}$$

where we have retained only terms which are linear in the perturbed displacement vector $\underline{\xi}$. Substituting Eq. (21) into Eq. (20) and using the relationship in Eq. (18)

$$\phi_1^L(\underline{R},t) \cong \phi_1^E(\underline{r}_o,t) + \underline{\xi}\cdot\underline{\nabla}_o\phi_o(\underline{r}_o,t) \ , \tag{22}$$

where $\underline{\nabla}_o$ indicates that that operator is defined at the position $\underline{r}_o$.

It is convenient, at this point, to relate the operator $\underline{\nabla}$ and the operator $\frac{\partial}{\partial t}$ at the position $\underline{R}$ to the same operators at the position $\underline{r}_o$. The coordinate and time transformations are defined through the relations

$$\underline{r}_o = \underline{R} - \underline{\xi}(\underline{R},t) \tag{23}$$

and

$$t_o = t. \tag{24}$$

It can then be easily shown that

$$\underline{\nabla} = \underline{\nabla}_o - \underline{\nabla}\underline{\xi}\cdot\underline{\nabla}_o \cong \underline{\nabla}_o - \underline{\nabla}_o\underline{\xi}\cdot\underline{\nabla}_o \tag{25}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_o} - \frac{\partial\underline{\xi}}{\partial t}\cdot\underline{\nabla}_o \cong \frac{\partial}{\partial t_o} - \frac{\partial\underline{\xi}}{\partial t_o}\cdot\underline{\nabla}_o \,. \tag{26}$$

We can now turn our attention to the generating equation for $\underline{\xi}$, which is obtained by subtracting the definition

$$\frac{d\underline{r}_o}{dt} \equiv \underline{v}_o(\underline{r}_o,t) \tag{27}$$

from Eq. (10):

$$\frac{d\underline{\xi}}{dt} = \underline{v}(\underline{R},t) - \underline{v}_o(\underline{r}_o,t) \equiv \underline{v}_1^L(\underline{R},t)\,. \tag{28}$$

The relation between the convective derivatives at the positions $\underline{R}$ and $\underline{r}_o$ is obtained from Eqs. (25), (26), and (28)

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underline{v}(\underline{R},t)\cdot\underline{\nabla} \tag{29}$$

$$= \frac{\partial}{\partial t_o} + \underline{v}_o(\underline{r}_o,t_o)\cdot\underline{\nabla}_o \equiv \frac{d}{dt_o}\,. \tag{30}$$

IV. EULERIAN PERTURBED FLUID EQUATIONS

The Eulerian perturbed fluid equations are obtained by writing all the variables in Eqs. (10) through (17) in terms of zeroth- plus first-order components. In what follows, Eulerian perturbation quantities will be indicated by a subscript 1

$$\phi(\underline{r}_o,t) = \phi_o(\underline{r}_o,t) + \phi_1(\underline{r}_o,t), \tag{31}$$

where $\phi_1 \ll \phi_o$. Next, the equations are split into zeroth- and first-order equations, where only terms that are linear in the ϕ_1's are retained in the first-

TABLE II ZEROTH-ORDER EQUATIONS	
$$\frac{d\underline{r}_o}{dt_o} = \underline{v}_o(\underline{r}_o t_o)$$	(32)
$$\frac{\partial \rho_o}{\partial t_o} + \underline{\nabla}_o \cdot (\rho_o \underline{v}_o) = 0$$	(33)
$$\rho_o\left(\frac{\partial \underline{v}_o}{\partial t} + \underline{v}_o \cdot \underline{\nabla}_o \underline{v}_o\right) = -\underline{\nabla}_o P_o$$	(34)
$$\left(\frac{\partial \varepsilon}{\partial T}\right)_{\rho_o} \frac{dT_o}{dt_o} = \left[\frac{P_o}{\rho_o^2} - \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{T_o}\right] \frac{d\rho_o}{dt_o} + \frac{1}{\rho_o} \underline{\nabla}_o \cdot \kappa_o \underline{\nabla} T_o + \frac{\dot{q}_o}{\rho_o}$$	(35)
$$P_o = P_o(\rho_o, T_o)$$	(36)
$$\varepsilon_o = \varepsilon_o(\rho_o, T_o)$$	(37)
$$\kappa_o = \kappa_o(\rho_o, T_o)$$	(38)
$$\dot{q}_o = \dot{q}_o(\rho_o, T_o)$$	(39)

order equations. This is possible only because the zeroth-order equations are satisfied identically with zeroth-order quantities.

The sets of zeroth- and first-order equations in Table II are readily obtained. In Table III, Eqs. (40) through (47) comprise the Eulerian perturbed fluid equations.

TABLE III
EULERIAN PERTURBED FLUID EQUATIONS

$$\frac{\partial \rho_1}{\partial t_o} + \underline{\nabla}_o \cdot (\rho_1 \underline{v}_o + \rho_o \underline{v}_1) = 0 \tag{40}$$

$$\rho_o\left(\frac{\partial \underline{v}_1}{\partial t_o} + \underline{v}_o \cdot \underline{\nabla}_o \underline{v}_1 + \underline{v}_1 \cdot \underline{\nabla}_o \underline{v}_o\right) = -\underline{\nabla}_o P_1 + \frac{\rho_1}{\rho_o} \underline{\nabla}_o P_o \tag{41}$$

$$\begin{aligned}\left(\frac{\partial \varepsilon}{\partial T}\right)_{\rho o}\left(\frac{dT_1}{dt_o} + \underline{v}_1 \cdot \underline{\nabla}_o T_o\right) + \left(\frac{\partial \varepsilon}{\partial T}\right)_{\rho 1} \frac{dT_o}{dt_o} &= \left[\frac{Po}{\rho_o 2} - \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{To}\right]\left(\frac{d\rho_1}{dt_o} + \underline{v}_1 \cdot \underline{\nabla}_o \rho_o\right) \\ &+ \left[\frac{P_1}{\rho_o 2} - \frac{2P_o}{\rho_o 3}\rho_1 - \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{T1}\right] \frac{d\rho_o}{dt_o} \\ &+ \frac{1}{\rho_o}\underline{\nabla}_o \cdot (\kappa_o \underline{\nabla}_o T_1 + \S_1 \underline{\nabla}_o T_o) \\ &+ \frac{\dot{q}_1}{\rho_o} - \frac{\rho_1}{\rho_o 2}\left(\dot{q}_o + \underline{\nabla}_o \cdot \kappa_o \underline{\nabla}_o T_o\right)\end{aligned} \tag{42}$$

$$P_1 = \left(\frac{\partial P}{\partial T}\right)_{\rho o} T_1 + \left(\frac{\partial P}{\partial \rho}\right)_{To} \rho_1 \tag{43}$$

$$\kappa_1 = \left(\frac{\partial \kappa}{\partial T}\right)_{\rho o} T_1 + \left(\frac{\partial \kappa}{\partial \rho}\right)_{To} \rho_1 \tag{44}$$

$$\dot{q}_1 = \left(\frac{\partial \dot{q}}{\partial T}\right)_{\rho o} T_1 + \left(\frac{\partial \dot{q}}{\partial \rho}\right)_{To} \rho_1 \tag{45}$$

$$\left(\frac{\partial \varepsilon}{\partial T}\right)_{\rho 1} = \left[\frac{\partial}{\partial T}\left(\frac{\partial \varepsilon}{\partial T}\right)\right]_{\rho\ \rho o} T_1 + \left[\left(\frac{\partial}{\partial \rho}\ \frac{\partial \varepsilon}{\partial T}\right)_{\rho}\right]_{To} \rho_1 \tag{46}$$

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{T1} = \left[\frac{\partial}{\partial T}\left(\frac{\partial \varepsilon}{\partial \rho}\right)_T\right]_{\rho o} T_1 + \left[\frac{\partial}{\partial \rho}\left(\frac{\partial \varepsilon}{\partial \rho}\right)_T\right]_{To} \rho_1 \tag{47}$$

V. LAGRANGIAN PERTURBED FLUID EQUATIONS

In Table IV we are applying Eq. (22) to each of the variables in Eqs. (40) through (47), immediately yielding the Lagrangian perturbed fluid equations. (See Appendix for algebra.)

TABLE IV
LAGRANGIAN PERTURBED FLUID EQUATIONS

$$\rho_1' = \rho_{1I}' - \rho_o(\underline{\nabla}_o\cdot\underline{\xi}) \tag{48}$$

$$\frac{d\underline{\xi}}{dt_o} = \underline{v}_1' \tag{49}$$

$$\rho_o\frac{d\underline{v}_1'}{dt_o} = -\underline{\nabla}_o P_1' + \frac{\rho_1'}{\rho_o}\underline{\nabla}_o P_o + \underline{\nabla}_o\underline{\xi}\cdot\underline{\nabla}_o P_o \tag{50}$$

$$\frac{\partial\varepsilon}{\partial T}_{\rho o}\frac{dT_1'}{dt_o} + \frac{\partial\varepsilon}{\partial T}_{\rho 1}\frac{dT_o}{dt_o} = \frac{P_o}{\rho_o^2} - \frac{\partial\varepsilon}{\partial\rho}_{To}\frac{d\rho_1'}{dt_o} + \frac{P_1'}{\rho_o^2} - \frac{2P_o}{\rho_o^3}\rho_1' - \frac{\partial\varepsilon}{\partial\rho}_{T_1}'\frac{d\rho_o}{dt_o}$$
$$+ \frac{1}{\rho_o}\underline{\nabla}_o\cdot[\kappa_o\underline{\nabla}_o T_1' + \kappa_1'\underline{\nabla}_o T_o] - \frac{\rho_1'}{\rho_o^2}\underline{\nabla}_o\cdot(\kappa_o\underline{\nabla}_o T_o)$$
$$+ \frac{1}{\rho_o}\dot{q}_1' - \frac{\rho_1'}{\rho_o^2}\dot{q}_o - \frac{1}{\rho_o}[\underline{\nabla}_o\cdot(\underline{\nabla}_o\underline{\xi}\cdot\kappa_o\underline{\nabla}_o T_o)$$
$$+ \underline{\nabla}_o\underline{\xi}:\underline{\nabla}_o(\kappa_o\underline{\nabla}_o T_o)] \tag{51}$$

$$P_1' = \left(\frac{\partial P}{\partial T}\right)_{\rho o} T_1' + \left(\frac{\partial P}{\partial\rho}\right)_{To}\rho_1' \tag{52}$$

$$\kappa_1' = \left(\frac{\partial\kappa}{\partial T}\right)_{\rho o} T_1' + \left(\frac{\partial\kappa}{\partial\rho}\right)_{To}\rho_1' \tag{53}$$

$$\dot{q}_1' = \left(\frac{\partial\dot{q}}{\partial T}\right)_{\rho o} T_1' + \left(\frac{\partial\dot{q}}{\partial\rho}\right)_{To}\rho_1' \tag{54}$$

$$\left(\frac{\partial\varepsilon}{\partial T}\right)_{\rho 1}' = \left[\frac{\partial}{\partial T}\left(\frac{\partial\varepsilon}{\partial T}\right)_\rho\right]_{\rho o} T_1' + \left[\frac{\partial}{\partial\rho}\left(\frac{\partial\varepsilon}{\partial T}\right)_\rho\right]_{To}\rho_1' \tag{55}$$

$$\left(\frac{\partial\varepsilon}{\partial\rho}\right)_{T1}' = \left[\frac{\partial}{\partial T}\left(\frac{\partial\varepsilon}{\partial\rho}\right)_T\right]_{\rho o} T_1' + \left[\frac{\partial}{\partial\rho}\left(\frac{\partial\varepsilon}{\partial\rho}\right)_T\right]_{To}\rho_1' \tag{56}$$

VI. SPHERICAL ANALYSIS AND HARMONIC DECOMPOSITION

Next, we specialize to the case in which the zeroth-order state of the fluid system is spherically symmetric. It will be advantageous, for computational reasons, to recast our perturbed equations in the following manner.

$$\underline{\nabla}_\Omega\cdot \equiv \frac{1}{r}\frac{\partial}{\partial\underline{\Omega}}\cdot \equiv \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\hat{e}_\theta\cdot) + \frac{\partial}{\partial\phi}(\hat{e}_\phi\cdot)\right] , \tag{57}$$

$$\underline{\xi} = \xi_r \hat{e}_r + r_o \underline{\Omega} \quad , \tag{58}$$

and
$$\underline{v}_1' = \dot{\xi}_r \hat{e}_r + v_o \underline{\Omega} + r_o \dot{\underline{\Omega}} \quad . \tag{59}$$

Operate on Eq. (49) with the operator $\underline{\nabla}_\Omega \cdot$:

$$\underline{\nabla}_\Omega \cdot \frac{d\underline{\xi}}{dt_o} = \underline{\nabla}_\Omega \cdot \underline{v}_1' \quad , \tag{60}$$

$$\frac{d}{dt_o} (\underline{\nabla}_\Omega \cdot \underline{\xi}) + \frac{v_o}{r} (\underline{\nabla}_\Omega \cdot \underline{\xi}) = \underline{\nabla}_\Omega \cdot \underline{v}_1' \quad , \tag{61}$$

$$\frac{d}{dt_o} (\underline{\nabla}_\Omega \cdot \underline{\xi}) + (\underline{\nabla}_\Omega \cdot v_o \underline{\Omega}) = \underline{\nabla}_\Omega \cdot \underline{v}_1' \quad , \tag{62}$$

and finally

$$\frac{d}{dt_o} (\underline{\nabla}_\Omega \cdot \underline{\xi}) = \underline{\nabla}_\Omega \cdot (\underline{v}_1' - v_o \underline{\Omega}) \quad . \tag{63}$$

To continue we calculate the time rate of change of $r^2 \underline{\nabla}_\Omega \cdot (\underline{v}_1' - v_o \underline{\Omega})$:

$$\frac{d}{dt_o} (r^2 \underline{\nabla}_\Omega \cdot (\underline{v}_1' - v_o \underline{\Omega})) = v_o \frac{\partial}{\partial \underline{\Omega}} \cdot (\underline{v}_1' - v_o \underline{\Omega}) + r \frac{\partial}{\partial \underline{\Omega}} \cdot \left[\frac{d\underline{v}_1'}{dt_o} - \frac{dv_o}{dt_o} \underline{\Omega} - v_o \dot{\underline{\Omega}} \right] \quad . \tag{64}$$

Using Eqs. (34), (50), and (59) and rearranging terms, we can recast Eq. (64) as follows.

$$\begin{aligned} \frac{d}{dt_o} (r^2 \underline{\nabla}_\Omega \cdot (\underline{v}_1' - v_o \underline{\Omega})) &= \frac{r^2}{\rho_o} \left[-\nabla_\Omega^2 P_1' + \underline{\nabla}_\Omega \cdot (\underline{\nabla}_o \underline{\xi} \cdot \underline{\nabla}_o P_o) + \frac{1}{r} \frac{\partial P_o}{\partial r} (\underline{\nabla}_\Omega \cdot \underline{\xi}) \right] . \\ &= \frac{r^2}{\rho_o} \left[-\nabla_\Omega^2 P_1' + \left(\nabla_\Omega^2 \xi_r - \frac{1}{r} (\underline{\nabla}_\Omega \cdot \underline{\xi}) \right) \frac{\partial P_o}{\partial r} + \frac{1}{r} \frac{\partial P_o}{\partial r} (\underline{\nabla}_\Omega \cdot \underline{\xi}) \right] \\ &= \frac{-1}{\rho_o} \left[r^2 \nabla_\Omega^2 P_1' - \frac{\partial P_o}{\partial r} r^2 \nabla_\Omega^2 \xi_r \right] \quad . \end{aligned} \tag{65}$$

Assuming the spherical nature of the zeroth-order fluid state, Eq. (51) can be recast as follows.

$$\left(\frac{\partial\varepsilon}{\partial T}\right)_{\rho o}\frac{dT_1'}{dt_o} \quad \left(\frac{\partial\varepsilon}{\partial T}\right)_{\rho 1}\frac{dT_o}{dt_o} = \left[\frac{P_o}{\rho_o^2} - \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{To}\right]\frac{d\rho_1'}{dt_o} + \left[\frac{P_1'}{\rho_o^2} - \frac{2P_o}{\rho_o^3}\rho_1' - \left(\frac{\partial\varepsilon}{\partial\rho}\right)'_{T1}\right]\frac{d\rho_o}{dt_o}$$
$$+ \frac{1}{\rho_o}\left[\frac{\partial\kappa_o}{\partial r}\frac{\partial T_1'}{\partial r} + \kappa_o\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_1'}{\partial r}\right) + \frac{\partial\kappa_1'}{\partial r}\frac{\partial T_o}{\partial r} + \frac{\kappa_1'}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_o}{\partial r}\right)\right.$$
$$\left. + \kappa_o\nabla_\Omega^2 T_1'\right] + \frac{\dot{q}_1}{\rho_o} - \frac{\rho_1'}{\rho_o^2}\dot{q}_o - \frac{1}{\rho_o}\left[\underset{\sim}{\nabla}_o\cdot\left(\underset{\sim}{\nabla}_o\xi\cdot\kappa_o\frac{\partial T_o}{\partial r}\hat{e}_r\right)\right.$$
$$\left. + \underset{\sim}{\nabla}_o\xi:\underset{\sim}{\nabla}_o\left(\kappa_o\frac{\partial T_o}{\partial r}\hat{e}_r\right)\right] . \tag{66}$$

The last two terms in Eq. (66) can be expanded and put in the following form.

$$-\frac{1}{\rho_o}\underset{\sim}{\nabla}_o\cdot\left(\underset{\sim}{\nabla}_o\xi\cdot\kappa_o\frac{\partial T_o}{\partial r}\hat{e}_r\right) + \underset{\sim}{\nabla}_o\xi:\underset{\sim}{\nabla}_o\left(\kappa_o\frac{\partial T_o}{\partial r}\hat{e}_r\right)\Bigg] = \frac{-1}{\rho_o}\left[\left(2\frac{\xi r}{r^2} - \frac{2}{r}\frac{\partial\xi r}{\partial r} + \frac{\partial^2\xi r}{\partial r^2} - \nabla_\Omega^2\xi r\right)\right.$$
$$\left.\kappa_o\frac{\partial T_o}{\partial r} + 2\frac{\partial\xi}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\kappa_o\frac{\partial T_o}{\partial r}\right)\right] . \tag{67}$$

The perturbed energy equation then becomes

$$\left(\frac{\partial\varepsilon}{\partial T}\right)_{\rho o}\frac{dT_1'}{dt_o} + \left(\frac{\partial\varepsilon}{\partial T}\right)'_{\rho 1}\frac{dT_o}{dt_o} = \left[\frac{P_o}{\rho_o^2} - \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{To}\right]\frac{d\rho_1'}{dt_o} + \left[\frac{P_1'}{\rho_o^2} - \frac{2P_o}{\rho_o^3}\rho_1' - \left(\frac{\partial\varepsilon}{\partial\rho}\right)'_{T1}\right]\frac{d\rho_o}{dt_o} .$$
$$+ \frac{1}{\rho_o}\left[\frac{\partial\kappa_o}{\partial r}\frac{\partial T_1'}{\partial r} + \frac{\kappa_o}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_1'}{\partial r}\right) + \frac{\partial\kappa_1'}{\partial r}\frac{\partial T_o}{\partial r} + \frac{\kappa_1'}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_o}{\partial r}\right)\right.$$
$$+ \kappa_o\nabla_\Omega^2 T_1' - \left(2\frac{\xi_r}{r^2} - \frac{2}{r}\frac{\partial\xi_r}{\partial r} + \frac{\partial^2\xi_r}{\partial r^2} + \nabla_\Omega^2\xi_r\right)$$
$$\left.\times\ \kappa_o\frac{\partial T_o}{\partial r} - 2\frac{\partial\xi_r}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\kappa_o\frac{\partial T_o}{\partial r}\right)\right] - \frac{\rho_1'}{\rho_o^2}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\kappa_o\frac{\partial T_o}{\partial r}\right)$$
$$+ \frac{\dot{q}_1}{\rho_o} - \frac{\rho_1'}{\rho_o^2}\dot{q}_o . \tag{68}$$

We now see that the angular variation only comes through angular divergences, thus allowing us to replace the thirteen equations, Eqs. (48) through (56), with the eleven equations that are displayed in Table V.

TABLE V
PERTURBED FLUID EQUATIONS

$$\rho_1' = \rho_{1I}' - \rho_o\left[\frac{2\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + (\underline{\nabla}_\Omega \cdot \underline{\xi})\right] \tag{69}$$

$$\frac{d\xi r}{dt_o} = v_{1r}' \tag{70}$$

$$\frac{d}{dt_o}(\underline{\nabla}_\Omega \cdot \underline{\xi}) = \underline{\nabla}_\Omega \cdot (\underline{v}_1' - v_o\underline{\Omega}) \tag{71}$$

$$\begin{aligned}\rho_o \frac{dv_{1r}'}{dt_o} = &-\left[\frac{\partial P_1'}{\partial r} + \left(\frac{2\xi r}{r} + (\underline{\nabla}_\Omega \cdot \underline{\xi})\right)\frac{\partial P_o}{\partial r}\right] \\ &+ \frac{\rho_{1I}'}{\rho_o}\frac{\partial P_o}{\partial r}\end{aligned} \tag{72}$$

$$\rho_o \frac{d}{dt_o}[r^2\underline{\nabla}_\Omega \cdot (\underline{v}_1' - v_o\underline{\Omega})] = -r^2\left[\nabla_\Omega^2 P_1' - \frac{\partial P_o}{\partial r}\nabla_\Omega^2 \xi_r\right] \tag{73}$$

$$\begin{aligned}\left(\frac{\partial \varepsilon}{\partial T}\right)_{\rho o}\frac{dT_1'}{dt_o} + \left(\frac{\partial \varepsilon}{\partial T}\right)_{\rho 1}'\frac{dT_o}{dt_o} = &\left(\frac{P_o}{\rho_o^2} - \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{To}\right)\frac{d\rho_1'}{dt_o} + \left(\frac{P_1'}{\rho_o^2} - \frac{2P_o}{\rho_o^3}\rho_1'\right. \\ &\left. - \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{T1}'\right)\frac{d\rho_o}{dt_o} + \frac{1}{\rho_o}\left[\frac{\partial \kappa_o}{\partial r}\frac{\partial T_1'}{\partial r} + \frac{\kappa_o}{r^2}\frac{\partial}{\partial r}\right. \\ &\times\left(r^2\frac{\partial T_1'}{\partial r}\right) + \frac{\partial \kappa_1'}{\partial r}\frac{\partial T_o}{\partial r} + \frac{\kappa_1'}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_o}{\partial r}\right) \\ &+ \kappa_o\nabla_\Omega^2 T_1' - \left(2\frac{\xi_r}{r^2} - \frac{2}{r}\frac{\partial \xi_r}{\partial r} + \frac{\partial^2 \xi_r}{\partial r^2}\right. \\ &\left.\left. + \nabla_\Omega^2 \xi_r\right)\kappa_o\frac{\partial T_o}{\partial r} - 2\frac{\partial \xi_r}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\kappa_o\frac{\partial T_o}{\partial r}\right)\right] \\ &- \frac{\rho_1'}{\rho_o^2}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\kappa_o\frac{\partial T_o}{\partial r}\right) + \frac{\dot{q}_1'}{\rho_o} - \frac{\rho_1'}{\rho_o^2}\dot{q}_o\end{aligned} \tag{74}$$

$$P_1' = \left(\frac{\partial P}{\partial T}\right)_{\rho o} T_1' + \left(\frac{\partial P}{\partial \rho}\right)_{To} \rho_1' \tag{75}$$

$$\kappa_1' = \left(\frac{\partial \kappa}{\partial T}\right)_{\rho o} T_1' + \left(\frac{\partial \kappa}{\partial \rho}\right)_{To} \rho_1' \tag{76}$$

$$\dot{q}_1' = \left(\frac{\partial \dot{q}}{\partial T}\right)_{\rho o} T_1' + \left(\frac{\partial \dot{q}}{\partial \rho}\right)_{To} \rho_1' \tag{77}$$

$$\left(\frac{\partial \varepsilon}{\partial T}\right)_{\rho 1}' = \left(\frac{\partial}{\partial T}\left(\frac{\partial \varepsilon}{\partial T}\right)_\rho\right)_{\rho o} T_1' + \left(\frac{\partial}{\partial \rho}\left(\frac{\partial \varepsilon}{\partial T}\right)_\rho\right)_{To} \rho_1' \tag{78}$$

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{T1}' = \left(\frac{\partial}{\partial T}\left(\frac{\partial \varepsilon}{\partial \rho}\right)_T\right)_{\rho o} T_1' + \left(\frac{\partial}{\partial \rho}\left(\frac{\partial \varepsilon}{\partial \rho}\right)_\rho\right)_{To} \rho_1' \tag{79}$$

At this point we perform a spherical harmonic decomposition of Eqs. (69) through (79). Since we are now considering only scalar quantities, all perturbed quantities can be written as

$$\psi_1'(r_o,\theta,\phi,t_o) = \sum_{\ell,m} \psi_1^{\ell,m}(r_o,t_o)\; Y_\ell^m(\theta,\phi) \quad . \tag{80}$$

Keeping in mind that the effect of the operator $\underline{\nabla}_\Omega^2$ on the spherical harmonics is simply

$$\nabla_\Omega^2 Y_\ell^m = -\frac{\ell(\ell+1)}{r^2} Y_\ell^m \quad , \tag{81}$$

we can write our Eqs. (69) through (79) in terms of their spherical harmonic amplitudes. If we define

$$A^\ell \equiv v_{1r}^{\ell,m} \quad , \tag{82}$$

$$B^\ell \equiv \xi_r^{\ell,m} \quad , \tag{83}$$

$$C^\ell \equiv [r^2 \underline{\nabla}_\Omega \cdot (\underline{v}_1' - v_o \underline{\Omega})]^{\ell,m} \quad , \tag{84}$$

and

$$D^\ell \equiv [\underline{\nabla}_\Omega \cdot \underline{\xi}]^{\ell,m} \quad , \tag{85}$$

we can rewrite our equations as given in Table VI.

Each of the variables with a superscript ℓ in Eqs. (86) through (96) represents the spherical harmonic amplitude of its respective perturbation variable. Therefore, all variables are now only a function of r_o, t_o, and ℓ. It is important to note that the equations in Table VI are independent of the m index of the spherical harmonic.

Equations (86) through (96) displayed in Table VI are the physical equations that are solved in the code DOC.

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1. R. L. McCrory, R. L. Morse, and K. A. Taggart, "Growth and Saturation of Instability of Spherical Implosions Driven by Laser or Charged Particle Beams," Nuclear Science and Engineering 64, 163-176 (1977).

TABLE VI

PERTURBED SPHERICAL HARMONIC EQUATIONS

$$\rho_1{}^{\ell} = \rho_{1I}{}^{\ell} - \rho_o\left[2\,\frac{B^{\ell}}{r} + \frac{\partial B^{\ell}}{\partial r} + D^{\ell}\right] \tag{86}$$

$$\frac{dB^{\ell}}{dt_o} = A^{\ell} \tag{87}$$

$$\frac{dD^{\ell}}{dt_o} = \frac{C^{\ell}}{r^2} \tag{88}$$

$$\frac{dA^{\ell}}{dt_o} = -\frac{1}{\rho_o}\left[\frac{\partial P_1{}^{\ell}}{\partial r} + \left(\frac{2B^{\ell}}{r} + D^{\ell}\right)\frac{\partial P_o}{\partial r}\right] + \frac{\rho_{1I}{}^{\ell}}{\rho_o{}^2}\frac{\partial P_o}{\partial r} \tag{89}$$

$$\frac{dC^{\ell}}{dt_o} = \frac{\ell(\ell+1)}{\rho_o}\left[P_1{}^{\ell} - B^{\ell}\frac{\partial P_o}{\partial r}\right] \tag{90}$$

$$\begin{aligned}\left(\frac{\partial\varepsilon}{\partial T}\right)_{\rho o}\frac{dT_1{}^{\ell}}{dt_o} + \left(\frac{\partial\varepsilon}{\partial T}\right)_{\rho 1}^{\ell}\frac{dT_o}{dt_o} = & -\frac{\rho_1{}^{\ell}}{\rho_o{}^2}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\kappa_o\frac{\partial T_o}{\partial r}\right) + \left[\frac{P_o}{\rho_o{}^2} - \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{To}\right]\frac{d\rho_1{}^{\ell}}{dt_o} \\ & + \left[\frac{P_1{}^{\ell}}{\rho_o{}^2} - 2\frac{P_o}{\rho_o{}^3}\rho_1{}^{\ell} - \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{T1}^{\ell}\right]\frac{d\rho_o}{dt_o} \\ & + \frac{1}{\rho} + \frac{1}{\rho_o}\ \frac{\partial\kappa_o}{\partial r}\ \frac{\partial T_1{}^{\ell}}{\partial r} + \frac{\kappa_o}{r^2}\frac{\partial}{\partial r}\ r^2\frac{\partial T_1{}^{\ell}}{\partial r} + \frac{\partial\kappa_1{}^{\ell}}{\partial r}\ \frac{\partial T_o}{\partial r} \\ & + \frac{\kappa_1{}^{\ell}}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_o}{\partial r}\right) - \kappa_o\frac{\ell(\ell+1)}{r^2}T_1{}^{\ell} + \dot{q}_1{}^{\ell} \\ & \times \frac{\rho_1{}^{\ell}}{\rho_o}\dot{q}_o - \left(2\frac{B^{\ell}}{r^2} - \frac{2}{r}\frac{\partial B^{\ell}}{\partial r} + \frac{\partial^2 B^{\ell}}{\partial r^2} - \frac{\ell(\ell+1)}{r^2}B^{\ell}\right) \\ & \times \kappa_o\frac{\partial T_o}{\partial r} - \left[2\frac{\partial B^{\ell}}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\kappa_o\frac{\partial T_o}{\partial r}\right)\right]\end{aligned} \tag{91}$$

$$P_1 = \left(\frac{\partial P}{\partial T}\right)_{\rho o}T_1{}^{\ell} + \left(\frac{\partial P}{\partial\rho}\right)_{To}\rho_1{}^{\ell} \tag{92}$$

$$\kappa_1{}^{\ell} = \left(\frac{\partial\kappa}{\partial T}\right)_{\rho o}T_1{}^{\ell} + \left(\frac{\partial\kappa}{\partial\rho}\right)_{To}\rho_1{}^{\ell} \tag{93}$$

$$\dot{q}_1{}^{\ell} = \left(\frac{\partial\dot{q}}{\partial T}\right)_{\rho o}T_1{}^{\ell} + \left(\frac{\partial\dot{q}}{\partial\rho}\right)_{To}\rho_1{}^{\ell} \tag{94}$$

$$\left(\frac{\partial\varepsilon}{\partial T}\right)_{\rho 1}^{\ell} = \left[\frac{\partial}{\partial T}\left(\frac{\partial\varepsilon}{\partial T}\right)_{\rho}\right]_{\rho o}T_1{}^{\ell} + \left[\frac{\partial}{\partial\rho}\left(\frac{\partial\varepsilon}{\partial T}\right)_{\rho}\right]_{To}\rho_1{}^{\ell} \tag{95}$$

$$\left(\frac{\partial\varepsilon}{\partial\rho}\right)_{T1}^{\ell} = \left[\frac{\partial}{\partial T}\left(\frac{\partial\varepsilon}{\partial\rho}\right)_{T}\right]_{\rho o}T_1{}^{\ell} + \left[\frac{\partial}{\partial\rho}\left(\frac{\partial\varepsilon}{\partial\rho}\right)_{T}\right]_{To}\rho_1{}^{\ell} \tag{96}$$

APPENDIX

ALGEBRA FOR LAGRANGIAN PERTURBED FLUID EQUATIONS

The Lagrangian perturbation equations are obtained from Eqs. (40) through (47) by expressing the Eulerian perturbations ϕ_1 in terms of Lagrangian perturbations using Eq. (22). In what follows Lagrangian perturbation quantities are indicated by a subscript $_1$ and a prime. Making the appropriate substitutions in Eq. (40) yields

$$\frac{\partial \rho_1'}{\partial t_o} - \frac{\partial}{\partial t_o}(\underline{\xi}\cdot\underline{\nabla}_o\rho_o) + \underline{\nabla}_o\cdot[\rho_1'\underline{v}_o + \rho_o\underline{v}_1' - (\underline{\xi}\cdot\underline{\nabla}_o\rho_o)\underline{v}_o - \rho_o\underline{\xi}\cdot\underline{\nabla}_o\underline{v}_o] = 0 \ . \quad \text{(A1)}$$

The last two terms in Eq. (A1) can be rewritten as

$$-\underline{\nabla}_o\cdot[(\underline{\xi}\cdot\underline{\nabla}_o\rho_o)\underline{v}_o] = -(\underline{v}_o\cdot\underline{\nabla}_o\underline{\xi})\cdot\nabla_o\rho_o - (\underline{\xi}\cdot\underline{\nabla}_o)[\underline{\nabla}_o\cdot(\rho_o\underline{v}_o)] + (\underline{\xi}\cdot\underline{\nabla}_o\underline{v}_o)\cdot\underline{\nabla}_o\rho_o$$
$$+ \rho_o(\underline{\xi}\cdot\underline{\nabla}_o)(\underline{\nabla}_o\cdot\underline{v}_o) \ , \quad \text{(A2)}$$

and

$$-\underline{\nabla}_o\cdot[\rho_o\underline{\xi}\cdot\underline{\nabla}_o\underline{v}_o] = -\rho_o(\underline{\xi}\cdot\underline{\nabla}_o)(\underline{\nabla}_o\cdot\underline{v}_o) - (\underline{\xi}\cdot\underline{\nabla}_o\underline{v}_o)\cdot\underline{\nabla}_o\rho_o - \rho_o\underline{\nabla}_o\underline{\xi}:\underline{\nabla}\,\underline{v}_o \ . \quad \text{(A3)}$$

Substituting Eqs. (A2) and (A3) into Eq. (A1) gives

$$\frac{\partial \rho_1'}{\partial t_o} - \frac{\partial \underline{\xi}}{\partial t_o}\cdot\underline{\nabla}_o\rho_o - \underline{\xi}\cdot\underline{\nabla}_o\frac{\partial \rho_o}{\partial t_o} + \underline{\nabla}_o\cdot[\rho_1'\underline{v}_o + \rho_o\underline{v}_1'] - (\underline{v}_o\cdot\underline{\nabla}_o\underline{\xi})\cdot\underline{\nabla}_o\rho_o$$
$$-(\underline{\xi}\cdot\underline{\nabla}_o)(\underline{\nabla}_o\cdot\rho_o\underline{v}_o) - \rho_o\underline{\nabla}_o\xi:\underline{\nabla}_o\underline{v}_o = 0 \ . \quad \text{(A4)}$$

Using the generating equation for $\underline{\xi}$ in terms of the Lagrangian perturbed velocity $\underline{v}_1'$ and collecting terms, Eq. (A4) can be recast into the form

$$\frac{d\rho_1'}{dt_o} - \frac{\rho_1'}{\rho_o}\frac{d\rho_o}{dt_o} + \rho_o\underline{\nabla}_o\cdot\underline{v}_1' - \rho_o\underline{\nabla}_o\underline{\xi}:\underline{\nabla}_o\underline{v}_o = 0 \ . \quad \text{(A5)}$$

However,

$$\underline{\nabla}_o\cdot\underline{v}_1' = \frac{\partial}{\partial t_o}(\underline{\nabla}_o\cdot\underline{\xi}) + \underline{\nabla}_o\cdot[\underline{v}_o\cdot\underline{\nabla}_o\underline{\xi}] \ , \quad \text{(A6)}$$

or

$$\underline{\nabla}_o \cdot \underline{v}_1' = \frac{\partial}{\partial t_o} (\underline{\nabla}_o \cdot \underline{\xi}) + \underline{v}_o \cdot \underline{\nabla}_o (\underline{\nabla}_o \cdot \underline{\xi}) + \underline{\nabla}_o \underline{\xi} : \underline{\nabla}_o \underline{v}_o \ . \tag{A7}$$

Substituting Eq. (A7) into Eq. (A5) and dividing by ρ_o .

$$\frac{d}{dt_o} \left(\frac{\rho_1'}{\rho_o} + (\underline{\nabla}_o \cdot \underline{\xi}) \right) = 0 \tag{A8}$$

or

$$\rho_1' = \rho_{1I}' - \rho_o (\underline{\nabla}_o \cdot \underline{\xi}) \ . \tag{A9}$$

Writing ρ_1' in terms of $\underline{\xi}$ now necessitates that we solve the generating equation for $\underline{\xi}$ also. In the Lagrangian formalism

$$\frac{d\underline{\xi}}{dt_o} = \underline{v}_1' \ . \tag{A10}$$

Thus we have, in the Lagrangian perturbation scheme, replaced Eq. (41) with the two equations, Eqs. (A9) and (A10).

Next, replace the Eulerian perturbation quantities in Eq. (41) by their Lagrangian counterparts and we obtain the following equation.

$$\begin{aligned} \rho_o \frac{d\underline{v}_1'}{dt_o} + \rho_o \Bigg[(\underline{\xi} \cdot \underline{\nabla}_o) \left(\frac{1}{\rho_o} \underline{\nabla}_o P_o \right) - (\underline{\xi} \cdot \underline{\nabla}_o) \cdot \underline{\nabla}_o \underline{v}_o - (\underline{v}_o \cdot \underline{\nabla}_o)(\underline{\xi} \cdot \underline{\nabla}_o \underline{v}_o) + \underline{v}_1' \cdot \underline{\nabla}_o \underline{v}_o \\ - \frac{\partial \underline{\xi}}{\partial t_o} \cdot \underline{\nabla}_o \underline{v}_o \Bigg] = - \underline{\nabla}_o P_1' + \frac{\rho_1'}{\rho_o} \underline{\nabla}_o P_o + \underline{\nabla}_o (\underline{\xi} \cdot \underline{\nabla}_o P_o) - (\underline{\xi} \cdot \underline{\nabla}_o \rho_o) \frac{1}{\rho_o} \underline{\nabla}_o P_o \ . \end{aligned} \tag{A11}$$

However,

$$-(\underline{\xi} \cdot \underline{\nabla}_o \underline{v}_o) \cdot \underline{\nabla}_o \underline{v}_o - (\underline{v}_o \cdot \underline{\nabla}_o)(\underline{\xi} \cdot \underline{\nabla}_o \underline{v}_o) = - (\underline{\xi} \cdot \underline{\nabla}_o)(\underline{v}_o \cdot \underline{\nabla}_o \underline{v}_o) - (\underline{v}_o \cdot \underline{\nabla}_o \underline{\xi}) \cdot \underline{\nabla}_o \underline{v}_o \ . \tag{A12}$$

Substituting Eq. (A12) and using Eq. (A10) yields, after collecting terms,

$$\rho_o \frac{d\underline{v}_1'}{dt_o} = - \underline{\nabla}_o P_1' + \frac{\rho_1'}{\rho_o} \underline{\nabla}_o P_o + \underline{\nabla}_o (\underline{\xi} \cdot \underline{\nabla}_o P_o) - (\underline{\xi} \cdot \underline{\nabla}_o) \underline{\nabla}_o P_o \ . \tag{A13}$$

The last two terms combine as follows:

$$\underline{\nabla}_o(\underline{\xi}\cdot\underline{\nabla}_o P_o) - (\underline{\xi}\cdot\underline{\nabla}_o)\underline{\nabla}_o P_o = \underline{\nabla}_o\underline{\xi}\cdot\underline{\nabla}_o P_o \quad , \tag{A14}$$

which finally produces the Lagrangian perturbed momentum equation,

$$\rho_o \frac{d\underline{v}_1'}{dt_o} = -\underline{\nabla}_o P' + \frac{\rho_1'}{\rho_o}\underline{\nabla}_o P_o + \underline{\nabla}_o\underline{\xi}\cdot\underline{\nabla}_o P_o \quad . \tag{A15}$$

Before transforming the energy equation, Eq. (42), we immediately see how the given expressions, Eqs. (43) through (47), transform. Each is of the form

$$\psi_1 = \left(\frac{\partial\psi}{\partial T}\right)_{\rho_o} T_1 + \left(\frac{\partial\psi}{\partial\rho}\right)_{To} \rho_1 \quad . \tag{A16}$$

Once again, using Eq. (22) yields

$$\psi_1' - \underline{\xi}\cdot\underline{\nabla}_o\psi_o = \left(\frac{\partial\psi}{\partial T}\right)_{\rho o}(T_1' - \underline{\xi}\cdot\underline{\nabla}_o T_o) + \left(\frac{\partial\psi}{\partial\rho}\right)_{To}(\rho_1' - \underline{\xi}\cdot\underline{\nabla}_o\rho_o) \quad . \tag{A17}$$

However, if $\psi_o = \psi_o(\rho_o, T_o)$, then

$$\underline{\nabla}_o\psi_o = \left(\frac{\partial\psi}{\partial T}\right)_{\rho o}\underline{\nabla}_o T_o + \left(\frac{\partial\psi}{\partial\rho}\right)_{To}\underline{\nabla}_o\rho_o \quad , \tag{A18}$$

and

$$\underline{\xi}\cdot\underline{\nabla}_o\psi_o = \left(\frac{\partial\psi}{\partial T}\right)_{\rho o}(\underline{\xi}\cdot\underline{\nabla}_o T_o) + \left(\frac{\partial\psi}{\partial\rho}\right)_{To}(\underline{\xi}\cdot\underline{\nabla}_o\rho_o) \quad . \tag{A19}$$

Substituting Eq. (A19) into Eq. (A17), we find that

$$\psi_1' = \left(\frac{\partial\psi}{\partial T}\right)_{\rho o} T_1' + \left(\frac{\partial\psi}{\partial\rho}\right)_{To}\rho_1' \quad . \tag{A20}$$

Therefore, the Lagrangian equations corresponding to the Eulerian relationships Eqs. (43) through (47), are obtained by replacing all Eulerian perturbation quantities by their Lagrangian counterparts. The procedure yields

$$P_1' = \left(\frac{\partial P}{\partial T}\right)_{\rho o} T_1' + \left(\frac{\partial P}{\partial \rho}\right)_{To} \rho_1' \quad , \tag{A21}$$

$$\kappa_1' = \left(\frac{\partial \kappa}{\partial T}\right)_{\rho o} T_1' + \left(\frac{\partial \kappa}{\partial \rho}\right)_{To} \rho_1' \quad , \tag{A22}$$

$$\dot{q}_1' = \left(\frac{\partial \dot{q}}{\partial T}\right)_{\rho o} T_1' + \left(\frac{\partial \dot{q}}{\partial \rho}\right)_{To} \rho_1' \quad , \tag{A23}$$

$$\left(\frac{\partial \varepsilon}{\partial T}\right)_{\rho 1}' = \left(\frac{\partial}{\partial T}\left(\frac{\partial \varepsilon}{\partial T}\right)_\rho\right)_{\rho o} T_1' + \left(\frac{\partial}{\partial \rho}\left(\frac{\partial \varepsilon}{\partial T}\right)_\rho\right)_{To} \rho_1' \quad , \tag{A24}$$

and

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{T1}' = \left(\frac{\partial}{\partial T}\left(\frac{\partial \varepsilon}{\partial \rho}\right)_T\right)_{\rho o} T_1' + \left(\frac{\partial}{\partial \rho}\left(\frac{\partial \varepsilon}{\partial \rho}\right)_T\right)_{To} \rho_1' . \tag{A25}$$

At this point we turn our attention to the transformation of the perturbed Eulerian energy equation, Eq. (42). Transforming the Eulerian perturbed variables as before yields

$$\left(\frac{\partial \varepsilon}{\partial T}\right)_{\rho o}\left(\frac{dT_1'}{dt_o} - \frac{d}{dt_o}(\underline{\xi}\cdot\underline{\nabla}_o T_o) + \underline{v}_1'\cdot\underline{\nabla}_o T_o - (\underline{\xi}\cdot\underline{\nabla}_o\underline{v}_o)\cdot\underline{\nabla}_o T_o\right) + \left(\left(\frac{\partial \varepsilon}{\partial T}\right)_{\rho 1}'\right.$$

$$\left. - \underline{\xi}\cdot\underline{\nabla}_o\left(\frac{\partial \varepsilon}{\partial T}\right)_{\rho o}\right)\frac{dT_o}{dt_o} = \left(\frac{P_o}{\rho_o^2} - \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{To}\right)\left(\frac{d\rho_1'}{dt_o} - \frac{d}{dt_o}(\underline{\xi}\cdot\underline{\nabla}_o\rho_o) + \underline{v}_1'\cdot\underline{\nabla}_o\rho_o\right.$$

$$\left. -(\underline{\xi}\cdot\underline{\nabla}_o\underline{v}_o)\cdot\underline{\nabla}_o\rho_o\right) + \left(\frac{P_1'}{\rho_o^2} - \frac{\underline{\xi}\cdot\underline{\nabla}_o P_o}{\rho_o^2} - 2\frac{P_o}{\rho_o^3}\rho_1' + 2\frac{P_o}{\rho_o^3}(\underline{\xi}\cdot\underline{\nabla}_o\rho_o)\right.$$

$$\left. - \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{T1}' + (\underline{\xi}\cdot\underline{\nabla}_o)\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{To}\right)\frac{d\rho_o}{dt_o} + \frac{1}{\rho_o}\underline{\nabla}_o\cdot(\kappa_o\underline{\nabla}_o T_1' + \kappa_1'\underline{\nabla}_o T_o)$$

$$- \frac{1}{\rho_o} \underline{\nabla}_o \cdot [\kappa_o \underline{\nabla}_o (\underline{\xi}\cdot\underline{\nabla}_o T_o) + (\underline{\xi}\cdot\underline{\nabla}_o \kappa_o)\underline{\nabla}_o T_o] - \frac{\rho_1'}{\rho_o^2}\underline{\nabla}_o\cdot\kappa_o\underline{\nabla}_o T_o$$

$$+ \frac{(\underline{\xi}\cdot\underline{\nabla}_o\rho_o)}{\rho_o^2}\underline{\nabla}_o\cdot\kappa_o\underline{\nabla}_o T_o + \frac{\dot{q}_1'}{\rho_o} - \frac{(\underline{\xi}\cdot\underline{\nabla}_o \dot{q}_o)}{\rho_o} - \frac{\rho_1'}{\rho_o^2}\dot{q}_o \qquad \text{(A26)}$$

$$+ \frac{(\underline{\xi}\cdot\underline{\nabla}_o\rho_o)}{\rho_o^2}\dot{q}_o \quad .$$

It can easily be shown that

$$\frac{d}{dt_o}(\underline{\xi}\cdot\underline{\nabla}_o\psi_o) = (\underline{\xi}\cdot\underline{\nabla}_o)\frac{d\psi_o}{dt_o} + \frac{d\underline{\xi}}{dt_o}\cdot\underline{\nabla}_o\psi_o - (\underline{\xi}\cdot\underline{\nabla}_o\underline{V}_o)\cdot\underline{\nabla}_o\psi_o \quad , \qquad \text{(A27)}$$

and

$$\frac{-1}{\rho_o}\underline{\nabla}_o\cdot[\kappa_o\underline{\nabla}_o(\underline{\xi}\cdot\underline{\nabla}_o T_o) + (\underline{\xi}\cdot\underline{\nabla}_o\kappa_o)\underline{\nabla}_o T_o] = \frac{-1}{\rho_o}(\underline{\xi}\cdot\underline{\nabla}_o)[\underline{\nabla}_o\cdot\kappa_o\underline{\nabla}_o T_o]$$

$$\frac{-1}{\rho_o}[\underline{\nabla}_o\cdot(\underline{\nabla}_o\underline{\xi}\cdot\kappa_o\underline{\nabla}_o T_o) \qquad \text{(A28)}$$

$$+ \underline{\nabla}_o\underline{\xi} : \underline{\nabla}_o(\kappa_o\underline{\nabla}_o T_o)] \quad .$$

Substituting Eqs. (A27) and (A28) into (A26), and making use of the zeroth-order Eq. (35) and Eq. (A10) yields, after collecting terms, the perturbed Lagrangian energy equation

$$\left(\frac{\partial\varepsilon}{\partial T}\right)_{\rho o}\frac{dT_1'}{dt_o} + \left(\frac{\partial\varepsilon}{\partial T}\right)_{\rho_1}\frac{dT_o}{dt_o} = \left[\frac{P_o}{\rho_o^2} - \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{To}\right]\frac{d\rho_1'}{dt_o} + \left[\frac{P_1'}{\rho_o^2} - 2\frac{P_o}{\rho_o^3}\rho_1' - \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{T1}'\right]\frac{d\rho_o}{dt_o}$$

$$+ \frac{1}{\rho_o}\underline{\nabla}_o\cdot[\kappa_o\underline{\nabla}_o T_1' + \kappa_1'\underline{\nabla}_o T_o] - \frac{\rho_1'}{\rho_o^2}\underline{\nabla}_o\cdot\kappa_o\underline{\nabla}_o T_o + \frac{1}{\rho_o}\dot{q}_1'$$

$$- \frac{\rho_1'}{\rho_o^2}\dot{q}_o - \frac{1}{\rho_o}[\underline{\nabla}_o\cdot(\underline{\nabla}_o\underline{\xi}\cdot\kappa_o\underline{\nabla}_o T_o) + \underline{\nabla}_o\underline{\xi}:\nabla_o\kappa_o\underline{\nabla}_o T_o)] . \text{(A29)}$$

We now have a complete set of Lagrangian perturbed fluid equations, which are displayed in Table IV.

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